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GRADED COMMUTATIVE ALGEBRAS: EXAMPLES, CLASSIFICATION, OPEN PROBLEMS

SOPHIE MORIER-GENOUD VALENTIN OVSIENKO

Abstract. We consider Γ-graded commutative algebras, where Γ is an abelian group. Starting from a remarkable example of the classical algebra of quaternions and, more generally, an arbitrary Clifford algebra, we develop a general viewpoint on the subject. We then give a recent classification result and formulate an open problem.

Keywords: Graded commutative algebras, quaternions, Clifford algebras
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1. Introduction

The algebra of quaternions. Our first example of graded commutative algebra is the classical algebra of quaternions, $\mathbb{H}$. This is a 4-dimensional associative algebra with the basis $\{1, i, j, k\}$ and relations expressed by the celebrated formula of Hamilton:

$$i^2 = j^2 = k^2 = i \cdot j \cdot k = -1.$$

It turns out that $\mathbb{H}$ is commutative in the following sense. Associate the "triple degree" to the basis elements:

$$\begin{aligned} \bar{\varepsilon} &= (0,0,0), \\ \bar{i} &= (0,1,1), \\ \bar{j} &= (1,0,1), \\ \bar{k} &= (1,1,0), \end{aligned} \tag{1}$$

viewed as an element of the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, see [8]. The usual product of quaternions then satisfies the condition

$$a\,b = (-1)^{\langle \bar{a}, \bar{b} \rangle}\, b\,a, \tag{2}$$

where a, b are homogeneous (i.e., proportional to the basis elements) and where $\langle\, ,\, \rangle$ is the usual *scalar product* of 3-vectors [1].

The purpose of this talk. Motivated by the above example, we define the notion of Γ-commutative algebra for an abelian group Γ. We compare our definition with other known versions of generalized commutativity: the classical supercommutativity, as wekk as with more recent notions of β-commutativity and of twisted commutative algebra.

We formulate a recent classification result characterizing the Clifford algebras as the only simple associative Γ-commutative algebras (see [9]).

[1] Indeed, $\langle \bar{i}, \bar{j} \rangle = 1$ and similarly for k, so that i, j and k anticommute with each other. But, $\langle \bar{i}, \bar{i} \rangle = 0$, so that i, j, k commute with themselves.

Our main goal is to attract an interest to this subject of a wide number of mathematicians working in different areas. We formulate an open problem and some perspectives of further development. The subject is on the crossroads of algebra and differential geometry. It is closely related to the theory of Lie superalgebras and geometry of supermanifolds.

2. What is a commutative algebra?

This question is not as naive as it seems to be.

The definition. The way we understand commutativity is as follows.

Definition 2.1. Let $(\Gamma, +)$ be an abelian group and

$$\langle\, ,\, \rangle : \Gamma \times \Gamma \to \mathbb{Z}_2 \tag{3}$$

a bilinear map. An algebra $\mathcal{A}$ will be called a Γ-graded commutative (or Γ-commutative) if $\mathcal{A}$ is Γ-graded:

$$\mathcal{A} = \bigoplus_{\gamma\in\Gamma} \mathcal{A}_\gamma, \qquad \mathcal{A}_\gamma \cdot \mathcal{A}_{\gamma'} \subset \mathcal{A}_{\gamma+\gamma'}$$

and the condition (2) is satisfied for all homogeneous elements $a, b \in \mathcal{A}$ of degree $\bar{a}, \bar{b}$, respectively.

Let us compare this definition with other ways to understand commutativity.

Classical supercommutativity. The classical notion of supercommutativity is a particular case of Definition 2.1 corresponding to $\Gamma = \mathbb{Z}_2$ and $\langle\, ,\, \rangle$ the standard product. A $\mathbb{Z}_2$-commutative algebra is called a *supercommutative algebra*.

The main examples of associative supercommutative algebras are the algebras of functions on supermanifolds. Note that these algebras cannot be simple (i.e., they always contain a non-trivial proper ideal).

The classical notion of supercommutativity is too rigid for our purpose.

The notion of β-commutative algebra. A more general notion of commutativity has been considered recently. Instead of the bilinear map (3) with values in $\mathbb{Z}_2$, one considers an arbitrary bilinear function $\beta : \Gamma \times \Gamma \to \mathbb{K}$ (a bicharacter) and assume the condition

$$a\, b = \beta(\bar{a}, \bar{b})\, b\, a.$$

If β is symmetric, then $\mathcal{A}$ is called β-commutative.

Simple β-commutative algebras were studied and classified in [3]. This notion is less restrictive than Definition 2.1.

Let us mention that an application of β-commutative algebras to deformation quantization is recently proposed in [6].

Twisted commutative algebras. A different framework due to S. Majid is related to Hopf algebra viewpoint. We cite [1, 2] as the most relevant references. Given a commutative associative algebra $\mathcal{C}$ and a function $F : \mathcal{C} \times \mathcal{C} \to \mathbb{K}^*$, define a *twisted* algebra structure $\mathcal{A} = (\mathcal{C}, \cdot_F)$, where the new product is given by

$$a \cdot_F b = F(a, b)\, ab. \tag{4}$$

It is very easy to check that $\mathcal{A}$ is associative if and only if F is a 2-cocycle:

$$F(ab, c)\, F(a, b) = F(a, bc)\, F(b, c). \tag{5}$$

The relation to the previous definitions is as follows. Consider a function F of the form $F(a,b) = q^{f(a,b)}$, where $q \in \mathbb{K}^*$. If f is bilinear (i.e., $f(a+b,c) = f(a,c)+f(b,c)$ and similarly on the second argument) then F obviously satisfies (5). If, in addition, the algebra $\mathcal{C}$ is Γ-graded and f depends only on the grading, i.e., $f(a,b) = f(\bar{a},\bar{b})$, then $\mathcal{A}$ is a β-commutative algebra structure with $\beta(\bar{a},\bar{b}) = q^{f(\bar{a},\bar{b})-f(\bar{b},\bar{a})}$.

3. A classification result

All the algebras and vector spaces we will consider are defined over the ground field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.

Universality of $(\mathbb{Z}_2)^n$-grading. It was proved in [9] that $\Gamma = (\mathbb{Z}_2)^n$ is the only group relevant for the notion of Γ-commutative algebra. Furthermore, one can always assume that the bilinear map $\langle\, , \rangle$ is the usual scalar product.

Theorem 1. *(i) If the abelian group Γ is finitely generated, then for an arbitrary Γ-commutative algebra $\mathcal{A}$, there exists n such that $\mathcal{A}$ is $(\mathbb{Z}_2)^n$-commutative.*

(ii) The bilinear map $\langle\, , \rangle$ can be chosen as the usual scalar product.

It is important to stress that there is no additional assumption for the algebra $\mathcal{A}$. We think that this preliminary result is useful for all kind of classification problems.

Statement of the problem. We are interested in simple Γ-commutative algebras. Recall that there are two different ways to understand simplicity.

(1) An algebra is called *simple* if it has no proper (two-sided) ideal.
(2) An algebra is called *graded-simple* if it has no proper (two-sided) ideal which itself is a graded subalgebra.

We will essentially use the (most classical) definition (1), but we will also take into account the definition (2) which is less restrictive.

In order to obtain a classification result, we assume that:

- $\dim \mathcal{A} < \infty$;
- $\mathcal{A}$ is associative.

The best known examples of a simple finite-dimensional associative algebras are of course the algebras M_n of $n \times n$-matrices. One is therefore led to the following natural question. *Is there a Γ-grading on the algebra M_n such that this algebra can be viewed as a Γ-commutative algebra (for a suitable abelian group Γ)?* Let us mention that gradings on the algebras of matrices M_n and more generally on associative algebras is an important subject, see [4, 5, 7] and references therein. However, the Γ-commutativity condition has not been studied thoroughly.

The Frobenius theorem. A simple associative and commutative (in the usual sense) algebra is necessarily a division algebra [2]. The associative division algebras are classified by (a particular case of) the classical Frobenius theorem. The result is well-known.

- In the complex case, there is a unique simple commutative associative algebra, namely $\mathbb{C}$ itself.
- In the real case, there is are exactly two simple commutative associative algebras: $\mathbb{R}$ and $\mathbb{C}$.

[2] The proof is elementary and nice, the reader is encouraged to find it.

The following result can be understood as a "graded version" of the Frobenius theorem.

Classification in the associative case. Here is the main result of [9].

Theorem 2. *Every finite-dimensional simple associative* Γ*-commutative algebra over* $\mathbb{C}$ *or over* $\mathbb{R}$ *is isomorphic to a Clifford algebra.*

The well-known classification of simple Clifford algebras (cf. [11]) readily gives a complete list:

(1) The algebras $\mathrm{Cl}_{2m}(\mathbb{C})$ are the only simple associative Γ-commutative algebras over $\mathbb{C}$. Note that $\mathrm{Cl}_{2m}(\mathbb{C})$ is isomorphic to the algebra $\mathrm{M}_{2^m}(\mathbb{C})$ of complex $2^m \times 2^m$-matrices.
(2) The real Clifford algebras $\mathrm{Cl}_{p,q}$ with $p-q \neq 4k+1$ and the algebras $\mathrm{Cl}_{2m}(\mathbb{C})$ viewed as algebras over $\mathbb{R}$ are the only real simple associative Γ-commutative algebras. Note that all these Clifford algebras are isomorphic to the algebras $\mathrm{M}_{2^m}(\mathbb{R}), \mathrm{M}_{2^m}(\mathbb{C}), \mathrm{M}_{2^m}(\mathbb{H})$

In particular, Theorem 2 answers the above question.

Corollary 3.1. *The algebra* M_n *can be realized as* Γ*-commutative algebra, if and only if* $n = 2^m$*.*

Let us show here that Clifford algebras are indeed Γ-commutative. We refer to [9] for a complete proof of Theorem 2.

Clifford algebras as $(\mathbb{Z}_2)^{n+1}$-commutative algebras. Recall that a real Clifford algebra $\mathrm{Cl}_{p,q}$ is an associative algebra with unit ε and $n = p + q$ generators $\alpha_1, \ldots, \alpha_n$ subject to the relations

$$\alpha_i\alpha_j = -\alpha_j\alpha_i, \qquad \alpha_i^2 = \begin{cases} \varepsilon, & 1 \leq i \leq p \\ -\varepsilon, & p < i \leq n. \end{cases}$$

The complex n-generated Clifford algebra $\mathrm{Cl}_n = \mathrm{Cl}_{p,q} \otimes \mathbb{C}$ can be defined by the same formulæ, but one can always choose the generators in such a way that $\alpha_i^2 = \varepsilon$ for all i.

Assign the following degree to every basis element (see [9]).

$$\begin{aligned} \overline{\alpha_1} &= (1, 0, 0, \ldots, 0, 1), \\ \overline{\alpha_2} &= (0, 1, 0, \ldots, 0, 1), \\ &\ldots \\ \overline{\alpha_n} &= (0, 0, 0 \ldots, 1, 1). \end{aligned} \tag{6}$$

One then has $\langle \alpha_i, \alpha_j \rangle = 1$ so that the anticommuting generators α_i and α_j become commuting in the $(\mathbb{Z}_2)^{n+1}$-grading sense. If follows that an n-generated Clifford algebra is $(\mathbb{Z}_2)^{n+1}$-commutative algebra, the bilinear map $\langle\, , \,\rangle$ being the usual scalar product.

Albuquerque-Majid [2] showed that Cl_n can actually be viewed as the group algebra $\mathbb{K}[(\mathbb{Z}_2)^n]$ twisted by some 2-cocycle. In particular, Cl_n is β-commutative over $(\mathbb{Z}_2)^n$, but the bilinear map β is more complicated than the scalar product.

Clifford algebras as twisted group algebras. Consider the algebra $(\mathbb{K}[(\mathbb{Z}_2)^n], \cdot_F)$ twisted by the 2-cocycle $F(a,b) = (-1)^{f(a,b)}$

$$F(a,b) = (-1)^{\sum_{i>j} a_i\, b_j}. \tag{7}$$

The cocycle condition (5) is obviously satisfied since f is bilinear.

The cocycle (7) has a nice and simple combinatorial meaning. The product in $\mathbb{K}[(\mathbb{Z}_2)^n]$ is a simple addition of n-tuples of 0 or 1. The twisted multiplication has an additional *sign rule*. Whenever we exchange "left-to-right" two units, we put the "$-$" sign [3]. For instance, $(0,1,0)\cdot(1,0,0) = -(1,1,0)$.

Proposition 3. [2] *In the complex case,* $\mathrm{Cl}_n \cong (\mathbb{K}[(\mathbb{Z}_2)^n], \cdot_F)$.

To obtain the real Clifford algebra $\mathrm{Cl}_{0,n}$, we will proceed in a slightly different way. Consider the *even subgroup* $(\mathbb{Z}_2)_0^{n+1} \subset (\mathbb{Z}_2)^{n+1}$ consisting in $(n+1)$-tuples of $0,1$ with even number of 1-entries.

Proposition 4. *The real Clifford algebra* $\mathrm{Cl}_{0,n} \cong \left(\mathbb{K}[(\mathbb{Z}_2)_0^{n+1}], \cdot_F\right)$.

Example 3.2. The algebra of quaternions $\mathbb{H}$ is the algebra of even triplets. For instance, one has:

$$i \cdot j \leftrightarrow (0,1,1)\cdot(1,0,1) = (1,1,0) \leftrightarrow k,$$

since the total number of exchanges is even and

$$j \cdot i \leftrightarrow (1,0,1)\cdot(0,1,1) = -(1,1,0) \leftrightarrow -k,$$

since the total number of exchanges is odd. In this way, one recovers the complete multiplication table of $\mathbb{H}$.

4. From graded algebras to geometry

Among a huge number of open problems we selected one challenging problem that looks particularly promising.

A large class of algebras can be viewed as Γ-commutative. This opens a possibility to apply algebraic geometry in order to define geometric objects. The key notions to be investigated are that of spectrum of a Γ-commutative algebra. We think that such a theory could in particular bring a new viewpoint to the Clifford analysis.

It worse noticing that, in the simplest $\mathbb{Z}_2$-commutative case, the theory is very well developed. However, even the notion of supermanifold is far of being obvious and involves a sophisticated technique of algebraic geometry. This notion is still intensively discussed in the literature.

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[3] We encourage the reader to check this.

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