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## A VERY ABSOLUTE $\Pi_2^1$ REAL SINGLETON

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### 0. Introduction

By Shoenfield absoluteness theorem, the  $\Pi_2^1$  real singleton are the simplest, non constructible, projective reals. Very few  $\Pi_2^1$  non constructible real singletons exist in the literature.

One of them is the real  $0^\#$  introduced by Solovay in [9]. It is ‘provably  $\Pi_2^1$  singleton’, that means that there is a  $\Pi_2^1$  formula  $\varphi$  such that for any model  $M$  of ZF that contains  $0^\#$  as an element,  $0^\#$  is — in  $M$  — the only solution of  $\varphi$ . The existence of  $0^\#$  needs a large cardinal property as Ramsey or measurable and  $0^\#$  is not generic over  $L$  by mean of a constructible set of conditions.

Other ones have been constructed by forcing over  $L$  (see Jensen and Solovay [8] and Jensen [5]). They are not ‘provably  $\Pi_2^1$  singleton’; they only are  $\Pi_2^1$  singleton ‘at home’, that means there is a  $\Pi_2^1$  formula  $\varphi$  such that the real  $r$  is the only solution of  $\varphi$  in  $L(r)$ , but (except in trivial cases — for instance if  $M$  and  $L(r)$  have the same reals) it is not true that if  $M$  is another model containing  $r$  as an element,  $r$  remains — in  $M$  — the only solution of  $\varphi$ . Of course, by Shoenfield theorem,  $r$  remains a solution of  $\varphi$  but not necessarily the only one.

Later Jensen and Johnsbraten constructed, also by forcing, a new and more absolute  $\Pi_2^1$  singleton  $r$ :  $r$  is the only solution — in  $M$  — of some  $\Pi_2^1$  formula  $\varphi$  as far as  $M$  satisfies:  $\omega_1 = \omega_1^L$ .

In the sense of absoluteness, this real is the ‘best’  $\Pi_2^1$  singleton known, except  $0^\#$ . Other  $\Pi_2^1$  singleton with various properties have been constructed (see [2, 3, 4]), but — in view of absoluteness — since they are constructed by use of the Jensen and Johnsbraten method, they are not different from that one.

A natural question is: find a ‘provably’  $\Pi_2^1$  non constructible singleton  $r$  such that  $0^\# \notin L(r)$ ? It seems to be a difficult problem. This paper is devoted to the proof of a theorem which is — perhaps — a step to a solution of this problem.

**Theorem 1.** *Let  $M$  be a transitive model of  $\text{ZF} + V = L$ . There is an  $M$  definable*

class  $P$  of forcing conditions and a  $\Pi_2^1$  formula  $\varphi$  such that, if  $N$  is a  $P$  generic extension of  $M$ , then:

- (1)  $N$  and  $M$  have the same cardinals and the same cofinality function.
- (2) There is a real  $r$  in  $N$  such that  $N$  satisfies:

$$\text{ZF} + \neg 0^\# + V = L(r) + r \notin L + \varphi(r) + \exists! x \varphi(x).$$

- (3) If  $\bar{N}$  is a generic extension of  $N$  (by a set of conditions), then  $\bar{N}$  satisfies:  $\exists! x \varphi(x)$ .

**Note.** (1) The real given by the theorem is not provably  $\Pi_2^1$  singleton. I shall discuss and make some comments at the end of this paper.

(2) If  $0^\#$  exists, I may assume that this real  $r$  is such that  $r \in L(0^\#)$ ; but  $r$  is not  $\Pi_2^1$  singleton in  $L(0^\#)$ .

(3) The conclusion of (3) in the theorem remains true for some other extensions of  $N$ : for instance if  $\bar{N}$  is a generic extension of  $N$  by use of the class  $P$  of conditions where  $P$  is:

- (i) the forcing to add a Cohen subset of  $[\alpha, \alpha^+]$  for all the cardinals  $\alpha$ ,
- (ii) the forcing to deny the generalized continuum hypothesis.

On the other way there are classes  $Q$  of conditions such that  $r$  does not remain  $\Pi_2^1$  singleton in a  $Q$  generic extension of  $N$ .

The proof of this theorem will use the method developed by Jensen in his ‘Coding the universe by a real’ [6]. A basic and important modification has to be done, so the entire proof — and not only one or two points — has to be reexamined. It will be of course too long and fastidious — and useless — to rewrite the whole proof in this new context. So, in a first step I shall prove a theorem which is the basic idea of the theorem and the ‘building block’ of it. Then, while proving the theorem, I shall invite the reader to work with a copy — and/or a good knowledge — of Jensen’s theorem by himself.

## 1. The building block

This section is devoted to the proof of the following:

**Theorem 2.** *Let  $M$  be a transitive model of  $\text{ZF} + V = L$ . There is an  $M$  definable set  $P$  of conditions and a  $\Pi_2^1$  formula  $\varphi$  such that if  $N$  is a  $P$  generic extension of  $M$ , then:*

- (1)  $N$  and  $M$  have the same cardinals and the same cofinality function.
- (2) There is a real  $a$  in  $N$  such that  $N$  satisfies:

$$V = L(a) + \varphi(a) + \exists! x \varphi(x).$$

- (3) If  $\bar{N}$  is a generic extension of  $N$  by use of a set of conditions with the strong

$\omega_2$ -chain condition (i.e.:  $\forall x \subset C (\bar{x} = \omega_2 \Rightarrow \exists y \subset x (\bar{y} = \omega_2 \text{ and } \forall c, c' \in y, c \text{ and } c' \text{ are compatible})))$ , then  $\bar{N}$  satisfies:  $\exists! x \varphi(x)$ .

This real comes from something as a three steps forcing extension. I work in a model  $M_0 = M$  of  $V = L$ .

### 1.1. The trees

I define a sequence  $(T(n))_{n \in \omega}$  of Suslin trees of height  $\omega_2$  in the following way (it is an easy generalisation of Jensen's construction of a Suslin tree of height  $\omega_1$ ). If  $T$  is a tree I shall denote  $T^\alpha$  the  $\alpha$ th level of  $T$  and  $T \upharpoonright \alpha = \bigcup_{\beta < \alpha} T^\beta$ . The  $T(n)$  will be such that:  $x \in T^\alpha(n) \Rightarrow x : \alpha \rightarrow 2$ . The levels of the  $T(n)$  are defined by induction as follows:

$$x \in T^0(n) \text{ iff } x = \emptyset,$$

$$x \in T^{\alpha+1}(n) \text{ iff } x = y \cup \{(\alpha, i) \mid i = 0 \text{ or } 1\} \text{ for some } y \in T^\alpha(n_2)$$

for limit  $\alpha$ ,

$$\text{cf}(\alpha) = \omega : x \in T^\alpha(n) \text{ iff } x : \alpha \rightarrow 2 \text{ \& } \forall \lambda < \alpha \ x \upharpoonright \lambda \in T^\lambda(n) \\ (\text{since } \bar{\omega}_2^\omega = \omega_1, \bar{T}^\lambda(n) = \omega_1)$$

for limit  $\lambda$ ,

$$\text{cf}(\lambda) = \omega_1.$$

The elements of  $T^\lambda(n)$  are branches in  $T(n) \upharpoonright \lambda$  choosen (in a classical way) by the following forcing:

Let  $\eta_\lambda$  be the least  $\eta$  such that  $(T(n) \upharpoonright \lambda)_{n \in \omega}$  is in  $L_\eta$ , and  $L_\eta \models \text{ZF}^- + \bar{\lambda} = \omega_1$ . Let

$$F_\lambda = \{(\alpha, f) \in L_{\eta_\lambda} \mid \alpha < \omega_1 \text{ \& } f = (f_n)_n : \alpha \rightarrow T(n) \upharpoonright \lambda \text{ s.t.}$$

$$\forall n, m \in \omega \ \forall \beta, \bar{\beta} < \alpha \mid f_n(\beta) = |f_m(\bar{\beta})|\},$$

where  $|x|$  is the level of  $x$ . Set  $(\alpha, f) \leq (\bar{\alpha}, \bar{f})$  iff  $\alpha \geq \bar{\alpha}$  and

$$\forall n \in \omega \ \forall \beta < \bar{\alpha} \ f_n(\beta) \geq \bar{f}_n(\beta)$$

(in the ordering of  $T(n) \upharpoonright \lambda$ ).

Because of the definition of  $T(n)$  in the case of cofinality  $\omega$ , it is clear that  $F_\lambda$  is an  $\omega$ -closed set of conditions and so there is — in  $\mathbb{L}$  — an  $F_\lambda$  generic over  $L_{\eta_\lambda}$ . The  $L$ -least one gives  $\omega_2$  branches in  $T(n) \upharpoonright \lambda$ . They are the elements of  $T^\lambda(n)$ . This achieves the definition of the  $T(n)$ .

**Lemma.** Let  $a$  be a finite subset of  $\omega$ , and  $T_a = \prod_{n \notin a} T(n)$ . That is: a condition is a sequence  $(x_n)_{n \in \omega - a}$  such that  $x_n \in T(n)$ ; and  $(x_n)_n \leq (y_n)_n$  iff  $\forall n \in \omega - a \ x_n \geq y_n$ . Then  $T_a$  is  $\omega$ -closed and satisfies the  $\omega_2$ -chain condition. Moreover let  $G$  be  $T_a$  generic over  $L$  and  $i \in a$ , then  $L(G)$  satisfies:  $T(i)$  is Suslin.

**Proof.** Although the proof is classical, I shall give it since I shall refer to it later.  $T_a$  is clearly  $\omega$ -closed. Let  $A$  be a maximal antichain in  $T_a$ . Let  $X$  be the  $L$ -least such that:

$$X < L_{\omega_3} \ \& \ \{A\} \cup \omega_1 \subset X \text{ and if } \lambda = \omega_2 \cap X, \text{ then } \lambda \in \omega_2 \text{ and } \text{cf}(\lambda) = \omega_1$$

(clearly such an  $X$  exists). Let  $\Pi: L_\beta \xrightarrow{\cong} X$ , so  $\lambda = \omega_2 \cap X = \Pi^{-1}(\omega_2)$ . Let  $g$  be a function from  $\omega - a$  into  $\omega_1$  and

$$D_g = \{(\alpha, f) \in F_\lambda \mid (f_n(g(n)))_{n \in \omega - a} \text{ is compatible with some element of } A\}.$$

It is easy to see that  $D_g$  is a dense subset of  $F_\lambda$  and that  $D_g \in L_{\eta_\lambda}$  (since  $\eta_\lambda > \beta$  because  $L_\beta \models \lambda = \omega_2$  whereas  $L_{\eta_\lambda} \models \bar{\lambda} = \omega_1$ ). It follows easily that  $A = A \cap \lambda$  and so  $\bar{A} \leq \omega_1$ .

Assume now  $i \in a$ , set  $b = a - \{i\}$ ; then  $T_b = T_a * T(i)$  (where  $*$  means the iteration of forcing) satisfies the  $\omega_2$ -chain condition and so  $T(i)$  satisfies in  $L(G)$  the  $\omega_2$ -chain condition.  $\square$

## 1.2. Coding the branches

For  $n \in \omega$  and  $B_n$  a branch in  $T(n)$  (in some generic extension of  $L$ ). I shall define a set  $Q_n$  of conditions to code  $B_n$  by a subset of  $\omega_1$ . It is a slight modification of the coding by use of almost disjoint sets and one of the basic idea of proof of the theorem.

### 1.2.1

Let  $\alpha < \beta < \omega_1$  be such that:  $L_\beta \models \alpha = \omega_1 + \text{ZF}^- + \forall x (\bar{x} \leq \omega_1) + \forall \lambda (\text{cf}(\lambda) = \alpha \Rightarrow \eta_\lambda \text{ exists})$  (I shall denote  $\Theta(\alpha, \beta)$  this property). I can construct in  $L_\beta$  the trees  $T(n)$  exactly as I construct in  $L_{\omega_2}$  the right ones. I shall denote  $T_\beta(n)$  these trees.

**Claim 1.** Assume  $\Theta(\alpha, \beta)$  and  $\Theta(\alpha, \bar{\beta})$  and  $\beta \leq \bar{\beta}$ , then

$$T_\beta(n) = T_{\bar{\beta}}(n) \cap L_\beta.$$

This is an easy consequence of:

**Claim 2.** For  $\gamma < \beta$ :

- (1) if  $L_{\bar{\beta}} \models \text{cf}(\gamma) = \omega$ , then  $L_\beta \models \text{cf}(\gamma) = \omega$ ;
- (2) if  $f \in L_{\bar{\beta}} : \omega \xrightarrow{\text{inj.}} T(n) \restriction \gamma$ , then  $f \in L_\beta$ .

**Proof.** (1) Let  $f: \omega \xrightarrow{\text{cof., inj.}} \gamma \in L_{\bar{\beta}}$ ;  $g \in L_\beta : \gamma \xrightarrow{\text{bij.}} \alpha$ , then  $g \circ f \in L_{\bar{\beta}}$  and  $g \circ f: \omega \rightarrow \alpha$  so  $g \circ f \in L_\alpha \subset L_\beta$  so  $f = ((g \circ f)_0^{-1} g)^{-1} \in L_\beta$ .

- (2) let  $g \in L_\beta : T(n) \restriction \gamma \xrightarrow{\text{bij.}} \alpha$ ; then  $g \circ f \in L_\alpha$  and so  $f \in L_\beta$ .  $\square$

## 1.2.2

Fix  $n$ ; let  $B_n$  be a branch in  $T(n)$ . I define the forcing  $Q_n$  in the following way. (I shall omit the subscript  $n$  when not necessary.)

**Claim 1.** There is a definable function  $S: T \rightarrow P(\omega_1)$  such that:

- (1) if  $x \neq y$   $\overline{S(x) \cap S(y)} \leq \omega$ ;
- (2) if  $\Pi: L_\beta \rightarrow L_{\omega_2}$  is an elementary embedding such that  $\pi(\alpha) = \omega_1$ , then  $\Pi(S(x) \cap \alpha) = S(\Pi(x))$ .

**Proof.** Easy.

A condition in  $Q_n$  is a pair  $(p, p^*)$  such that:

- (1)  $p: |p| < \omega_1 \rightarrow 2$  such that for  $\alpha \leq |p|$  and  $\beta$  if  $\Theta(\alpha, \beta)$  and  $L_\beta(p \upharpoonright \alpha) \models \text{ZF}^- + \alpha = \omega_1 = \omega_1^L$ , then:  $\{x \in T_\beta \mid S(x) \cap \tilde{p} \cap \alpha \cap E \text{ is bounded in } \alpha\}$  is a cofinal branch in  $T_\beta$ ; (where  $\tilde{p} = \{\gamma < |p| \mid p(\gamma) = 1\}$  and  $E = \{\lambda + 2n \mid \lambda \text{ limit; } n \in \omega\}$ ).
- (2)  $\dot{p} \subset B \times \omega_2$ ;  $\tilde{p}^* \leq \omega$  and for  $(x, \eta) \in p^*$   $(S(x) - \eta) \cap \tilde{p} = \emptyset$ . Set  $(p, p^*) \leq (q, q^*)$  if  $p \supset q$  and  $p^* \supset q^*$ .

**Note.** So  $Q$  is the usual forcing to code  $B$  by a subset of  $\omega_1$ , except the fact that I ensure that this subset will have enough properties when later I shall use a condensation argument.

**Lemma 1.** Let  $G$  be  $Q$  generic over  $L(B)$ ; then there is a subset  $A$  of  $\omega_1$  such that:

- (1)  $L(B)(G) = L(A)$ ;
- (2) for  $x \in T$ ,  $x \in B$  iff  $S(x) \cap A \cap E$  is bounded in  $\omega_1$ ;
- (3) if  $\Theta(\alpha, \beta)$  and  $L_\beta(A \cap \alpha) \models \text{ZF}^- + \alpha = \omega_1$ , then  $\{x \in T_\beta \mid S(x) \cap A \cap E \text{ is bounded in } \alpha\}$  is a cofinal branch in  $T_\beta$ .

**Proof.** The only non usual fact is that for any condition  $(p, p^*)$  and  $|p| \leq \gamma < \omega_1$  there is a  $q: \gamma \rightarrow 2$  such that  $(q, q^*)$  is a condition stronger than  $(p, p^*)$  (it is for that reason, I code  $B$  only on the even ordinals (to keep place for other things). Also note that since — later — I shall work with the  $Q_n$  all together, I have in fact to use a partition of  $\omega_1$  into  $\omega$  parts).

To do that it is enough to put on the odd ordinals between  $|p|$  and  $|p| + \omega$  an order of type  $\gamma$  so that there are no  $\alpha \in ]|p|, \gamma]$  and  $\beta$  such that  $L_\beta(q \upharpoonright \alpha) \models \text{ZF}^- + \alpha = \omega_1$ .  $\square$

The following is crucial:

**Lemma 2.**  $Q$  is  $\omega$ -distributive in  $L(B)$ .

**Proof.**  $Q$  is equivalent to the two steps forcing.

- (1) Add a subset  $A_0$  of  $\omega_1 \cap E$  to code  $B$  in the usual way. This is trivially distributive.

(2) The forcing  $\bar{Q}$  defined as follows: a condition is a function  $p: |p| \cap \omega_2 - E \rightarrow 2$  such that  $|p| < \omega_1$  and for  $\alpha \leq |p|$  and  $\beta$ , if  $\Theta(\alpha, \beta)$  and  $L_\beta(A_0 \cap \alpha, p \restriction \alpha) \models ZF^- + \alpha = \omega_1$ , then  $\{x \in T_\beta \mid S(x) \cap A_0 \cap \alpha \text{ is bounded in } \alpha\}$  is a cofinal branch in  $T_\beta$ .

It is clearly enough to show:

**Claim 2.**  $\bar{Q}$  is  $\omega$ -distributive in  $L(B)(A_0) = L(A_0)$ .

**Proof.** Let  $(\Delta_i)_{i < \omega}$  be a sequence of strongly dense subsets of  $\bar{Q}$  and  $p_0 \in \bar{Q}$ . Define  $(X_i)_i$  by:

$X_0 =$  the smallest  $X < L_{\omega_2}(A_0)$  such that  $p_0, (\Delta_i)_{i < \omega} \in X$ ,

$X_{i+1} =$  the smallest  $X < L_{\omega_2}(A_0)$  s.t.  $\{X_i\} \cup X_i \subset X$ ,

$X_\omega = \bigcup_{i < \omega} X_i$ .

Let

$\sigma_i: L_{\beta_i}(A_0 \cap \alpha_i) \xrightarrow{\cong} X_i$  for  $i \leq \omega$ .

Define  $(p_i)_i$  by:  $p_{i+1} =$  the least  $p \leq p_i$  such that  $p \in \Delta_i$  and  $|p| \geq \alpha_i$ . It is enough to show that  $p = \bigcup_{i < \omega} p_i$  is a condition. It is easy to see that for  $i < \omega$   $|p_i| < \alpha_i$  and so  $|p| = \alpha_\omega$ ; so it remains to show that for  $\beta > \alpha_\omega$  if  $(*)$ :  $\Theta(\alpha_\omega, \beta)$  and  $L_\beta(A_0 \cap \alpha_\omega, p) \models ZF^- + \alpha_\omega = \omega_1$ , then  $\{x \in T_\beta \mid S(x) \cap A_0 \cap \alpha_\omega \text{ is bounded in } \alpha_\omega\}$  is a cofinal branch in  $T_\beta$ .

For  $\beta \leq \beta_\omega$  this is trivial since  $\sigma_\omega: L_{\beta_\omega}(A_0 \cap \alpha_\omega) \rightarrow L_{\omega_2}(A_0)$  is an elementary embedding and  $A_0$  really codes a branch in  $T$  (it also uses Claim 1 in Section 1.2.1 and Claim 1 in Section 1.2.2).

For  $\beta > \beta_\omega$  there is nothing to prove since  $(*)$  does not occur: In  $L_\beta(A_0 \cap \alpha_\omega)$  we can define the sequence  $(\alpha_i)_{i < \omega}$  using  $L_{\beta_\omega}(A_0 \cap \alpha_\omega)$  instead of  $L_{\omega_2}(A_0)$  so  $L_\beta(A_0 \cap \alpha_\omega) \models \bar{\alpha}_\omega = \omega$ .  $\square$

### 1.3.

For  $n \in \omega$  define  $P_n = T(n) * Q_n$ , i.e.  $P_n$  is the forcing that adds a branch in  $T(n)$  and then codes it ‘nicely’ by a subset of  $\omega_1$ .

**Lemma 1.** Let  $P = \prod_{n \in \omega} P_n$ ; then  $P$  is  $\omega$ -distributive and satisfies the  $\omega_2$ -chain condition.

**Proof.** It is clear from the previous results.  $\square$

Let  $M_1$  be a  $P$  generic extension of  $M_0$ . New work in  $M_1$ .

**Lemma 2.** *There is a sequence  $(A_n)_{n < \omega}$  of subsets of  $\omega_1$  such that  $M_1$  satisfies:*

$$V = L((A_n)_{n \in \omega}).$$

I shall now define the last forcing  $C$  that gives the real  $a$  that will be  $\Pi_2^1$  singleton;  $a$  will code (by use of almost disjoint sets) the  $A_n$  in such a way that  $A_{2n} \in L(a)$  iff  $a(n) = 1$  and  $A_{2n+1} \in L(a)$  iff  $a(n) = 0$ . It will be shown that in  $L(a)$  (as in some extension of this model)  $T(2n)$  (resp.  $T(2n+1)$ ) is Suslin iff  $a(n) = 0$  (resp.  $= 1$ ).

For  $s \in {}^{<\omega}2$  define  $C_s$  by:  $r$  is a condition iff  $r$  is a pair  $(r_0, r_1)$  such that:

- (1)  $r_0: |r| < \omega \rightarrow 2; r_0 \supset s;$
- (2)  $\bar{r}_1 < \omega$  and

$$r_1 \subset \omega \times \{(\alpha, 2i) \mid r_0(i) = 1 \text{ and } \alpha \in A_{2i}\} \\ \cup \omega \times \{(\alpha, 2i+1) \mid r_0(i) = 0 \text{ and } \alpha \in A_{2i+1}\};$$

- (3) if  $(n, \beta) \in r_1$ , then:

$$(S(\beta) - n) \cap \bar{r}_0 = \emptyset;$$

where  $\bar{r}_0 = \{i \mid r_0(i) = 1\}$  and  $\beta \rightarrow S(\beta)$  is some nice function giving almost disjoint subsets of  $\omega$ .

**Lemma 3.** *For  $s \in {}^{<\omega}2$ ,  $C_s$  satisfies the  $\omega_1$  chain condition.*

Let  $M_2$  be a  $C_\emptyset$  generic extension of  $M_1$ .

**Lemma 4.** *There is a real  $a$  in  $M_2$  such that, setting*

$$D_n = \{\alpha \mid S((\alpha, n)) \cap \bar{a} \text{ is finite}\},$$

*then:*

*if  $n = 2i$ , then:  $a(i) = 1 \Rightarrow D_n = A_n$  and  $a(i) = 0 \Rightarrow D_n = \emptyset$ ,*

*if  $n = 2i+1$ , then:  $a(i) = 1 \Rightarrow D_n = \emptyset$  and  $a(i) = 0 \Rightarrow D_n = A_n$ .*

**Lemma 5.** *Let  $M_3 = L(a)$  ( $M_3$  is strictly included in  $M_2$ ); then  $M_3$  satisfies:  $T(2i)$  is Suslin iff  $a(i) = 0$  and  $T(2i+1)$  is Suslin iff  $a(i) = 1$ .*

**Proof.** The part 'if' is trivial since if  $a(i) = 1$  (resp. 0), then  $A_{2i}$  (resp.  $A_{2i+1}$ ) and so  $B_{2i}$  (resp.  $B_{2i+1}$ ) is in  $L(a)$ .

For the 'only if' direction I first need:

**Claim 1.**  $M_2$  is generic over  $M_0$  by the following set of conditions:

$F = P * C_\emptyset = \{(x_n)_n, (p_n, p_n'), (r_0, r_1) \mid \text{where } x_n \in T(n), (p_n, p_n') \text{ satisfies the definition of } Q_n \text{ where } B_n \text{ is replaced by } \{y \in T(n) \mid y \leq x_n\} \text{ and } (r_0; r_1) \text{ satisfies the definition of } C_\emptyset \text{ where } A_j \text{ is replaced by } \bar{p}_j\}.$



*Proof.* Trivial.  $\square$

**Lemma 6.** Let  $f = (p, (r_0, r_1)) \in F$ ; then  $F_f = \{g \in F \mid g \leq f\}$  is isomorphic to the product  $F' \times F''$  where

$$F' = \left[ \prod_{\substack{r_0(n)=1 \\ \text{or} \\ n \geq |r_0|}} P_{2n} \times \prod_{\substack{r_0(n)=0 \\ \text{or} \\ n \geq |r_0|}} P_{2n+1} \right] * C_{r_0};$$

$$F'' = \prod_{r_0(n)=0} P_{2n} \times \prod_{r_0(n)=1} P_{2n+1}.$$

*Proof.* Clear.  $\square$

Now suppose Lemma 5 is false; there is an  $i$  and  $s = (p, (r_0, r_1))$  in the generic such that  $r_0(i) = 0$  and  $s \Vdash T(2i)$  is not Suslin in  $L(a)$ . Using Lemma 6, write the generic subset of  $F = (H', H'')$ ; it is clear that  $a$  is in  $L(H')$ , so it remains to show:

**Claim 2.**  $T(2i)$  is Suslin in  $L(H')$ .

*Proof.* It is easy to see that

$$F' \simeq \left( \prod_{\substack{r_0(n)=1 \\ \text{or} \\ n \geq |r_0|}} Q_{2n} \times \prod_{\substack{r_0(n)=0 \\ \text{or} \\ n \geq |r_0|}} Q_{2n+1} \right) * C_{r_0}.$$

(This comes from the fact that forcing with  $T(n)$  does not add subsets of  $\omega_1$ .)  $\square$

It is then enough to show that for each step  $T(2i)$  remains Suslin. For the first one, it is exactly the lemma in Section 1.1. For the other ones it comes from the following:

**Lemma 7.** Let  $N$  be a model of ZF and  $T$  be in  $N$  a Suslin tree of height  $\omega_2$ ; let  $P$  be a notion of forcing with the strong  $\omega_2$ -chain condition; then if  $\bar{N}$  is a  $P$  generic extension of  $N$ ,  $\bar{N}$  satisfies:  $T$  is Suslin.

*Proof.* Let  $A \in \bar{N}$  be an antichain in  $T$ ; let  $\check{A}$  be a name for  $A$  and  $p$  in the generic such that:  $p \Vdash \check{A}$  is an antichain in  $\check{T}$ .

Let  $B = \{x \in T \mid \exists q \leq p, q \Vdash \check{x} \in \check{A}\}$ . Then  $B \in N$  and  $A \subset B$ ; so it suffices to show that  $\bar{B} \leq \omega_1$  (in  $N$ ). Now work in  $N$ . For each  $x \in B$ , pick  $q_x \leq p$  such that  $q_x \Vdash \check{x} \in \check{A}$ ; note that if  $x \neq x'$  and  $q_x$  and  $q_{x'}$  are compatible, then  $x$  and  $x'$  are incompatible. Suppose  $B$  has cardinality  $\omega_2$ ; let  $C = \{q_x \mid x \in B\}$ .

If  $C$  has cardinality  $\omega_1$ , then for some  $q$  the set  $\{x \mid q \Vdash \check{x} \in \check{A}\}$  is an antichain of  $T$  of cardinality  $\omega_2$ ; this is impossible.

So  $C$  contains (by the strong  $\omega_2$ -chain condition) a subset  $D$  of pairwise

compatible conditions,  $\bar{D} = \omega_2$ ; but then the set  $\{x \mid q_x \in D\}$  is an antichain of  $T$  of cardinality  $\omega_2$ ; and the proof is complete.  $\square$

#### 1.4.

Let  $\varphi(x)$  be the formula:

$\forall \alpha < \beta < \omega_1$  if  $(L_\beta \models \text{ZF}^- + \alpha = \omega_1 + \forall x (\bar{x} \leq \omega_1) + \forall \lambda (\text{cf}(\lambda) = \alpha \Rightarrow \eta_\lambda \text{ exists})$  and  $L_\beta(x) \models \text{ZF}^- + \alpha = \omega_1$ , then: assume you decode  $x$  into  $\omega$  subsets of  $\alpha$ , which themselves are decoded into subsets  $B_n$  of  $T_\beta(n)$ , then:

If  $x(i) = 1$ , then  $B_{2i}$  is a cofinal branch in  $T_\beta(2i)$ .

If  $x(i) = 0$ , then  $B_{2i+1}$  is a cofinal branch in  $T_\beta(2i+1)$ .

**Claim.**  $\varphi$  is equivalent to a  $\Pi_2^1$  formula.

**Lemma 1.** (1)  $L(a) \models \varphi(a)$ .

(2) For any real  $x$ , if  $L(x) \models \varphi(x)$ , then:

If  $x(i) = 1$ ,  $T(2i)$  is not Suslin in  $L(x)$ .

If  $x(i) = 0$ ,  $T(2i+1)$  is not Suslin in  $L(x)$ .

**Proof.** (1) Assume  $\alpha, \beta$  satisfy the hypothesis in  $\varphi$ ; when  $a$  is decoded the sequence  $(A_i \cap \alpha)_{i < \omega}$  is obtained and then the conclusion follows from the Lemma 1 in Section 1.2.2.

(2) It follows immediately from the Lowenheim–Skolem theorem.  $\square$

Now Theorem 2 follows easily from all the previous results.

#### 1.5.

The idea of the proof of Theorem 1 is simply: Do the same thing with the trees  $T(n)$  not only on  $\omega_2$  but on all the cardinals, and choose a real  $a$  that codes (by Jensen's method, with enough condensation properties) branches in the  $T(n)$  in such a way that  $a$  can be recovered by looking whether  $T(n)$  is Suslin or not. Doing that is only a slight modification of Jensen's proof; the important one comes from the following:

One of the important fact in the proof of Theorem 2 is the lemma which essentially says that for  $n \neq m$   $T(n)$  remains Suslin when forcing with  $P_m$ ; this comes first from the property of the sequence  $(T(n))_n$  (that is easy to extend to greater cardinals) and secondly from the fact that  $Q_m$  is small in view of  $T(n)$  (the strong chain condition).

In the general case, when  $P_m$  will be a class of conditions,  $P_m$  will not be 'small' in view of  $T(n)$ ; of course  $P_m$  can be cut into its small and big part and one has the lemma:

**Lemma.** Let  $M$  be a model of ZF,  $\alpha$  a cardinal in  $M$  and  $T$  a Suslin tree of height

$\alpha^+$  in  $M$ ; let  $P$  be a notion of forcing which is  $\alpha$  closed (i.e.: every decreasing chain of conditions of length  $\alpha$  has a lower bound). Then if  $N$  is a  $P$  generic extension of  $M$ , then  $N$  satisfies:  $T$  is Suslin.

Unfortunately the big part of  $P_m$  only is  $\alpha$  distributive and this is not sufficient to preserve the Suslinity of a tree (since forcing with the tree itself is distributive!). Happily there is a solution for that: the idea is the following: Work with the  $T(n)$  defined in Section 1.1; recall that the level  $\lambda$  for  $\text{cf}(\lambda) = \omega_1$  has been defined by use of forcing over some  $L_{\eta_\lambda}$ . Assume  $C$  is some forcing which preserves the cardinals and adds a subset  $D$  of  $\omega_2$  such that for each  $\lambda$ ,  $\text{cf}(\lambda) = \omega_1$ ,  $D \cap \lambda \in L_{\eta_\lambda}$ ; it follows easily from the proof of the lemma in Section 1.1 that the  $T(n)$  remain Suslin in  $L(D)$ .

The essential — but not so difficult! — fact that has to be examined in the iteration is:

**Claim.** In the Jensen's conditions, the level of constructibility of the conditions can be controlled.

## 2. The iteration

### 2.1.

**Definition 1.** Let  $\alpha$  be a cardinal; I define  $\chi_n(\alpha)$  by:

$$\chi_0(\alpha) = \alpha,$$

$$\chi_{n+1}(\alpha) = \chi_n(\alpha)^+.$$

**Definition 2.** Let  $\alpha$  be a cardinal; for  $\xi \in [\alpha, \alpha^+]$  I define by induction the ordinals  $\mu_\xi^i$  ( $i \leq \omega$ ) as follows:

$$\mu_\xi^0 = \sup(\alpha \cup \{\mu_\eta^\omega \mid \eta \in [\alpha, \xi]\});$$

$$\mu_\xi^{i+1} = \text{the least } \mu > \mu_\xi^i \text{ such that:}$$

$$(i) \quad L_\mu \models \text{ZF}^- + \forall x \bar{x} \leq \alpha,$$

$$(ii) \quad \text{if } \alpha \text{ is a successor cardinal } \text{cf}(\mu) = \alpha;$$

$$\mu_\xi^\omega = \sup(\mu_\xi^i \mid i < \omega).$$

**Definition 3.** Let  $\alpha$  be a limit cardinal and  $n \in \omega$ , I define one Suslin tree  $T(n, \alpha)$  of height  $\chi_{n+2}(\alpha)$  in the same way the sequence  $T(n)$  has been defined in Section 1. The levels of the tree are defined by induction; for limit  $\lambda$  such that  $\text{cf}(\lambda) = \chi_{n+1}(\alpha)$  the level is defined by forcing over  $L_{\mu_\lambda^1}$ .

**Note.** (1) If  $\beta$  is an ordinal such that  $L_\beta \models \text{ZF}^- +$  there is a greatest cardinal and

$$\forall \xi < \beta \quad \forall i \leq \omega \quad \mu_\xi^i < \beta;$$

then the trees  $T(n, \gamma)$  can be defined in  $L_\beta$  — by the same definition as the right ones — for any  $\gamma$  such that  $L_\beta$  satisfies:  $\gamma$  is a limit cardinal such that  $\chi_{n+1}(\gamma)$  exists.

(2) If  $\beta \leq \bar{\beta}$  satisfy the hypothesis of (1) and  $L_\beta$  and  $L_{\bar{\beta}}$  have the same cardinals, then the  $T_\beta(n, \gamma)$  and the  $T_{\bar{\beta}}(n, \gamma)$  are the same if  $\chi_{n+1}(\gamma)$  is not the greatest cardinal; and if  $L_\beta$  and  $L_{\bar{\beta}}$  satisfy:  $\chi_{n+1}(\gamma)$  is the greatest cardinal, then:

$$T_\beta(n, \gamma) = T_{\bar{\beta}}(n, \gamma) \cap L_\beta.$$

(3) Since now I shall work on a class of cardinals one tree by level will be enough and simpler.

(4) If  $|x|$  is the level of  $x$  in  $T$ , note that  $x \in L_{\mu_{|x|}^2}$  for any  $x$  and that:

$$\forall x \in T(n, \alpha) \text{ (cf}(|x|) < \chi_{n+1}(\alpha) \Rightarrow x \in L_{\mu_{|x|}^1}).$$

(This will be often used without mention.)

## 2.2.

Now I am going to define, for each  $n$ , a forcing  $P(n)$  which looks like the final forcing  $P_\omega$  of Jensen. Fix  $n$  up to the end of this section (the subscript  $n$  will be often omitted). In the following I shall also follow Jensen's notations (see [1]).

**Definition 1.** Let  $\alpha$  be a cardinal  $\geq \omega$ ; define  $S_\alpha$  as follows:

(I) If  $\alpha$  is not  $\chi_{n+1}(\gamma)$  for some limit cardinal  $\gamma$ , then  $s \in S_\alpha$  iff  $s : [\alpha, |s|] \rightarrow 2$  such that  $|s| < \alpha^+$  and  $\forall \xi \leq |s|$ :

(1)  $s \restriction \xi \in L_{\mu_\xi^1}$ ;

(2)  $\forall \beta L_\beta(s \restriction \xi)$  satisfies: “if  $\Theta(\alpha, \xi, s \restriction \xi)$ , then  $\Gamma(\alpha, \xi, s \restriction \xi)$ ” where  $\Theta$  is “ $\text{ZF}^- + \xi = \alpha^+ +$  there is a greatest cardinal (not necessarily  $\xi!$ ) +  $\forall i \leq \omega \forall \nu \mu_\nu^i$  exists” and  $\Gamma$  is: “when  $s \restriction \xi$  is decoded by Jensen's method, a cofinal branch is founded in each  $T_\beta(n, \gamma)$  such that  $L_\beta \models \gamma$  is a limit cardinal such that  $\chi_{n+1}(\gamma)$  exists”.

(II) If  $\alpha$  is  $\chi_{n+1}(\gamma)$  for some limit  $\gamma$ , then (1) is replaced by:

(1')  $s \restriction Z_0$  is a branch in  $T(n, \gamma)$  (recall  $T$  is a binary tree);

(1'')  $s \restriction \xi \in L_{\mu_\xi^1}(s \restriction Z_0 \cap \xi)$ ;

where as in [1]  $Z_\nu = \{(\eta, \nu) \mid \eta \in \text{Ord}\}$  and  $(\cdot, \cdot)$  is the Gödel pairing function.

**Note.** In [1] for  $s \in S_\alpha$ , ordinals  $\mu_s^i$  are defined.

**Claim.** The  $\mu_s^i$  defined in [1] are the  $\mu_{|s|}^i$  of my Definition 2.

**Proof.** In the case (I) this is trivial since  $s \in L_{\mu_s^1}$ .

In the case (II) I only have to show that  $b = L_{\mu_{|s|}^1}(s \restriction Z_0)$  satisfies the conditions  $a \rightarrow d$  of Jensen's definition of the  $\mu_s^1$  in the case  $\text{cf}(|s|) = \chi_{n+1}(\gamma)$  (by Note 4 in Definition 3); but then  $s \restriction Z_0$  is (by the construction of  $T$ ) generic over  $L_{\mu_{|s|}^1}$  and the result follows easily.  $\square$

**Definition 2.** For  $\alpha$  a cardinal and  $s \in S_{\alpha^+}$ ,  $R_s$  is the set of pairs  $(r, r')$  such that:

(i)  $r \in S_\alpha$ ;

(ii)  $(r, r')$  codes  $s$  as in [1].

**Note.** Here there is no  $A \cap \alpha^+$ : so the generic will be a Cohen in  $Z_0$  if  $\alpha$  is not  $\chi_{n+1}(\gamma)$  for some limit cardinal  $\gamma$  and a branch in  $T(n, \gamma)$  in the other case. Define also the  $\mathcal{A}_s^i$  ( $i \leq \omega$ ;  $s \in S_\alpha$ ) as in [1].

**Claim.** (1) For  $i \neq 1$   $\mathcal{A}_s^i = L_{\mu_{|s|}^i}$ .

(2) For  $i = 1$   $\mathcal{A}_s^1 = L_{\mu_{|s|}^1}$  if  $\alpha$  is not  $\chi_{n+1}(\gamma)$  for some limit  $\gamma$  and  $\mathcal{A}_s^1 = L_{\mu_{|s|}^1}(s \restriction Z_0)$  in the other case.

**Lemma 1.**  $R_s$  is  $\alpha$  distributive in  $\mathcal{A}_s^1$ .

**Proof.** Forcing with  $R_s$  is equivalent to the three following forcing operations

(1)  $R_0$  to code  $s$  on  $Z_1$  and so add  $D_0$  a subset of  $[\alpha, \alpha^+[\cap Z_1$  in such a way that:

$$\forall \xi \in [\alpha, \alpha^+[\ D_0 \cap \xi \in L_{\mu_\xi^1}.$$

**Claim.**  $R_0$  is  $\alpha$  distributive in  $\mathcal{A}_s^1$ .

**Proof.** Assume  $(\Delta_i)_{i < \alpha}$  is a sequence of strongly dense subsets of  $R_0$ ,  $(\Delta_i)_{i < \alpha} \in \mathcal{A}_s^1$  and  $p_0 \in R_0$ ; set  $b = \mathcal{A}_s^2 = L_{\mu_{|s|}^2}$ .

Define a sequence  $(X_i)_{i \leq \alpha}$  as follows:

$$X_0 = \text{the least } X < b \text{ such that } \alpha \cup \{p_0, (\Delta_i)_i\} \subset X;$$

$$X_{i+1} = \text{the least } X < b \text{ such that } X_i \cup \{X_i\} \subset X;$$

$$X_\lambda = \bigcup_{i < \lambda} X_i \text{ for limit } \lambda.$$

Set  $\sigma_i : b_i = L_{\beta_i} \xrightarrow{\sim} X_i$  and  $\alpha_i = X_i \cap \alpha^+ = \sigma_i^{-1}(\alpha^+)$ .

Define a sequence  $(p_i)_{i \leq \alpha}$  of conditions by:

$$p_{i+1} = \text{the least } p \leq p_i \text{ such that } p \in \Delta_i \text{ and } |p| \geq \alpha_i$$

(I shall prove later — in Section 2.4 — that such a  $p$  exists);

$$p_\lambda = \bigcup_{i < \lambda} p_i \text{ for limit } \lambda \text{ if } \bigcup p_i \text{ is a condition, undefined if not.}$$

As in [1] the only problem is to show that for limit  $\lambda$   $p = \bigcup_{i < \lambda} p_i$  is a condition. And for that I must show that  $p \in L_{\mu_{\alpha_\lambda}^1}$ . But as in [1] it is easy to see that  $p \in L_{\beta_{\lambda+1}}$  ( $p$  can be defined in  $L_{\beta_{\lambda+1}}$  by use of  $b_\lambda$  instead of  $b$ ), and since  $\beta_\lambda < \mu_{\alpha_\lambda}^1$  we are done. We now work in  $\mathcal{A}_s^1(D_0) = L_{\mu_{|s|}^1}(D_0)$ .  $\square$

(2) The second step is (if  $\alpha$  is  $\chi_{n+1}(\gamma)$  for some limit  $\gamma$ ): add a branch  $B$  in  $T(n, \gamma)$ ; this is trivially  $\alpha$  distributive since  $T$  remains Suslin in  $L_{\mu_s^1}(D_0)$ .

(3) The third has to ensure the property (2) in the definition of  $S_\alpha$ . A condition is a function  $r$ :

$$[\alpha, |r|] [\cap \bigcup_{\beta \in I} Z_\nu \rightarrow 2$$

such that  $|r| < \alpha^+$  and  $I = [2, \alpha^+$  if (2) has occurred (resp.  $\{0\} \cup [2, \alpha^+$  if not) and  $\forall \xi \in [\alpha, |r|]$

(i)  $r \restriction \xi \in L_{\mu_\xi^1}(B \cap \xi)$  (resp.  $L_{\mu_\xi^1}$ )

(ii)  $\forall \beta L_\beta(D_0 \cap \xi, B \cap \xi, r \restriction \xi)$  (resp.  $L_\beta(D_0 \cap \xi, r \cap \xi)$ ) satisfies: “if  $\Theta(\alpha, \xi, (D_0, B, r) \restriction \xi)$ , then  $\Gamma(-----)$ ” (resp. “if  $\Theta(\alpha, \xi, (D_0, r) \restriction \xi)$ , then  $\Gamma(-----)$ ”). The proof that this forcing is  $\alpha$  distributive in  $L_{\mu_s^1}(D_0, B)$  (resp.  $L_{\mu_s^1}(D_0)$ ) uses essentially the same argument as in Lemma 2 in Section 1.2, but since some care is needed, I shall repeat it.

I shall develop the argument in the first case (when  $B$  occurs). It is the same in the other case.

Begin as in the proof for  $R_0$ , using  $b = L_{\mu_s^1}(D_0, B)$  and define  $X_i, \alpha_i, p_i$  ( $i \leq \alpha$ ) in the same way; as usual I have to prove (1'') and (2) in the definition of  $S_\alpha$  only for  $\xi = \alpha_\lambda = |p|$  when  $\lambda$  is limit and  $p = \bigcup_{i < \lambda} p_i$ .

(1'') is as for  $R_0$ : it is easy to see that

$$p \in L_{\beta_\lambda+1}(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda) \subset L_{\mu_{\alpha_\lambda}^1}(B \cap \alpha_\lambda).$$

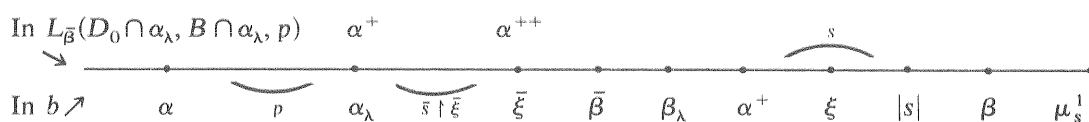
I now prove (2): choose  $\bar{\beta}$  such that  $L_{\bar{\beta}}(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda, p) \models \Theta$ .

Since  $(\alpha_i)_{i < \lambda} \in L_{\beta_\lambda+1}(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda)$ ,  $\bar{\beta}$  is not greater than  $\beta_\lambda$  set  $\bar{s} = \sigma_\lambda^{-1}(s)$  (recall  $s \in b$ ).

(A) Assume first there is a  $\bar{\xi}$  such that  $L_{\bar{\beta}} \models \bar{\xi} = \alpha^{++}$ , then  $\bar{\beta} < \beta_\lambda$  (since  $\sigma_\lambda : L_{\beta_\lambda} \xrightarrow{\sim} L_{\mu_s^1}$  and the definition of  $\mu_s^1$  ensures that  $L_{\mu_s^1} \models \alpha^{++}$  does not exist).

**Claim.**  $\bar{\xi} \leq |\bar{s}|$ .

If  $|\bar{s}| < \bar{\xi}$ , then  $\mu_{\bar{s}}^1 < \bar{\xi} < \bar{\beta} < \beta_\lambda$  and so (by  $\sigma_\lambda$ )  $\mu_s^1$  exists in  $b$ ; a contradiction. Set  $\xi = \sigma_\lambda(\bar{\xi})$  (so  $\xi \leq |s|$ ) and  $\beta = \sigma_\lambda(\bar{\beta})$ ; the picture is:



Now work in  $L_{\bar{\beta}}(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda, p)$ :  $B \cap \alpha_\lambda$  really is a branch; if you decode you first find  $\bar{s}$  and  $L_{\bar{\beta}}(\bar{s} \restriction \bar{\xi})$  satisfies  $\Theta(\alpha_\lambda, \bar{\xi}, \bar{s} \restriction \bar{\xi})$  so (by  $\sigma_\lambda$ )  $L_\beta(s \restriction \xi)$  satisfies  $\Theta(\alpha^+, \xi, s \restriction \xi)$ ; since  $s \in S_{\alpha^+}$  and (by  $\sigma_\lambda$ )  $L_{\bar{\beta}}(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda, \bar{s} \restriction \bar{\xi})$  satisfies  $\Gamma$ ; and this is exactly what we need.

(B) Assume now  $L_{\bar{\beta}} \models \alpha_\lambda$  is the greatest cardinal.

**Claim.**  $\bar{\beta} \leq \bar{s}$ .

If  $|\bar{s}| = \bar{\xi} < \bar{\beta}$ , then by the definition of  $\Theta$ ,  $\mu_{\bar{s}}^1 < \bar{\beta} \leq \beta_\lambda$  and so by  $\sigma_\lambda$ ,  $\mu_s^1$  exists in  $b$ ; a contradiction. So in  $L_{\bar{\beta}}(D_0 \cap \alpha_\lambda, B \cap \alpha_\lambda, p)$  we can recover  $\bar{s} \restriction \bar{\beta}$  and so by

the elementarity of  $\sigma_\lambda \bar{s} \restriction \bar{\beta} \cap Z_0$  is a branch in the tree (if it has to be one) (note that this does not occur when (B) appears — since that last tree has height  $\alpha_\lambda$  — however, I give the argument, since when (B) does not appear, this case can occur). This achieves the proof of the lemma.  $\square$

Defining  $S_\alpha^+$  and  $R^D$  as in [1] (see Definition 2, p. 37) we have:

**Lemma 2.**  $R^D$  satisfies the  $\alpha^{++}$ -chain condition in  $\mathcal{A}_D$ .

**Proof.** Each step of the iteration satisfies it — and even the strong chain condition, except for adding a branch in the tree.  $\square$

### 2.3.

The other definitions and lemmas used to define what is called  $P_\omega$  in [1] are exactly as in [1], except one point: It is more convenient to define  $\rho_{s\tau}^i$  to be a code for  $\mathcal{A}_{s, \chi_{n+2}(\tau)}^i$  (instead of  $\mathcal{A}_{\sigma\tau}^i$  in [1]). That does not modify the proof in [1] but for me it ensures that there is no intrusion of the coding at limit cardinals with the branches. Now what I call  $P(n)$  (remind  $n$  was fixed) is what it called  $P_\omega$  in [1].

I have to show the lemmas that say:

- (1) a condition can be arbitrarily extended;
- (2)  $P_\tau(n)$  is  $\tau$ -distributive.

I shall prove that later and now show how to conclude. At the moment the following will be clear.

**Lemma 1.** Let  $N$  be a  $P(n)$  generic extension of  $M_0$ ; then there is a subset  $A_n$  of  $\omega_1$  such that  $N$  satisfies:

- (1)  $ZF + V = L(A_n)$ .
- (2)  $\forall \alpha < \omega_1 \forall \beta$  if  $L_\beta(A_n \cap \alpha) \models ZF^-$  + there is a greatest cardinal  $\alpha = \omega_1 = \omega_1^L$ , then if you decode  $A_n \cap \alpha$  by Jensen's method you find a cofinal branch in each  $T_\beta(n, \gamma)$  such that  $L_\beta \models \gamma$  is a limit cardinal and  $\chi_{n+1}(\gamma)$  exists.

Let  $M_1$  be a  $\prod_{n \in \omega} P(n)$  generic extension of  $M_0$ . Define the forcing  $C_s$  as in Section 1. Let  $M_2$  be a  $C_\emptyset$  generic extension of  $M_1$ . This gives the real  $a$  for which I must prove.

**Lemma 2.** For all the limit  $\gamma$ ,  $L(a)$  satisfies:

- $T(2i, \gamma)$  is Suslin iff  $a(i) = 0$ ,
- $T(2i+1, \gamma)$  is Suslin iff  $a(i) = 1$ .

This follows — as in Section 1 — from Lemmas 3 to 5.

**Lemma 3.** Let  $f = (p(n))_{n \in \omega}, (r_0, r_1)$  be a condition in

$$F = \prod_{n \in \omega} P(n) * C_\emptyset,$$

then  $F_f = \{g \in F \mid g \leq f\}$  is isomorphic to the product  $F' \times F''$  where

$$F' = \left( \prod_{\substack{r_0(n)=1 \\ \text{or} \\ n \geq |r_0|}} P(2n) \times \prod_{\substack{r_0(n)=0 \\ \text{or} \\ n \geq |r_0|}} P(2n+1) \right) * C_{r_0},$$

$$F'' = \left( \prod_{r_0(n)=0} P(2n) \times \prod_{r_0(n)=1} P(2n+1) \right).$$

**Proof.** Trivial.  $\square$

**Lemma 4.** Let  $\gamma$  be a limit cardinal,  $a$  be a finite subset of  $\omega$  and  $n \in a$ . Let  $N$  be a  $\prod_{m \notin a} P(m)$  generic extension of  $M_0$ , then  $N$  satisfies:  $T(n, \gamma)$  is Suslin.

**Proof.** Set  $\alpha = \chi_{n+1}(\gamma)$ ; still using the notation in [1], there are subsets  $D_m$  ( $m \notin a$ ) of  $[\alpha, \alpha^+]$  such that:  $P(m) = P_\alpha(m) * P_{\omega^m}^{D_m}(m)$ .

**Claim.**

$$\prod_{m \notin a} P(m) \simeq \left( \prod_{m \notin a} P_\alpha(m) \right) * \left( \prod_{m \notin a} P_{\omega^m}^{D_m}(m) \right).$$

**Proof.** This follows easily from the fact that  $\prod_{m \notin a} P_\alpha(m)$  is  $\alpha$  distributive and  $\alpha$  is a successor cardinal.  $\square$

So  $\prod_{m \notin a} P(m)$  is as a two step forcing:

The first one gives a model that satisfies:

$$V = L((D_m)_{m \notin a}) + \forall m \notin a \ D_m \subset [\alpha, \alpha^+][\forall \xi \in [\alpha, \alpha^+][\forall m \ D_m \upharpoonright \xi \in L_{\mu_\xi^1}].$$

So  $T(n, \gamma)$  is Suslin in  $L((D_m)_{m \notin a})$ ; this follows from the Lemma 6 below.

The second one satisfies the  $\alpha^+$  strong chain condition, so  $T(n, \gamma)$  remains Suslin by Lemma 5 below.  $\square$

**Lemma 5.** Let  $M$  be a model of ZF,  $\alpha$  a cardinal and  $T$  a Suslin tree of height  $\alpha^+$ ; let  $P$  be a notion of forcing that satisfies the  $\alpha^+$  strong chain condition and  $N$  be a  $P$  generic extension of  $M$ ; then  $N$  satisfies:  $T$  is Suslin.

**Proof.** As in Lemma 7 in Section 1.3.  $\square$

**Lemma 6.** Let  $M$  be a model of:  $\text{ZF} + V = L(D)$ ,  $D \subset \alpha^+$  for some regular cardinal  $\alpha$ . Let  $T$  be the Suslin tree of height  $\alpha^+$  defined in  $L$  as in Section 1 (by forcing over  $L_{\mu_\xi^1}$  for  $\text{cf}(\xi) = \alpha$ ). Assume:  $\forall \xi < \alpha^+ (\text{cf}(\xi) = \alpha \rightarrow D \upharpoonright \xi \in L_{\mu_\xi^1})$ . Then  $T$  is Suslin in  $M$ .

**Proof.** This is an easy corollary to the proof of the (generalized) lemma in Section 1.1.  $\square$



**Definition.** Let  $\varphi$  be the formula:  $\forall \alpha < \beta < \omega_1$  if  $L_\beta(x) \models \text{ZF}^- + \alpha = \omega_1 = \omega_1^L +$  there is a greatest cardinal  $+\forall \xi \forall i < \omega \mu_\xi^i$  exists, then if you decode  $x$  by Jensen's method you find a cofinal branch in  $T_\beta(n, \gamma)$  for each  $\gamma$  such that  $L_\beta \models \gamma$  is a limit cardinal and for each  $n$  such that:

$$n = 2i \quad \text{and} \quad x(i) = 1$$

or

$$n = 2i + 1 \quad \text{and} \quad x(i) = 0.$$

Now the theorem follows easily from all the previous results; the argument for (3) in the theorem being: (by use of Lemma 5) you cannot kill all the  $T(n, \gamma)$  by forcing with a set of conditions.

It remains now to prove that conditions can be arbitrarily extended and then distributivity.

#### 2.4.

**Lemma 1.** Let  $\psi$  be a formula;  $\alpha$  a cardinal; let  $C$  be the following notion of forcing: a condition is a function  $c : [\alpha, |c|] \rightarrow 2$  such that  $|c| < \alpha^+$  and for  $\xi \leq |c|$

$$(1) \quad c \upharpoonright \xi \in L_{\mu_\xi^+},$$

$$(2) \quad \forall \beta L_\beta(c \upharpoonright \xi) \models (\text{ZF}^- + \xi = \alpha^+ \Rightarrow \psi),$$

then for any condition  $c_0$  and  $\gamma \in [|c|, \alpha^+]$  there is a condition  $c \leq c_0$  such that  $|c| = \gamma$ .

As a corollary of this lemma we immediately have:

**Lemma 2.** Any condition in  $R_s$  can be arbitrarily extended.

**Proof of Lemma 1.** By induction on  $\gamma$ ; for  $\gamma$  successor this is trivial, so let  $\gamma$  be a limit ordinal.

The proof of  $\square_\alpha$  (see for instance [1, Section 6]) shows that there is a closed unbounded set  $D$  of  $[\alpha, \gamma]$  such that  $|D| \leq \alpha$  (the order type of  $D$ ) and for all  $\nu \in D \cup \{\gamma\}$ ,  $D \cap \nu \in L_{\mu_\nu^+}$ .

Let  $(\gamma_i)_{i < \lambda}$  be the monotone enumeration of  $D$ ; I may assume  $\gamma_0 = |c_0|$  and  $\gamma_i$  is a limit ordinal for  $i < \lambda$ .

For any ordinal  $\beta = \xi + n$  where  $\xi$  is limit set  $f(\beta) = \xi + 2n + 1$ .

Define  $c_i$  ( $i \leq \lambda$ ) by induction as follows:

$c_{i+1} = c_i \cup \chi_E$  where  $\chi_E$  is the characteristic function of the set  $E = \{\gamma_i\} \cup \{f(\beta) \mid \beta \in c'\}$  where  $c'$  is the least  $c$  such that  $c \subset [\gamma_i, \gamma_{i+1}]$  and  $c_i \cup \chi_c$  is a condition.

$c_\eta = \bigcup_{i < \eta} c_i$  for limit  $\eta$  if  $\bigcup c_i$  is a condition, undefined if not.

As usual it is enough to show that for limit  $\eta$   $c = \bigcup_{i < \eta} c_i$  is a condition and for that to see that (1) and (2) in the definition of  $C$  is true for  $\xi = \gamma_\eta$ . (2) is trivial

since  $D \cap \gamma_\eta$  can be recovered from  $c$  so there is no  $\beta$  such that

$$L_\beta(c) \models \gamma_\eta = \alpha^+.$$

(1) since  $D \cap \gamma_\eta \in L_{\mu_{\gamma_\eta}^1}$  it follows easily that  $c \in L_{\mu_{\gamma_\eta}^1}$ .  $\square$

The other lemma that has to be proved is lemma I.6 in [1] with the new condition that control the level of constructibility.

**Definition 1.** Let  $\alpha$  be a cardinal; define  $\tilde{S}_\alpha$  to be the set of functions  $s : [\alpha, |s|] \rightarrow 2$  such that  $|s| < \alpha^+$  and for  $\xi \leq |s|$   $s \upharpoonright \xi \in L_{\mu_\xi^1}$ .

**Definition 2.** Let  $\kappa$  be a limit cardinal,  $A$  a subset of  $[\kappa, \kappa^+]$  such that:  $\forall \xi < \kappa^+ A \cap \xi \in L_{\mu_\xi^1}$ ; let  $I$  be the set of successor cardinals less than  $\kappa$ . For  $\tau$  a cardinal less than  $\kappa$  define  $\tilde{P}_\tau$  by:

A condition is a sequence  $p = (p_\gamma)_{\gamma \in I \cap [\tau, \kappa]}$  such that:

(1)  $\forall \gamma p_\gamma \in \tilde{S}_\gamma$ .

(2) If  $|p|$  is defined by:  $|p|$  = the least  $\xi$  such that  $p \in L_{\mu_\xi^1}$ , then for  $\xi < |p|$  there is a  $\nu \in I$  such that:  $\forall \gamma \in [\nu, \kappa] \cap I$

$$p_\gamma(\rho_{\xi\gamma}) = \begin{cases} 1 & \text{if } \xi \in A, \\ 0 & \text{if not} \end{cases}$$

(where the  $\rho_{\xi\gamma}$  are defined as in [5]).

**Lemma 3.** For any  $p \in \tilde{P}_\tau$  and  $\xi \geq |p|$  there is a  $q \leq p$  such that  $|q| = \xi$ .

The proof follows the proof in [1]: I shall need:

**Lemma 4.** Let  $\gamma$  be a successor cardinal;  $\xi \in [\gamma, \gamma^+]$  such that  $\text{cf}(\xi) < \gamma$ ;  $\mu \in ]\xi, \gamma^+[$  such that  $\text{cf}(\mu) = \gamma$  and  $L_\mu \models \text{ZF}^- + \forall x (\bar{x} \leq \gamma)$ ;  $X \subset \xi$  such that  $X \in L$  and  $\forall \eta < \xi$   $X \cap \eta \in L_\mu$ . Then  $X \in L_\mu$ .

**Proof.** Set  $\lambda = \text{cf}(\xi) < \gamma$ ; the proof follows immediately from the two claims.

**Claim 1.**

$$L_\mu \models \text{cf}(\xi) = \lambda.$$

Let  $f \in L$ ;  $f : \lambda \rightarrow \xi$  be cofinal, injective;  $g \in L_\mu$ ;  $g : \xi \rightarrow \gamma$  bijective, then  $g \circ f : \lambda \rightarrow \gamma$  injective, so since  $\gamma$  is regular  $g \circ f \in L_\gamma \subset L_\mu$  and so  $f \in L_\mu$ .

**Claim 2.** Let  $f \in L_\mu : \lambda \rightarrow \xi$  be cofinal, increasing and  $x_i = X \cap f(i)$ ; then  $(x_i)_{i < \lambda} \in L_\mu$ .

Let  $\varphi : \mu \rightarrow L_\mu$  be the canonical enumeration of  $L_\mu$ ; set  $\eta_i = \varphi^{-1}(x_i)$ ; it is

enough to show:  $(\eta_i)_{i<\lambda} \in L_\mu$  but this follows by the same argument as in Claim 1 from  $\text{cf}(\mu) = \gamma > \lambda$ .

**Proof of Lemma 3.** The successor case is as in [1] (I of course use Lemma 1 to extend  $p$  up to  $\rho_{\xi_\gamma}$ ).

For the limit case, I extend  $p$  using a closed unbounded subset of  $[|p|, \xi[$  coming from a  $\square$  sequence. As usual it is enough to prove that for a limit  $\lambda$  if  $p = \bigcup_{i<\lambda} p^i$ , then for  $\gamma \in I$   $p_\gamma \in L_{\mu_{p_\gamma}^1}$ . It follows easily from Lemma 4 and the fact I may always assume that  $\lambda < \gamma$  (if the order type of the c.u.b. is less than  $\kappa$  this is easy; if not, then  $\kappa$  is regular and the  $p^i$  are defined in such a way that  $p_\gamma^\lambda = \bigcup_{i<\lambda} p_\gamma^i$  is a non trivial union only for  $\gamma > \lambda$  (remind  $\gamma$  is a successor)).

Now the proof of the extendability of the conditions is exactly as in [1], using the previous lemmas and specially the Lemma 4.

The only (slight) different point is in Lemma 2.14.3 of [1] when the  $r_{ij}, \tilde{q}_{ij}$  are defined. I must be more careful since for limit  $i$   $(r'_j)_{j<i}$  is not necessarily in  $\mathcal{A}_r^1$ ; so  $r_{i1}$  has to be defined in the following way:

Let  $\xi_i$  be the least such that  $x_i \in L_{\mu_{\xi_i}^1}$  where  $x_i = (p \upharpoonright \beta_i; (r_{h0} \mid h \leq i); \tilde{q} \upharpoonright \beta_i)$ ; define  $r_{i1}$  by:  $r_{i0} \cup \{(\eta_i, 5)\} \cup r'$  where  $r'$  is such that:

- (1)  $\eta_i$  is s.t.  $L_{\mu_{\xi_i}^1} \models x_i$  has number  $\eta_i$  in the canonical well ordering of  $V$ ;
  - (2)  $r'$  is the least  $r \subset [r_{i0}, \mu_{\xi_i}[ \cap Z_4$  such that  $r_{i1} \in S_{\beta_i}$ . It is then easy to see that
- It is then easy to see that

$$r_i^* = \bigcup_{j<\omega} r_{ij} \in S_{\beta_i}$$

and the end of the lemma is the same.  $\square$

## 2.5.

The distributivity is proved as in [1]. I shall only mention the points where something has to be changed or where some care is needed. Lemma 3.7 in [1] has to be replaced by:

**Lemma.** Assume distributivity holds up to  $\alpha$ ; let  $s \in S_{\alpha^+}$ ,  $\tau \leq \alpha$  and let  $D \subset [\tau, \alpha^+[$  be  $P_\tau^s$  generic over  $\mathcal{A}_s^1$  then:

- (1)  $s, D \in L_{\mu_s^1}(D_\tau)$ ;
- (2) for  $\gamma \in [\tau, \alpha]$  and  $\xi \in [\gamma, \gamma^+[$  then
  - (i)  $D_\gamma \upharpoonright \xi \in L_{\mu_\xi^1}$  (resp. if  $\gamma = \chi_{n+1}(\nu)$  for some limit cardinal  $\nu$   $D_\gamma \cap Z_0$  is a branch in  $T(n, \nu)$  and  $D_\gamma \upharpoonright \xi \in L_{\mu_\xi^1}(D_\gamma \cap Z_0 \cap \xi)$ );
  - (ii)  $\forall \beta L_\beta(D_\tau \cap \xi)$  satisfies: “if  $\Theta$ , then  $I$ ”.

**Proof.** As in [1].  $\square$

The only other point is in the proof of Lemma 3.22(a) in [1]. I have to verify the additional property of  $S_\gamma$ .

There are two things to look at: the usual facts in [1] and the branches in the trees. Moreover we have to show that  $\prod_n P_\tau^s(n)$  is  $\tau$ -distributive, not only that for each  $n$   $P_\tau^s(n)$  is  $\tau$ -distributive, since the product of  $\tau$ -distributive forcings is not necessarily  $\tau$ -distributive.

What about branches?

Note that

$$\prod_n P_\tau^s(n) \simeq P_{\tau^+}^s(n) * \prod_n R^{D_n} \quad \text{for some generic sets } D_n.$$

With the first forcing there are no problems with the branches since all the trees involved in this forcing are  $\tau$ -closed.

For the second one there are no problems too with the branches except for the  $n$  (if there is one) such that:

$$\tau = \chi_{n+1}(\beta) \quad \text{for some limit } \beta.$$

But  $T(n, \beta)$  is Suslin in  $L_{\mu_s^*}(D_m \mid m < \omega)$ , since  $L_{\mu_s^*}$  and  $L_{\mu_s^*}(D_m \mid m < \omega)$  have the same subsets of  $\tau^+$  and it is well known that forcing with a Suslin tree of height  $\tau^+$  is  $\tau$  distributive.

So I can ‘forget’ the branches and only have to look at the properties (1) and (2) in  $S_\gamma$ : We start the construction with  $\mathcal{A}_s^2 = L_{\mu_s}$  instead of  $\mathcal{A}_s^1$  in [1].

The  $b$  appearing in the lemma is some  $L_{\mu'}$  with  $\mu' < \mu_{|p_\gamma^\lambda|}$  but since the sequence  $(p^i)_{i < \lambda}$  is definable from  $b$  as it was defined from  $\mathcal{A}_s^2$  it follows that  $p_\gamma^\lambda \in L_{\mu_{|p_\gamma^\lambda|}}$ .

The second property of  $S_\gamma$  comes from the fact that  $p_\gamma^\lambda$  is generic over the imitation of  $P_\tau^s$  in  $b$ , so the result follows for a  $\beta \leq \alpha^*$ ; it remains to show it for  $\beta \in ]\alpha^*, \mu'[($  (since  $L_{\mu'+1} \models |p_\gamma^\lambda| = \gamma$ ) but this is exactly as in Lemma 1 in Section 2.2.

This achieves the proof of Theorem 1.

## 2.6.

(1) Following exactly the proof of Theorem 0.2 in [1] we see that the real in my theorem I may be assumed to be in  $L(0^\#)$ .

(2) To prove that the real is not provably  $\Pi_2^1$  singleton, do as follows: Start from  $M_1$  (the  $\prod_n P(n)$  generic extension of  $M_0$ ) and choose  $M_2'$  to be a  $C_\emptyset \times C_\emptyset$  generic extension of  $M_1$ ; then there are two different reals  $a$  and  $a'$  that satisfy  $\varphi$  so  $L(a, a') \not\models \exists! x \varphi(x)$ .

(3) The following gives a slight improvement of Theorem 1: I may assume — as in [7] — that the trees are such that: if  $\alpha = |T|$  and  $B_0, B_1$  are distinct, cofinal branches in  $T$ , then  $\bar{\alpha}^{L(B_0, B_1)} < \alpha$  (simply use an elementary, saturated — in the sense of model theory — extension of the Rational numbers); I may assume too that the first level in  $T$  only has two points: the right and the left one.

When forcing with the  $P(n)$  I may assume that all the branches are left branches (i.e.: their first point is the left one) and so in  $\varphi$  it can be said that the branches are left ones.

Now in  $L(a)$  some trees remain Suslin; add in these trees right branches. Let  $N$  be this extension. In  $N$   $a$  remains  $\Pi_2^1$  singleton since if  $x$  is another real satisfying  $\varphi$ , then for some  $i$ ,  $T(i)$  will have two branches: a right and a left one, and this is impossible.

More generally if  $\bar{N}$  is an extension of  $N$  and if there is another real  $x$  in  $\bar{N}$  that satisfies  $\varphi$ , then a class of cardinals are collapsed (in the context I prove the theorem it is  $\{\chi_{n+2}(\beta) \mid \beta \text{ limit cardinal and } a(n) \neq x(n)\}$ ). I may assume that this class is  $\{\alpha^+ \mid \alpha \text{ regular}\}$ .

It would be more interesting if I could assume that all the successor cardinals were collapsed since then an extension of  $N$  where  $a$  is not  $\Pi_2^1$  would have to contain  $0^\#$ ; but it seems difficult to do that: it would be necessary to build trees of length successors of singular cardinals: it does not seem difficult to build such trees which are Suslin and have at most one branch (as far as  $|\tau|$  is not collapsed); but in the proof I also need the fact that: (1) the trees are closed enough (all the branches of 'small' length have extension in the tree); (2) the tree remain Suslin in some extension of  $L \cdots$  and I do not know how to do that!

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