Congested traffic dynamics, weak flows and very degenerate elliptic equations

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Abstract

Starting from a model of traffic congestion, we introduce a minimal-flow–like variational problem whose solution is characterized by a very degenerate elliptic PDE. We precisely investigate the relations between these two problems, which can be done by considering some weak notion of flow for a related ODE. We also prove regularity results for the degenerate elliptic PDE, which enables us in some cases to apply the DiPerna-Lions theory.

Keywords: traffic congestion, weak flows, superposition solutions, DiPerna-Lions theory, degenerate PDE's, regularity.

AMS subject classifications: 35D10, 35J70, 49K10.

1 Introduction

Traffic congestion issues have received a lot of attention from engineers since the 50's mainly in network models (see [22], [5] and the references therein). In such (finite-dimensional) network models, congestion effects are captured through the fact that the travel time of each arc of the network is an increasing function of the flow on this arc. In [22], Wardrop defined a concept of equilibrium for such congested networks that has been very popular since. Roughly speaking, a Wardrop equilibrium is a flow configuration that satisfies natural mass preservation constraints (Kirchhoff's law and compatibility with the given distribution of sources and sinks) and such that every actually used (i.e. where the flow is positive) path connecting a source and destination should be a shortest path (taking into account the congestion effects). Recently, in [8], a model of continuous congested traffic equilibrium has been proposed as well as a generalization of Wardrop's equilibrium to a continuous setting. In this model, an equilibrium is a probability measure over a set of paths that gives full mass to geodesics for a metric that itself depends on the measure due to congestion effects.

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One aim of the present paper is to construct such equilibria as measures supported in some sense on the integral curves of some non-autonomous vector field (not regular in general). For the sake of completeness and to motivate what follows, we will briefly explain the model and some of the results of [8]. In [8], a domain $\Omega \subset \mathbb{R}^N$ and two probability measures μ_0 and μ_1 on $\overline{\Omega}$ are given (distribution of sources and sinks or residents and services, say, in a urban region). In the framework of [8], an equilibrium is in fact a probability measure Q on $C([0,1];\overline{\Omega})$ that solves the following variational problem:

$$\inf_{Q\in\mathcal{Q}^p(\mu_0,\mu_1)} \int_\Omega H(i_Q(x))\ dx \tag{1.1}$$

where $H : \mathbb{R}_+ \to \mathbb{R}_+$ is some convex increasing function with a $p-$th power growth at infinity, $\mathcal{Q}^p(\mu_0,\mu_1)$ is the set of probability measures on $C([0,1];\overline{\Omega})$ concentrated on absolutely continuous curves satisfying compatibility conditions with the distributions of sources and sinks (i.e. $(e_0)_\#Q = \mu_0$ and $(e_1)_\#Q = \mu_1$, where the maps $e_t : C([0,1];\overline{\Omega}) \to \overline{\Omega}$ are the evaluation maps at time t) and such that i_Q is an L^p function, where i_Q is the *traffic intensity* associated to Q defined by

$$\int_{\overline{\Omega}} \varphi(x)\, di_Q(x) := \int_{C([0,1];\overline{\Omega})} \Big(\int_0^1 \varphi(\gamma(t))|\gamma'(t))|dt\Big) dQ(\gamma),\ \forall \varphi \in C(\overline{\Omega}).$$

In this formulation, i_Q represents the total cumulated traffic and H is defined by $H(0) = 0$, $H'(i) = g(i)$ where g is an increasing function that models the congestion effect (that is, in some sense, if the intensity of traffic is i_Q then the congested metric is $g(i_Q)$). Once again, we refer to [8] for more details and in particular the existence of a solution to (1.1) as soon as $\mathcal{Q}^p(\mu_0,\mu_1) \neq \emptyset$ and the precise sense in which the Euler-Lagrange equation of (1.1) corresponds to the fact that Q is a Wardrop equilibrium (i.e. Q-a.e. γ is a geodesic for the metric $g(i_Q)$, a metric which, by the way, is typically given by an L^q function only, with $q = p/(p-1)$, so that one has to properly define distances and geodesics in such a non-continuous setting, and this is one of the main issues solved by [8]).

As already mentioned, one aim of this paper is to construct solutions of (1.1). A first ingredient to achieve this goal, is to introduce a minimal-flow–like problem and to relate it to the scalar problem (1.1) as follows. First, for $Q \in \mathcal{Q}^p(\mu_0,\mu_1)$, define the vector-valued measure σ_Q by:

$$\int_{\overline{\Omega}} \varphi(x)\, d\sigma_Q(x) := \int_{C([0,1];\overline{\Omega})} \Big(\int_0^1 \varphi(\gamma(t))\cdot\gamma'(t))dt\Big) dQ(\gamma),\ \forall \varphi \in C(\overline{\Omega},\mathbb{R}^N),$$

i.e. sort of a vector version of i_Q. It is immediate to check that $|\sigma_Q| \le i_Q$ so that $\sigma_Q \in L^p(\Omega,\mathbb{R}^N)$ and that

$$\mathrm{div}\sigma_Q = \mu_0 - \mu_1,\ \ \sigma_Q\cdot\nu = 0 \text{ on } \partial\Omega$$

since H is increasing, this implies that the infimum of (1.1) is larger than that of the minimal flow problem:

$$\inf_{\sigma\in L^p(\Omega;\mathbb{R}^N)} \left\{ \int_\Omega \mathcal{H}(\sigma)\ dx\ :\ \mathrm{div}\sigma = \mu_0 - \mu_1,\ \sigma\cdot\nu = 0 \text{ on } \partial\Omega \right\}, \tag{1.2}$$

where $\mathcal{H}(\sigma) := H(|\sigma|)$. In the sequel, problem (1.2) will be often referred to as the "vector" problem while problem (1.1) as the "scalar" problem.

If, conversely, σ solves (1.2) and if we are able to construct $Q \in \mathcal{Q}^p(\mu_0, \mu_1)$ such that $i_Q = |\sigma|$ then Q will be a solution to (1.1). Heuristically (i.e. ignoring regularity issues) a natural candidate Q is $Q := \delta_{X(.,x)} \otimes \mu_0$ where $X(., x)$ is the flow of the non-autonomous ODE:

$$\partial_t X(t,x) = \widehat{\sigma}(X(t,x),t),\ X(0,x) = x,\ \ \widehat{\sigma}(x,t) := \frac{\sigma(x)}{(1-t)\mu_0(x) + t\mu_1(x)} \tag{1.3}$$

with σ solving (1.2), according to a deformation argument which essentially dates back to Moser (see [18]) and which has also been exploited by Evans and Gangbo ([14]) in the context of mass transportation problems. If $\widehat{\sigma}$ is Lipschitz, this flow can be defined in a classical sense and the situation is relatively easy to understand. This leads us to the study of the regularity of $\widehat{\sigma}$ and hence of σ and we will see that unfortunately, requiring σ to be Lipschitz will be unrealistic for the models of traffic congestion we are interested in. Formally (see section 2 for details and precise assumptions), by duality, the solution of (1.2) is $\sigma = \nabla\mathcal{H}^*(\nabla u)$ where $\mathcal{H}^*$ is the Legendre transform of $\mathcal{H}$ and u solves the PDE:

$$\left\{ \begin{array}{rcll} \mathrm{div}\nabla\mathcal{H}^*(\nabla u) & = & \mu_0 - \mu_1, & \text{in } \Omega, \\ \nabla\mathcal{H}^*(\nabla u)\cdot\nu & = & 0, & \text{on } \partial\Omega, \end{array} \right. \tag{1.4}$$

Hence, the question immediately becomes a question on regularity properties for the solutions of this equation.

For instance, if one takes $\mathcal{H}(\sigma) = |\sigma|^p/p$, then it is easy to see that we have $\nabla\mathcal{H}^*(z) = |z|^{q-2}z$, so that (1.4) simply becomes a homogeneous Neumann problem for the q-Laplacian operator. This degenerate elliptic equation has been widely studies in literature and in general one cannot hope for better results than $C^{1,\alpha}$ regularity for u (i.e. $\sigma \in C^{0,\alpha}$, see for instance [11, 16]).

Yet, the situation in the cases which are motivated by traffic congestion is even worse. Indeed, let us recall that $H' = g$ where g is the congestion function related the metric to the traffic intensity. It is therefore natural to have $g(0) > 0$: the metric is positive even if there is no traffic, so that the radial function $\mathcal{H}$ is not differentiable at 0 and then its subdifferential at 0 contains a ball. By duality, this implies $\nabla\mathcal{H}^* = 0$ on this ball which makes (1.4) very degenerate. A reasonable model of congestion is $g(t) = \lambda + t^{p-1}$ for $t \geq 0$, with $p > 1$ and $\lambda > 0$, so that

$$\mathcal{H}(\sigma) = \frac{1}{p}|\sigma|^p + \lambda|\sigma|,\ \mathcal{H}^*(z) = \frac{1}{q}(|z| - \lambda)_+^q,\ \text{with } q = \frac{p}{p-1}. \tag{1.5}$$

In this very degenerate case, one will not look for the regularity of u but only of $\sigma = \nabla\mathcal{H}^*(\nabla u)$. Regularity for this term should not be astonishing, as far as one notices that Ω can be, roughly speaking, divided into two zones, one where $\sigma = 0$, the other where the equation is less degenerate (but obviously the two regions are not open sets and one has to make rigorous this idea). Assuming μ_0 and μ_1 are Lipschitz functions bounded from below by positive constants, if one can prove Sobolev regularity of $\nabla\mathcal{H}^*(\nabla u)$ as well as an L^∞ bound on ∇u for the PDE (1.4), then one can define a flow for (1.3) in the sense of the DiPerna-Lions theory. Such regularity results have, in our opinion, their own interest and are proved respectively in section 4 (Sobolev regularity of $\nabla\mathcal{H}^*(\nabla u)$) and 5 (global Lipschitz regularity of u). In general, as explained in subsection 3.2, when very little regularity is

available on the velocity field $\widehat{\sigma}$, it is still possible to relate (1.1) to (1.2) and (1.3) by using the notion of superposition solutions and the superposition principle (see [3]). Surprisingly, the results which are less expected in the framework of Elliptic Regularity theory are those of Section 4, but they are proven by simple variants of usual schemes for the $p-$Laplacian; on the other hand, the L^∞ result of Section 5 could be easily guessed by subsolution arguments, but, for the sake of completeness, we provide here a self-contained proof with some ideas that to our knowledge are new.

Section 2 is devoted to a precise characterization of the minimal flow problem. In section 3, different notions of flows for (1.3) are considered and the precise connection between the scalar problem (1.1) to (1.2) and (1.3) is given as well as the proof of the equality of the values of (1.1) and (1.2) by using superposition solutions. Then, focusing on the case of (1.5), we prove Sobolev regularity of $\nabla\mathcal{H}^*(\nabla u)$ in section 4 and Lipschitz regularity of u in section 5 for the degenerate PDE (1.4).

2 Minimal flow model

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary and let $\mu_0, \mu_1 \in \mathscr{P}(\overline{\Omega})$ be two given probability measures over its closure. We consider the following minimization problem

$$\inf_{\sigma \in L^p(\Omega;\mathbb{R}^N)} \left\{ \int_\Omega \mathcal{H}(\sigma(x))\, dx \ : \ \mathrm{div}\sigma = \mu_0 - \mu_1,\ \sigma\cdot\nu = 0 \text{ on } \partial\Omega \right\}, \tag{2.1}$$

where $\mathcal{H} : \mathbb{R}^N \to \mathbb{R}$ satisfies:

(i) $\mathcal{H}$ is a strictly convex radially symmetric function, with $\mathcal{H}(0) = 0$;

(ii)
$$a|\sigma|^p \le \mathcal{H}(\sigma) \le b(|\sigma|^p + 1),\ \sigma \in \mathbb{R}^N,$$
for some $p \in (1,\infty)$ and a, b positive constants;

(iii) $\mathcal{H}$ is differentiable in $\mathbb{R}^N \setminus \{0\}$ and there exists a positive constant c such that
$$|\nabla\mathcal{H}(\sigma)| \le c(|\sigma|^{p-1} + 1),\ \sigma \in \mathbb{R}^N \setminus \{0\}.$$

Example 1. Taking $\mathcal{H}(\sigma) = |\sigma|$, then (2.1) becomes the *continuous transportation model*

$$\inf_{\sigma \in \mathcal{M}(\Omega;\mathbb{R}^N)} \left\{ \|\sigma\|_{L^1} \ : \ \mathrm{div}\sigma = \mu_0 - \mu_1,\ \sigma\cdot\nu = 0 \text{ on } \partial\Omega \right\},$$

which is nothing but an equivalent formulation of the Monge's problem, with cost equal to the distance (see [4, 21]).

Example 2. Another interesting case, more related to the case of congested dynamics, is given by the choice $H(\sigma) = |\sigma|^2$, for which the minimal value (2.1) is given by (see [9] for the details)

$$C(\mu_0,\mu_1) = \begin{cases} \|\mu_0 - \mu_1\|^2_{X^*}, & \text{if } \mu_0 - \mu_1 \in X^* \\ +\infty, & \text{otherwise.} \end{cases}$$

where X^* indicates the dual of the Hilbert space $X = W^{1,2}_\diamond(\Omega) = \{\varphi \in W^{1,2}(\Omega) \ : \ \int_\Omega \varphi = 0\}$, equipped with the scalar product

$$\langle \varphi, \psi \rangle_X = \int \nabla\varphi \cdot \nabla\psi.$$

Even for this simple problem with quadratic cost, it is only thanks to the results in the present paper that one gets a rigorous equivalence between the "vector" problem used in [9] and the models suggested by Beckmann ([4]) which are better interpreted with a "scalar" construction.

In what follows, we will mainly confine our analysis to the case in which

$$\mathcal{H}(\sigma) = \frac{1}{p}|\sigma|^p + \lambda|\sigma|, \ \sigma \in \mathbb{R}^N, \tag{2.2}$$

with $p \in (1, 2]$ and $\lambda > 0$ a positive constant. The reasons for the restriction on the exponent p are twofolds: on the one hand, the scalar problem of [8] is fully understood under the extra assumption $p < N/(N-1)$ (i.e. $p < 2$ in two dimensions, which is the most relevant case in applications); on the other hand we will see extra difficulties arise concerning elliptic regularity whenever we are in the singular case $q = p/(p-1) < 2$.

Theorem 2.1. *Suppose that the infimum in* (2.1) *is finite and let* σ_0 *be its unique optimizer, then there exists* $\varphi_0 \in W^{1,q}(\Omega)$ *such that*

$$\sigma_0 = \nabla\mathcal{H}^*(\nabla\varphi_0), \tag{2.3}$$

and φ_0 *is a weak solution of*

$$\begin{cases} \mathrm{div}\nabla\mathcal{H}^*(\nabla u) &= \mu_0 - \mu_1, & \text{in } \Omega, \\ \nabla\mathcal{H}^*(\nabla u)\cdot\nu &= 0, & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $\mathcal{H}^*$ *is the Legendre transform of* $\mathcal{H}$ *and* $q = p/(p-1)$.

Proof. We first observe that problem (2.1) consists in minimizing a strictly convex and coercive functional on L^p subject to a convex and closed constraint: then an optimizer σ_0 exists and must be unique.

It is well known that problem (2.1) has a dual formulation, given by the convex analysis formula (see for instance [13])

$$\sup\left\{\int_\Omega \varphi \ d(\mu_1 - \mu_0) - \int_\Omega \mathcal{H}^*(\nabla\varphi) \ dx\right\} = \inf\left\{\int_\Omega \mathcal{H}(\sigma) \ : \ \mathrm{div}\sigma = \mu_0 - \mu_1, \ \sigma\cdot\nu = 0\right\}.$$

Due to the superlinear growth and the strict convexity of $\mathcal{H}$, we get that $\mathcal{H}^* \in C^1$ and it verifies the following growth conditions

$$B(|z|^q - 1) \le \mathcal{H}^*(z) \le A|z|^q,$$

where $q = p/(p-1)$, then using the Direct Methods of the Calculus of Variations it is not difficult to show that the dual problem admits at least a solution φ_0 belonging to $W^{1,q}_\diamond(\Omega)$, where

$$W^{1,q}_\diamond = \left\{ \varphi \in W^{1,q}(\Omega) \ : \ \int_\Omega \varphi(x)\ dx = 0 \right\}.$$

We observe further that the Euler-Lagrange equation of

$$\mathcal{F}(\varphi) = \int_\Omega \mathcal{H}^*(\nabla\varphi(x))\ dx - \int_\Omega \varphi(x)\ d(\mu_1 - \mu_0),$$

is given by (2.4), so that φ_0 solves it, in distributional sense. Moreover, φ_0 and σ_0 verifies

$$\begin{aligned} \int_\Omega \mathcal{H}(\sigma_0) &= \int_\Omega \varphi_0(\mu_1 - \mu_0) - \int_\Omega \mathcal{H}^*(\nabla\varphi_0) \\ &= \int_\Omega \nabla\varphi_0 \cdot \sigma_0 - \int_\Omega \mathcal{H}^*(\nabla\varphi_0), \end{aligned}$$

where we have used the fact that $\mathrm{div}\sigma_0 = \mu_0 - \mu_1$ and $\sigma_0 \cdot \nu = 0$. The previous can be written as

$$\int_\Omega \mathcal{H}(\sigma_0) + \int_\Omega \mathcal{H}^*(\nabla\varphi_0) = \int_\Omega \nabla\varphi_0 \cdot \sigma_0,$$

which, by means of the so called Legendre reciprocity formula, implies that

$$\sigma_0(x) \in \partial\mathcal{H}^*(\nabla\varphi_0(x)), \quad \text{for } \mathscr{L}^N\text{-a.e. } x \in \Omega.$$

Using the fact that $\mathcal{H}^* \in C^1$, we obtain that actually the subgradient set $\partial\mathcal{H}^*$ is made of just an element, namely the gradient $\nabla\mathcal{H}^*$, concluding the proof. □

3 Different meanings and equivalences

In this section we discuss how to connect the "scalar" problem on measures on paths to the "vector" problem on fields with prescribed divergence: in which sense and when they are equivalent and how to pass from one minimizer to the other.

3.1 Cauchy-Lipschitz flow

Let us consider a non-autonomous vector field $\mathbf{v} : [0,1]\times\Omega \to \mathbb{R}^N$ such that $\mathbf{v}\cdot\nu = 0$, where ν stands for the outer normal vector to $\partial\Omega$. It is well-known that if $\mathbf{v}$ is sufficiently smooth, say Lipschitz with respect to the spatial variable, then for every μ_0 the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}\mu(t,x) + \mathrm{div}_x(\mathbf{v}(t,x)\mu(t,x)) &= 0, \qquad (t,x) \in [0,1]\times\Omega, \\ \mu(0,x) &= \mu_0(x), \qquad x \in \Omega, \end{cases} \tag{3.1}$$

is given by

$$\mu(t,\cdot) = (X(t,\cdot))_\sharp \mu_0, \tag{3.2}$$

where $X : [0,1] \times \overline{\Omega} \to \overline{\Omega}$ is the *flow* of $\mathbf{v}$, that is X is the map that to every $(s,x) \in [0,1] \times \overline{\Omega}$ assigns the position at time s of the curve γ satisfying

$$\begin{cases} \gamma'(s) &= \mathbf{v}(s,\gamma(s)) \\ \gamma(0) &= x \end{cases} \tag{3.3}$$

This is a particular case of the *method of characteristics* which basically says that the solution of (3.1) is given by the evolution, through the flow of $\mathbf{v}$, of the initial measure μ_0 (see [3] for a clarifying exposition of this theory).

We now take two probability measures μ_0 and μ_1 on Ω, absolutely continuous w.r.t to $\mathscr{L}^N$ and having density given by f_0 and f_1, respectively.

Using the above remarks on ODEs and the continuity equation, we now illustrate our general strategy to prove the equivalence between the two problems

$$\inf_{Q \in \mathcal{Q}^p(\mu_0,\mu_1)} \int_\Omega H(i_Q)\, dx \quad \text{and} \quad \inf_{\sigma \in L^p(\Omega;\mathbb{R}^N)} \left\{ \int_\Omega \mathcal{H}(\sigma)\, dx \ :\ \mathrm{div}\sigma = f_0 - f_1,\ \sigma \cdot \nu = 0 \right\}. \tag{3.4}$$

We already know that in general the value of the vector minimization problem (right hand side of (3.4)) is less than or equal to the value of the scalar one. The key point is to show that, given the optimizer σ of the vector problem, we can construct a $Q \in \mathcal{Q}^p(\mu_0,\mu_1)$ such that $|\sigma| \le i_Q$. Then (3.4) is a straightforward consequence on the monotonicity assumptions on H.

As we already mentioned, the main idea will be the use of the deformation argument due to Moser and used later by Evans and Gangbo: for the moment we make the further assumption that f_0 and f_1 are Lipschitz continuous and bounded from below, that is $f_0, f_1 \ge c > 0$ on Ω.

If σ is the unique solution of the convex optimization problem (2.1), we construct the non-autonomous vector field

$$\widehat{\sigma}(t,x) = \frac{\sigma(x)}{(1-t)f_0(x) + t f_1(x)},\ (t,x) \in [0,1] \times \Omega. \tag{3.5}$$

The latter will not have any Lipschitz continuity property in general, unless the optimizer σ itself is regular: anyway, if we assume that one can prove $\sigma \in \mathrm{Lip}(\Omega)$, then the flow $X : [0,1] \times \Omega \to \Omega$ of $\widehat{\sigma}$ is well-defined and we can take μ_t as in (3.2). In this way, we have obtained the solution of (3.1), with $\mathbf{v} = \widehat{\sigma}$ and initial datum f_0. Moreover, the same Cauchy problem is solved by the linear interpolating curve

$$\rho_t(\cdot) := (1-t)f_0(\cdot) + t f_1(\cdot), \tag{3.6}$$

which implies, due to well-posedness of (3.1), that ρ_t and μ_t must coincide. This in turn yields that

$$(X(1,\cdot))_\sharp f_0 = f_1, \tag{3.7}$$

which ensures that $X(1,\cdot)$ transports μ_0 on μ_1. If we now consider the probability measure concentrated on the flow, i.e.

$$Q = \delta_{X(\cdot,x)} \otimes \mu_0,$$

then thanks to (3.7) Q is admissible and it is not difficult to see that $i_Q = |\sigma|$ (we will give all the details in Theorem 3.2 below), which finally implies that the minimum of the two problems coincide. Moreover, this construction provides a transport map (that is $X(1,\cdot)$) from μ_0 to μ_1, whose transport "rays" evidently do not cross and which is monotone on transport "rays" (as a consequence of Cauchy-Lipschitz Theorem).

Remark 1. We point out that in this setting, where everything is sufficiently smooth, property (3.7) can be proved at a Lagrangian level, without mentioning the well-posedness of the continuity equation: indeed one can use a trick of Dacorogna and Moser (see [10]) to show that the quantity

$$h(t,x) = \det \nabla_x X(t,x)[(1-t)f_0(X(t,x)) + tf_1(X(t,x))],$$

is actually constant in time. Then using the fact that $X(0,x) = x$ we get that

$$f_0(x) = f_1(X(1,x)) \det \nabla_x X(1,x),$$

which in turn implies (3.7) by means of the area formula.

Anyway, recalling the optimality condition for σ provided by Theorem 2.1, the reader can easily convince himself that our choice for the function H rules out any possibility of Lipschitz regularity for σ. So the previous construction of Q is purely formal: we will see in the next subsections how (and in what sense) one can still construct a flow X and make this construction a rigorous one.

Remark 2. On the contrary, when one takes $\mathcal{H}(z) = |z|^2$, standard elliptic theory allows to prove Lipschitz regularity for σ and this concept of Cauchy-Lipschitz flows may be used.

3.2 Superposition of flows

For a general vector field $\mathbf{v}$ under very mild assumptions, the most general meaning that we can give to the flow of $\mathbf{v}$ is in terms of the so-called *superposition principle*, that we now explain in some details. As far as we can see, this provides a very weak concept of flow, which anyway is strong enough to still give sense to the construction of the previous subsection.

Definition 1. Let $Q \in \mathscr{P}(C([0,1];\overline{\Omega}))$ be concentrated on the absolutely continuous solutions of (3.3), in the sense that

$$\int_{C([0,1];\overline{\Omega})} \left| \gamma(t) - \gamma(0) - \int_0^t \mathbf{v}(s,\gamma(s))\, ds \right| \, dQ(\gamma) = 0. \tag{3.8}$$

If we define the curve of measures μ_t^Q through

$$\int_{\overline{\Omega}} \varphi(x)\, d\mu_t^Q(x) := \int_{C([0,1];\overline{\Omega})} \varphi(\gamma(t))\, dQ(\gamma) \quad \text{for every } \varphi \in C(\overline{\Omega}), \tag{3.9}$$

then this curve μ_t^Q is called *superposition solution* of Problem (3.1): μ_t^Q is actually a distributional solution of the continuity equation, with initial datum $\mu_0 = \mu_0^Q$.

Remark 3. It is not hard to see that when $\mathbf{v}$ is smooth, formula (3.9) is exactly equivalent to (3.2). Indeed in this case, for every $x \in \Omega$, there exists a unique curve $X(\cdot, x)$ solving (3.3), so that $Q = Q^x \otimes \mu_0$ with Q^x a Dirac mass concentrated on this curve, that is

$$Q^x = \delta_{X(\cdot,x)},$$

and (3.9) now becomes

$$\int_{\overline{\Omega}} \varphi(x)\ d\mu_t^Q(x) = \int_{\overline{\Omega}} \varphi(X(x,t))\ d\mu_0(x) = \int_{\overline{\Omega}} \varphi(x)\ d(X(t,\cdot))_\sharp \mu_0(x).$$

In this way, we can think of the concept of superposition solutions as a *probabilistic* version of the method of characteristics.

The most valuable fact of this theory is that every positive measure-valued distributional solution of (3.1) can be realized as a superposition solution: a proof can be found in [3] (Theorem 12).

Theorem 3.1 (Superposition principle)**.** *Let μ_t be a positive measure-valued solution of the continuity equation*

$$\frac{\partial}{\partial t}\mu_t + \operatorname{div}(\mathbf{v}\mu_t) = 0,$$

with the vector field $\mathbf{v}$ satisfying the following condition

$$\int_0^1 \int_\Omega \frac{|\mathbf{v}(t,x)|}{1+|x|}\ d\mu_t(x)\ dt < +\infty, \tag{3.10}$$

then μ_t is a superposition solution.

Using the concept of superposition solution, it is now a straightforward fact to provide a rigorous proof of the equivalence between the two problems in (3.4).

Theorem 3.2. *Let $\mu_0, \mu_1 \in \mathscr{P}(\Omega)$ having L^p density w.r.t. to $\mathscr{L}^N$, given by f_0 and f_1, respectively. Then equality of values of the two problems in* (3.4) *holds true.*

Proof. As before, we take the minimizer σ of the vector problem and we consider the non-autonomous vector field defined by (3.5). We point out that the L^p assumption on the densities has been chosen in order to guarantee finiteness of the infima of both problems (see [8]). With this choice of $\widehat{\sigma}$, the linear interpolating curve $\mu_t = (1-t)\mu_0 + t\mu_1$ is a positive measure-valued distributional solution of the continuity equation

$$\frac{\partial}{\partial t}\mu_t + \operatorname{div}(\widehat{\sigma}\mu_t) = 0,$$

with initial datum μ_0. Moreover, $\widehat{\sigma}$ satisfies hypothesis (3.10), so that μ_t is a superposition solution: this means that there exists a probability measure $Q \in \mathscr{P}(C([0,1];\overline{\Omega}))$ such that (3.8) holds and

$$\int_\Omega \varphi(x)\ d\mu_t(x) = \int_\Omega \varphi(x)\ d\mu_t^Q(x), \quad \text{for every } \varphi \in C(\overline{\Omega}),$$

with μ_t^Q given by (3.9) (observe that in the Cauchy-Lipschitz case, this amounted to say that ρ_t defined by (3.6) had to coincide with the solution given by (3.2)). This Q is admissible, that is $Q \in \mathcal{Q}(\mu_0, \mu_1)$ and moreover, using Fubini Theorem and the disintegration $Q = Q^x \otimes \mu_0$, we get

$$\begin{aligned}
\int_{\overline{\Omega}} \varphi(x)\ di_Q(x) &= \int_C \int_0^1 \varphi(\gamma(t))|\gamma'(t)|\ dt\ dQ(\gamma) \\
&= \int_0^1 \int_C \varphi(\gamma(t))|\gamma'(t)|\ dQ(\gamma)\ dt \\
&= \int_0^1 \int_\Omega \int_C \varphi(\gamma(t))|\gamma'(t)|\ dQ^x(\gamma) d\mu_0(x) dt \\
&= \int_0^1 \int_\Omega \varphi(x)|\widehat{\sigma}(t,x)|\ d\mu_t(x)\ dt = \int_0^1 \int_\Omega \varphi(x)|\sigma(x)|\ dx\ dt,
\end{aligned}$$

so that

$$\int_\Omega \varphi(x)\ di_Q(x) = \int_\Omega \varphi(x)|\sigma(x)|\ dx, \quad \text{for every } \varphi \in C(\overline{\Omega}).$$

This clearly implies that $i_Q = |\sigma|$ and thus $Q \in \mathcal{Q}^p(\mu_0, \mu_1)$ and it solves the scalar problem in (3.4), concluding the proof. □

Notice that the regularity of the curves which are charged by the measure Q corresponding to a superposition solution is very poor. On the contrary, if one knows that $\mathbf{v}$ is continuous, these curves are C^1 and they solve their ODE in a classical sense. The forthcoming paper [20] will prove a C^0 result in two spatial dimensions for the vector field we are interested in. Obviously, continuity without Lipschitz continuity or similar conditions is not sufficient for ensuring any kind of uniqueness result. We will see in a while that some kind of uniqueness may be recovered by an intermediate concept of solution.

3.3 DiPerna-Lions flow

As far as now, we have seen that everything goes well if we face a Lipschitz vector field $\mathbf{v}$ and that we can at least prove equality of the minima if, instead, $\mathbf{v}$ is only integrable. In the latter case, it is not evident to add anything else to this equality and in particular one has no real clue to construct a minimizer for the scalar problem from a minimizer for the vector one. The problem is mainly linked to the lack of uniqueness. We will see in this section an intermediate concept, for vector fields which are not Lipschitz but much better than just integrable.

If $\mathbf{v}(t,\cdot) \in W^{1,1}(\Omega)$ and the vector field has bounded divergence, we can enforce the conclusion of Theorem 3.2 and guarantee that the optimal Q associated to the optimizer σ is actually concentrated on a uniquely defined flow X (possibly in a.e. sense), trasporting μ_0 to μ_1.

In fact in this setting, it is still possible to give sense to formula (3.2), through the DiPerna-Lions theory of flows of weakly differentiable vector fields: we recall the following fundamental result (see Theorem III.2 of [12]; the same results are also presented in [3] where the language is more similar to ours).

Theorem 3.3. *Let* $\mathbf{v} \in L^1([0,1]; W^{1,1}(\Omega))$ *and such that* $\mathrm{div}_x \mathbf{v} \in L^1([0,1]; L^\infty(\Omega))$. *Then there exists a unique* $X \in C^0([0,1]\times[0,1]; L^1(\Omega;\mathbb{R}^N))$ *which leaves* $\overline{\Omega}$ *invariant and such that:*

(i) if we set $A(t) = \int_0^t \|\mathrm{div}_x \mathbf{v}(\tau,\cdot)\|_\infty \, d\tau$, *then*

$$e^{-|A(t)-A(s)|}\mathscr{L}^N \le (X(t,s,\cdot))_\sharp \mathscr{L}^N \le e^{|A(t)-A(s)|}\mathscr{L}^N, \quad \text{for every } t \in [0,1];$$

(ii) X satisfies the group property

$$X(t_3,t_1,x) = X(t_3,t_2,X(t_2,t_1,x)), \quad \text{for } \mathscr{L}^N\text{-a.e. } x\in\Omega, \quad \text{for every } t_1<t_2<t_3\in[0,1];$$

(iii) for every $s \ge 0$ and for $\mathscr{L}^N$-a.e. $x \in \Omega$, X is an absolutely continuous integral solution of (3.3), *that is*

$$X(t,s,x) = x + \int_s^t \mathbf{v}(r, X(r,s,x))\, dr, \quad \text{for } \mathscr{L}^N\text{-a.e. } x\in\Omega,\ t\ge s.$$

Moreover, if $\mu_0 = \rho_0 \mathscr{L}^N$ *with* $\rho_0 \in L^p(\Omega)$, *then for every* $s \in [0,1)$

$$\mu(t,\cdot) = X(t,s,\cdot)_\sharp \mu_0, \ \ s \le t \in [0,1],$$

is the unique renormalized solution in $C^0([s,1]; L^p(\Omega))$ *of the continuity equation, with initial datum* $\mu(s,x) = \mu_0(x)$.

Definition 2. We recall that μ is said to be a *renormalized solution* of the continuity equation if there holds

$$\frac{\partial}{\partial t}\beta(\mu) + \mathbf{v}\cdot\nabla_x\beta(\mu) + (\mathrm{div}_x\mathbf{v})\mu\beta'(\mu) = 0, \quad \text{in } (0,1)\times\Omega, \tag{3.11}$$

in the sense of distributions, for every $\beta \in C^1(\mathbb{R})$.

Clearly, every renormalized solution is a distributional solution (just take $\beta \equiv 1$ in (3.11)), while in general the converse does not hold true. It is a remarkable fact of the DiPerna-Lions theory that when $\mathbf{v}$ has a Sobolev regularity in x, then $\mathbf{v}$ has the *renormalization property*, that is every distributional solution is actually a renormalized one. Moreover, renormalized solutions are the right class in which existence, uniqueness and stability of solutions to the continuity equation can be proved: *this is crucial for our construction*. Indeed, as already observed in the subsection on Cauchy-Lipschitz flow, well-posedness of the continuity equation guarantees that the flow at time 1 transports μ_0 on μ_1, so that the measure Q associated to σ is admissible.

Finally, we just point out that the renormalization property can be proved also for vector fields with BV regularity (with respect to the space variable), as shown by Ambrosio ([2]): some L^∞ bounds on the divergence of the vector field are again essential.

Due to the previous facts, the rest of the paper is devoted to provide Sobolev and L^∞ estimates for the optimizer σ under the following assumptions:

(i) $\mu_i = f_i \mathscr{L}^N$, with $f_i \in \text{Lip}\,(\Omega)$ and $f_i \geq c > 0$, for $i = 0, 1$;

(ii) Ω open connected bounded subset of $\mathbb{R}^N$ having Lipschitz boundary.

In fact with these assumptions, the vector field $\widehat{\sigma}$ given by (3.5) is well-defined and satisfies the hypotheses of DiPerna-Lions Theorem, once we know that $\sigma \in W^{1,r} \cap L^\infty$, for some $r \geq 1$. Indeed, the Sobolev regularity of $\widehat{\sigma}$ is equivalent to that of σ, once f_0 and f_1 are Lipschitz. For the condition on the divergence one may see that we have

$$\text{div}\widehat{\sigma} = \frac{\text{div}\sigma}{\rho_t} - \frac{\sigma \cdot \nabla \rho_t}{\rho_t^2}.$$

Lipschitz regularity and lower bounds on $\rho_t = (1-t)f_0 + tf_1$ (i.e. on f_0 and f_1) and L^∞ on σ seem compulsory for getting the assumption on the divergence of $\widehat{\sigma}$.

Moreover, (i) guarantees that then $(1-t)\mu_0 + t\mu_1$ is a renormalized solution of (3.1) and so it must coincide with $X_\sharp \mu_0$.

We will achieve these results strongly relying on the optimality condition for σ provided by Theorem 2.1, which ensures that

$$\sigma = \nabla \mathcal{H}^*(\nabla u),$$

where $u \in W^{1,q}(\Omega)$ is a distributional solution of the degenerate elliptic equation

$$\text{div}(\nabla \mathcal{H}^*(\nabla u)) = f_0 - f_1, \tag{3.12}$$

under homogeneus Neumann boundary conditions.

4 Sobolev regularity of the vector field

In order to apply the DiPerna-Lions theory, first of all we have to show that σ is weakly differentiable: we will indeed show that $\sigma \in W^{1,r}(\Omega)$, for a suitable r. Observe that in general no more than $C^{0,1}$ regularity should be expected for solutions of equation (3.12). Actually, with the choice

$$\mathcal{H}(\sigma) = \frac{1}{p}|\sigma|^p + |\sigma|,\ \sigma \in \mathbb{R}^N,$$

we get

$$\nabla \mathcal{H}^*(z) = (|z| - 1)_+^{q-1} \frac{z}{|z|},\ z \in \mathbb{R}^N,$$

so that every $1-$Lipschitz function is a solution of the homogeneous equation. Moreover, we have

$$\frac{(|\nabla u| - 1)_+^{q-1}}{|\nabla u|}|\xi|^2 \leq \langle D^2 \mathcal{H}^*(\nabla u)\xi, \xi\rangle \leq (q-1)(|\nabla u| - 1)_+^{q-2}|\xi|^2,\ \xi \in \mathbb{R}^N,$$

that is the ellipticity constants degenerate in the region $\{|\nabla u| \leq 1\}$.

We will confine our analysis to the non-singular case $q \geq 2$, which is anyway relevant for the applications to minimization problems in traffic congestion.

First of all, we need the following pointwise inequalities. This is the main point where the precise structure of $\mathcal{H}^*$ plays a role.

Lemma 4.1. *For every $q \geq 2$, let us define the following vector field*

$$G(z) = |\nabla \mathcal{H}^*(z)|^{\frac{p}{2}} \frac{z}{|z|} = (|z| - 1)_+^{\frac{q}{2}} \frac{z}{|z|}, \ z \in \mathbb{R}^N. \tag{4.1}$$

Then for every $z, w \in \mathbb{R}^N$ we get

$$(\nabla \mathcal{H}^*(z) - \nabla \mathcal{H}^*(w)) \cdot (z - w) \geq \frac{4}{q^2} |G(z) - G(w)|^2, \tag{4.2}$$

$$|\nabla \mathcal{H}^*(z) - \nabla \mathcal{H}^*(w)| \leq (q-1) \left(|G(z)|^{\frac{q-2}{q}} + |G(w)|^{\frac{q-2}{q}} \right) |G(z) - G(w)|. \tag{4.3}$$

Proof. We first observe that if

$$\max\{|z|, |w|\} \leq 1,$$

then (4.2) and (4.3) are trivially true. Secondly, in the case

$$\min\{|z|, |w|\} \leq 1,$$

supposing for example that $|w| \leq 1$ and $|z| > 1$, using Cauchy-Schwarz inequality we get

$$\begin{aligned} \nabla \mathcal{H}^*(z) \cdot (z - w) &= (|z| - 1)_+^{q-1} \frac{z}{|z|} \cdot (z - w) \\ &\geq (|z| - 1)_+^{q-1} |z| - (|z| - 1)_+^{q-1} = (|z| - 1)_+^q, \end{aligned}$$

which proves (4.2), while (4.3) is easily seen to be true in this case, too.

Let us now suppose that $|z| > 1$ and $|w| > 1$. Now, we recall the inequality (see [17])

$$(|s|^{q-2} s - |t|^{q-2} t) \cdot (s - t) \geq \frac{4}{q^2} \left| |s|^{\frac{q-2}{2}} s - |t|^{\frac{q-2}{2}} t \right|^2, \ s, t \in \mathbb{R}^N, \tag{4.4}$$

and we see that if we are able to prove the following

$$(|s|^{q-2} s - |t|^{q-2}) \cdot \left((|s| + 1) \frac{s}{|s|} - (|t| + 1) \frac{t}{|t|} \right) \geq (|s|^{q-2} s - |t|^{q-2} t) \cdot (s - t), \tag{4.5}$$

then choosing

$$s = (|z| - 1)_+ \frac{z}{|z|}, \ t = (|w| - 1)_+ \frac{w}{|w|},$$

and using (4.5) in combination with (4.4), we obtain (4.2). So, let us prove inequality (4.5): one sees that this is equivalent to

$$|s|^{q-1} + |t|^{q-1} - s \cdot t \left[\frac{|s|^{q-2}}{|t|} + \frac{|t|^{q-2}}{|s|} \right] \geq 0,$$

which is just a simple consequence of Cauchy-Schwarz inequality $s \cdot t \leq |s||t|$.

In order to prove (4.3), it is enough to start from the inequality

$$||s|^{q-2}s - |t|^{q-2}t| \leq (q-1)(|s|^{\frac{q-2}{2}} + |t|^{\frac{q-2}{2}}) \left||s|^{\frac{q-2}{2}}s - |t|^{\frac{q-2}{2}}t\right|,$$

which is valid for every $t, s \in \mathbb{R}^N$ and then take s and t as before. □

We are ready to prove the main result of this section: the proof is an adaption of an argument originally used by Bojarski and Iwaniec (see [6]) for the p-Laplacian operator.

Theorem 4.2. *Let us suppose that Ω has a Lipschitz boundary and take $f \in W^{1,p}_{\diamond}(\Omega)$, with $p = q/(q-1)$. If $u \in W^{1,q}_{\diamond}(\Omega)$ is a weak solution of the following Neumann boundary problem*

$$\begin{cases} -\mathrm{div}\,(\nabla\mathcal{H}^*(\nabla u)) &= f, \quad \text{in } \Omega, \\ \nabla\mathcal{H}^*(\nabla u)\cdot\nu &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

then we get $\mathcal{G} \in W^{1,2}(\Omega)$, where the function $\mathcal{G}$ is defined by

$$\mathcal{G}(x) := G(\nabla u(x)) = (|\nabla u(x)| - 1)^{\frac{q}{2}}_{+} \frac{\nabla u(x)}{|\nabla u(x)|}, \ x \in \Omega. \tag{4.7}$$

Proof. First of all, we observe that if u is a weak solution of (4.6), then

$$\int_\Omega \nabla\mathcal{H}^*(\nabla u)\cdot\nabla\varphi\ dx = \int_\Omega f\varphi\ dx, \quad \text{for every } \varphi \in W^{1,q}(\Omega).$$

Then, if for a given function g we denote

$$g_h(x) := g(x+h), \ x \in \mathbb{R}^N,$$

for every $h \in \mathbb{R}^N$ such that

$$\mathscr{L}^N(\Omega \cap (\Omega - h)) > 0,$$

and for every $\varphi \in W^{1,q}(\Omega)$ having support in $\Omega \cap (\Omega - h)$, we have

$$\int_\Omega \nabla\mathcal{H}^*(\nabla u_h)\cdot\nabla\varphi\ dx = \int_\Omega f_h\varphi\ dx,$$

hence subtracting, we obtain

$$\int_\Omega (\nabla\mathcal{H}^*(\nabla u_h) - \nabla\mathcal{H}^*(\nabla u))\cdot\nabla\varphi\ dx = \int_\Omega (f_h - f)\varphi\ dx, \tag{4.8}$$

for every φ supported in $\Omega \cap (\Omega - h)$.

We now exploit (4.8) in order to prove (4.7): we need to select a suitable test function φ.

We confine our analysis to $W^{1,2}$ estimates near the boundary of Ω: then it should be clear how to apply the same techniques in order to obtain interior $W^{1,2}$ estimates. Let us fix $\delta > 0$ small enough and define

$$\overline{\Omega}_\delta = \{x \in \overline{\Omega} \ : \ \mathrm{dist}(x, \partial\Omega) \leq \delta\},$$

then we can cover this set with a finite number of balls, having center on $\partial\Omega$. Let $B(x_0, 4\rho)$ be one of these balls: we set

$$B^+(x_0, 4\rho) = B(x_0, 4\rho) \cap \overline{\Omega} = \{x \in \overline{\Omega} \ : \ |x - x_0| < 4\rho\}.$$

We consider a smooth cut-off function $\zeta \in C_0^1(B^+(x_0, 2\rho))$, such that

(i) $0 \leq \zeta \leq 1$;

(ii) $\zeta \equiv 1$ on $B^+(x_0, \rho)$;

(iii) $\|\nabla\zeta\|_\infty \leq C\rho^{-1}$.

Then we make the following choice for φ

$$\varphi(x) = \zeta^2(x)(u_h(x) - u(x)), \ x \in \Omega,$$

for every $h \in \mathbb{R}^N$ such that $|h| < \rho$ and $B^+(x_0, 2\rho) + h \subset \Omega$: the existence of at least such a vector h is a consequence of having assumed $\partial\Omega$ Lipschitz. Moreover, the set of these h contains at least a cone at the origin with non empty interior, for the same reason. We observe that

$$\mathrm{supp}\ \varphi \subset \Omega \cap (\Omega - h),$$

so that (4.8) is valid in this case.

We now develop φ and use Cauchy-Schwarz inequality, getting

$$\begin{aligned}\int_\Omega (\nabla\mathcal{H}^*(\nabla u_h) - \nabla\mathcal{H}^*(\nabla u)) \cdot (\nabla u_h - \nabla u)\zeta^2 \ dx \leq 2 \int_\Omega |\nabla\mathcal{H}^*(\nabla u_h) - \nabla\mathcal{H}^*(\nabla u)| \\ \times \zeta|\nabla\zeta||u_h - u| \ dx \\ + \int_\Omega \zeta^2|f - f_h||u_h - u| \ dx.\end{aligned}$$

An application of the pointwise inequalities (4.2) and (4.3) yields

$$\begin{aligned}\int_\Omega |\mathcal{G}_h - \mathcal{G}|^2 \zeta^2 \ dx \leq C \int_\Omega \left(|\mathcal{G}_h|^{\frac{q-2}{q}} + |\mathcal{G}|^{\frac{q-2}{q}}\right) |\mathcal{G}_h - \mathcal{G}|\, \zeta|\nabla\zeta||u_h - u| \ dx \\ + \left(\int_\Omega \zeta^p |f_h - f|^p \ dx\right)^{\frac{1}{p}} \left(\int_\Omega \zeta^q |u_h - u|^q\right)^{\frac{1}{q}}\end{aligned}$$

where the constant C depends on q only. We can apply Hölder inequality with exponents

$$\frac{1}{2} + \frac{1}{q} + \frac{q-2}{2q} = 1,$$

obtaining

$$\begin{aligned}\int_\Omega |\mathcal{G}_h-\mathcal{G}|^2\zeta^2\,dx \le C&\left(\int_{B^+(x_0,2\rho)}\left(|\mathcal{G}_h|^{\frac{q-2}{q}}+|\mathcal{G}|^{\frac{q-2}{q}}\right)^{\frac{2q}{q-2}}\,dx\right)^{\frac{q-2}{2q}}\\ &\times\left(\int_\Omega|\nabla\zeta|^q|u_h-u|^q\,dx\right)^{\frac{1}{q}}\left(\int_\Omega|\mathcal{G}_h-\mathcal{G}|^2\zeta^2\,dx\right)^{\frac{1}{2}}\\ &+\left(\int_\Omega\zeta^p|f_h-f|^p\,dx\right)^{\frac{1}{p}}\left(\int_\Omega\zeta^q|u_h-u|^q\right)^{\frac{1}{q}}.\end{aligned}\tag{4.9}$$

It is now sufficient to observe that

$$\begin{aligned}\left(\int_{B^+(x_0,2\rho)}\left(|\mathcal{G}|^{\frac{q-2}{q}}+|\mathcal{G}_h|^{\frac{q-2}{q}}\right)^{\frac{2q}{q-2}}\,dx\right)^{\frac{q-2}{2q}}&\le\left(\int_{B^+(x_0,2\rho)}|\mathcal{G}|^2\,dx\right)^{\frac{q-2}{2q}}\\ &+\left(\int_{B^+(x_0,2\rho)}|\mathcal{G}_h|^2\,dx\right)^{\frac{q-2}{2q}}\\ &\le 2\left(\int_\Omega|\mathcal{G}|^2\,dx\right)^{\frac{q-2}{2q}},\end{aligned}$$

so that inserting the latter in (4.9), we easily get

$$\begin{aligned}\int_{B^+(x_0,2\rho)}|\mathcal{G}_h-\mathcal{G}|^2\zeta^2\,dx\le\frac{C}{\rho}&\left(\int_\Omega|\mathcal{G}|^2\,dx\right)^{\frac{q-2}{2q}}\left(\int_{B^+(x_0,2\rho)}|u_h-u|^q\,dx\right)^{\frac{1}{q}}\\ &\times\left(\int_{B^+(x_0,2\rho)}|\mathcal{G}_h-\mathcal{G}|^2\zeta^2\,dx\right)^{\frac{1}{2}}\\ &+\left(\int_{B^+(x_0,2\rho)}|f_h-f|^p\,dx\right)^{\frac{1}{p}}\left(\int_{B^+(x_0,2\rho)}|u_h-u|^q\right)^{\frac{1}{q}},\end{aligned}$$

which in turn implies the following

$$\begin{aligned}\left(\int_{B^+(x_0,\rho)}|\mathcal{G}_h-\mathcal{G}|^2\,dx\right)^{\frac{1}{2}}\le\frac{C}{\rho}&\left(\int_\Omega|\mathcal{G}|^2\,dx\right)^{\frac{q-2}{2q}}\left(\int_{B^+(x_0,2\rho)}|u_h-u|^q\,dx\right)^{\frac{1}{q}}\\ &+\left(\int_{B^+(x_0,2\rho)}|f_h-f|^p\,dx\right)^{\frac{1}{2p}}\left(\int_{B^+(x_0,2\rho)}|u_h-u|^q\,dx\right)^{\frac{1}{2q}},\end{aligned}$$

We now divide both members by $|h|$, so to obtain

$$\left(\int_{B^+(x_0,\rho)}\left|\frac{\mathcal{G}_h-\mathcal{G}}{h}\right|^2\,dx\right)^{\frac{1}{2}}\le\frac{C}{\rho}\left(\int_\Omega|\mathcal{G}|^2\,dx\right)^{\frac{q-2}{2q}}\left(\int_{B^+(x_0,2\rho)}\left|\frac{u_h-u}{h}\right|^q\,dx\right)^{\frac{1}{q}}$$
$$+\left(\int_{B^+(x_0,2\rho)}\left|\frac{f_h-f}{h}\right|^p\,dx\right)^{\frac{1}{2p}}\left(\int_{B^+(x_0,2\rho)}\left|\frac{u_h-u}{h}\right|^q\,dx\right)^{\frac{1}{2q}}$$

Finally, we just observe that, by means of the characterization of Sobolev spaces in terms of integrated difference quotients (see [7]), we have

$$\int_{B^+(x_0,2\rho)}\left|\frac{u_h-u}{h}\right|^q\,dx\le C_N\int_\Omega|\nabla u|^q\,dx,$$

and

$$\int_{B^+(x_0,2\rho)}\left|\frac{f_h-f}{h}\right|^p\,dx\le C_N\int_\Omega|\nabla f|^p\,dx,$$

this allows us to conclude that

$$\left(\int_{B^+(x_0,\rho)}\left|\frac{\mathcal{G}_h-\mathcal{G}}{h}\right|^2\,dx\right)^{\frac{1}{2}}\le\frac{C}{\rho}\left(\int_\Omega|\nabla u|^q\,dx\right)^{\frac{1}{2}}+C_N\left(\int_\Omega|\nabla f|^p\right)^{\frac{1}{2p}}\left(\int_\Omega|\nabla u|^q\right)^{\frac{1}{2q}},\qquad(4.10)$$

that is $\mathcal{G}$ has a square-integrable weak derivative along the direction given by h. The same result applies choosing N linear independent direction $h_1,\dots,h_N$ satisfying $B^+(x_0,2\rho)+\lambda h_i\subset\Omega$, for $\lambda>0$ small enough. This finally gives $\mathcal{G}\in W^{1,2}(B^+(x_0,\rho))$, concluding the proof. □

As a consequence of Theorem 4.2 we get the following:

Corollary 4.3.

$$\sigma=\nabla\mathcal{H}^*(\nabla u)=|\mathcal{G}|^{\frac{q-2}{q}}\mathcal{G}\in W^{1,r}(\Omega),\qquad(4.11)$$

for suitable exponents $r=r(N,q)$ given by

$$r(N,q)=\begin{cases}2, & \text{if } N=q=2,\\ \text{any value } <2, & \text{if } N=2,\ q>2,\\ \frac{Nq}{(N-1)q+2-N}, & \text{if } N>2.\end{cases}$$

Proof. This is a consequence of the fact that the composition of a $W^{1,2}$ function (in our case $\mathcal{G}$) with a locally Lipschitz continuous function $F:\mathbb{R}^N\to\mathbb{R}^N$ that satisfies an estimate of the type $|\nabla F(z)|\le C|z|^\alpha$ (in our case $z\mapsto|z|^{1-2/q}z$), belongs to a Sobolev space whose summability may be computed explicitely thanks to the summability of the original function (which is in fact in L^{2^*}) and of its gradient.
Notice also that, should $\mathcal{G}$ be bounded, one would automatically get $\sigma\in W^{1,2}$. □

Remark 4. We observe that if we take $q > N-2$, then Theorem 4.2 and Sobolev Imbedding Theorems give that $u \in C^{0,\alpha}$, with $\alpha = 1-(N-2)/q$. Indeed (4.7) implies that $\mathcal{G} \in L^{2^*}(\Omega)$ and then

$$\int_\Omega (|\nabla u(x)|-1)_+^{\frac{qN}{N-2}}\ dx = \int_\Omega |\mathcal{G}(x)|^{\frac{2N}{N-2}}\ dx < +\infty,$$

which ensures that $\nabla u \in L^{\frac{qN}{N-2}}(\Omega)$.

Remark 5. The same arguments in the proof of Theorem 4.2 may obviously be applied to the case of uniformly elliptic equations, such as $\mathrm{div}(\nabla \mathcal{K}(\nabla u)) = f$ with $cI_N \le D^2\mathcal{K} \le CI_N$. In this case they provide global H^2 regularity results on Lipschitz domains.

Remark 6. Notice that we asked for a stronger regularity assumption on f than what is usually considered in standard elliptic regularity (where ∇u is Sobolev as soon as $f \in L^p$). Actually, in non-degenerate equations, when we arrive to the term $\int (f_h - f)(u_h - u)$, we can pass all the increments on the function u, thus getting something that may be estimated again by the norm of $\nabla u_h - \nabla u$ (but to the power of one, while at the left hand side it is to the power of two). Yet, here this is no more useful, since ∇u may not be obtained from $\mathcal{G}$. This is why we asked for a better regularity on f which could somehow shock the reader who is more familiar with the standard theory.

5 L^∞ estimate for the gradient

In this section we will prove that every solution $u \in W^{1,q}_\diamond(\Omega)$ of (4.6) is actually a Lipschitz function. The main ingredient is, as usual in Elliptic regularity, the fact that partial derivatives of u solve a linear elliptic equation and that convex functions of these partial derivatives are subsolutions of a similar equation. In this case the interesting quantity will be $(\partial u/\partial x_1 - 2)_+$, so that the equation becomes uniformly elliptic. We develop in this section all the techniques we need, so as to have a self-contained exposition and to show some interesting tools that allow to give elliptic regularity results and to expose them in a simple way.

We will start by approximating the problem (and hence the equation). The goal is twofold: get non-quantified regularity on the solution, so that one is allowed to differentiate it, and simplify the shape of Ω. Indeed, by taking only horizontal and vertical boundaries, we will directly consider the partial derivatives $\partial u/\partial x_i$ of u, instead of considering "local" derivative fields such as $b \cdot \nabla u$, for a vector field b which is either tangent or normal to $\partial\Omega$. This second method (using b) is usual in Neumann regularity but in our case, we saw that we got some extra terms in the equation whose summability was not known a priori. In order to get the L^∞ result we wanted, we would have been forced to add assumptions on the exponent q and 2^*, getting in the end restrictions on the dimension N. The method we give is on the contrary dimension-independent.

Lemma 5.1. *Let $\Omega_\varepsilon \supset \Omega$ be a sequence of domains converging to Ω in the sense $|\Omega_\varepsilon \setminus \Omega| \to 0$, all contained in a large bounded domain $U \subset \mathbb{R}^d$. Let $\tilde{\mathcal{H}}^*_\varepsilon$ be a sequence of convex functions increasingly converging to $\mathcal{H}^*$ and whose second derivatives are bounded from above, and set $\mathcal{K}_\varepsilon(z) = \varepsilon|z|^2 + \tilde{\mathcal{H}}^*_\varepsilon(z)$*

and f_ε a sequence of functions in $L^2(U)$ converging to $f1_\Omega$. Then, the functionals J_ε defined on $L^2(U)$ by

$$J_\varepsilon(u) = \begin{cases} \int_{\Omega_\varepsilon} \mathcal{K}_\varepsilon(\nabla u) - f_\varepsilon u & \text{if } u = 0 \text{ on } \Omega_\varepsilon^c \text{ and } u \text{ is } W^{1,2} \text{ on } \Omega_\varepsilon \\ +\infty & \text{otherwise} \end{cases}$$

$\Gamma-$converge with respect to the strong convergence in $L^2(U)$ to the functional

$$J(u) = \begin{cases} \int_\Omega \mathcal{H}^*(\nabla u) - fu & \text{if } u = 0 \text{ on } \Omega^c \text{ and } u \text{ is } W^{1,2} \text{ on } \Omega \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. For the $\Gamma - \liminf$ inequality take a sequence $u_\varepsilon \to u$: if each u_ε vanishes on Ω_ε^c is is evident by pointwise convergence that $u = 0$ in Ω^c. By strong convergence, it is clear that $\int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon \to \int_\Omega fu$. Moreover, one can use

$$\int_{\Omega_\varepsilon} \mathcal{K}_\varepsilon(\nabla u_\varepsilon) \geq \int_\Omega \mathcal{H}^*_{\varepsilon_0}(\nabla u_\varepsilon)$$

to get, thanks to the semicontinuity of this integral functional on Ω (which is a consequence of the convexity with respect to the gradient) and to the fact that $u_\varepsilon \to u$ in $L^2(\Omega)$,

$$\liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \mathcal{K}_\varepsilon(\nabla u_\varepsilon) \geq \int_\Omega \mathcal{H}^*_{\varepsilon_0}(\nabla u).$$

This proves that $u \in W^{1,2}(\Omega)$ and, passing to supremum in $\varepsilon_0 \to 0$ and combining with the previous observations, one gets

$$\liminf_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) \geq J(u).$$

For the opposite inequality, one takes a function u such that $J(u) < +\infty$ and observe that this clearly means that $u \in W^{1,q}(\Omega)$. By standard extension theorems (see [1], Theorem 4.32), being Ω regular enough, one may find a function $\tilde{u} \in W^{1,q}(\mathbb{R}^d)$ such that $\tilde{u} = u$ on Ω. Hence take $u_\varepsilon = \tilde{u}1_{\Omega_\varepsilon}$, where 1_{Ω_ε} stands for the characteristic function of Ω_ε. Since $\tilde{u} \in W^{1,2}(U)$ it is easy to see that $\varepsilon \int_{\Omega_\varepsilon} |\nabla \tilde{u}|^2 \to 0$ and that $u_\varepsilon \to u$ in $L^2(U)$: this implies the convergence of the term $\int f_\varepsilon u_\varepsilon$ to $\int fu$. For the other terms, notice that $\int_{\Omega_\varepsilon \setminus \Omega} \mathcal{H}^*_\varepsilon(\nabla u_\varepsilon) \leq \int_{\Omega_\varepsilon \setminus \Omega} \mathcal{H}^*(\nabla \tilde{u}) \to 0$ and that $\int_\Omega \mathcal{H}^*_\varepsilon(\nabla u_\varepsilon) \leq \int_\Omega \mathcal{H}^*(\nabla u)$. In the end one gets

$$\begin{aligned} \limsup_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) &\leq \lim_{\varepsilon\to 0} \varepsilon \int_{\Omega_\varepsilon} |\nabla \tilde{u}|^2 - \lim_{\varepsilon\to 0} \int_U f_\varepsilon u_\varepsilon + \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon \setminus \Omega} \mathcal{H}^*_\varepsilon(\nabla u_\varepsilon) + \limsup_{\varepsilon\to 0} \int_\Omega \mathcal{H}^*_\varepsilon(\nabla u_\varepsilon) \\ &\leq -\int_\Omega fu + \int_\Omega \mathcal{H}^*(\nabla u) = J(u), \end{aligned}$$

concluding. □

The previous Lemma allows us to pass to the limit in estimates which are valid under additional assumptions on Ω (which will be replaced by Ω_ε with boundaries on a square grid) and on the regularity of u (since the solution u_ε of the perturbated problems will be regular).

The other tools we need are the following.

Lemma 5.2. *Suppose $\partial\Omega$ is composed by faces which are parallel to the coordinate axes and that u is a solution of a problem*

$$\begin{cases} \operatorname{div}(\nabla\mathcal{K}(\nabla u)) = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

where $\mathcal{K}$ is a C^∞ convex function with $cI_N \le D^2\mathcal{K} \le CI_N$ and $D^2\mathcal{K}(z) \ge c_1 I_N$ for $|z| \ge 2$ and $f \in C^\infty(\Omega)$.

Then u is $C^\infty(\Omega) \cap H^2(\overline{\Omega})$ and its norms $||\partial u/\partial x_i||_{L^\infty(\Omega)}$ are uniformly bounded by a constant depending on $||\nabla u||_{L^2(\Omega)}$, $||f||_{Lip(\Omega)}$, on c_1 and on the constant c_Ω of the immersion of $W^{1,2}(\Omega)$ into $L^{\bar p}(\Omega)$ (for a fixed exponent $\bar p \le 2^*$).

Proof. The regularity of u comes from standard elliptic theory (see for instance [15]), while H^2 regularity can be obtainde by the same methods as in the previous section (see Remark 5) In particular, ∇u has a trace on the boundary and the Neumann condition is true in a pointwise almost everywhere meaning.

Now consider $v = \partial u/\partial x_1$ (or, equivalently, any other coordinate derivative): this function satisfies an elliptic boundary problem of the form

$$\begin{cases} \operatorname{div}(a\nabla v)) = f' & \text{in } \Omega \\ v\frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

The mixed boundary conditions come from the fact that either the direction of ν is the direction of x_1 (and in this case we have Dirichlet, say on a part $\Gamma_D \subset \partial\Omega$ of the boundary), either the direction is orthogonal, and this means that

$$\frac{\partial}{\partial \nu} v = \frac{\partial}{\partial x_1}\frac{\partial}{\partial \nu} u = 0$$

(i.e. Neumann on the remainding part Γ_N). The function f' is the derivative of f with respect to x_1 and $a(x)$ is the Hessian matrix of $\mathcal{K}$ computed at $\nabla u(x)$. In particular one has $a(x) \ge c_1 I_N$ if $v(x) \ge 2$. As usual, convex functions of the solution are subsolutions of the same equation in the sense that, if $w = h(v)$ and h is convex, one has

$$\int_\Omega a\nabla w \cdot \psi \le \int -f' h'(v)\psi, \qquad \text{for all } \psi \ge 0,\ \psi = 0 \text{ on } \Gamma_D.$$

Now take a function $h \ge 0$ so that $h(t) = 0$ for $t \le 2$. This implies two important facts: first, the matrix a will be considered in its ellipticity region only, since $w = 0$ on $a < c_1 I_N$; second, the test function $\psi = w$ itself is admissible. By considering $w_p = (v-2)_+^p$ (setting $w = w_1$ and $w_p = w^p$) one gets (using $h'(v) = p w_{p-1}$)

$$c_1 \int |\nabla w_p|^2 \le p \int |f'| w_{p-1} w_p \le p\ C \int w^{2p-1}.$$

Then one can use Sobolev Imbedding Theorem to get

$$\begin{aligned} ||w_p||^2_{L^{\bar p}} \le C(c_1, c_\Omega)||w_p||^2_{W^{1,2}(\Omega)} &\le p\ C(c_1, c_\Omega, ||f||_{Lip(\Omega)}) \left(||w||^{2p-1}_{L^{2p-1}} + ||w_p||^2_{L^2} \right) \\ &\le p\ C(c_1, c_\Omega, ||f||_{Lip(\Omega)}) ||w||^{2p}_{L^{2p}}, \end{aligned}$$

for an exponent $\bar{p} > 2$ admitted by the Sobolev immersion, that is $\bar{p} \leq 2^* = 2N/(N-2)$. This in turn implies

$$||w||^{2p}_{L^{p\bar{p}}} \leq p\ C||w||^{2p}_{L^{2p}}.$$

Taking a sequence p_n with $p_{n+1} = p_n\bar{p}/2$ and $p_0 = 2$, setting $y_n = \log ||w||_{L^{p_n}}$ one gets

$$y_{n+1} \leq \frac{\log p_n + C}{p_n} + y_n$$

and proving an L^∞ bound on w (and then on v and on $|\nabla u|$) means proving that these sequences $(y_n)_n$ are uniformly bounded. To do this, the summability of

$$\sum_n \frac{\log p_n + C}{p_n} < +\infty$$

is sufficient and this comes from the exponential behavior of p_n. The dependence of the bound on the L^2 norm comes from the starting datum y_0. □

Remark 7. The same proof could have worked under weaker assumptions on f, but $f' \in L^\infty$ was the easiest one to get easily a result which only requires $2^* > 2$.

Definition 3. Once fixed an exponent $\bar{p} \leq 2^*$, we say that Ω has the *bounded Sobolev cubic approximation property* (BSCAP for short) if the family of approximating sets Ω_ε has a bounded Sobolev immersion constant for $\varepsilon << 1$, i.e. there exists a constant C such that

$$||w||_{L^{\bar{p}}(\Omega_\varepsilon)} \leq C||w||_{W^{1,2}(\Omega_\varepsilon)} \quad \text{for all } w \in W^{1,2}(\Omega_\varepsilon),$$

where the sets Ω_ε are defined as follows:

$$\Omega_\varepsilon = \bigcup_{\{j \in \mathbb{Z}^N\ :\ \varepsilon j + \varepsilon[0,1]^N \cap \Omega \neq \emptyset\}} \varepsilon j + \varepsilon[0,1]^N.$$

From now on, we will denote by $Grid_\varepsilon$ the set of all cubes of size ε having their vertices on the regular grid $\varepsilon\mathbb{Z}^N$.

The following proofs and definitions will be devoted to getting a uniform bounds on the constant c_{Ω_ε} of the Sobolev immersions for the domains Ω_ε. This will be accomplished under the assumption of cone condition on Ω. Notice that it is well-known (see [1], Lemma 5.10) that the constant in the Sobolev immersion for a domain Ω which satisfies such a condition may be chosen so that it only depends on this cone. Yet, it is not so easy to get a uniform cone condition on all the Ω_ε from the condition on Ω. This is why we will develop a slightly different strategy, where obviously the cone condition will play a crucial role.

Definition 4. For given $J \in \mathbb{N}$, we call *Tetris piece of order J* any connected union of cubes in $Grid_1$ which is included in $[0, J]^N$ ("connected" meaning that the interior is connected). We will denote by T_J the set of all Tetris pieces of order J.

We may prove that an interior cone condition is sufficient for having the BSCAP: in order to do this, we first need a technical result.

Lemma 5.3. *Suppose that $\Omega \subset \mathbb{R}^N$ satisfies the interior cone condition: then there exists a number $J \in \mathbb{N}$ such that, if $Q \in Grid_\varepsilon$ is a cube of size ε with $Q \cap \Omega \neq \emptyset$, then at least one Tetris piece $S \in T_J$ has the property that, once translated and dilated so that Q coincides with a cube of εS, then all the other cubes of εS are included in Ω_ε and at least one of them is included in Ω.*

Proof. This fact is quite obvious, once we know that Ω satisfies the cone condition for a certain cone $\mathcal{C}$. Indeed, for every $\varepsilon > 0$ small enough, there exists a number k such that $\mathcal{C} \cap B(0, k\varepsilon)$ includes a ball of size $2\sqrt{N}\varepsilon$ in its interior and hence any possible rototranslation of $\mathcal{C} \cap B(0, k\varepsilon)$ includes at least one cube of $Grid_\varepsilon$: then take $J = k + 1$. Hence, if $Q \in Grid_\varepsilon$ is such that a point x_0 belongs to $Q \cap \Omega$ (and hence it is included in Ω_ε), just take the Tetris piece S composed by those cubes in $Grid_\varepsilon$ that intersect $x_0 + \varepsilon(\mathcal{C} \cap B(0, k))$. All the cubes in this configuration intersect this cone and hence Ω and one of them is included in $x_0 + \mathcal{C} \subset \Omega$. □

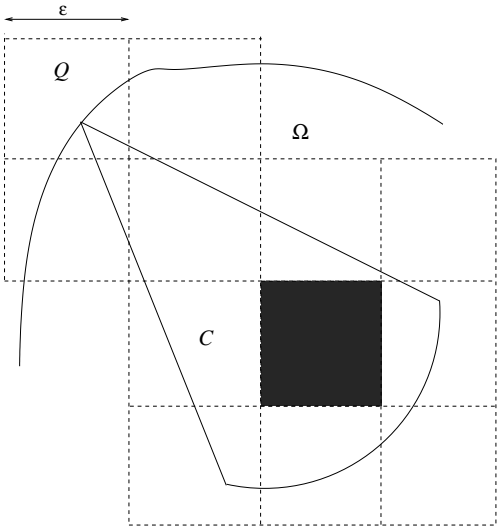

Figure 1: A cube Q and a Tetris piece (of order 4) containing it as in Lemma 5.3.

In order to go on, we will notice that, for every $2 \leq \bar{p} \leq 2^*$, there exist a universal constant $C = C(J)$ such that the following Sobolev-type inequality holds:

$$||u||_{L^{\bar{p}}(rS)} \leq Cr^{N/\bar{p}-N/2+1}||\nabla u||_{L^2(rS)} + C||u||_{L^{\bar{p}}(rQ)}, \quad \text{for every } u \in W^{1,2}(rS), \tag{5.1}$$

for any tetris piece $S \in T_J$, any cube $Q \in Grid_1$ with $Q \subset S$ and any $r > 0$. This is true because one can prove (5.1) for any pair of domains $(S, Q \subset S)$ and then take the worst constant among all pairs (S, Q) we are interested in. The dependence on r may be easily got by scaling.

To prove (5.1) on a fixed pair of bounded domains, there are several strategies: for $\bar{p} < 2^*$ one can argue by contradiction, taking advantage of the compact immersion of $W^{1,2}$ into $L^{\bar{p}}$. For $\bar{p} = 2^*$ one can first use the standard Sobolev inequality (with the L^2 norm of u on the whole domain at the right hand side) and then compose with a Poincaré-type inequality to get rid of the norm on the whole domain.

Hence we can prove the following:

Proposition 5.4. *Suppose that $\Omega \subset \mathbb{R}^N$ satisfies an interior cone condition: then Ω has the BSCAP.*

Proof. Fix $\varepsilon > 0$ and let u belong to $W^{1,2}(\Omega_\varepsilon)$: for any cube $Q_i \in Grid_\varepsilon$ composing Ω_ε take the rescaled Tetris piece εS_i coming from Lemma 5.3 and use inequality (5.1) to get

$$\int_{\varepsilon S_i} |u|^{\bar{p}} \leq C\varepsilon^{N+\bar{p}-N\bar{p}/2} \left(\int_{\varepsilon S_i} |\nabla u|^2 \right)^{\bar{p}/2} + C \int_{Q_{k(i)}} |u|^{\bar{p}},$$

choosing the cube $Q_{k(i)}$ so that it is included in Ω. Then sum up over i and notice that each cube of Ω_ε is used at most $(2J-1)^N$ times by the tetris pieces εS_i and each cube of Ω is used at most $(2J-1)^N$ times as a cube $Q_{k(i)}$ as well. On the contrary, each cube Q_i of Ω_ε is entirely covered by at least one tetris piece (exactly by S_i), and then one may write (with constants C dendending on J and N only)

$$\int_{\Omega_\varepsilon} |u|^{\bar{p}} \leq C\varepsilon^{N+\bar{p}-N\bar{p}/2} \sum_i \left(\int_{\varepsilon S_i} |\nabla u|^2 \right)^{\bar{p}/2} + C \int_{\Omega} |u|^{\bar{p}}.$$

By using the elementary inequality $\sum_i t_i^{\bar{p}/2} \leq (\sum_i t_i)^{\bar{p}/2}$ (which holds true since $\bar{p}/2 > 1$), and, again, the fact that every cube is used no more than $(2J-1)^N$ times, one has

$$\sum_i \left(\int_{\varepsilon S_i} |\nabla u|^2 \right)^{\bar{p}/2} \leq \left(\sum_i \int_{\varepsilon S_i} |\nabla u|^2 \right)^{\bar{p}/2} \leq C(J) \left(\int_{\Omega_\varepsilon} |\nabla u|^2 \right)^{\bar{p}/2} \leq C(J) ||u||^{\bar{p}}_{W^{1,2}(\Omega_\varepsilon)}.$$

Then one goes on with $\varepsilon^{N+\bar{p}-N\bar{p}/2} \leq 1$ (due to $\bar{p} \leq 2^*$) and $||u||_{L^{\bar{p}}(\Omega)} \leq c_\Omega ||u||_{W^{1,2}(\Omega)}$, and gets

$$\int_{\Omega_\varepsilon} |u|^{\bar{p}} \leq C||u||^{\bar{p}}_{W^{1,2}(\Omega_\varepsilon)} + C||u||^{\bar{p}}_{W^{1,2}(\Omega)}$$

which proves the thesis. □

Theorem 5.5. *Suppose that Ω has Lipschitz boundary and that f is Lipschitz continuous: then any solution of problem (4.6) is a Lipschitz function.*

Proof. First define the sets Ω_ε as in Definition 3 and the functions K_ε as in Lemma 5.1. Notice that, since Ω is Lipschitz, then one obviously has $|\Omega_\varepsilon \setminus \Omega| \to 0$.
On the approximating problems one may apply Lemma 5.2 and get uniform estimates on the solutions u_ε. These estimates are uniform since they give

$$||\nabla u_\varepsilon||_{L^\infty} \leq C(||f||_{Lip}, c_{\Omega_\varepsilon}, ||\nabla u_\varepsilon||_{L^2})$$

and all these quantities are bounded as $\varepsilon \to 0$, thanks to the $W^{1,2}$ bounds on u_ε and to the bound on c_{Ω_ε} (here we use the fact that Ω has the BSCAP, by means of Lemma 5.4).

After that, one notices that u_ε converge in $L^2(U)$ to a solution $\bar{u}$ of the limit problem, whose Euler-Lagrange equation in given by (4.6). The latter in general is not uniquely solvable: anyway, all the solutions of (4.6) share the same $\nabla\mathcal{H}^*(\nabla u)$. In particular if $\nabla\bar{u} \in L^\infty(\Omega)$, then the same happens for any other solutions. □

Acknowledgements G.C. and F.S. gratefully acknowledge the support of the *Agence Nationale de la Recherche* via the research project OTARIE and the three authors are thankful to the support of the Université Franco-Italienne via the mobility program *Galilée* "Modèles de réseaux de transport: phénomènes de congestion et de branchement" as well of the Università di Pisa through the program *Cooperazione Accademica Internazionale* "Optimal trasportation and related topics" .

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