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FACTORIZATION OF THE CANONICAL BASES FOR HIGHEST WEIGHT MODULES IN AFFINE TYPE A

NICOLAS JACON AND CÉDRIC LECOUVEY

Abstract. We show that the canonical basis associated to any highest weight $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module can be decomposed on the canonical basis of its corresponding $\mathcal{U}_v(\mathfrak{sl}_\infty)$-module. We establish that the transition matrix associated to this decomposition is unitriangular with coefficients in $\mathbb{Z}[v]$ and give a procedure to compute them. We conjecture these coefficients are in fact in $\mathbb{N}[v]$. This provides a natural quantization of a theorem by Geck and Rouquier on the factorization of decomposition matrices associated to Ariki-Koike algebras.

1. Introduction

In the classification of finite complex reflection groups by Shephard and Todd [22], there is a single infinite family of groups $G(lp,p,n)$ parametrized by the triplets $(l,p,n)\in\mathbb{N}^3$ and 34 other "exceptional" groups. If $p=1$, the group $G(l,1,n)$ is in fact the wreath product of the cyclic group of order l with the symmetric group $\mathfrak{S}_n$. It generalizes both the Weyl group of type A (corresponding to the case $l=1$) and the Weyl group of type B_n ($l=2$). To $G(l,1,n)$ is associated its Hecke algebra over the ring $A:=\mathbb{C}[q^{\pm1},Q_0^{\pm1},\dots,Q_{l-1}^{\pm1}]$ where $(q,Q_0,\dots,Q_{l-1})$ is an $l+1$-tuple of indeterminates. This algebra can be seen as a deformation of the group algebra of $G(l,1,n)$ and is connected to the modular representation theory of finite reductive groups (see for example the survey [21]). As an A-algebra, it has a set of generators $\{T_0,\dots,T_{n-1}\}$ with corresponding defining relations

$$\prod_{i=0}^{l-1}(T_0-Q_i)=0,\ (T_i-q)(T_i+1)=0\ i=1,\dots,n-1.$$

We denote by $\mathcal{H}_A$ this algebra. An interesting and deep problem concerns the representation theory of the Hecke algebras $\mathcal{H}_A$. First, if we consider the field of fractions $K=\mathbb{C}(q,Q_0,\dots,Q_{l-1})$ of A and if we extend the scalars of $\mathcal{H}_A$ to K, we obtain the algebra $\mathcal{H}_K:=K\otimes_A\mathcal{H}_A$ whose representation theory is well understood. For example, we know how to classify the irreducible representations, what are their dimensions etc. The theory is far more difficult in the modular case. Let $\theta:A\to\mathbb{C}$ be a ring homomorphism and let $\mathcal{H}_\mathbb{C}:=\mathbb{C}\otimes_A\mathcal{H}_A$ be the associated Hecke algebra. Due to results of Dipper and Mathas [6], one can in fact reduce to the case where $\theta(q)=\eta_e:=\exp(\frac{2i\pi}{e})$ is a e-th root of 1 ($e\in\mathbb{Z}_{\geq2}$) and there exists an l-tuple $(s_0,\dots,s_{l-1})\in\mathbb{Z}^l$ such that $\theta(Q_j)=\eta_e^{s_j}$. To this datum corresponds a decomposition map, which controls the representation theory of $\mathcal{H}_\mathbb{C}$. As $\mathcal{H}_A$ is a cellular algebra [12], this map is easily defined as follows. Let $V_K\in\mathrm{Irr}(\mathcal{H}_K)$. Then there exists an $\mathcal{H}_A$-module V_A such that $V_K=K\otimes_A V_A$. We can then associate to V_A an $\mathcal{H}_\mathbb{C}$-module $V_\mathbb{C}$. Following [10], we obtain a well defined map at the level of

Grothendieck groups $R_0(\mathcal{H}_K)$ (resp. $R_0(\mathcal{H}_{\mathbb{C}})$) of finitely generated $\mathcal{H}_K$-modules (resp. $\mathcal{H}_{\mathbb{C}}$-modules):

$$\begin{array}{rccc} d_\theta : & R_0(\mathcal{H}_K) & \to & R_0(\mathcal{H}_{\mathbb{C}}) \\ & [V_K] & \mapsto & [V_{\mathbb{C}}] \end{array}$$

and a decomposition matrix D_e which encodes this map. Although there exists explicit algorithms to compute the map d_θ, it remains difficult to describe in general. However, a nice result by Geck and Rouquier gives informations on the matrix D_e by showing the decomposition map factorizes. Let $\theta^q : A \to \mathbb{C}(q)$ be the specialization defined by setting $\theta^q(Q_i) = q^{s_i}$ for $i = 0, \ldots, l-1$. Denote by $\mathcal{H}_{\mathbb{C}(q)} := \mathbb{C}(q) \otimes_A \mathcal{H}_A$ the associated Hecke algebra. As above, we have a decomposition map

$$d_{\theta^q} : R_0(\mathcal{H}_K) \to R_0(\mathcal{H}_{\mathbb{C}(q)})$$

and an associated decomposition matrix D_∞.

Theorem 1.1 (Geck-Rouquier)**.** *There exists a unique additive map*

$$d^\theta_{\theta^q} \colon R_0(\mathcal{H}_{\mathbb{C}(q)}) \to R_0(\mathcal{H}_{\mathbb{C}})$$

such that the following diagram commutes:

$$\begin{tikzcd} R_0(\mathcal{H}_K) \arrow[rr, "d_\theta"] \arrow[dr, "d_{\theta^q}"'] & & R_0(\mathcal{H}_{\mathbb{C}}) \\ & R_0(\mathcal{H}_{\mathbb{C}(q)}) \arrow[ur, "d^\theta_{\theta^q}"'] & \end{tikzcd}$$

Thus, we also have a factorization of the corresponding decomposition matrices as $D_e = D_\infty.D^e_\infty$ where D^e_∞ is a matrix whose entries are non-negative integers.

In some sense, this result shows that a part of the representation theory of $\mathcal{H}_{\mathbb{C}}$ does not depend on e but only on the representation theory of $\mathcal{H}_{\mathbb{C}(q)}$, which is "easier" to understand (for example, there is closed formulae for the entries of D_∞ when $l = 2$ [20]). This has been in particular recently used to make explicit the connections between the various classifications of simple modules arising from the theory of canonical basic sets in type B_n [11].

As noted above, there is an algorithm for computing the decomposition matrices of $\mathcal{H}_{\mathbb{C}}$ and $\mathcal{H}_{\mathbb{C}(q)}$. This algorithm relies on Ariki's proof [2] of the Lascoux-Leclerc-Thibon conjecture [19]. Ariki's theorem asserts that the D_e (resp. D_∞) is equal to the evaluation at $v = 1$ of the matrix $D_e(v)$ (resp. $D_\infty(v)$) of the canonical basis corresponding to a highest weight $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module (resp. $\mathcal{U}_v(\mathfrak{sl}_\infty)$-module). Theorem 1.1 thus implies the existence of a matrix D^e_∞ such that $D_e(1) = D_\infty(1).D^e_\infty$. It is then natural to ask whether the matrix D^e_∞ has also a natural quantization. Namely, is there a matrix $D^e_\infty(v)$ such that $D_e(v) = D_\infty(v).D^e_\infty(v)$? In other words, does the matrix of the canonical basis for $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-modules factorize through the matrix of the canonical basis for $\mathcal{U}_v(\mathfrak{sl}_\infty)$-modules ? If so, which interpretation(s) can be given of $D^e_\infty(v)$ and is there an easy way to compute it ?

In this note, we answer positively to the above questions. We show the existence of the a matrix $D^e_\infty(v)$ with entries in $\mathbb{Z}[v]$ compatible with the above factorization. We also provide an algorithm for computing $D^e_\infty(v)$ (and thus an algorithm for computing D^e_∞). In addition, we conjecture that the coefficients of $D^e_\infty(v)$ are in $\mathbb{N}[v]$. These results follow from strong connections between the representation

theories of $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ and $\mathcal{U}_v(\mathfrak{sl}_\infty)$. Note that a similar phenomenon has already been observed in [17] at the level of crystal graphs. Observe also that recent conjectures and results suggest that the matrix $D_e(v)$ might appeas in the context of the representation theory of Hecke algebras [3], [4], [5]. It should be interesting to give an interpretation of the factorization of $D_e(v)$ at this level.

2. Background on canonical bases

We refer to [18] and to [2] for a detailed review on canonical and crystal basis theory. [8, §7] also gives a nice survey on some of the problems we will consider. Consider v an indeterminate and $e > 1$ an integer. Let $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ be the quantum group of affine type $A^{(1)}_{e-1}$. This is an associative $\mathbb{Q}(v)$-algebra with generators e_i, f_i, t_i, t_i^{-1}, $i \in \mathbb{Z}/e\mathbb{Z}$ and ∂. We refer to [23, §2.1] for a complete presentation. The bar involution $\overline{}$ is the ring automorphism of $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ such that $\overline{v} = v^{-1}, \overline{\partial} = \partial$ and for any $i \in \mathbb{Z}/e\mathbb{Z}$

$$\overline{e_i} = e_i, \quad \overline{t_i} = t_i \text{ and } \overline{t_i} = t_i.$$

We denote by $\mathcal{U}'_v(\widehat{\mathfrak{sl}_e})$ the subalgebra generated by e_i, f_i, t_i, t_i^{-1}, $i \in \mathbb{Z}/e\mathbb{Z}$. By slightly abuse of notation, we identify the elements of $\mathbb{Z}/e\mathbb{Z}$ with their corresponding labels in $\{0, \dots, e-1\}$ when there is no risk of confusion. Write $\{\Lambda_0, \dots, \Lambda_{e-1}, \delta\}$ for the set of fundamental weights of $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$. Let $l \in \mathbb{Z}_{\geq 1}$ and consider $\mathbf{s} = (s_0, \dots, s_{l-1}) \in \mathbb{Z}^l$. We set $\mathfrak{s} = (s_0 (\text{mod } e), \dots, s_{l-1} (\text{mod } e)) \in (\mathbb{Z}/e\mathbb{Z})^l$ and $\Lambda_{\mathfrak{s}} := \Lambda_{s_0 (\text{mod } e)} + \dots + \Lambda_{s_{l-1} (\text{mod } e)}$. We denote by $V_e(\mathbf{s})$ the $\mathcal{U}'_v(\widehat{\mathfrak{sl}_e})$-highest weight module of weight $\Lambda_{\mathfrak{s}}$.

Similarly, let $\mathcal{U}_v(\mathfrak{sl}_\infty)$ be the quantum group of type A_∞. This is an associative $\mathbb{Q}(v)$-algebra with generators E_j, F_j, T_j, T_j^{-1}, $j \in \mathbb{Z}$. We also denote by $\overline{}$ the ring automorphism of $\mathcal{U}_v(\mathfrak{sl}_\infty)$ such that $\overline{v} = v^{-1}$ and for any $j \in \mathbb{Z}$

$$\overline{E_j} = E_j, \quad \overline{T_j} = T_j \text{ and } \overline{T_j} = T_j.$$

Write $\{\omega_j, j \in \mathbb{Z}\}$ for its set of fundamental weights. To $\mathbf{s} = (s_0, \dots, s_{l-1}) \in \mathbb{Z}^l$, we associate the dominant weight $\Lambda_{\mathbf{s}} := \omega_{s_0} + \dots + \omega_{s_{l-1}}$. We denote by $V_\infty(\mathbf{s})$ the $\mathcal{U}_v(\mathfrak{sl}_\infty)$-highest weight module of weight $\Lambda_{\mathbf{s}}$.

2.1. Fock spaces.

Let $\Pi_{l,n}$ be the set of l-partitions with rank n, that is the set of sequences $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(l-1)})$ of partitions such that $\left|\lambda^{(0)}\right| + \dots + \left|\lambda^{(l-1)}\right| = n$. Set $\Pi_l = \cup_{n \geq 0} \Pi_{l,n}$. As a $\mathbb{C}(v)$-vector space, the Fock space $\mathfrak{F}$ of level l admits the set of all l-partitions as a natural basis. Namely the underlying vector space is

$$\mathfrak{F} = \bigoplus_{n \geq 0} \bigoplus_{\boldsymbol{\lambda} \in \Pi_{l,n}} \mathbb{Q}(v) \boldsymbol{\lambda}.$$

The Fock space $\mathfrak{F}$ can be endowed with a structure of $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ and $\mathcal{U}_v(\mathfrak{sl}_\infty)$-modules. Let $\boldsymbol{\lambda}$ be an l-partition (identified with its Young diagram). Then, the nodes of $\boldsymbol{\lambda}$ are the triplets $\gamma = (a, b, c)$ where $c \in \{0, \dots, l-1\}$ and a, b are respectively the row and column indices of the node γ in $\lambda^{(c)}$. The content of γ is the integer $c(\gamma) = b - a + s_c$ and the residue $\text{res}(\gamma)$ of γ is the element of $\mathbb{Z}/e\mathbb{Z}$ such that

$$\text{res}(\gamma) \equiv c(\gamma) (\text{mod } e). \tag{1}$$

Observe that the partition $\lambda^{(c)}$ contains at most one node with fixed content $c \in \mathbb{Z}$. For any $i \in \mathbb{Z}/e\mathbb{Z}$, we will say that γ is an i-node of $\boldsymbol{\lambda}$ when $\text{res}(\gamma) \equiv i (\text{mod } e)$. For any $j \in \mathbb{Z}$, γ is a j-node of $\boldsymbol{\lambda}$ when $c(\gamma) = j$. The node γ is removable

when $\gamma = (a, b, c) \in \boldsymbol{\lambda}$ and $\boldsymbol{\lambda}\backslash\{\gamma\}$ is an l-partition. Similarly γ is addable when $\gamma = (a, b, c) \notin \boldsymbol{\lambda}$ and $\boldsymbol{\lambda} \cup \{\gamma\}$ is an l-partition.

Let $i \in \mathbb{Z}/e\mathbb{Z}$. We define a total order on the set of i-nodes of $\boldsymbol{\lambda}$. Consider $\gamma_1 = (a_1, b_1, c_1)$ and $\gamma_2 = (a_2, b_2, c_2)$ two nodes in $\boldsymbol{\lambda}$. We set

$$\gamma_1 \prec_{\mathbf{s}} \gamma_2 \Longleftrightarrow \begin{cases} c(\gamma_1) < c(\gamma_2) \text{ or} \\ c(\gamma_1) = c(\gamma_2) \text{ and } c_1 > c_2. \end{cases}$$

Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be two l-partitions of rank n and $n+1$ such that $[\boldsymbol{\mu}] = [\boldsymbol{\lambda}] \cup \{\gamma\}$ where γ is an i-node. Define

$$\begin{aligned} N_i^{\succ}(\boldsymbol{\lambda}, \boldsymbol{\mu}) =& \sharp\{\text{addable } i\text{-nodes } \gamma \text{ of } \boldsymbol{\lambda} \text{ such that } \gamma' \succ_{\mathbf{s}} \gamma\} \\ &- \sharp\{\text{removable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\mu} \text{ such that } \gamma' \succ_{\mathbf{s}} \gamma\}, \qquad (2) \\ N_i^{\prec}(\boldsymbol{\lambda}, \boldsymbol{\mu}) =& \sharp\{\text{addable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\lambda} \text{ such that } \gamma' \prec_{\mathbf{s}} \gamma\} \\ &- \sharp\{\text{removable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\mu} \text{ such that } \gamma' \prec_{\mathbf{s}} \gamma\}, \qquad (3) \\ N_i(\boldsymbol{\lambda}) =& \sharp\{\text{addable } i\text{-nodes of } \boldsymbol{\lambda}\} \\ &- \sharp\{\text{removable } i\text{-nodes of } \boldsymbol{\lambda}\} \text{ for} \\ \text{and } M_0(\boldsymbol{\lambda}) =& \sharp\{0(\text{mod e-nodes of } \boldsymbol{\lambda}\}. \end{aligned}$$

Theorem 2.1. [16] *Let $\mathbf{s} \in \mathbb{Z}^l$. The Fock space $\mathcal{F}$ has a structure of an integrable $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module $\mathcal{F}_e^{\mathbf{s}}$ defined by*

$$e_i\boldsymbol{\lambda} = \sum_{res([\boldsymbol{\lambda}]/[\boldsymbol{\mu}])=i} v^{-N_i^{\succ}(\boldsymbol{\mu},\boldsymbol{\lambda})}\boldsymbol{\mu}, \qquad f_i\boldsymbol{\lambda} = \sum_{res([\boldsymbol{\mu}]/[\boldsymbol{\lambda}])=i} v^{N_i^{\prec}(\boldsymbol{\lambda},\boldsymbol{\mu})}\boldsymbol{\mu},$$

$$t_i\boldsymbol{\lambda} = v^{N_i(\boldsymbol{\lambda})}\boldsymbol{\lambda}, \qquad \partial\boldsymbol{\lambda} = -(\Delta + M_0(\boldsymbol{\lambda}))\boldsymbol{\lambda}, \qquad (i \in \mathbb{Z}/e\mathbb{Z})$$

where Δ is a rational number defined in [16, Thm 2.1]. *The module structure on $\mathcal{F}_e^{\mathbf{s}}$ depends on $\mathbf{s}$ and e.*

The Fock space $\mathcal{F}$ also admits the structure of a $\mathcal{U}_v'(\widehat{\mathfrak{sl}_e})$-module that we will also denote $\mathcal{F}_e^{\mathbf{s}}$ by a slight abuse of notation.

Let $j \in \mathbb{Z}$. Now assume $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are l-partitions of rank n and $n+1$ such that $[\boldsymbol{\mu}] = [\boldsymbol{\lambda}] \cup \{\gamma\}$ where γ is a j-node. We define $N_j^{\succ}(\boldsymbol{\lambda}, \boldsymbol{\mu})$, $N_j^{\prec}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ and $N_j(\boldsymbol{\lambda})$ as in (2) except we consider j-nodes rather than i-nodes.

Theorem 2.2. [16] *Let $\mathbf{s} \in \mathbb{Z}^l$. The Fock space $\mathcal{F}$ has a structure of an integrable $\mathcal{U}_v(\mathfrak{sl}_\infty)$-module $\mathcal{F}_\infty^{\mathbf{s}}$ defined by*

$$E_j\boldsymbol{\lambda} = \sum_{c([\boldsymbol{\lambda}]/[\boldsymbol{\mu}])=j} v^{-N_j^{\succ}(\boldsymbol{\mu},\boldsymbol{\lambda})}\boldsymbol{\mu}, \qquad F_j\boldsymbol{\lambda} = \sum_{c([\boldsymbol{\mu}]/[\boldsymbol{\lambda}])=j} v^{N_j^{\prec}(\boldsymbol{\lambda},\boldsymbol{\mu})}\boldsymbol{\mu},$$

$$T_j\boldsymbol{\lambda} = v^{N_j(\boldsymbol{\lambda})}\boldsymbol{\lambda}, \qquad j \in \mathbb{Z}.$$

The module structure on $\mathcal{F}_e^{\mathbf{s}}$ depends on e.

From the above theorems, one can easily derive the following crucial result which is implicit in [16, Prop 3.5].

Proposition 2.3. *The module structures $\mathcal{F}_e^{\mathbf{s}}$ and $\mathcal{F}_\infty^{\mathbf{s}}$ are compatible. More precisely for any l-partition $\boldsymbol{\lambda}$, we have for any $i \in \mathbb{Z}/e\mathbb{Z}$*

$$\begin{aligned} e_i\boldsymbol{\lambda} &= \sum_{j\in\mathbb{Z}, j\equiv i (mod\ e)} \left(\prod_{r\geq 1} T_{j+re}^{-1}\right) E_j\ \boldsymbol{\lambda}, \\ f_i\boldsymbol{\lambda} &= \sum_{j\in\mathbb{Z}, j\equiv i (mod\ e)} \left(\prod_{r\geq 1} T_{j-re}\right) F_j\ \boldsymbol{\lambda}, \\ t_i\boldsymbol{\lambda} &= \prod_{j\in\mathbb{Z}, j\equiv i (mod\ e)} T_j\ \boldsymbol{\lambda} \end{aligned}$$

Remark 2.4. The infinite sums and products in the previous proposition reduce in fact to finite ones since the number of nodes in $\boldsymbol{\lambda}$ is finite.

The empty multipartition $\emptyset$ is a highest weight vector in $\mathcal{F}_e^{\mathbf{s}}$ and $\mathcal{F}_\infty^{\mathbf{s}}$ of weight $\Lambda_{\mathbf{s}}$ and $\Lambda_{\mathbf{s}}$, respectively. We will identify $V_e(\mathbf{s})$ and $V_\infty(\mathbf{s})$ with $\mathcal{U}'_v(\widehat{\mathfrak{sl}_e}).\emptyset$ and $\mathcal{U}_v(\mathfrak{sl}_\infty).\emptyset$, respectively. Thanks to this identification, it follows from the previous proposition that $V_\infty(\mathbf{s})$ is endowed with the structure of a $\mathcal{U}'_v(\widehat{\mathfrak{sl}_e})$-module. Moreover $V_e(\mathbf{s})$ then coincides with the $\mathcal{U}'_v(\widehat{\mathfrak{sl}_e})$-irreducible component of $V_\infty(\mathbf{s})$ with highest weight vector $\emptyset$.

2.2. **Uglov's canonical bases.** We now briefly recall Uglov's construction of canonical basis of the Fock spaces. Let $\mathbb{A}(v)$ be the ring of rational functions without pole at $v = 0$. Set

$$\begin{aligned} \mathcal{L} &:= \bigoplus_{n\geq 0} \bigoplus_{\boldsymbol{\lambda}\in\Pi_{l,n}} \mathbb{A}(v)\boldsymbol{\lambda} \text{ and} \\ \mathcal{B} &:= \{\boldsymbol{\lambda} \ (\text{mod } v\mathcal{L}) \mid \boldsymbol{\lambda} \in \Pi_l\}. \end{aligned}$$

Theorem 2.5. *The pair $(\mathcal{L}, \mathcal{B})$ is a crystal basis for $\mathcal{F}_e^{\mathbf{s}}$ and $\mathcal{F}_\infty^{\mathbf{s}}$.*

Observe that the crystal basis of the Fock space is the same for $\mathcal{F}_e^{\mathbf{s}}$ and $\mathcal{F}_\infty^{\mathbf{s}}$. Nevertheless, the crystal structures $\mathcal{B}_e$ and $\mathcal{B}_\infty$ on $\mathcal{B}$ do not coincide for $\mathcal{F}_e^{\mathbf{s}}$ and $\mathcal{F}_\infty^{\mathbf{s}}$. The crystal structure $\mathcal{B}_e$ is obtained as follows. Starting from any l-partition $\boldsymbol{\lambda}$ and any $i \in \mathbb{Z}/e\mathbb{Z}$, consider its set of addable and removable i-nodes. Let w_i be the word obtained first by writing the addable and removable i-nodes of $\boldsymbol{\lambda}$ in increasing order with respect to $\prec_{\mathbf{s}}$ next by encoding each addable i-node by the letter A and each removable i-node by the letter R. Write $\widetilde{w}_i = A^p R^q$ for the word derived from w_i by deleting as many of the factors RA as possible. If $p > 0$, let γ be the rightmost addable i-node in $\widetilde{w}_i$. When $\widetilde{w}_i \neq \emptyset$, the node γ is called the good i-node. Now, the crystal $\mathcal{B}_e$ is the graph with

- vertices: the l-partitions,
- edges: $\boldsymbol{\lambda} \xrightarrow{i} \boldsymbol{\mu}$ if and only if $\boldsymbol{\mu}$ is obtained by adding to $\boldsymbol{\lambda}$ a good i-node.

The connected component $B_e(\mathbf{s})$ of $\mathcal{B}_e$ with highest weight vertex $\emptyset$ will be identify with the crystal graph of $V_e(\mathbf{s})$.
The graph structure $\mathcal{B}_\infty$ is obtained similarly by using j-nodes ($j \in \mathbb{Z}$) instead of i-nodes. We identify the crystal graph of $V_\infty(\mathbf{s})$ with $B_\infty(\mathbf{s})$, the connected component of $\mathcal{B}_\infty$ with highest weight vertex $\emptyset$.

We now consider the $\mathbb{Z}$-form of $\mathcal{L}$

$$\mathcal{L}_{\mathbb{Z}} := \bigoplus_{n\geq 0} \bigoplus_{\boldsymbol{\lambda}\in\Pi_{l,n}} \mathbb{Z}[v]\boldsymbol{\lambda}.$$

In [23], Uglov introduces an involution $\overline{}$ on $\mathcal{F}_e^{\mathbf{s}}$ verifying

$$\overline{u.f} = \overline{u}.\overline{f} \text{ for any } u \in \mathcal{U}_v'(\widehat{\mathfrak{sl}_e}) \text{ and any } f \in \mathcal{F}_e^{\mathbf{s}}.$$

Similarly, there exists an involution also denoted $\overline{}$ on $\mathcal{F}_\infty^{\mathbf{s}}$ verifying

$$\overline{u.f} = \overline{u}.\overline{f} \text{ for any } u \in \mathcal{U}_v(\mathfrak{sl}_\infty) \text{ and any } f \in \mathcal{F}_\infty^{\mathbf{s}}.$$

Despite the notation which could be confusing, the two bar involutions so defined on $\mathcal{F}^{\mathbf{s}}$ do not coincide.

Theorem 2.6. [23] *Consider* $\mathbf{s} \in \mathbb{Z}^l$ *and* $e \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. *There exists a unique base* $\mathcal{G}_e(\mathbf{s}) = \{G_e(\boldsymbol{\lambda}, \mathbf{s}) \mid \boldsymbol{\lambda} \in \Pi_l\}$ *of* $\mathcal{F}_e^{\mathbf{s}}$ *verifying the two following conditions*

(1) $\overline{G_e(\boldsymbol{\lambda}, \mathbf{s})} = G_e(\boldsymbol{\lambda}, \mathbf{s})$,
(2) $G_e(\boldsymbol{\lambda}, \mathbf{s}) = \boldsymbol{\lambda}$ *(mod* $v\mathcal{L}_{\mathbb{Z}}$*)*.

The basis $\mathcal{G}_e(\mathbf{s})$ is called the canonical basis of $\mathcal{F}_e^{\mathbf{s}}$. It strongly depends on $e \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. This terminology is justified by the following result of Uglov which permits to identify the Kashiwara canonical basis of $V_e(\mathbf{s})$ with the vectors of $\mathcal{G}_e(\mathbf{s})$ which belong to $V_e(\mathbf{s})$.

Theorem 2.7. [23] *Consider* $\mathbf{s} \in \mathbb{Z}^l$ *and* $e \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. *Set*

$$\mathcal{G}_e^\circ(\mathbf{s}) = \{G_e(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{G}_e(\mathbf{s}) \mid G_e(\boldsymbol{\lambda}, \mathbf{s}) \in V_e(\mathbf{s})\}.$$

Then $\mathcal{G}_e^\circ(\mathbf{s})$ *coincides with the canonical basis of the irreducible highest weight* $\mathcal{U}_v'(\widehat{\mathfrak{sl}_e})$*-module* $V_e(\mathbf{s})$. *Moreover* $G_e(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{G}_e^\circ(\mathbf{s})$ *if and only if* $\boldsymbol{\lambda} \in B_e(\mathbf{s})$.

3. Compatibility of canonical bases

In this section, we establish that for any $e \in \mathbb{Z}_{\geq 2}$ the canonical basis $\mathcal{G}_e^\circ(\mathbf{s})$ decomposes on the canonical basis $\mathcal{G}_\infty^\circ(\mathbf{s})$. We show that the transition matrix is unitriangular with coefficients in $\mathbb{Z}[v]$.

3.1. Decomposition property.

We first begin with the following proposition which is a consequence of Uglov's results on Fock spaces and Kashiwara's theory of canonical bases for highest weight modules.

Proposition 3.1. *Consider* $\mathbf{s} \in \mathbb{Z}^l$ *and* $e \in \mathbb{Z}_{\geq 2}$. *For any* $\boldsymbol{\lambda} \in B_e(\mathbf{s})$, *the canonical basis vector* $G_e(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{G}_e^\circ(\mathbf{s})$ *decomposes on the basis* $\mathcal{G}_\infty^\circ(\mathbf{s})$ *on the form*

$$G_e(\boldsymbol{\lambda}, \mathbf{s}) = \sum_{\boldsymbol{\mu}\in B_\infty(\mathbf{s})} a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) G_\infty(\boldsymbol{\mu}, \mathbf{s}) \tag{4}$$

where $a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \in \mathbb{Q}[v]$.

Proof. As already observed, Proposition 2.3 implies that $V_e(\mathbf{s})$ can be regarded as the irreducible component with highest weight vector $\emptyset$ in the restriction of $V_e(\mathbf{s})$ from $\mathcal{U}_v(\mathfrak{sl}_\infty)$ to $\mathcal{U}_v'(\widehat{\mathfrak{sl}_e})$. Moreover, $\mathcal{G}_\infty(\mathbf{s})^\circ$ is a $\mathbb{Q}(v)$-basis of $V_\infty(\mathbf{s})$. This implies we can decompose $G_e(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{G}_e^\circ(\mathbf{s})$ on the form

$$G_e(\boldsymbol{\lambda}, \mathbf{s}) = \sum_{\boldsymbol{\mu}\in B_\infty(\mathbf{s})} a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) G_\infty(\boldsymbol{\mu}, \mathbf{s}) \tag{5}$$

where $a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \in \mathbb{Q}(v)$.
For any $e \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$, set

$$L_e(\mathbf{s}) := \bigoplus_{\boldsymbol{\lambda} \in B_e(\mathbf{s})} \mathbb{A}(v) G_e(\boldsymbol{\lambda}, \mathbf{s}).$$

It follows from Theorem 2.6 that the crystal lattice of the whole Fock space satisfies

$$\mathcal{L} = \bigoplus_{\boldsymbol{\lambda} \in \Pi_l} \mathbb{A}(v) G_e(\boldsymbol{\lambda}, \mathbf{s}).$$

By Theorem 2.7 and by unicity of the crystal basis of the irreducible highest weight module $V_e(\mathbf{s})$, $L_e(\mathbf{s})$ is the crystal lattice of $V_e(\mathbf{s})$. Moreover $L_e(\mathbf{s}) = \mathcal{L} \cap V_e(\mathbf{s})$ since for any multipartition $\boldsymbol{\lambda}$ we have the equivalence

$$G_e(\boldsymbol{\lambda}, \mathbf{s}) \in V_e(\mathbf{s}) \Longleftrightarrow \boldsymbol{\lambda} \in B_e(\mathbf{s}).$$

Now for any $e \in \mathbb{Z}_{\geq 2}$, $V_e(\mathbf{s})$ is a subspace of $V_\infty(\mathbf{s})$. This implies that $L_e(\mathbf{s}) = \mathcal{L} \cap V_e(\mathbf{s})$ is a $\mathbb{A}(v)$-sublattice of $L_\infty(\mathbf{s}) = \mathcal{L} \cap V_\infty(\mathbf{s})$. In particular (5) holds in $L_\infty(\mathbf{s})$ and thus $a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \in \mathbb{A}(v)$.

For any $e \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$, the $\mathbb{Q}$-form of $V_e(\mathbf{s})$ is defined as

$$V_{e,\mathbb{Q}}(\mathbf{s}) := \bigoplus \mathbb{Q}[v, v^{-1}] f_{i_1}^{(p_1)} \cdots f_{i_r}^{(p_r)} \cdot \emptyset$$

where the sum runs over all the products of divided powers of the Chevalley generators f_i's. By Theorem 2.7 and the unicity of Kashiwara's global basis in highest weight modules, we must have

$$V_{e,\mathbb{Q}}(\mathbf{s}) := \bigoplus_{\boldsymbol{\lambda} \in B_e(\mathbf{s})} \mathbb{Q}[v, v^{-1}] G_e(\boldsymbol{\lambda}, \mathbf{s}).$$

Set

$$\mathcal{L}_{\mathbb{Q}} := \bigoplus_{\boldsymbol{\lambda} \in \Pi_l} \mathbb{Q}[v, v^{-1}] \boldsymbol{\lambda}.$$

We deduce from Theorem 2.6 that

$$\mathcal{L}_{\mathbb{Q}} = \bigoplus_{\boldsymbol{\lambda} \in \Pi_l} \mathbb{Q}[v, v^{-1}] G_e(\boldsymbol{\lambda}, \mathbf{s}).$$

By using similarly Theorem 2.7, this imposes that $V_{e,\mathbb{Q}}(\mathbf{s}) = \mathcal{L}_{\mathbb{Q}} \cap V_e(\mathbf{s})$ for any $e \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. We thus obtain that $V_{e,\mathbb{Q}}(\mathbf{s})$, $e \in \mathbb{Z}_{\geq 2}$ is a $\mathbb{Q}[v, v^{-1}]$ sublattice of $V_{\infty,\mathbb{Q}}(\mathbf{s})$. In particular (5) holds in $V_{\infty,\mathbb{Q}}(\mathbf{s})$ and thus $a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \in \mathbb{Q}[v, v^{-1}]$. Since we have already established that $a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ belong to $\mathbb{A}(v)$, we obtain $a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \in \mathbb{Q}[v]$. □

For each $\boldsymbol{\lambda} \in B_e(\mathbf{s})$, there exist polynomials $d^e_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \in \mathbb{Z}[v]$ (with $\boldsymbol{\mu} \in \Pi_{l,n}$) such that

$$G_e(\boldsymbol{\lambda}, \mathbf{s}) = \sum_{\boldsymbol{\mu} \in \Pi_{l,n}} d^e_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \boldsymbol{\mu}$$

Similarly, for each $\boldsymbol{\lambda} \in B_\infty(\mathbf{s})$, there exist polynomials $d^\infty_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \in \mathbb{Z}[v]$ (with $\boldsymbol{\mu} \in \Pi_{l,n}$) such that

$$G_\infty(\boldsymbol{\lambda}, \mathbf{s}) = \sum_{\boldsymbol{\mu} \in \Pi_{l,n}} d^\infty_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \boldsymbol{\mu}$$

If we denote

$$D_e(v) = (d^e_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v))_{\boldsymbol{\lambda} \in B_e(\mathbf{s}), \boldsymbol{\mu} \in \Pi_{l,n}} \text{ and } D_\infty(v) = (d^\infty_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v))_{\boldsymbol{\lambda} \in B_\infty(\mathbf{s}), \boldsymbol{\mu} \in \Pi_{l,n}}$$

The above theorem asserts that there exists a matrix

$$D_e^{\infty}(v) = (a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v))_{\boldsymbol{\lambda}\in B_e(\mathbf{s}),\boldsymbol{\mu}\in B_\infty(\mathbf{s})}$$

with coefficients in $\mathbb{Q}[v]$, which is called the adjustment matrix, such that

$$D_e(v) = D_\infty(v)D_\infty^e(v).$$

Remark 3.2. If $l = 1$, then the matrix $D_\infty^e(v)$ is the identity and the adjustment matrix is equal to $D_e(v)$.

3.2. **The computation of $\mathcal{G}_e^\circ(\mathbf{s})$ and $\mathcal{G}_\infty^\circ(\mathbf{s})$.** There are several algorithms available for the computation of the canonical basis $\mathcal{G}_e^\circ(\mathbf{s})$ and $\mathcal{G}_\infty^\circ(\mathbf{s})$. The ones of Uglov [23] and Yvonne [24] compute in fact the canonical basis for the whole Fock space : $\mathcal{G}_e(\mathbf{s})$ and $\mathcal{G}_\infty(\mathbf{s})$. As we are here only interested in the computation of $\mathcal{G}_e^\circ(\mathbf{s})$ and $\mathcal{G}_\infty^\circ(\mathbf{s})$, we focus on the algorithm given in [19] and in [14] which are more suited for our purpose.

Consider $e \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. Assume first that $0 \leq s_0 \leq s_1 \leq \cdots \leq s_{l-1} < e$. For each $\boldsymbol{\lambda} \in B_e(\mathbf{s})$, the above papers first construct a sequence of elements in $\mathbb{Z}/e\mathbb{Z}$

$$\underbrace{k_1, \cdots, k_1}_{u_1}, \underbrace{k_2, \cdots, k_2}_{u_2}, \cdots, \underbrace{k_s, \cdots, k_s}_{u_s}$$

and their associated vectors

$$A(\boldsymbol{\lambda},\mathbf{s}) := f_{k_1}^{(u_1)} \cdots f_{k_s}^{(u_s)}.\emptyset \in V_e(\mathbf{s})$$

such that

$$\mathcal{A}^e(\mathbf{s}) = \{A(\boldsymbol{\lambda},\mathbf{s}) \mid \boldsymbol{\lambda} \in B_e(\mathbf{s})\}$$

is a basis of $V_e(\mathbf{s})$. Next, they provide a simple combinatorial algorithm for computing the coefficients $\gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \in \mathbb{Z}[v]$ satisfying

$$G_e(\boldsymbol{\lambda}, \mathbf{s}) = \sum_{\boldsymbol{\mu}\in B_e(\mathbf{s})} \gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) A(\boldsymbol{\mu}, \mathbf{s}) \tag{6}$$

When $e \in \mathbb{Z}_{\geq 2}$, there is a natural action of the (extended) affine symmetric group $\widehat{\mathfrak{S}}_l$ on $\mathbb{Z}^l$ (see [17, §5.1]). The set

$$\mathcal{B}^l := \left\{(s_0, \ldots, s_{l-1}) \in \mathbb{Z}^l \mid 0 \leq s_0 \leq \cdots \leq s_{l-1} < e\right\}$$

is a fundamental domain for this action. Hence, for any $\mathbf{v} := (v_0, \ldots, v_{l-1}) \in \mathbb{Z}^l$, there exist $\mathbf{s} := (s_0, \ldots, s_{l-1}) \in \mathcal{B}^l$ and $w \in \widehat{\mathfrak{S}}_l$ such that $\mathbf{v} = w.\mathbf{s}$. Since $\mathbf{v}$ and $\mathbf{s}$ yield the same dominant weight, we have an isomorphism $\phi_{\mathbf{s},\mathbf{v}}$ from $V^e(\mathbf{s})$ to $V^e(\mathbf{v})$. We can assume that $\phi_{\mathbf{s},\mathbf{v}}(\emptyset) = \emptyset$. For any $\boldsymbol{\lambda} \in B_e(\mathbf{s})$, set

$$A_e(\boldsymbol{\lambda}, \mathbf{v}) = f_{k_1}^{(u_1)} \cdots f_{k_s}^{(u_s)}.\emptyset \in V_e(\mathbf{v}).$$

The modules structures on $\mathfrak{F}_e^{\mathbf{s}}$ and $\mathfrak{F}_e^{\mathbf{v}}$ do not coincide, thus $A_e(\boldsymbol{\lambda}, \mathbf{v}) \neq A_e(\boldsymbol{\lambda}, \mathbf{s})$ in general. Nevertheless, we have $\phi_{\mathbf{s},\mathbf{v}}(A_e(\boldsymbol{\lambda}, \mathbf{s}) = A_e(\boldsymbol{\lambda}, \mathbf{v})$. Kashiwara's definition of crystal basis also implies that $\phi_{\mathbf{s},\mathbf{v}}(G_e(\boldsymbol{\lambda}, \mathbf{s}) = G_e(\varphi_{\mathbf{s},\mathbf{v}}(\boldsymbol{\lambda}), \mathbf{v})$ where $\varphi_{\mathbf{s},\mathbf{v}}$ is the

crystal isomorphism from $B^e(\mathbf{s})$ to $B^e(\mathbf{v})$ (see [17] for a combinatorial description of $\varphi_{\mathbf{s},\mathbf{v}}$). By applying $\phi_{\mathbf{s},\mathbf{v}}$ to both sides of (6), we thus have for any $\boldsymbol{\nu} \in B^e(\mathbf{v})$

$$\begin{aligned} G_e(\boldsymbol{\nu},\mathbf{v}) &= \sum_{\boldsymbol{\mu}\in B_e(\mathbf{s})} \gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) A(\boldsymbol{\mu},\mathbf{v}) \text{ with } \boldsymbol{\lambda} = \varphi^{-1}_{\mathbf{s},\mathbf{v}}(\boldsymbol{\nu}) \\ \text{and } \mathcal{G}_e(\mathbf{v}) &= \left\{ \sum_{\boldsymbol{\mu}\in B_e(\mathbf{s})} \gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) A(\boldsymbol{\mu},\mathbf{v}) \mid \boldsymbol{\lambda} \in B_e(\mathbf{s}) \right\}. \end{aligned}$$

This means that the algorithm of [19] computes the canonical basis $\mathcal{G}_e(\mathbf{v})$ for any $\mathbf{v} = (v_0, \dots, v_{l-1}) \in \mathbb{Z}^l$ (not only when $0 \le v_0 \le \cdots \le v_{l-1} < e$). Finally, observe that another algorithm has been recently proposed by Fayers [7] for computing the canonical basis of the highest weight $\mathcal{U}'_v(\widehat{\mathfrak{sl}_e})$-modules when these modules are regarded as irreducible components in tensor products of level 1 Fock spaces.

3.3. **Triangularity and transition matrix.** We here want to study the unitirangularity of the "adjustment matrix"

$$D^\infty_e(v) := (a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v))_{\boldsymbol{\lambda}\in B_e(\mathbf{s}),\boldsymbol{\mu}\in B_\infty(\mathbf{s})}$$

To do this, we need to introduce a partial order which does not depend on e. Given $\boldsymbol{\lambda}$ an l-partition, we set

$$r_{\boldsymbol{\lambda}} := \min\{j \in \mathbb{N} \mid \ \lambda^{(i)}_{j+1} = 0 \text{ for any } i = 0, \dots, l-1\}$$

Consider $k \in \mathbb{N}$ and $\mathbf{u} = (u_1, \dots, u_k) \in \mathbb{Q}^k, \mathbf{v} = (v_1, \dots, v_k) \in \mathbb{Q}^k$. We write $\mathbf{u} \rhd \mathbf{v}$ when $\mathbf{u} \neq \mathbf{v}$ and

$$\sum_{a=1}^{i} u_a \ge \sum_{a=1}^{i} v_i \text{ for any } a = 1, \dots, k.$$

Now fix $(\alpha_0, \dots, \alpha_{l-1}) \in \mathbb{Q}^l$ such that $0 < \alpha_0 < \cdots < \alpha_{l-1} < 1$. Given any l-partition $\boldsymbol{\lambda}$, write $\gamma(\boldsymbol{\lambda})$ for the sequence obtained by collecting the elements

$$\lambda^{(i)}_j - j + s_i - \alpha_i, \ j = 1, \dots, r_{\boldsymbol{\lambda}}, \ i = 0, \dots, l-1$$

in increasing order.

Definition 3.3. Given $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ two l-partitions of n, we set $\boldsymbol{\lambda} \succ \boldsymbol{\mu}$ if and only if $\gamma(\boldsymbol{\lambda}) \rhd \gamma(\boldsymbol{\mu})$.

Observe that the above (partial) order depends on $\mathbf{s}$ but not on the fixed l-tuple $(\alpha_0, \dots, \alpha_{l-1})$ providing $0 < \alpha_0 < \cdots < \alpha_{l-1} < 1$.

Theorem 3.4.

(1) *For all $\boldsymbol{\lambda} \in B_e(\mathbf{s})$, there exist polynomials $d^e_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \in \mathbb{Z}_{\ge 0}[v]$ (with $\boldsymbol{\mu} \in \Pi_{l,n}$) such that*

$$G_e(\boldsymbol{\lambda},\mathbf{s}) = \boldsymbol{\lambda} + \sum_{\boldsymbol{\lambda}\succ\boldsymbol{\mu}} d^e_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)\boldsymbol{\mu}$$

(2) *For all $\boldsymbol{\lambda} \in B_\infty(\mathbf{s})$, there exist polynomials $d^\infty_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \in \mathbb{Z}_{\ge 0}[v]$ (with $\boldsymbol{\mu} \in \Pi_{l,n}$) such that*

$$G_\infty(\boldsymbol{\lambda},\mathbf{s}) = \boldsymbol{\lambda} + \sum_{\boldsymbol{\lambda}\succ\boldsymbol{\mu}} d^\infty_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)\boldsymbol{\mu}$$

We do not give here the proof of the unitriangularity of these decompositions, the arguments being the same as those used in [23] and [14]. The main ingredient is the unitriangularity of the matrix of the bar involution on $\mathcal{F}_e^{\mathbf{s}}$ and $\mathcal{F}_\infty^{\mathbf{s}}$ with respect to our partial order $\succ$. This directly follows from the explicit description of the bar involution obtained in [23, Prop 3.16]. Note also that Uglov proved that the coefficients $d^e_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ are parabolic affine Kazhdan Lusztig polynomials which are known to have nonnegative integer coefficients. We can now establish our main result:

Theorem 3.5. *For all $\boldsymbol{\lambda} \in B_e(\mathbf{s})$ and $\boldsymbol{\nu} \in B_\infty(\mathbf{s})$, in the decomposition*

$$G_e(\boldsymbol{\lambda}, \mathbf{s}) = \sum_{\boldsymbol{\mu} \in B_\infty(\mathbf{s})} a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) G_\infty(\boldsymbol{\mu}, \mathbf{s}) \tag{7}$$

we have :

(1) $a_{\boldsymbol{\lambda},\boldsymbol{\lambda}}(v) = 1$,
(2) $a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \in v\mathbb{Z}[v]$ *if* $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$
(3) $a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \neq 0$ *only if* $\boldsymbol{\lambda} \succeq \boldsymbol{\mu}$.

Proof. Assume there is an l-partition $\boldsymbol{\mu}$ in $B_\infty(\mathbf{s})$ such that $\boldsymbol{\lambda} \not\succ \boldsymbol{\mu}$ and $a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \neq 0$. Any partition $\boldsymbol{\nu} \succ \boldsymbol{\mu}$ also verifies $\boldsymbol{\lambda} \not\succ \boldsymbol{\nu}$. Thus, we can assume without loss of generality that $\boldsymbol{\mu}$ is maximal, that is there is no l-partition $\boldsymbol{\nu}$ in $B_\infty(\mathbf{s})$ such that $\boldsymbol{\nu} \succ \boldsymbol{\mu}$ and $a_{\boldsymbol{\lambda},\boldsymbol{\nu}}(v) \neq 0$. By Assertion 2 of Theorem 3.4, the partition $\boldsymbol{\mu}$ appears with a coefficient 1 in the expansion of $G_\infty(\boldsymbol{\mu},\mathbf{s})$. Since $\boldsymbol{\mu}$ is maximal, it also cannot appear in the expansion of a vector $G_\infty(\boldsymbol{\nu},\mathbf{s})$ with $\boldsymbol{\nu}, \neq \boldsymbol{\mu}$ and $a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \neq 0$. But then (7) implies that $\boldsymbol{\mu}$ appears with the nonzero coefficient $a_{\boldsymbol{\lambda},\boldsymbol{\nu}}(v)$ in $G_e(\boldsymbol{\lambda},\mathbf{s})$ which violates assertion 1 of Theorem 3.4. This means that $a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \neq 0$ only if $\boldsymbol{\lambda} \succeq \boldsymbol{\mu}$. We obtain similarly $a_{\boldsymbol{\lambda},\boldsymbol{\lambda}}(v) = 1$ since $\boldsymbol{\lambda}$ can only appear in $G_\infty(\boldsymbol{\lambda},\mathbf{s})$ and then with a coefficient equal to 1. Since $G_e(\boldsymbol{\lambda},\mathbf{s}) = \boldsymbol{\lambda} \pmod{v\mathcal{L}_\mathbb{Z}}$, we have $a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \in v\mathbb{Q}[v]$ when $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$. It remains to establish that our polynomials $a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ have integer coefficients. This is an immediate consequence of the algorithm for computing the decomposition (4) detailed below. □

The procedure for computing the coefficients $a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ is based on the algorithms of §3.2 yielding the matrices $D_e(v)$ and $D_\infty(v)$. We proceed as follows :

(1) Let $\boldsymbol{\lambda} \in B_e(\mathbf{s})$ then, by Theorem 3.5, $a_{\boldsymbol{\lambda},\boldsymbol{\lambda}}(v) = 1$.
(2) We then consider the expansion of

$$G_e(\boldsymbol{\lambda},\mathbf{s}) - G_\infty(\boldsymbol{\lambda},\mathbf{s}) = \sum_{\kappa} b_{\boldsymbol{\lambda},\boldsymbol{\kappa}}(v)\boldsymbol{\kappa}$$

on the basis of all l-partitions. By (1) of Theorem 3.4, $b_{\boldsymbol{\lambda},\boldsymbol{\kappa}}(v) \in \mathbb{Z}[v]$. Let $\boldsymbol{\lambda}^1$ be any l-partition maximal with respect to $\succ$ among those for which $b_{\boldsymbol{\lambda},\boldsymbol{\kappa}}(v) \neq 0$. Theorem 3.5 implies there should exist $\boldsymbol{\mu}^1 \in B_\infty(\mathbf{s})$ maximal such that $\boldsymbol{\lambda}^1$ appears in $G_\infty(\boldsymbol{\mu}^1, \mathbf{s})$ and $a_{\boldsymbol{\lambda},\boldsymbol{\mu}^1}(v) \neq 0$. As explained in the previous proof, this implies that $b_{\boldsymbol{\lambda},\boldsymbol{\mu}^1}(v) = a_{\boldsymbol{\lambda},\boldsymbol{\mu}^1}(v) \neq 0$. Moreover, $\boldsymbol{\mu}^1 \succeq \boldsymbol{\lambda}^1$. Since $\boldsymbol{\lambda}^1$ is maximal such that $b_{\boldsymbol{\lambda},\boldsymbol{\kappa}}(v) \neq 0$ we must have $\boldsymbol{\mu}^1 = \boldsymbol{\lambda}^1$. This shows that $\boldsymbol{\lambda}^1 \in B_\infty(\mathbf{s})$ and $b_{\boldsymbol{\lambda},\boldsymbol{\lambda}^1}(v) = a_{\boldsymbol{\lambda},\boldsymbol{\lambda}^1}(v)$.
(3) We then consider the element $G_e(\boldsymbol{\lambda},\mathbf{s}) - G_\infty(\boldsymbol{\lambda},\mathbf{s}) - a_{\boldsymbol{\lambda},\boldsymbol{\lambda}^1}(v)G_\infty(\boldsymbol{\lambda}^1,\mathbf{s})$. By continuing in this way with maximal l-partitions, we obtain by induction the expansion of $G_e(\boldsymbol{\lambda})$ in terms of the canonical basis $\mathcal{G}_\infty(\mathbf{s})$. Clearly all the coefficients considered belong to $\mathbb{Z}[v]$.

Based on numerous explicit computations, we conjecture in fact the following positivity property

Conjecture 3.6. *For all $\boldsymbol{\lambda} \in B_e(\mathbf{s})$ and $\boldsymbol{\nu} \in B_\infty(\mathbf{s})$, we have $a_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \in \mathbb{N}[v]$.*

3.4. **Example.** We set $e = 2$, Then the matrix $D_e(v)$ of the canonical basis of the $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module $\mathcal{M}_e^{(0,0)}$ is

$$\begin{array}{c} ((3),\emptyset) \\ (\emptyset,(3)) \\ ((2),(1)) \\ ((1),(2) \\ ((2.1),\emptyset) \\ (\emptyset,(2.1)) \\ ((1.1),(1)) \\ ((1),(1.1)) \\ ((1.1.1),\emptyset) \\ (\emptyset,(1.1.1)) \end{array} \begin{pmatrix} 1 & . & . \\ v & . & . \\ v & 1 & . \\ v^2 & v & . \\ . & . & 1 \\ . & . & v \\ v & v^2 & . \\ v^2 & v^3 & . \\ v^2 & . & . \\ v^3 & . & . \end{pmatrix}$$

where a dot means 0. The matrix $D_\infty(v)$ of the canonical basis of the $\mathcal{U}_v(\mathfrak{sl}_\infty)$-module $V_\infty(0,0)$ is

$$\begin{array}{c} ((3),\emptyset) \\ (\emptyset,(3)) \\ ((2),(1)) \\ ((1),(2) \\ ((2.1),\emptyset) \\ (\emptyset,(2.1)) \\ ((1.1),(1)) \\ ((1),(1.1)) \\ ((1.1.1),\emptyset) \\ (\emptyset,(1.1.1)) \end{array} \begin{pmatrix} 1 & . & . & . & . \\ v & . & . & . & . \\ . & 1 & . & . & . \\ . & v & . & . & . \\ . & . & 1 & . & . \\ . & . & v & . & . \\ . & . & . & 1 & . \\ . & . & . & v & . \\ . & . & . & . & 1 \\ . & . & . & . & v \end{pmatrix}$$

The adjustment matrix $D_e^\infty(v)$ obtained from our algorithm is

$$\begin{pmatrix} 1 & . & . \\ v & 1 & . \\ . & . & 1 \\ v & v^2 & . \\ v^2 & . & . \end{pmatrix}$$

and one can check that we have

$$D_e(v) = D_\infty(v).D_\infty^e(v)$$

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