Optimal double stopping time

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Abstract

We consider the optimal double stopping time problem defined for each stopping time S by $v(S) = \text{ess sup}\{\, E[\psi(\tau_1, \tau_2) \,|\, \mathcal{F}_S], \tau_1, \tau_2 \geq S \,\}$. Following the optimal one stopping time problem, we study the existence of optimal stopping times and give a method to compute them. The key point is the construction of a *new reward* ϕ such that the value function $v(S)$ satisfies $v(S) = \text{ess sup}\{\, E[\phi(\tau) \,|\, \mathcal{F}_S], \tau \geq S \,\}$. Finally, we give an example of an american option with double exercise time.

Key words : Optimal stopping; optimal multiple stopping; stopping times; hitting times; american options with double exercise time.

AMS 1991 subject classifications : 60G40.

Introduction

Our present work on the optimal double stopping times problem consists, following the optimal one stopping time problem, in proving the existence of the maximal reward, finding necessary or sufficent conditions for the existence of optimal stopping times, and giving a method to compute these optimal stopping times.

The results are well known in the case of the optimal one stopping time problem. Consider a *reward* given by a RCLL postive adapted process $(\phi_t,\ 0 \leq t \leq T)$ on $\mathbb{F} = (\Omega, \mathcal{F}, (\mathcal{F}_t,)\,_{0 \leq t \leq T}, P)$, $\mathbb{F}$ satisfying the usual conditions, and look for the maximal reward

$$v(0) = \sup\{\, E[\phi_\tau],\ \tau \in T_0 \,\},$$

where T_0 is the set of stopping times lesser than T. In order to compute $v(0)$ we introduce for each $S \in T_0$ the *value function* $v(S) = \text{ess sup}\{\, E[\phi_\tau \,|\, \mathcal{F}_S], \tau \in T_S \,\}$, where T_S is the set of stopping times in T_0 greater than S. The family $\{\, v(S), S \in T_0 \,\}$ can be *aggregated* in a RCLL adapted process $(v_t,\ 0 \leq t \leq T)$ such that $v_S = v(S)$. The process $(v_t,\ 0 \leq t \leq T)$ is the *Snell enveloppe* of $(\phi_t,\ 0 \leq t \leq T)$, that is the smallest supermartingale that dominates ϕ.

Moreover, when the reward $(\phi_t,\ 0 \leq t \leq T\,\}$ is continuous an optimal stopping time is given by

$$\theta^*(S) = \inf\{\, t \geq S,\ v_t = \phi_t \,\}. \tag{0.1}$$

We show in the present work that computing the value function for the optimal double stopping times problem

$$v(S) = \text{ess sup}\{\, E[\psi(\tau_1, \tau_2) \,|\, \mathcal{F}_S],\ \tau_1, \tau_2 \in T_S \,\},$$

reduces to computing the value function for an optimal one stopping time problem

$$u(S) = \text{ess sup}\{\, E[\phi(\theta) \,|\, \mathcal{F}_S],\ \theta \in T_S \,\},$$

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where the *new reward* ϕ is no longer a RCLL process but a family $\{\phi(\theta), \theta \in T_0\}$ of positive random variables which satisfy some compatibility properties.

In section 1, we revisit the optimal one-stopping time problem for admissible families. In section 2, we solve the optimal two-stopping times problem. In section 3, we give an example of american exchange option with double exercise time.

Let $\mathbb{F} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions of right continuity and augmentation by the null sets of $\mathcal{F} = \mathcal{F}_T$. We suppose that $\mathcal{F}_0$ contains only sets of probability 0 or 1. The time horizon is a fixed constant $T \in [0, \infty[$. We denote by T_0 the collection of stopping times of $\mathbb{F}$ with values in $[0, T]$. More generally, for any stopping times S, we denote by T_S (resp. T^S) the class of stopping times $\theta \in T_0$ with $S \leq \theta$ a.s (resp. $\theta \leq S$ a.s). Also, we will use the following notation: we note $t_n \uparrow t$ if $\lim_{n\infty} t_n = t$ with $t_n \leq t$ for each n.

1 The optimal one stopping time problem revisited

Definition 1.1. *We say that a family $\{\phi(\theta), \theta \in T_0\}$ is* admissible *if it satisfies the following conditions*

      1) for all $\theta \in T_0$ $\phi(\theta)$ is a $\mathcal{F}_\theta$-measurable positive random variable (r.v.),

      2) for all $\theta, \theta' \in T_0$, $\phi(\theta) = \phi(\theta')$ on $\{\theta = \theta'\}$,

Remark 1.1. *Note that if (ϕ_t) is a progressive process, then the family defined by $\{\phi(\theta) = \phi_\theta\}$, $\theta \in T_0\}$ is admissible.*

The *value function at time S*, where $S \in T_0$, is given by

$$v(S) = \operatorname{ess\,sup}_{\theta \in T_S} E[\phi(\theta) \,|\, \mathcal{F}_S]. \tag{1.2}$$

Proposition 1.1. *Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family, then the family of r.v $\{v(S), S \in T_0\}$ is admissible and is a* supermartingale *system, that is for any stopping times $\theta, \theta' \in T_S$ such that $\theta \geq \theta'$, $E[v(\theta) \,|\, \mathcal{F}_{\theta'}] \leq v(\theta')$ a.s,.*

Definition 1.2. *An admissible family $\{\phi(\theta), \theta \in T_0\}$ is said to be* right (resp. left) continuous *along stopping times in expectation if for any $\theta \in T_0$ and for any $\theta_n \downarrow \theta$ (resp. $\theta_n \uparrow \theta$) a.s. one has $E[\phi(\theta)] = \lim_{n \to \infty} E[\phi(\theta_n)]$.*

Recall the following classical lemma (see El Karoui (1979))

Lemma 1.1. *Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family right continuous along stopping times in expectation such that $E[\operatorname{ess\,sup}_{\theta \in T_0} \phi(\theta)] < \infty$. Then, the family $\{v(S), S \in T_0\}$ is right continuous along stopping times in expectation.*

We will now state the existence of of an optimal stopping time under quite weak assumptions. For each $S \in T_0$, let us introduce the following $\mathcal{F}_S$-measurable random variable $\theta^*(S)$ defined by

$$\theta^*(S) := \operatorname{ess\,inf}\{\theta \in T_S \,, v(\theta) = \phi(\theta) \text{ a.s}\} \tag{1.3}$$

Note that $\theta^*(S)$ is a stopping time. Indeed, for each $S \in T_0$, one can easily show that the set $\mathcal{T}_S = \{\theta \in T_S \,, v(\theta) = \phi(\theta) \text{ a.s}\}$ is closed under pairwise maximization. By a classical result, there exists a sequence $(\theta^n)_{n \in \mathbb{N}}$ of stopping times in $\mathcal{T}_S$ such that $\theta_n \downarrow \theta^*(S)$ a.s. Futhermore, we state the following theorem which generalizes the classical existence result of optimal stopping times to the case of a reward given by an admissible family of r.v. (instead of a RCLL adapted process).

Theorem 1.1. *Suppose that $E[\text{ess sup}_{\theta \in T_0} \phi(\theta)] < \infty$.*

Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family, right and left continuous along stopping times in expectation. Let $\{v(S), S \in T_0\}$ be the family of value function defined by (1.2). Then, for each $S \in T_0$, the stopping time $\theta^*(S)$ defined by (1.3) is optimal for $v(S)$ (i.e $v(S) = E[\phi(\theta^*(S)) \,|\, \mathcal{F}_S]$).

Remark 1.2. *When ϕ is given by a (RCLL) right continuous left limited adapted process, since the value function can be aggregated by a RCLL process $(v_t)_{t \in [0,T]}$, $\theta^*(S)$ satisfies equality (0.1) a.s.*

SKETCH OF THE PROOF : Fix $S \in T_0$. As in the case of a reward process, we begin by constructing a family of stopping times that are approximatively optimal. For $\lambda \in \,]0,1[$, define the $\mathcal{F}_S$-measurable random variable $\theta^\lambda(S)$ by

$$\theta^\lambda(S) := \text{ess inf}\{\,\theta \in T_S \,,\, \lambda v(\theta) \leq \phi(\theta) \text{ a.s}\,\} \tag{1.4}$$

For all $S \in T_0$, the function $\lambda \mapsto \theta^\lambda(S)$ is increasing on $\in\,]0,1[$ (and bounded above by $\theta^*(S)$). Let us now define, for $S \in T_0$, $\hat{\theta}(S) := \lim_{\lambda \uparrow 1} \uparrow \theta^\lambda(S)$.

Then , we prove that $\hat{\theta}(S)$ is optimal for $v(S)$ and that $\hat{\theta}(S) = \theta^*(S)$ by similar arguments as in the case of a reward process but the proof is shorter since it is no longer necessary to aggregate the value function. $\square$

2 The optimal two stopping times problem

We consider now the optimal two stopping times problem. Let us introduce the following definitions

Definition 2.3. *The family $\{\psi(\theta, S), \theta, S \in T_0\}$ is an* biadmissible family *if it satisfies*

  1) *for all $\theta, S \in T_0$, $\psi(\theta, S)$ is a $\mathcal{F}_{\theta \vee S}$-measurable positive random variable,*
  2) *for all $\theta, \theta', S, S' \in T_0$, $\quad \psi(\theta, S) = \psi(\theta', S')$ a.s on $\{\theta = \theta'\} \cap \{S = S'\}$,*

For each stopping time S, let us consider the value function

$$v(S) = \text{ess sup}_{\tau_1, \tau_2 \in T_S} E[\psi(\tau_1, \tau_2) \,|\, \mathcal{F}_S] \tag{2.5}$$

As in the case of one stopping time problem, the family $\{v(S), S \in T_0\}$ is an admissible family of postive r.v. and the family of positive r.v $\{v(S), S \in T_0\}$ is a supermartingale system.

2.1 Reduction to an optimal one stopping time problem

This optimal two-stopping times problem can be expressed in terms of one-stopping time problems as follows. For each stopping time $\theta \in T_S$, define the two $\mathcal{F}_\theta$-measurable random variables

$$u_1(\theta) = \text{ess sup}_{\tau_1 \in T_\theta} E[\psi(\tau_1, \theta) \,|\, \mathcal{F}_\theta], \quad u_2(\theta) = \text{ess sup}_{\tau_2 \in T_\theta} E[\psi(\theta, \tau_2) \,|\, \mathcal{F}_\theta]. \tag{2.6}$$

Note that by Proposition 1.1 since $\{\psi(\theta, S), \theta, S \in T_0\}$ is admissible, the families $\{u_1(\theta), \theta \in T_S\}$ and $\{u_2(\theta), \theta \in T_S\}$ are admissible. Put

$$\phi(\theta) = \max[u_1(\theta), u_2(\theta)], \tag{2.7}$$

the family $\{\phi(\theta), \theta \in T_S\}$, which is called the *new reward family*, is also clearly admissible. Define the $\mathcal{F}_S$-measurable variable

$$u(S) = \text{ess sup}_{\theta \in T_S} E[\phi(\theta) \,|\, \mathcal{F}_S] \quad \text{a.s} . \tag{2.8}$$

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Theorem 2.2. (Reduction) *Suppose that $\{\psi(\theta, S),\ \theta, S \in T_0\}$ is an admissible family. For each stopping time S, consider $v(S)$ defined by (2.5) and $u(S)$ defined by (2.6), (2.7), (2.8), then $v(S) = u(S)$ a.s.*

PROOF : Let S be a stopping time.
Step 1 : Let us show first that $v(S) \leq u(S)$ a.s.
Let $\tau_1, \tau_2 \in T_S$. Put $A = \{\tau_1 \leq \tau_2\}$. As A is in $\mathcal{F}_{\tau_1 \wedge \tau_2}$,

$$E[\psi(\tau_1, \tau_2)\,|\,\mathcal{F}_S] = E[\mathbf{1}_A E[\psi(\tau_1, \tau_2)\,|\,\mathcal{F}_{\tau_1 \wedge \tau_2}]\,|\,\mathcal{F}_S] + E[\mathbf{1}_{A^c} E[\psi(\tau_1, \tau_2)\,|\,\mathcal{F}_{\tau_1 \wedge \tau_2}]\,|\,\mathcal{F}_S].$$

By noticing that on A one has $E[\psi(\tau_1, \tau_2)\,|\,\mathcal{F}_{\tau_1 \wedge \tau_2}] = E[\psi(\tau_1, \tau_2)\,|\,\mathcal{F}_{\tau_1}] \leq u_2(\tau_1) \leq \phi(\tau_1 \wedge \tau_2)$ a.s, and similarly on A^c one has $E[\psi(\tau_1, \tau_2)\,|\,\mathcal{F}_{\tau_1 \wedge \tau_2}] = E[\psi(\tau_1, \tau_2)\,|\,\mathcal{F}_{\tau_2}] \leq u_1(\tau_2) \leq \phi(\tau_1 \wedge \tau_2)$ a.s, we get $E[\psi(\tau_1, \tau_2)\,|\,\mathcal{F}_S] \leq E[\phi(\tau_1 \wedge \tau_2)\,|\,\mathcal{F}_S] \leq u(S)$ a.s .
By taking the supremum over τ_1 and τ_2 in T_S we obtain step 1.
Step 2 : To simplify, suppose the existence of optimal stopping times for the three one-optimal stopping times problems. Let θ^* be an optimal stopping time for $u(S)$ *i.e*

$$u(S) = \text{ess sup}_{\theta \in T_S} E[\phi(\theta)\,|\,\mathcal{F}_S] = E[\phi(\theta^*)\,|\,\mathcal{F}_S] \quad \text{a.s},$$

and let θ_1^* and θ_2^* be optimal stopping times for $u_1(\theta^*)$ and $u_2(\theta^*)$, i.e. $u_1(\theta^*) = E[\psi(\theta_1^*, \theta^*)\,|\,\mathcal{F}_{\theta^*}]$ a.s. and $u_2(\theta^*) = E[\psi(\theta^*, \theta_2^*)\,|\,\mathcal{F}_{\theta^*}]$ a.s.
Put $B = \{u_1(\theta^*) \leq u_2(\theta^*)\}$. Let (τ_1^*, τ_2^*) be the stopping times defined by

$$\tau_1^* = \theta^* \mathbf{1}_B + \theta_1^* \mathbf{1}_{B^c}; \quad \tau_2^* = \theta_2^* \mathbf{1}_B + \theta^* \mathbf{1}_{B^c}, \tag{2.9}$$

As B is $\mathcal{F}_{\theta^*}$ measurable, and since $u(S) = E[\phi(\theta^*)\,|\,\mathcal{F}_S]$, we have

$$u(S) \quad = \quad E[\mathbf{1}_B u_2(\theta^*) + \mathbf{1}_{B^c} u_1(\theta^*)\,|\,\mathcal{F}_S].$$

Hence, we have $u(S) = E[\mathbf{1}_B \psi(\theta^*, \theta_2^*) + \mathbf{1}_{B^c} \psi(\theta_1^*, \theta^*)\,|\,\mathcal{F}_S] = E[\psi(\tau_1^*, \tau_2^*)\,|\,\mathcal{F}_S]$.

Hence, $u(S) \leq v(S)$ a.s. Note that in the general case, Step 2 can be also be obtained, for example by using optimizing sequences of stopping times. $\square$

Remark 2.3. *Note that step 2 gives a constructive method of optimal stopping times. Indeed, let θ^* be optimal for $u(S)$, let θ_2^* be optimal for $u_2(\theta^*)$ and let θ_1^* be optimal for $u_1(\theta^*)$. Put $B = \{u_1(\theta^*) \leq u_2(\theta^*)\}$, then the pair of stopping times (τ_1^*, τ_2^*) defined by (2.9) is optimal for $v(S)$. Note that step 1 gives also a necessary condition of optimality (but not a characterization in general).*

Before studing the problem of existence of optimal stopping times, we have to state some regularity properties of the new reward family $\{\phi(\theta),\ \theta \in T_0\}$.

2.2 Regularity properties of the new reward family

As in the case of one stopping time, we will need some regularity properties on the reward family ψ. Let us introduce the following definition,

Definition 2.4. *Suppose that $E[\text{ess sup}_{\theta, S \in T_0} \psi(\theta, S)] < \infty$.*
An admissible family $\{\psi(\theta, S),\ \theta, S \in T_0\}$ is said to be uniformly right (resp. left) continuous in expectation along stopping times *if for each $\theta, S \in T_0$ and each sequences (θ_n), (S_n) of T_0 such that $\theta_n \downarrow \theta$ a.s. and $S_n \uparrow S$ a.s., we have $\lim_{n\infty} E\left(\text{ess sup}_{\theta \in T_0} |\psi(\theta, S) - \psi(\theta, S_n)|\right) = 0$ and $\lim_{n\infty} E\left(\text{ess sup}_{\theta \in T_0} |\psi(S, \theta) - \psi(S_n, \theta)|\right) = 0$. (resp. $\lim_{n\infty} E\left(\text{ess sup}_{S \in T_0} |\psi(\theta, S) - \psi(\theta_n, S)|\right) = 0$ and $\lim_{n\infty} E\left(\text{ess sup}_{S \in T_0} |\psi(S, \theta) - \psi(S, \theta_n)|\right) = 0$).*

The following continuity property holds true for the new reward family :

Theorem 2.3. *Suppose that the admissible family $\{\,\psi(\theta, S),\ \theta, S \in T_0\,\}$ is uniformly right continuous(resp. left continuous) in expectation along stopping times.*
Then, the family $\{\,\phi(S),\ S \in T_0\,\}$ defined by (2.7) is right continuous (resp. left continuous) in expectation along stopping times.

PROOF : Let us show the right continuity property. As $\phi(\theta) = \max[u_1(\theta), u_2(\theta)]$, it is sufficent to show the right continuity property for the family $\{\,u_1(\theta),\ \theta \in T_0\,\}$.
Let us introduce the following value function for each $S, \theta \in T_0$,

$$U_1(\theta, S) = \operatorname{ess\,sup}_{\tau_1 \in T_\theta} E[\psi(\tau_1, S)\,|\,\mathcal{F}_\theta] \quad \text{a.s.} \tag{2.10}$$

As for all $\theta \in T_0$, $u_1(\theta) = U_1(\theta, \theta)$ a.s, it is sufficient to prove that $\{\,U_1(\theta, S), \theta, S \in T_0\,\}$ is right continuous in expectation along stopping times.
Let $\theta \in T_0$ and $(\theta_n)_n$ a nonincreasing sequence of stopping times in T_θ that converges to θ a.s.
Let $S \in T_0$ and $(S_n)_n$ a nonincreasing sequence of stopping times in T_S that converges to S a.s.
We have,

$$|E[U_1(\theta, S)] - E[U_1(\theta_n, S_n)]| \quad \leq \quad |E[U_1(\theta, S)] - E[U_1(\theta_n, S)]| + |E[U_1(\theta_n, S)] - E[U_1(\theta_n, S_n)]|.$$

The first term of the right hand side converges a.s to 0 as n tends to ∞ by the right continuity in expectation of the value function $\{U_1(\theta, S), \theta \in T_0\}$ by lemma 1.1.
Also, the seond term converges to 0 since $\{\,\psi(\theta, S),\ \theta, S \in T_0\,\}$ is uniformly right continuous in expectation along stopping times.
For the left continuity property of the new reward, the arguments are the same. $\quad\square$
We will now turn to the problem of existence of optimal stopping times.

2.3 Existence of optimal stopping times

Under uniform right and left continuity in expectation along stopping times of the admissible family $\{\,\psi(\theta, S),\ \theta, S \in T_0\,\}$, we can establish the existence of optimal stopping times.

Theorem 2.4. *Let $\{\,\psi(\theta, S),\ \theta, S \in T_0\,\}$ be an admissible family which is uniformly right and left continuous in expectation along stopping times. Suppose that there exists $p > 1$ such that $E[\operatorname{ess\,sup}_{\theta, S \in T_0}\psi(\theta, S)^p] < \infty$.*
Then, there exists a pair (τ_1^*, τ_2^*) of optimal stopping times for $v(S)$ defined by (2.5), that is

$$v(S) = \operatorname{ess\,sup}_{\tau_1, \tau_2 \in T_S} E[\psi(\tau_1, \tau_2)\,|\,\mathcal{F}_S] = E[\psi(\tau_1^*, \tau_2^*)\,|\,\mathcal{F}_S].$$

PROOF : By Theorem 2.3 the admissible family of positive r.v $\{\,\phi(\theta),\ \theta \in T_0\,\}$ defined by (2.7) is right and left continuous in expectation along stopping times. By Theorem 1.1 the stopping time

$$\theta^*(S) = \operatorname{ess\,inf}\{\,\theta \in T_S \mid v(\theta) = \phi(\theta)\,\text{a.s}\,\},$$

is optimal for $u(S) = v(S)$, that is $u(S) = \operatorname{ess\,sup}_{\theta \in T_S} E[\phi(\theta)\,|\,\mathcal{F}_S] = E[\phi(\theta^*)\,|\,\mathcal{F}_S]$ a.s .
Moreover, the families $\{\,\psi(\theta, \theta^*),\ \theta \in T_{\theta^*}\,\}$ and $\{\,\psi(\theta^*, \theta),\ \theta \in T_{\theta^*}\,\}$ are clearly admissible and are right and left continuous in expectation along stopping times. Consider the following optimal stopping time problem :

$$v_1(S) = \operatorname{ess\,sup}_{\theta \in T_S} E[\psi(\theta, \theta^*)\,|\,\mathcal{F}_S] \quad \text{and} \quad v_2(S) = \operatorname{ess\,sup}_{\theta \in T_S} E[\psi(\theta^*, \theta)\,|\,\mathcal{F}_S].$$

Let θ_1^* and θ_2^* are optimal stopping times given by Theorem 1.1 for $v_1(\theta^*)$ and $v_2(\theta^*)$. Note that $v_1(\theta^*) = u_1(\theta^*)$ and $v_2(\theta^*) = u_2(\theta^*)$.
The existence of optimal stopping times (τ_1^*, τ_2^*) for problem (2.5) follows by Remark 2.3. $\quad\square$

3 Example: an exchange option

Suppose that the market contains two risky assets with price processes $X_t = (X_t^1, X_t^2)$ which satisfy $dX_t^1 = rX_t^1 dt + \sigma_1 X_t^1 dW_t^1$ and $dX_t^2 = rX_t^2 dt + \sigma_2 X_t^2 dW_t^2$, where $W = (W^1, W^2)$ is a $\mathbb{R}^2$-valued brownian motion , and $(\mathcal{F}_t)_{t \geq 0}$ is the associated standard filtration. Without loss of generality, we first suppose that the interest rate r is equal to 0 (the general case can be derived by considering discounted prices). The buyer of the option chooses two stopping times τ_1 (for the first asset) and τ_2 (for the second one) smaller than T. At time $\tau_1 \vee \tau_2$, he is allowed to exchange the first asset against the second one; in other terms, he receives the amount: $(X_{\tau_1}^1 - X_{\tau_2}^2)^+$.

Given the initial data $(s, x) = (s, x_1, x_2) \in [0, T] \times \mathbb{R}^2$, the price at time s of this option is given by

$$v(s, x) = \operatorname*{ess\,sup}_{\tau_1, \tau_2 \in T_s} E[(X_{\tau_1}^{1, s, x_1} - X_{\tau_2}^{2, s, x_2})^+ | \mathcal{F}_s], \tag{3.11}$$

Given initial data $(s, x_1) \in [0, T] \times \mathbb{R}$ consider also $v^1(s, x_1, y_2) = \operatorname*{ess\,sup}_{\tau_1 \in T_s} E_{s, x_1}[(X_{\tau_1}^1 - y_2)^+]$, for each fixed parameter $y_2 \in \mathbb{R}$. We have clearly that $v^1(s, x_1, y_2) = y_2 C_1(s, \frac{x_1}{y_2})$, where for initial data $(s, z) \in [0, T] \times \mathbb{R}$, $C_1(s, z) = E_{s, z}[(X_T^1 - 1)^+]$, which is explicitely given by the Black-Sholes formula. Similarly, given initial data $(t, y_2) \in [0, T] \times \mathbb{R}$, consider $v^2(x_1, t, y_2) = \operatorname*{ess\,sup}_{\tau_2 \in T_t} E_{t, y_2}[(x_1 - X_{\tau_2}^2)^+]$ for each fixed parameter $x_1 \in \mathbb{R}$. We have clearly that $v^2(x_1, t, y_2) = x_1 P_2(t, \frac{y_2}{x_1})$, where for initial data $(s, z) \in [0, T] \times \mathbb{R}$, $P_2(t, z) = E_{t, z}[(1 - X_T^2)^+]$. Note that the function $P_2(s, z)$ corresponds to the price of an european put option. We set

$$\phi(s, x_1, x_2) = \max\left(x_2 C_1(s, \frac{x_1}{x_2}), x_1 P_2(s, \frac{x_2}{x_1}) \right), \text{ for } s, x_1, x_2 \in [0, T] \times \mathbb{R}^2.$$

By the previous results, the price function v is the value function of the following optimal stopping time problem:

$$v(s, x_1, x_2) = \operatorname{ess\,sup}_{\theta \in T_s} E_{s, x_1, x_2}[\phi(\theta, X_\theta^1, X_\theta^2)].,$$

It corresponds to the price of an American option with maturity T and payoff $(\phi(t, X_t^1, X_t^2))_{0 \leq t \leq T}$. By classical results, it is a viscosity solution of an obstacle problem. By Remark 2.3, we can easily construct a pair (τ_1^*, τ_2^*) of optimal stopping times for the exchange option (3.11).

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