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A VERY ABSOLUTE Π_2^1 REAL SINGLETON

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0. Introduction

By Shoenfield absoluteness theorem, the Π_2^1 real singletons are the simplest, non constructible, projective reals. Very few Π_2^1 non constructible real singletons exist in the literature.

One of them is the real $0^\#$ introduced by Solovay in [9]. It is ‘provably Π_2^1 singleton’, that means that there is a Π_2^1 formula φ such that for any model M of ZF that contains $0^\#$ as an element, $0^\#$ is—in M —the only solution of φ . The existence of $0^\#$ needs a large cardinal property as Ramsey or measurable and $0^\#$ is not generic over L by means of a constructible set of conditions.

Other ones have been constructed by forcing over L (see Jensen and Solovay [8] and Jensen [5]). They are not ‘provably Π_2^1 singleton’; they only are Π_2^1 singleton ‘at home’, that means there is a Π_2^1 formula φ such that the real r is the only solution of φ in $L(r)$, but (except in trivial cases—for instance if M and $L(r)$ have the same reals) it is not true that if M is another model containing r as an element, r remains—in M —the only solution of φ . Of course, by Shoenfield theorem, r remains a solution of φ but not necessarily the only one.

Later Jensen and Johnsraten constructed, also by forcing, a new and more absolute Π_2^1 singleton r : r is the only solution—in M —of some Π_2^1 formula φ as far as M satisfies: $\omega_1 = \omega_1^L$.

In the sense of absoluteness, this real is the ‘best’ Π_2^1 singleton known, except $0^\#$. Other Π_2^1 singletons with various properties have been constructed (see [2, 3, 4]), but—in view of absoluteness—since they are constructed by use of the Jensen and Johnsraten method, they are not different from that one.

A natural question is: find a ‘provably’ Π_2^1 non constructible singleton r such that $0^\# \notin L(r)$? It seems to be a difficult problem. This paper is devoted to the proof of a theorem which is—perhaps—a step to a solution of this problem.

Theorem 1. *Let M be a transitive model of $ZF + V = L$. There is an M definable*

class P of forcing conditions and a Π_2^1 formula φ such that, if N is a P generic extension of M , then:

- (1) N and M have the same cardinals and the same cofinality function.
- (2) There is a real r in N such that N satisfies:

$$\text{ZF} + \neg 0^\# + V = L(r) + r \notin L + \varphi(r) + \exists! x \varphi(x).$$

- (3) If \bar{N} is a generic extension of N (by a set of conditions), then \bar{N} satisfies: $\exists! x \varphi(x)$.

Note. (1) The real given by the theorem is not provably Π_2^1 singleton. I shall discuss and make some comments at the end of this paper.

(2) If $0^\#$ exists, I may assume that this real r is such that $r \in L(0^\#)$; but r is not Π_2^1 singleton in $L(0^\#)$.

(3) The conclusion of (3) in the theorem remains true for some other extensions of N : for instance if \bar{N} is a generic extension of N by use of the class P of conditions where P is:

- (i) the forcing to add a Cohen subset of $[\alpha, \alpha^+]$ for all the cardinals α ,
- (ii) the forcing to deny the generalized continuum hypothesis.

On the other way there are classes Q of conditions such that r does not remain Π_2^1 singleton in a Q generic extension of N .

The proof of this theorem will use the method developed by Jensen in his ‘Coding the universe by a real’ [6]. A basic and important modification has to be done, so the entire proof — and not only one or two points — has to be reexamined. It will be of course too long and fastidious — and useless — to rewrite the whole proof in this new context. So, in a first step I shall prove a theorem which is the basic idea of the theorem and the ‘building block’ of it. Then, while proving the theorem, I shall invite the reader to work with a copy — and/or a good knowledge — of Jensen’s theorem by himself.

1. The building block

This section is devoted to the proof of the following:

Theorem 2. *Let M be a transitive model of $\text{ZF} + V = L$. There is an M definable set P of conditions and a Π_2^1 formula φ such that if N is a P generic extension of M , then:*

- (1) N and M have the same cardinals and the same cofinality function.
- (2) There is a real a in N such that N satisfies:

$$V = L(a) + \varphi(a) + \exists! x \varphi(x).$$

- (3) If \bar{N} is a generic extension of N by use of a set of conditions with the strong

ω_2 -chain condition (i.e.: $\forall x \subset C (\bar{x} = \omega_2 \Rightarrow \exists y \subset x (\bar{y} = \omega_2 \text{ and } \forall c, c' \in y, c \text{ and } c' \text{ are compatible}))$), then \bar{N} satisfies: $\exists! x \varphi(x)$.

This real comes from something as a three steps forcing extension. I work in a model $M_0 = M$ of $V = L$.

1.1. The trees

I define a sequence $(T(n))_{n \in \omega}$ of Suslin trees of height ω_2 in the following way (it is an easy generalisation of Jensen's construction of a Suslin tree of height ω_1). If T is a tree I shall denote T^α the α th level of T and $T \upharpoonright \alpha = \bigcup_{\beta < \alpha} T^\beta$. The $T(n)$ will be such that: $x \in T^\alpha(n) \Rightarrow x : \alpha \rightarrow 2$. The levels of the $T(n)$ are defined by induction as follows:

$$x \in T^0(n) \text{ iff } x = \emptyset,$$

$$x \in T^{\alpha+1}(n) \text{ iff } x = y \cup \{(\alpha, i) \mid i = 0 \text{ or } 1\} \text{ for some } y \in T^\alpha(n_2)$$

for limit α ,

$$\text{cf}(\alpha) = \omega : x \in T^\alpha(n) \text{ iff } x : \alpha \rightarrow 2 \ \& \ \forall \lambda < \alpha \ x \upharpoonright \lambda \in T^\lambda(n) \\ (\text{since } \bar{\omega}_2^\omega = \omega_1, \bar{T}^\lambda(n) = \omega_1)$$

for limit λ ,

$$\text{cf}(\lambda) = \omega_1.$$

The elements of $T^\lambda(n)$ are branches in $T(n) \upharpoonright \lambda$ chosen (in a classical way) by the following forcing:

Let η_λ be the least η such that $(T(n) \upharpoonright \lambda)_{n \in \omega}$ is in L_η , and $L_\eta \models \text{ZF}^- + \bar{\lambda} = \omega_1$. Let

$$F_\lambda = \{(\alpha, f) \in L_{\eta_\lambda} \mid \alpha < \omega_1 \ \& \ f = (f_n)_n \ f_n : \alpha \rightarrow T(n) \upharpoonright \lambda \text{ s.t.} \\ \forall n, m \in \omega \ \forall \beta, \bar{\beta} < \alpha \ |f_n(\beta)| = |f_m(\bar{\beta})|\},$$

where $|x|$ is the level of x . Set $(\alpha, f) \leq (\bar{\alpha}, \bar{f})$ iff $\alpha \geq \bar{\alpha}$ and

$$\forall n \in \omega \ \forall \beta < \bar{\alpha} \ f_n(\beta) \supseteq \bar{f}_n(\beta)$$

(in the ordering of $T(n) \upharpoonright \lambda$).

Because of the definition of $T(n)$ in the case of cofinality ω , it is clear that F_λ is an ω -closed set of conditions and so there is — in \mathbb{L} — an F_λ generic over L_{η_λ} . The L -least one gives ω_2 branches in $T(n) \upharpoonright \lambda$. They are the elements of $T^\lambda(n)$. This achieves the definition of the $T(n)$.

Lemma. Let a be a finite subset of ω , and $T_a = \prod_{n \neq a} T(n)$. That is: a condition is a sequence $(x_n)_{n \in \omega - a}$ such that $x_n \in T(n)$; and $(x_n)_n \leq (y_n)_n$ iff $\forall n \in \omega - a \ x_n \supseteq y_n$. Then T_a is ω -closed and satisfies the ω_2 -chain condition. Moreover let G be T_a generic over L and $i \in a$, then $L(G)$ satisfies: $T(i)$ is Suslin.

Proof. Although the proof is classical, I shall give it since I shall refer to it later. T_a is clearly ω -closed. Let A be a maximal antichain in T_a . Let X be the L -least such that:

$$X < L_{\omega_3} \ \& \ \{A\} \cup \omega_1 \subset X \text{ and if } \lambda = \omega_2 \cap X, \text{ then } \lambda \in \omega_2 \text{ and } \text{cf}(\lambda) = \omega_1$$

(clearly such an X exists). Let $\Pi: L_\beta \cong X$, so $\lambda = \omega_2 \cap X = \Pi^{-1}(\omega_2)$. Let g be a function from $\omega - a$ into ω_1 and

$$D_g = \{(\alpha, f) \in F_\lambda \mid (f_n(g(n)))_{n \in \omega - a} \text{ is compatible with some element of } A\}.$$

It is easy to see that D_g is a dense subset of F_λ and that $D_g \in L_{\eta_\lambda}$ (since $\eta_\lambda > \beta$ because $L_\beta \vDash \lambda = \omega_2$ whereas $L_{\eta_\lambda} \vDash \bar{\lambda} = \omega_1$). It follows easily that $A = A \cap \lambda$ and so $\bar{A} \leq \omega_1$.

Assume now $i \in a$, set $b = a - \{i\}$; then $T_b = T_a * T(i)$ (where $*$ means the iteration of forcing) satisfies the ω_2 -chain condition and so $T(i)$ satisfies in $L(G)$ the ω_2 -chain condition. \square

1.2. Coding the branches

For $n \in \omega$ and B_n a branch in $T(n)$ (in some generic extension of L). I shall define a set Q_n of conditions to code B_n by a subset of ω_1 . It is a slight modification of the coding by use of almost disjoint sets and one of the basic ideas of proof of the theorem.

1.2.1

Let $\alpha < \beta < \omega_1$ be such that: $L_\beta \vDash \alpha = \omega_1 + \text{ZF}^- + \forall x (\bar{x} \leq \omega_1) + \forall \lambda (\text{cf}(\lambda) = \alpha \Rightarrow \eta_\lambda \text{ exists})$ (I shall denote $\Theta(\alpha, \beta)$ this property). I can construct in L_β the trees $T(n)$ exactly as I construct in L_{ω_2} the right ones. I shall denote $T_\beta(n)$ these trees.

Claim 1. Assume $\Theta(\alpha, \beta)$ and $\Theta(\alpha, \bar{\beta})$ and $\beta \leq \bar{\beta}$, then

$$T_\beta(n) = T_{\bar{\beta}}(n) \cap L_\beta.$$

This is an easy consequence of:

Claim 2. For $\gamma < \beta$:

- (1) if $L_{\bar{\beta}} \vDash \text{cf}(\gamma) = \omega$, then $L_\beta \vDash \text{cf}(\gamma) = \omega$;
- (2) if $f \in L_{\bar{\beta}} f: \omega \xrightarrow{\text{inj.}} T(n) \upharpoonright \gamma$, then $f \in L_\beta$.

Proof. (1) Let $f: \omega \xrightarrow{\text{cof., inj.}} \gamma f \in L_{\bar{\beta}}$; $g \in L_\beta g: \gamma \xrightarrow{\text{bij.}} \alpha$, then $g \circ f \in L_{\bar{\beta}}$ and $g \circ f: \omega \rightarrow \alpha$ so $g \circ f \in L_\alpha \subset L_\beta$ so $f = ((g \circ f)_0^{-1} g)^{-1} \in L_\beta$.

- (2) let $g \in L_\beta g: T(n) \upharpoonright \gamma \xrightarrow{\text{bij.}} \alpha$; then $g \circ f \in L_\alpha$ and so $f \in L_\beta$. \square

1.2.2

Fix n ; let B_n be a branch in $T(n)$. I define the forcing Q_n in the following way. (I shall omit the subscript n when not necessary.)

Claim 1. There is a definable function $S: T \rightarrow P(\omega_1)$ such that:

- (1) if $x \neq y$ $\overline{S(x) \cap S(y)} \leq \omega$;
- (2) if $\Pi: L_\beta \rightarrow L_{\omega_2}$ is an elementary embedding such that $\pi(\alpha) = \omega_1$, then $\Pi(S(x) \cap \alpha) = S(\Pi(x))$.

Proof. Easy.

A condition in Q_n is a pair (p, p') such that:

(1) $p: |p| < \omega_1 \rightarrow 2$ such that for $\alpha \leq |p|$ and β if $\Theta(\alpha, \beta)$ and $L_\beta(p \upharpoonright \alpha) \models \text{ZF}^- + \alpha = \omega_1 = \omega_1^L$, then: $\{x \in T_\beta \mid S(x) \cap \bar{p} \cap \alpha \cap E \text{ is bounded in } \alpha\}$ is a cofinal branch in T_β ; (where $\bar{p} = \{\gamma < |p| \mid p(\gamma) = 1\}$ and $E = \{\lambda + 2n \mid \lambda \text{ limit; } n \in \omega\}$).

(2) $\dot{p} \subset B \times \omega_2$; $\bar{p}' \leq \omega$ and for $(x, \eta) \in p'$ $(S(x) - \eta) \cap \bar{p} = \emptyset$. Set $(p, p') \leq (q, q')$ if $p \supset q$ and $p' \supset q'$.

Note. So Q is the usual forcing to code B by a subset of ω_1 , except the fact that I ensure that this subset will have enough properties when later I shall use a condensation argument.

Lemma 1. Let G be Q generic over $L(B)$; then there is a subset A of ω_1 such that:

- (1) $L(B)(G) = L(A)$;
- (2) for $x \in T$, $x \in B$ iff $S(x) \cap A \cap E$ is bounded in ω_1 ;
- (3) if $\Theta(\alpha, \beta)$ and $L_\beta(A \cap \alpha) \models \text{ZF}^- + \alpha = \omega_1$, then $\{x \in T_\beta \mid S(x) \cap A \cap E \text{ is bounded in } \alpha\}$ is a cofinal branch in T_β .

Proof. The only non usual fact is that for any condition (p, p') and $|p| \leq \gamma < \omega_1$ there is a $q: \gamma \rightarrow 2$ such that (q, q') is a condition stronger than (p, p') (it is for that reason, I code B only on the even ordinals (to keep place for other things). Also note that since — later — I shall work with the Q_n all together, I have in fact to use a partition of ω_1 into ω parts).

To do that it is enough to put on the odd ordinals between $|p|$ and $|p| + \omega$ an order of type γ so that there are no $\alpha \in]|p|, \gamma]$ and β such that $L_\beta(q \upharpoonright \alpha) \models \text{ZF}^- + \alpha = \omega_1$. \square

The following is crucial:

Lemma 2. Q is ω -distributive in $L(B)$.

Proof. Q is equivalent to the two steps forcing.

(1) Add a subset A_0 of $\omega_1 \cap E$ to code B in the usual way. This is trivially distributive.

(2) The forcing \bar{Q} defined as follows: a condition is a function $p: |p| \cap \omega_2 - E \rightarrow 2$ such that $|p| < \omega_1$ and for $\alpha \leq |p|$ and β , if $\Theta(\alpha, \beta)$ and $L_\beta(A_0 \cap \alpha, p \upharpoonright \alpha) \Vdash \text{ZF}^- + \alpha = \omega_1$, then $\{x \in T_\beta \mid S(x) \cap A_0 \cap \alpha \text{ is bounded in } \alpha\}$ is a cofinal branch in T_β .

It is clearly enough to show:

Claim 2. \bar{Q} is ω -distributive in $L(B)(A_0) = L(A_0)$.

Proof. Let $(\Delta_i)_{i < \omega}$ be a sequence of strongly dense subsets of \bar{Q} and $p_0 \in \bar{Q}$. Define $(X_i)_i$ by:

$X_0 =$ the smallest $X < L_{\omega_2}(A_0)$ such that $p_0, (\Delta_i)_{i < \omega} \in X$,

$X_{i+1} =$ the smallest $X < L_{\omega_2}(A_0)$ s.t. $\{X_i\} \cup X_i \subset X$,

$X_\omega = \bigcup_{i < \omega} X_i$.

Let

$\sigma_i: L_{\beta_i}(A_0 \cap \alpha_i) \xrightarrow{\cong} X_i$ for $i \leq \omega$.

Define $(p_i)_i$ by: $p_{i+1} =$ the least $p \leq p_i$ such that $p \in \Delta_i$ and $|p| \geq \alpha_i$. It is enough to show that $p = \bigcup_{i < \omega} p_i$ is a condition. It is easy to see that for $i < \omega$ $|p_i| < \alpha_i$ and so $|p| = \alpha_\omega$; so it remains to show that for $\beta > \alpha_\omega$ if $(*)$: $\Theta(\alpha_\omega, \beta)$ and $L_\beta(A_0 \cap \alpha_\omega, p) \Vdash \text{ZF}^- + \alpha_\omega = \omega_1$, then $\{x \in T_\beta \mid S(x) \cap A_0 \cap \alpha_\omega \text{ is bounded in } \alpha_\omega\}$ is a cofinal branch in T_β .

For $\beta \leq \beta_\omega$ this is trivial since $\sigma_\omega: L_{\beta_\omega}(A_0 \cap \alpha_\omega) \rightarrow L_{\omega_2}(A_0)$ is an elementary embedding and A_0 really codes a branch in T (it also uses Claim 1 in Section 1.2.1 and Claim 1 in Section 1.2.2).

For $\beta > \beta_\omega$ there is nothing to prove since $(*)$ does not occur: In $L_\beta(A_0 \cap \alpha_\omega)$ we can define the sequence $(\alpha_i)_{i < \omega}$ using $L_{\beta_\omega}(A_0 \cap \alpha_\omega)$ instead of $L_{\omega_2}(A_0)$ so $L_\beta(A_0 \cap \alpha_\omega) \Vdash \bar{\alpha}_\omega = \omega$. \square

1.3.

For $n \in \omega$ define $P_n = T(n) * Q_n$, i.e. P_n is the forcing that adds a branch in $T(n)$ and then codes it 'nicely' by a subset of ω_1 .

Lemma 1. Let $P = \prod_{n \in \omega} P_n$; then P is ω -distributive and satisfies the ω_2 -chain condition.

Proof. It is clear from the previous results. \square

Let M_1 be a P generic extension of M_0 . New work in M_1 .

Lemma 2. *There is a sequence $(A_n)_{n < \omega}$ of subsets of ω_1 such that M_1 satisfies:*

$$V = L((A_n)_{n \in \omega}).$$

I shall now define the last forcing C that gives the real a that will be Π_2^1 singleton; a will code (by use of almost disjoint sets) the A_n in such a way that $A_{2n} \in L(a)$ iff $a(n) = 1$ and $A_{2n+1} \in L(a)$ iff $a(n) = 0$. It will be shown that in $L(a)$ (as in some extension of this model) $T(2n)$ (resp. $T(2n+1)$) is Suslin iff $a(n) = 0$ (resp. $= 1$).

For $s \in {}^{<\omega}2$ define C_s by: r is a condition iff r is a pair (r_0, r_1) such that:

- (1) $r_0: |r| < \omega \rightarrow 2; r_0 \supset s;$
- (2) $\bar{r}_1 < \omega$ and

$$r_1 \subset \omega \times \{(\alpha, 2i) \mid r_0(i) = 1 \text{ and } \alpha \in A_{2i}\}$$

$$\cup \omega \times \{(\alpha, 2i+1) \mid r_0(i) = 0 \text{ and } \alpha \in A_{2i+1}\};$$

- (3) if $(n, \beta) \in r_1$, then:

$$(S(\beta) - n) \cap \bar{r}_0 = \emptyset;$$

where $\bar{r}_0 = \{i \mid r_0(i) = 1\}$ and $\beta \rightarrow S(\beta)$ is some nice function giving almost disjoint subsets of ω .

Lemma 3. *For $s \in {}^{<\omega}2$, C_s satisfies the ω_1 chain condition.*

Let M_2 be a C_\emptyset generic extension of M_1 .

Lemma 4. *There is a real a in M_2 such that, setting*

$$D_n = \{\alpha \mid S((\alpha, n)) \cap \bar{a} \text{ is finite}\},$$

then:

$$\text{if } n = 2i, \text{ then: } a(i) = 1 \Rightarrow D_n = A_n \text{ and } a(i) = 0 \Rightarrow D_n = \emptyset,$$

$$\text{if } n = 2i+1, \text{ then: } a(i) = 1 \Rightarrow D_n = \emptyset \text{ and } a(i) = 0 \Rightarrow D_n = A_n.$$

Lemma 5. *Let $M_3 = L(a)$ (M_3 is strictly included in M_2); then M_3 satisfies: $T(2i)$ is Suslin iff $a(i) = 0$ and $T(2i+1)$ is Suslin iff $a(i) = 1$.*

Proof. The part 'if' is trivial since if $a(i) = 1$ (resp. 0), then A_{2i} (resp. A_{2i+1}) and so B_{2i} (resp. B_{2i+1}) is in $L(a)$.

For the 'only if' direction I first need:

Claim 1. M_2 is generic over M_0 by the following set of conditions:

$F = P * C_\emptyset = \{(x_n)_n, (p_n, p'_n)_n, (r_0, r_1) \mid \text{where } x_n \in T(n), (p_n, p'_n) \text{ satisfies the definition of } Q_n \text{ where } B_n \text{ is replaced by } \{y \in T(n) \mid y \leq x_n\} \text{ and } (r_0; r_1) \text{ satisfies the definition of } C_\emptyset \text{ where } A_j \text{ is replaced by } \bar{p}_j\}.$

Proof. Trivial. \square

Lemma 6. Let $f = (p, (r_0, r_1)) \in F$; then $F_f = \{g \in F \mid g \leq f\}$ is isomorphic to the product $F' \times F''$ where

$$F' = \left[\prod_{\substack{r_0(n)=1 \\ \text{or} \\ n \geq |r_0|}} P_{2n} \times \prod_{\substack{r_0(n)=0 \\ \text{or} \\ n \geq |r_0|}} P_{2n+1} \right] * C_{r_0};$$

$$F'' = \prod_{r_0(n)=0} P_{2n} \times \prod_{r_0(n)=1} P_{2n+1}.$$

Proof. Clear. \square

Now suppose Lemma 5 is false; there is an i and $s = (p, (r_0, r_1))$ in the generic such that $r_0(i) = 0$ and $s \Vdash T(2i)$ is not Suslin in $L(a)$. Using Lemma 6, write the generic subset of $F = (H', H'')$; it is clear that a is in $L(H')$, so it remains to show:

Claim 2. $T(2i)$ is Suslin in $L(H')$.

Proof. It is easy to see that

$$F' \simeq \left(\prod_{\substack{r_0(n)=1 \\ \text{or} \\ n \geq |r_0|}} Q_{2n} \times \prod_{\substack{r_0(n)=0 \\ \text{or} \\ n \geq |r_0|}} Q_{2n+1} \right) * C_{r_0}.$$

(This comes from the fact that forcing with $T(n)$ does not add subsets of ω_1 .) \square

It is then enough to show that for each step $T(2i)$ remains Suslin. For the first one, it is exactly the lemma in Section 1.1. For the other ones it comes from the following:

Lemma 7. Let N be a model of ZF and T be in N a Suslin tree of height ω_2 ; let P be a notion of forcing with the strong ω_2 -chain condition; then if \bar{N} is a P generic extension of N , \bar{N} satisfies: T is Suslin.

Proof. Let $A \in \bar{N}$ be an antichain in T ; let \dot{A} be a name for A and p in the generic such that: $p \Vdash \dot{A}$ is an antichain in \check{T} .

Let $B = \{x \in T \mid \exists q \leq p, q \Vdash \check{x} \in \dot{A}\}$. Then $B \in N$ and $A \subset B$; so it suffices to show that $\bar{B} \leq \omega_1$ (in N). Now work in N . For each $x \in B$, pick $q_x \leq p$ such that $q_x \Vdash \check{x} \in \dot{A}$; note that if $x \neq x'$ and q_x and $q_{x'}$ are compatible, then x and x' are incompatible. Suppose B has cardinality ω_2 ; let $C = \{q_x \mid x \in B\}$.

If C has cardinality ω_1 , then for some q the set $\{x \mid q \Vdash \check{x} \in \dot{A}\}$ is an antichain of T of cardinality ω_2 ; this is impossible.

So C contains (by the strong ω_2 -chain condition) a subset D of pairwise

compatible conditions, $\bar{D} = \omega_2$; but then the set $\{x \mid q_x \in D\}$ is an antichain of T of cardinality ω_2 ; and the proof is complete. \square

1.4.

Let $\varphi(x)$ be the formula:

$\forall \alpha < \beta < \omega_1$ if $(L_\beta \models \text{ZF}^- + \alpha = \omega_1 + \forall x (\bar{x} \leq \omega_1) + \forall \lambda (\text{cf}(\lambda) = \alpha \Rightarrow \eta_\lambda \text{ exists})$ and $L_\beta(x) \models \text{ZF}^- + \alpha = \omega_1$, then: assume you decode x into ω subsets of α , which themselves are decoded into subsets B_n of $T_\beta(n)$, then:

If $x(i) = 1$, then B_{2i} is a cofinal branch in $T_\beta(2i)$.

If $x(i) = 0$, then B_{2i+1} is a cofinal branch in $T_\beta(2i+1)$.

Claim. φ is equivalent to a Π_2^1 formula.

Lemma 1. (1) $L(a) \models \varphi(a)$.

(2) For any real x , if $L(x) \models \varphi(x)$, then:

If $x(i) = 1$, $T(2i)$ is not Suslin in $L(x)$.

If $x(i) = 0$, $T(2i+1)$ is not Suslin in $L(x)$.

Proof. (1) Assume α, β satisfy the hypothesis in φ ; when a is decoded the sequence $(A_i \cap \alpha)_{i < \omega}$ is obtained and then the conclusion follows from the Lemma 1 in Section 1.2.2.

(2) It follows immediately from the Lowenheim–Skolem theorem. \square

Now Theorem 2 follows easily from all the previous results.

1.5.

The idea of the proof of Theorem 1 is simply: Do the same thing with the trees $T(n)$ not only on ω_2 but on all the cardinals, and choose a real a that codes (by Jensen's method, with enough condensation properties) branches in the $T(n)$ in such a way that a can be recovered by looking whether $T(n)$ is Suslin or not. Doing that is only a slight modification of Jensen's proof; the important one comes from the following:

One of the important fact in the proof of Theorem 2 is the lemma which essentially says that for $n \neq m$ $T(n)$ remains Suslin when forcing with P_m ; this comes first from the property of the sequence $(T(n))_n$ (that is easy to extend to greater cardinals) and secondly from the fact that Q_m is small in view of $T(n)$ (the strong chain condition).

In the general case, when P_m will be a class of conditions, P_m will not be 'small' in view of $T(n)$; of course P_m can be cut into its small and big part and one has the lemma:

Lemma. Let M be a model of ZF, α a cardinal in M and T a Suslin tree of height

α^+ in M ; let P be a notion of forcing which is α closed (i.e.: every decreasing chain of conditions of length α has a lower bound). Then if N is a P generic extension of M , then N satisfies: T is Suslin.

Unfortunately the big part of P_m only is α distributive and this is not sufficient to preserve the Suslinity of a tree (since forcing with the tree itself is distributive!). Happily there is a solution for that: the idea is the following: Work with the $T(n)$ defined in Section 1.1; recall that the level λ for $\text{cf}(\lambda) = \omega_1$ has been defined by use of forcing over some L_{η_λ} . Assume C is some forcing which preserves the cardinals and adds a subset D of ω_2 such that for each λ , $\text{cf}(\lambda) = \omega_1$, $D \cap \lambda \in L_{\eta_\lambda}$; it follows easily from the proof of the lemma in Section 1.1 that the $T(n)$ remain Suslin in $L(D)$.

The essential — but not so difficult! — fact that has to be examined in the iteration is:

Claim. In the Jensen's conditions, the level of constructibility of the conditions can be controlled.

2. The iteration

2.1.

Definition 1. Let α be a cardinal; I define $\chi_n(\alpha)$ by:

$$\begin{aligned}\chi_0(\alpha) &= \alpha, \\ \chi_{n+1}(\alpha) &= \chi_n(\alpha)^+.\end{aligned}$$

Definition 2. Let α be a cardinal; for $\xi \in [\alpha, \alpha^+]$ I define by induction the ordinals μ_ξ^i ($i \leq \omega$) as follows:

$$\begin{aligned}\mu_\xi^0 &= \sup(\alpha \cup \{\mu_\eta^\omega \mid \eta \in [\alpha, \xi]\}); \\ \mu_\xi^{i+1} &= \text{the least } \mu > \mu_\xi^i \text{ such that:} \\ &\quad \text{(i) } L_\mu \models \text{ZF}^- + \forall x \bar{x} \leq \alpha, \\ &\quad \text{(ii) if } \alpha \text{ is a successor cardinal } \text{cf}(\mu) = \alpha; \\ \mu_\xi^\omega &= \sup(\mu_\xi^i \mid i < \omega).\end{aligned}$$

Definition 3. Let α be a limit cardinal and $n \in \omega$, I define one Suslin tree $T(n, \alpha)$ of height $\chi_{n+2}(\alpha)$ in the same way the sequence $T(n)$ has been defined in Section 1. The levels of the tree are defined by induction; for limit λ such that $\text{cf}(\lambda) = \chi_{n+1}(\alpha)$ the level is defined by forcing over L_{μ_λ} .

Note. (1) If β is an ordinal such that $L_\beta \models \text{ZF}^- +$ there is a greatest cardinal and

$$\forall \xi < \beta \forall i \leq \omega \mu_\xi^i < \beta;$$

then the trees $T(n, \gamma)$ can be defined in L_β — by the same definition as the right ones — for any γ such that L_β satisfies: γ is a limit cardinal such that $\chi_{n+1}(\gamma)$ exists.

(2) If $\beta \leq \bar{\beta}$ satisfy the hypothesis of (1) and L_β and $L_{\bar{\beta}}$ have the same cardinals, then the $T_\beta(n, \gamma)$ and the $T_{\bar{\beta}}(n, \gamma)$ are the same if $\chi_{n+1}(\gamma)$ is not the greatest cardinal; and if L_β and $L_{\bar{\beta}}$ satisfy: $\chi_{n+1}(\gamma)$ is the greatest cardinal, then:

$$T_\beta(n, \gamma) = T_{\bar{\beta}}(n, \gamma) \cap L_\beta.$$

(3) Since now I shall work on a class of cardinals one tree by level will be enough and simpler.

(4) If $|x|$ is the level of x in T , note that $x \in L_{\mu_{|x|}^2}$ for any x and that:

$$\forall x \in T(n, \alpha) (\text{cf}(|x|) < \chi_{n+1}(\alpha) \Rightarrow x \in L_{\mu_{|x|}^1}).$$

(This will be often used without mention.)

2.2.

Now I am going to define, for each n , a forcing $P(n)$ which looks like the final forcing P_ω of Jensen. Fix n up to the end of this section (the subscript n will be often omitted). In the following I shall also follow Jensen's notations (see [1]).

Definition 1. Let α be a cardinal $\geq \omega$; define S_α as follows:

(I) If α is not $\chi_{n+1}(\gamma)$ for some limit cardinal γ , then $s \in S_\alpha$ iff $s: [\alpha, |s|[\rightarrow 2$ such that $|s| < \alpha^+$ and $\forall \xi \leq |s|$:

(1) $s \upharpoonright \xi \in L_{\mu_\xi^i}$;

(2) $\forall \beta L_\beta(s \upharpoonright \xi)$ satisfies: "if $\Theta(\alpha, \xi, s \upharpoonright \xi)$, then $\Gamma(\alpha, \xi, s \upharpoonright \xi)$ " where Θ is "ZF⁻ + $\xi = \alpha^+$ + there is a greatest cardinal (not necessarily $\xi!$) + $\forall i \leq \omega \forall \nu \mu_\nu^i$ exists" and Γ is: "when $s \upharpoonright \xi$ is decoded by Jensen's method, a cofinal branch is founded in each $T_\beta(n, \gamma)$ such that $L_\beta \models \gamma$ is a limit cardinal such that $\chi_{n+1}(\gamma)$ exists".

(II) If α is $\chi_{n+1}(\gamma)$ for some limit γ , then (1) is replaced by:

(1') $s \upharpoonright Z_0$ is a branch in $T(n, \gamma)$ (recall T is a binary tree);

(1'') $s \upharpoonright \xi \in L_{\mu_\xi^i}(s \upharpoonright Z_0 \cap \xi)$;

where as in [1] $Z_\nu = \{(\eta, \nu) \mid \eta \in \text{Ord}\}$ and (\cdot, \cdot) is the Gödel pairing function.

Note. In [1] for $s \in S_\alpha$, ordinals μ_s^i are defined.

Claim. The μ_s^i defined in [1] are the $\mu_{|s|}^i$ of my Definition 2.

Proof. In the case (I) this is trivial since $s \in L_{\mu_s^i}$.

In the case (II) I only have to show that $b = L_{\mu_{|s|}^1}(s \upharpoonright Z_0)$ satisfies the conditions $a \rightarrow d$ of Jensen's definition of the μ_s^i in the case $\text{cf}(|s|) = \chi_{n+1}(\gamma)$ (by Note 4 in Definition 3); but then $s \upharpoonright Z_0$ is (by the construction of T) generic over $L_{\mu_{|s|}^1}$ and the result follows easily. \square

Definition 2. For α a cardinal and $s \in S_{\alpha^+}$, R_s is the set of pairs (r, r') such that:

(i) $r \in S_\alpha$;

(ii) (r, r') codes s as in [1].

Note. Here there is no $A \cap \alpha^+$: so the generic will be a Cohen in Z_0 if α is not $\chi_{n+1}(\gamma)$ for some limit cardinal γ and a branch in $T(n, \gamma)$ in the other case. Define also the \mathcal{A}_s^i ($i \leq \omega$; $s \in S_\alpha$) as in [1].

Claim. (1) For $i \neq 1$ $\mathcal{A}_s^i = L_{\mu_{|s|}^i}$.

(2) For $i = 1$ $\mathcal{A}_s^1 = L_{\mu_{|s|}^1}$ if α is not $\chi_{n+1}(\gamma)$ for some limit γ and $\mathcal{A}_s^1 = L_{\mu_{|s|}^1}(s \upharpoonright Z_0)$ in the other case.

Lemma 1. R_s is α distributive in \mathcal{A}_s^1 .

Proof. Forcing with R_s is equivalent to the three following forcing operations

(1) R_0 to code s on Z_1 and so add D_0 a subset of $[\alpha, \alpha^+[\cap Z_1$ in such a way that:

$$\forall \xi \in [\alpha, \alpha^+[\ D_0 \cap \xi \in L_{\mu_\xi^1}.$$

Claim. R_0 is α distributive in \mathcal{A}_s^1 .

Proof. Assume $(\Delta_i)_{i < \alpha}$ is a sequence of strongly dense subsets of R_0 , $(\Delta_i)_{i < \alpha} \in \mathcal{A}_s^1$ and $p_0 \in R_0$; set $b = \mathcal{A}_s^2 = L_{\mu_{|s|}^2}$.

Define a sequence $(X_i)_{i < \alpha}$ as follows:

$$X_0 = \text{the least } X < b \text{ such that } \alpha \cup \{p_0, (\Delta_i)_i\} \subset X;$$

$$X_{i+1} = \text{the least } X < b \text{ such that } X_i \cup \{X_i\} \subset X;$$

$$X_\lambda = \bigcup_{i < \lambda} X_i \text{ for limit } \lambda.$$

Set $\sigma_i : b_i = L_{\beta_i} \xrightarrow{\sim} X_i$ and $\alpha_i = X_i \cap \alpha^+ = \sigma_i^{-1}(\alpha^+)$.

Define a sequence $(p_i)_{i < \alpha}$ of conditions by:

$$p_{i+1} = \text{the least } p \leq p_i \text{ such that } p \in \Delta_i \text{ and } |p| \geq \alpha_i$$

(I shall prove later — in Section 2.4 — that such a p exists);

$$p_\lambda = \bigcup_{i < \lambda} p_i \text{ for limit } \lambda \text{ if } \bigcup p_i \text{ is a condition, undefined if not.}$$

As in [1] the only problem is to show that for limit λ $p = \bigcup_{i < \lambda} p_i$ is a condition. And for that I must show that $p \in L_{\mu_{\alpha_\lambda}^1}$. But as in [1] it is easy to see that $p \in L_{\beta_{\lambda+1}}$ (p can be defined in $L_{\beta_{\lambda+1}}$ by use of b_λ instead of b), and since $\beta_\lambda < \mu_{\alpha_\lambda}^1$ we are done. We now work in $\mathcal{A}_s^1(D_0) = L_{\mu_{|s|}^1}(D_0)$. \square

(2) The second step is (if α is $\chi_{n+1}(\gamma)$ for some limit γ): add a branch B in $T(n, \gamma)$; this is trivially α distributive since T remains Suslin in $L_{\mu_\alpha^1}(D_0)$.

(3) The third has to ensure the property (2) in the definition of S_α . A condition is a function r :

$$[\alpha, |r|][\bigcap_{\beta \in I} Z_\nu \rightarrow 2$$

such that $|r| < \alpha^+$ and $I = [2, \alpha^+$ [if (2) has occurred (resp. $\{0\} \cup [2, \alpha^+$ [if not) and $\forall \xi \in [\alpha, |r|]$

(i) $r \upharpoonright \xi \in L_{\mu_\xi^1}(B \cap \xi)$ (resp. $L_{\mu_\xi^1}$)

(ii) $\forall \beta L_\beta(D_0 \cap \xi, B \cap \xi, r \upharpoonright \xi)$ (resp. $L_\beta(D_0 \cap \xi, r \cap \xi)$) satisfies: “if $\Theta(\alpha, \xi, (D_0, B, r) \upharpoonright \xi)$, then $\Gamma(\text{-----})$ ” (resp. “if $\Theta(\alpha, \xi, (D_0, r) \upharpoonright \xi)$, then $\Gamma(\text{-----})$ ”). The proof that this forcing is α distributive in $L_{\mu_s^1}(D_0, B)$ (resp. $L_{\mu_s^1}(D_0)$) uses essentially the same argument as in Lemma 2 in Section 1.2, but since some care is needed, I shall repeat it.

I shall develop the argument in the first case (when B occurs). It is the same in the other case.

Begin as in the proof for R_0 , using $b = L_{\mu_s^1}(D_0, B)$ and define X_i, α_i, p_i ($i \leq \alpha$) in the same way; as usual I have to prove (1'') and (2) in the definition of S_α only for $\xi = \alpha_\lambda = |p|$ when λ is limit and $p = \bigcup_{i < \lambda} p_i$.

(1'') is as for R_0 : it is easy to see that

$$p \in L_{\beta_\lambda+1}(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda) \subset L_{\mu_{\alpha_\lambda}^1}(B \cap \alpha_\lambda).$$

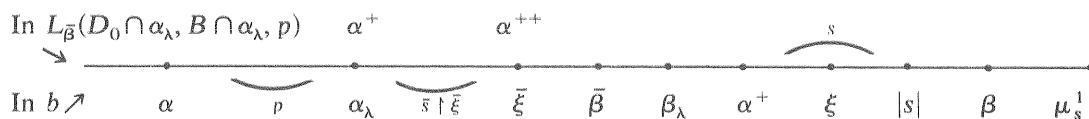
I now prove (2): choose $\bar{\beta}$ such that $L_{\bar{\beta}}(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda, p) \models \Theta$.

Since $(\alpha_i)_{i < \lambda} \in L_{\beta_\lambda+1}(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda)$, $\bar{\beta}$ is not greater than β_λ set $\bar{s} = \sigma_\lambda^{-1}(s)$ (recall $s \in b$).

(A) Assume first there is a $\bar{\xi}$ such that $L_{\bar{\beta}} \models \bar{\xi} = \alpha^{++}$, then $\bar{\beta} < \beta_\lambda$ (since $\sigma_\lambda : L_{\beta_\lambda} \xrightarrow{\sim} L_{\mu_s^1}$ and the definition of μ_s^1 ensures that $L_{\mu_s^1} \models \alpha^{++}$ does not exist).

Claim. $\bar{\xi} \leq |\bar{s}|$.

If $|\bar{s}| < \bar{\xi}$, then $\mu_{\bar{s}}^1 < \bar{\xi} < \bar{\beta} < \beta_\lambda$ and so (by σ_λ) μ_s^1 exists in b ; a contradiction. Set $\xi = \sigma_\lambda(\bar{\xi})$ (so $\xi \leq |s|$) and $\beta = \sigma_\lambda(\bar{\beta})$; the picture is:



Now work in $L_{\bar{\beta}}(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda, p)$: $B \cap \alpha_\lambda$ really is a branch; if you decode you first find \bar{s} and $L_{\bar{\beta}}(\bar{s} \upharpoonright \bar{\xi})$ satisfies $\Theta(\alpha_\lambda, \bar{\xi}, \bar{s} \upharpoonright \bar{\xi})$ so (by σ_λ) $L_\beta(s \upharpoonright \xi)$ satisfies $\Theta(\alpha^+, \xi, s \upharpoonright \xi)$; since $s \in S_{\alpha^+}$ and (by σ_λ) $L_{\bar{\beta}}(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda, \bar{s} \upharpoonright \bar{\xi})$ satisfies Γ ; and this is exactly what we need.

(B) Assume now $L_{\bar{\beta}} \models \alpha_\lambda$ is the greatest cardinal.

Claim. $\bar{\beta} \leq \bar{s}$.

If $|\bar{s}| = \bar{\xi} < \bar{\beta}$, then by the definition of Θ , $\mu_{\bar{s}}^1 < \bar{\beta} \leq \beta_\lambda$ and so by σ_λ , μ_s^1 exists in b ; a contradiction. So in $L_{\bar{\beta}}(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda, p)$ we can recover $\bar{s} \upharpoonright \bar{\beta}$ and so by

the elementarity of $\sigma_\lambda \bar{s} \upharpoonright \bar{\beta} \cap Z_0$ is a branch in the tree (if it has to be one) (note that this does not occur when (B) appears — since that last tree has height α_λ — however, I give the argument, since when (B) does not appear, this case can occur). This achieves the proof of the lemma. \square

Defining S_α^+ and R^D as in [1] (see Definition 2, p. 37) we have:

Lemma 2. R^D satisfies the α^{++} -chain condition in \mathcal{A}_D .

Proof. Each step of the iteration satisfies it — and even the strong chain condition, except for adding a branch in the tree. \square

2.3.

The other definitions and lemmas used to define what is called P_ω in [1] are exactly as in [1], except one point: It is more convenient to define $\rho_{s\tau}^i$ to be a code for $\mathcal{A}_{s, \chi_{n+2}(\tau)}^i$ (instead of $\mathcal{A}_{\sigma\tau}^i$ in [1]). That does not modify the proof in [1] but for me it ensures that there is no intrusion of the coding at limit cardinals with the branches. Now what I call $P(n)$ (remind n was fixed) is what it called P_ω in [1].

I have to show the lemmas that say:

- (1) a condition can be arbitrarily extended;
- (2) $P_\tau(n)$ is τ -distributive.

I shall prove that later and now show how to conclude. At the moment the following will be clear.

Lemma 1. Let N be a $P(n)$ generic extension of M_0 ; then there is a subset A_n of ω_1 such that N satisfies:

- (1) $ZF + V = L(A_n)$.
- (2) $\forall \alpha < \omega_1 \forall \beta$ if $L_\beta(A_n \cap \alpha) \models ZF^- +$ there is a greatest cardinal $\alpha = \omega_1 = \omega_1^L$, then if you decode $A_n \cap \alpha$ by Jensen's method you find a cofinal branch in each $T_\beta(n, \gamma)$ such that $L_\beta \models \gamma$ is a limit cardinal and $\chi_{n+1}(\gamma)$ exists.

Let M_1 be a $\prod_{n \in \omega} P(n)$ generic extension of M_0 . Define the forcing C_s as in Section 1. Let M_2 be a C_\emptyset generic extension of M_1 . This gives the real a for which I must prove.

Lemma 2. For all the limit γ , $L(a)$ satisfies:

- $T(2i, \gamma)$ is Suslin iff $a(i) = 0$,
- $T(2i+1, \gamma)$ is Suslin iff $a(i) = 1$.

This follows — as in Section 1 — from Lemmas 3 to 5.

Lemma 3. Let $f = (p(n))_{n \in \omega}, (r_0, r_1)$ be a condition in

$$F = \prod_{n \in \omega} P(n) * C_\emptyset,$$

then $F_f = \{g \in F \mid g \leq f\}$ is isomorphic to the product $F' \times F''$ where

$$F' = \left(\prod_{\substack{r_0(n)=1 \\ \text{or} \\ n \geq |r_0|}} P(2n) \times \prod_{\substack{r_0(n)=0 \\ \text{or} \\ n \geq |r_0|}} P(2n+1) \right) * C_{r_0},$$

$$F'' = \left(\prod_{r_0(n)=0} P(2n) \times \prod_{r_0(n)=1} P(2n+1) \right).$$

Proof. Trivial. \square

Lemma 4. Let γ be a limit cardinal, a be a finite subset of ω and $n \in a$. Let N be a $\prod_{m \neq a} P(m)$ generic extension of M_0 , then N satisfies: $T(n, \gamma)$ is Suslin.

Proof. Set $\alpha = \chi_{n+1}(\gamma)$; still using the notation in [1], there are subsets D_m ($m \notin a$) of $[\alpha, \alpha^+$ such that: $P(m) = P_\alpha(m) * P_\omega^{D_m}(m)$.

Claim.

$$\prod_{m \notin a} P(m) \simeq \left(\prod_{m \notin a} P_\alpha(m) \right) * \left(\prod_{m \notin a} P_\omega^{D_m}(m) \right).$$

Proof. This follows easily from the fact that $\prod_{m \notin a} P_\alpha(m)$ is α distributive and α is a successor cardinal. \square

So $\prod_{m \notin a} P(m)$ is as a two step forcing:

The first one gives a model that satisfies:

$$V = L((D_m)_{m \notin a}) + \forall m \notin a D_m \subset [\alpha, \alpha^+ + \forall \xi \in [\alpha, \alpha^+ [\forall m D_m \upharpoonright \xi \in L_{\mu_\xi}^1].$$

So $T(n, \gamma)$ is Suslin in $L((D_m)_{m \notin a})$; this follows from the Lemma 6 below.

The second one satisfies the α^+ strong chain condition, so $T(n, \gamma)$ remains Suslin by Lemma 5 below. \square

Lemma 5. Let M be a model of ZF, α a cardinal and T a Suslin tree of height α^+ ; let P be a notion of forcing that satisfies the α^+ strong chain condition and N be a P generic extension of M ; then N satisfies: T is Suslin.

Proof. As in Lemma 7 in Section 1.3. \square

Lemma 6. Let M be a model of: ZF + $V = L(D)$, $D \subset \alpha^+$ for some regular cardinal α . Let T be the Suslin tree of height α^+ defined in L as in Section 1 (by forcing over $L_{\mu_\xi}^1$ for $\text{cf}(\xi) = \alpha$). Assume: $\forall \xi < \alpha^+ (\text{cf}(\xi) = \alpha \rightarrow D \upharpoonright \xi \in L_{\mu_\xi}^1)$. Then T is Suslin in M .

Proof. This is an easy corollary to the proof of the (generalized) lemma in Section 1.1. \square

Definition. Let φ be the formula: $\forall \alpha < \beta < \omega_1$ if $L_\beta(x) \models \text{ZF}^- + \alpha = \omega_1 = \omega_1^L +$ there is a greatest cardinal $+\forall \xi \forall i < \omega \mu_\xi^i$ exists, then if you decode x by Jensen's method you find a cofinal branch in $T_\beta(n, \gamma)$ for each γ such that $L_\beta \models \gamma$ is a limit cardinal and for each n such that:

$$n = 2i \quad \text{and} \quad x(i) = 1$$

or

$$n = 2i + 1 \quad \text{and} \quad x(i) = 0.$$

Now the theorem follows easily from all the previous results; the argument for (3) in the theorem being: (by use of Lemma 5) you cannot kill all the $T(n, \gamma)$ by forcing with a set of conditions.

It remains now to prove that conditions can be arbitrarily extended and then distributivity.

2.4.

Lemma 1. Let ψ be a formula; α a cardinal; let C be the following notion of forcing: a condition is a function $c : [\alpha, |c|[\rightarrow 2$ such that $|c| < \alpha^+$ and for $\xi \leq |c|$

$$(1) \quad c \upharpoonright \xi \in L_{\mu_\xi^i},$$

$$(2) \quad \forall \beta L_\beta(c \upharpoonright \xi) \models (\text{ZF}^- + \xi = \alpha^+ \Rightarrow \psi),$$

then for any condition c_0 and $\gamma \in [|c|, \alpha^+[$ there is a condition $c \leq c_0$ such that $|c| = \gamma$.

As a corollary of this lemma we immediately have:

Lemma 2. Any condition in R_s can be arbitrarily extended.

Proof of Lemma 1. By induction on γ ; for γ successor this is trivial, so let γ be a limit ordinal.

The proof of \square_α (see for instance [1, Section 6]) shows that there is a closed unbounded set D of $[\alpha, \gamma[$ such that $|D| \leq \alpha$ (the order type of D) and for all $\nu \in D \cup \{\gamma\}$, $D \cap \nu \in L_{\mu_\nu^i}$.

Let $(\gamma_i)_{i < \lambda}$ be the monotone enumeration of D ; I may assume $\gamma_0 = |c_0|$ and γ_i is a limit ordinal for $i < \lambda$.

For any ordinal $\beta = \xi + n$ where ξ is limit set $f(\beta) = \xi + 2n + 1$.

Define c_i ($i \leq \lambda$) by induction as follows:

$c_{i+1} = c_i \cup \chi_E$ where χ_E is the characteristic function of the set $E = \{\gamma_i\} \cup \{f(\beta) \mid \beta \in c'\}$ where c' is the least c such that $c \subset [\gamma_i, \gamma_{i+1}[$ and $c_i \cup \chi_c$ is a condition.

$c_\eta = \bigcup_{i < \eta} c_i$ for limit η if $\bigcup c_i$ is a condition, undefined if not.

As usual it is enough to show that for limit η $c = \bigcup_{i < \eta} c_i$ is a condition and for that to see that (1) and (2) in the definition of C is true for $\xi = \gamma_\eta$. (2) is trivial

since $D \cap \gamma_\eta$ can be recovered from c so there is no β such that

$$L_\beta(c) \models \gamma_\eta = \alpha^+.$$

(1) since $D \cap \gamma_\eta \in L_{\mu_{\gamma_\eta}^1}$ it follows easily that $c \in L_{\mu_{\gamma_\eta}^1}$. \square

The other lemma that has to be proved is lemma I.6 in [1] with the new condition that control the level of constructibility.

Definition 1. Let α be a cardinal; define \tilde{S}_α to be the set of functions $s : [\alpha, |s|[\rightarrow 2$ such that $|s| < \alpha^+$ and for $\xi \leq |s|$ $s \upharpoonright \xi \in L_{\mu_\xi^1}$.

Definition 2. Let κ be a limit cardinal, A a subset of $[\kappa, \kappa^+[$ such that: $\forall \xi < \kappa^+ A \cap \xi \in L_{\mu_\xi^1}$; let I be the set of successor cardinals less than κ . For τ a cardinal less than κ define \tilde{P}_τ by:

A condition is a sequence $p = (p_\gamma)_{\gamma \in I \cap [\tau, \kappa[}$ such that:

(1) $\forall \gamma p_\gamma \in \tilde{S}_\gamma$.

(2) If $|p|$ is defined by: $|p| =$ the least ξ such that $p \in L_{\mu_\xi^1}$, then for $\xi < |p|$ there is a $\nu \in I$ such that: $\forall \gamma \in [\nu, \kappa[\cap I$

$$p_\gamma(\rho_{\xi\gamma}) = \begin{cases} 1 & \text{if } \xi \in A, \\ 0 & \text{if not} \end{cases}$$

(where the $\rho_{\xi\gamma}$ are defined as in [5]).

Lemma 3. For any $p \in \tilde{P}_\tau$ and $\xi \geq |p|$ there is a $q \leq p$ such that $|q| = \xi$.

The proof follows the proof in [1]: I shall need:

Lemma 4. Let γ be a successor cardinal; $\xi \in [\gamma, \gamma^+[$ such that $\text{cf}(\xi) < \gamma$; $\mu \in]\xi, \gamma^+[$ such that $\text{cf}(\mu) = \gamma$ and $L_\mu \models \text{ZF}^- + \forall x (\bar{x} \leq \gamma)$; $X \subset \xi$ such that $X \in L$ and $\forall \eta < \xi X \cap \eta \in L_\mu$. Then $X \in L_\mu$.

Proof. Set $\lambda = \text{cf}(\xi) < \gamma$; the proof follows immediately from the two claims.

Claim 1.

$$L_\mu \models \text{cf}(\xi) = \lambda.$$

Let $f \in L$; $f : \lambda \rightarrow \xi$ be cofinal, injective; $g \in L_\mu$; $g : \xi \rightarrow \gamma$ bijective, then $g \circ f : \lambda \rightarrow \gamma$ injective, so since γ is regular $g \circ f \in L_\gamma \subset L_\mu$ and so $f \in L_\mu$.

Claim 2. Let $f \in L_\mu : \lambda \rightarrow \xi$ be cofinal, increasing and $x_i = X \cap f(i)$; then $(x_i)_{i < \lambda} \in L_\mu$.

Let $\varphi : \mu \rightarrow L_\mu$ be the canonical enumeration of L_μ ; set $\eta_i = \varphi^{-1}(x_i)$; it is

enough to show: $(\eta_i)_{i<\lambda} \in L_\mu$ but this follows by the same argument as in Claim 1 from $\text{cf}(\mu) = \gamma > \lambda$.

Proof of Lemma 3. The successor case is as in [1] (I of course use Lemma 1 to extend p up to $\rho_{\xi\gamma}$).

For the limit case, I extend p using a closed unbounded subset of $[|p|, \xi[$ coming from a \square sequence. As usual it is enough to prove that for a limit λ if $p = \bigcup_{i<\lambda} p^i$, then for $\gamma \in I$ $p_\gamma \in L_{\mu_\gamma}$. It follows easily from Lemma 4 and the fact I may always assume that $\lambda < \gamma$ (if the order type of the c.u.b. is less than κ this is easy; if not, then κ is regular and the p^i are defined in such a way that $p_\gamma^\lambda = \bigcup_{i<\lambda} p_\gamma^i$ is a non trivial union only for $\gamma > \lambda$ (remind γ is a successor)).

Now the proof of the extendability of the conditions is exactly as in [1], using the previous lemmas and specially the Lemma 4.

The only (slight) different point is in Lemma 2.14.3 of [1] when the r_{ij}, \tilde{q}_{ij} are defined. I must be more careful since for limit i $(r'_j)_{j<i}$ is not necessarily in \mathcal{A}_i^1 ; so r_{i1} has to be defined in the following way:

Let ξ_i be the least such that $x_i \in L_{\mu_{\xi_i}^1}$ where $x_i = (p \upharpoonright \beta_i; (r_{h0} \upharpoonright h \leq i); \tilde{q} \upharpoonright \beta_i)$; define r_{i1} by: $r_{i0} \cup \{(\eta_i, 5)\} \cup r'$ where r' is such that:

- (1) η_i is s.t. $L_{\mu_{\xi_i}^1} \models x_i$ has number η_i in the canonical well ordering of V ;
 - (2) r' is the least $r \subset [r_{i0}, \mu_{\xi_i} \cap Z_4$ such that $r_{i1} \in S_{\beta_i}$. It is then easy to see that
- It is then easy to see that

$$r_i^* = \bigcup_{j<\omega} r_{ij} \in S_{\beta_i}$$

and the end of the lemma is the same. \square

2.5.

The distributivity is proved as in [1]. I shall only mention the points where something has to be changed or where some care is needed. Lemma 3.7 in [1] has to be replaced by:

Lemma. Assume distributivity holds up to α ; let $s \in S_{\alpha^+}$, $\tau \leq \alpha$ and let $D \subset [\tau, \alpha^+[$ be P_τ^s generic over \mathcal{A}_s^1 then:

- (1) $s, D \in L_{\mu_s^1}(D_\tau)$;
- (2) for $\gamma \in [\tau, \alpha]$ and $\xi \in [\gamma, \gamma^+[$ then
 - (i) $D_\gamma \upharpoonright \xi \in L_{\mu_\xi^1}$ (resp. if $\gamma = \chi_{n+1}(\nu)$ for some limit cardinal ν $D_\gamma \cap Z_0$ is a branch in $T(n, \nu)$ and $D_\gamma \upharpoonright \xi \in L_{\mu_\xi^1}(D_\gamma \cap Z_0 \cap \xi)$);
 - (ii) $\forall \beta L_\beta(D_\tau \cap \xi)$ satisfies: “if \emptyset , then Γ ”.

Proof. As in [1]. \square

The only other point is in the proof of Lemma 3.22(a) in [1]. I have to verify the additional property of S_γ .

There are two things to look at: the usual facts in [1] and the branches in the trees. Moreover we have to show that $\prod_n P_\tau^s(n)$ is τ -distributive, not only that for each n $P_\tau^s(n)$ is τ -distributive, since the product of τ -distributive forcings is not necessarily τ -distributive.

What about branches?

Note that

$$\prod_n P_\tau^s(n) \approx P_{\tau^+}^s(n) * \prod_n R^{D_n} \quad \text{for some generic sets } D_n.$$

With the first forcing there are no problems with the branches since all the trees involved in this forcing are τ -closed.

For the second one there are no problems too with the branches except for the n (if there is one) such that:

$$\tau = \chi_{n+1}(\beta) \quad \text{for some limit } \beta.$$

But $T(n, \beta)$ is Suslin in $L_{\mu_s}(D_m \mid m < \omega)$, since L_{μ_s} and $L_{\mu_s}(D_m \mid m < \omega)$ have the same subsets of τ^+ and it is well known that forcing with a Suslin tree of height τ^+ is τ distributive.

So I can 'forget' the branches and only have to look at the properties (1) and (2) in S_γ : We start the construction with $\mathcal{A}_s^2 = L_{\mu_s}$ instead of \mathcal{A}_s^1 in [1].

The b appearing in the lemma is some $L_{\mu'}$ with $\mu' < \mu_{|b|_\gamma}$ but since the sequence $(p^i)_{i < \lambda}$ is definable from b as it was defined from \mathcal{A}_s^2 it follows that $p_\gamma^\lambda \in L_{\mu_{|b|_\gamma}}$.

The second property of S_γ comes from the fact that p_γ^λ is generic over the imitation of P_τ^s in b , so the result follows for a $\beta \leq \alpha^*$; it remains to show it for $\beta \in]\alpha^*, \mu' [$ (since $L_{\mu'+1} \models |p_\gamma^\lambda| = \gamma$) but this is exactly as in Lemma 1 in Section 2.2.

This achieves the proof of Theorem 1.

2.6.

(1) Following exactly the proof of Theorem 0.2 in [1] we see that the real in my theorem I may be assumed to be in $L(0^\#)$.

(2) To prove that the real is not provably Π_2^1 singleton, do as follows: Start from M_1 (the $\prod_n P(n)$ generic extension of M_0) and choose M_2' to be a $C_\emptyset \times C_\emptyset$ generic extension of M_1 ; then there are two different reals a and a' that satisfy φ so $L(a, a') \not\models \exists! x \varphi(x)$.

(3) The following gives a slight improvement of Theorem 1: I may assume — as in [7] — that the trees are such that: if $\alpha = |T|$ and B_0, B_1 are distinct, cofinal branches in T , then $\bar{\alpha}^{L(B_0, B_1)} < \alpha$ (simply use an elementary, saturated — in the sense of model theory — extension of the Rational numbers); I may assume too that the first level in T only has two points: the right and the left one.

When forcing with the $P(n)$ I may assume that all the branches are left branches (i.e.: their first point is the left one) and so in φ it can be said that the branches are left ones.

Now in $L(a)$ some trees remain Suslin; add in these trees right branches. Let N be this extension. In N a remains Π_2^1 singleton since if x is another real satisfying φ , then for some i , $T(i)$ will have two branches: a right and a left one, and this is impossible.

More generally if \bar{N} is an extension of N and if there is another real x in \bar{N} that satisfies φ , then a class of cardinals are collapsed (in the context I prove the theorem it is $\{\chi_{n+2}(\beta) \mid \beta \text{ limit cardinal and } a(n) \neq x(n)\}$). I may assume that this class is $\{\alpha^+ \mid \alpha \text{ regular}\}$.

It would be more interesting if I could assume that all the successor cardinals were collapsed since then an extension of N where a is not Π_2^1 would have to contain $0^\#$; but it seems difficult to do that: it would be necessary to build trees of length successors of singular cardinals: it does not seem difficult to build such trees which are Suslin and have at most one branch (as far as $|\tau|$ is not collapsed); but in the proof I also need the fact that: (1) the trees are closed enough (all the branches of 'small' length have extension in the tree); (2) the tree remain Suslin in some extension of $L \dots$ and I do not know how to do that!

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