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LOCAL TIME OF A DIFFUSION IN A STABLE LÉVY ENVIRONMENT

ROLAND DIEL AND GUILLAUME VOISIN

ABSTRACT. We consider a one-dimensional diffusion in a stable Lévy environment. We show that the normalized local time process refocused at the bottom of the standard valley with height $\log t$, $(L_X(t, \mathfrak{m}_{\log t} + x)/t, x \in \mathbb{R})$, converges in law to a functional of two independent Lévy processes conditioned to stay positive. To prove this result, we show that the law of the standard valley is close to a two-sided Lévy process conditioned to stay positive. We also obtain the limit law of the supremum of the normalized local time. This result has been obtained by Andreoletti and Diel [1] in the case of a Brownian environment.

Key words : Diffusion process, Local time, Lévy process, Random environment
MSC 2000: 60G51 - 60J55 - 60J60

1. INTRODUCTION.

Let $(V(x), x \in \mathbb{R})$ be a real-valued càd-làg stochastic process such that $V(0) = 0$. A *diffusion in the environment* V is a process $(X(V, t), t \geq 0)$ (or $(X(t), t \geq 0)$ if there is no ambiguity about the environment) such that its generator conditionally on V is

$$\frac{1}{2}e^{V(x)} \frac{d}{dx} \left(e^{-V(x)} \frac{d}{dx} \right).$$

Let us remark that if V is almost surely differentiable, $(X(t), t \geq 0)$ is the solution of the following SDE :

$$\begin{cases} dX(t) = d\beta(t) - \frac{1}{2}V'(X(t))dt, \\ X(0) = 0 \end{cases}$$

where β is a standard Brownian motion independent of V . Historically, this model has been introduced by Schumacher [26]. It was mainly studied when the environment V is a Brownian motion (T. Brox [5] and K. Kawazu and H. Tanaka [18]), a drifted Brownian motion (K. Kawazu and H. Tanaka [19], Y. Hu, Z. Shi and M. Yor [17], A. Devulder [8] and M. Talet [30]) or a process which has the same asymptotic behaviour as a Brownian motion (Y. Hu and Z. Shi [14, 15]), this last case allows to study the discrete analogue of the diffusion in a Brownian environment, the Sinai's random walk . More recently, general Lévy random environment have been studied by Carmona [6], D. Cheliotis [7] and Singh [29].

The local time process L_X can be defined by the occupation time formula: $L_X = (L_X(t, x), t \geq 0, x \geq 0)$ is the unique process, almost surely continuous in the variable t , such that for all bounded Borel function f and for all $t \geq 0$,

$$(1) \quad \int_0^t f(X_s)ds = \int_{\mathbb{R}} f(x)L_X(t, x)dx.$$

The existence of such a local time process can easily be proved, see e.g. Shi [28].

The study of the local time process has been made in particular in the Brownian case, we cite for example Y. Hu and Z. Shi [14, 16], Z. Shi [28], A. Devulder [9] in the drifted Brownian case and P. Andreoletti and R. Diel [1]. In this paper, we generalize to the Lévy environment case the result of [1], however there are several difficulties for the environment: contrary to

the Brownian motion, the law of a Lévy process is not stable by time inversion and moreover the processes $(V(x) - \inf_{0 \leq y \leq x} V(y), x \geq 0)$, $(\sup_{0 \leq y \leq x} V(y) - V(x), x \geq 0)$ and $|V|$ have not the same law.

In what follows, we are interested in the local time of the diffusion when the environment V is an α -stable Lévy process on \mathbb{R} , that is to say there exists $\alpha > 0$, such that for all $c > 0$

$$(c^{-\frac{1}{\alpha}} V(cx), x \in \mathbb{R}) \stackrel{\mathcal{L}}{=} (V(x), x \in \mathbb{R})$$

where $\stackrel{\mathcal{L}}{=}$ is the equality in law.

One of the main result of this paper is the convergence in law of the supremum of the local time:

$$\forall t \geq 0, \quad L_X^*(t) := \sup_{x \in \mathbb{R}} L_X(t, x)$$

which measures the time passed by the diffusion in the most visited point.

To write the results of this paper, we need the notion of *valley* introduced in the Brownian case by Brox [5] (see also [22]).

We denote by \mathcal{V} the space of càd-làg functions $\omega : \mathbb{R} \rightarrow \mathbb{R}$, endowed with the Skorohod topology (the space \mathcal{V} is then Polish). Let $c > 0$, $\omega \in \mathcal{V}$ has a *c-minimum* in the point x_0 if there exists ξ and ζ such that $\xi < x_0 < \zeta$ and

- $\omega(\xi) \geq (\omega(x_0) \wedge \omega(x_0^-)) + c$,
- $\omega(\zeta^-) \geq (\omega(x_0) \wedge \omega(x_0^-)) + c$ and
- for all $x \in (\xi, \zeta)$, $x \neq x_0$, $\omega(x) \wedge \omega(x^-) > \omega(x_0) \wedge \omega(x_0^-)$.

Identically, ω has a *c-maximum* in x_0 if $-\omega$ has a *c-minimum* in x_0 . We denote by $M_c(\omega)$ the set of *c-extrema* of ω . If ω takes pairwise distinct values in its local extrema and is continuous at these points, then we can check that $M_c(\omega)$ has no accumulation point and that the *c-maxima* and the *c-minima* alternate. These assumptions are verified for α -stable Lévy processes with $\alpha \in [1, 2]$. Thus there exists a triplet $\Delta_c = \Delta_c(\omega) = (\mathfrak{p}_c, \mathfrak{m}_c, \mathfrak{q}_c)$ of successive elements of M_c such that

- \mathfrak{m}_c and 0 belongs to $[\mathfrak{p}_c, \mathfrak{q}_c]$,
- \mathfrak{p}_c and \mathfrak{q}_c are *c-maxima*,
- \mathfrak{m}_c is a *c-minimum*.

We call this triplet, the *c-standard valley* of ω .

We also introduce the notations ω^+ and ω^- :

$$(2) \quad (\omega^+(x), x \geq 0) = (\omega(x), x \geq 0) \quad , \quad (\omega^-(x), x \geq 0) = (\omega((-x)^-), x \geq 0)$$

Our main result gives the limit law of the local time process refocused on the minimum of the standard valley with height $\log t$ when t goes to infinity:

Theorem 1.1. *Let V be an α -stable Lévy process with $\alpha \in [1, 2]$ which is not a pure drift.*

The process $\left(\frac{L_X(t, \mathfrak{m}_{\log t} + x)}{t}, x \in \mathbb{R} \right)$ converges in law in the Skorohod sense to the process

$$\left(\frac{e^{-\tilde{V}(x)}}{\int_{-\infty}^{\infty} e^{-\tilde{V}(y)} dy}, x \in \mathbb{R} \right) \text{ where the law of } \tilde{V} \text{ is } \tilde{\mathbb{P}} \text{ and,}$$

under $\tilde{\mathbb{P}}(d\omega)$, ω^+ and ω^- are independent, and distributed respectively as \mathbb{P}^\dagger , the law of the Lévy process V conditioned to stay positive, and $\tilde{\mathbb{P}}^\dagger$ the law of the dual process V^- conditioned to stay positive.

Then we get the immediate corollary:

Corollary 1.2. *With the same assumptions, we get*

$$\frac{L_x^*(t)}{t} \xrightarrow{\mathcal{L}} \frac{1}{\int_{-\infty}^{\infty} e^{-\tilde{V}(y)} dy}$$

where \tilde{V} is the same process as in the previous theorem.

To prove this theorem, we need a result on the Lévy environment V . If we denote by $\underline{\tau}_c^+$ the time of the first rise of height c of V^+ above its infimum, by m_c^+ the infimum of V^+ on $[0, \underline{\tau}_c^+]$ and if we do the same for $\underline{\tau}_c^-$ and m_c^- using the process V^- , then one can show that the c -minimum \mathfrak{m}_c of the standard valley is equal to m_c^+ or to m_c^- . It is therefore sufficient to consider the environment on \mathbb{R}_+ . We get that:

- the law of the process $(V((m_c^+ - x)^-) - V(m_c^+)), 0 \leq x \leq m_c^+$ under \mathbb{P} is the law of the process $(V^\uparrow(x), 0 \leq x \leq \hat{m}_c^\uparrow)$ under $\hat{\mathbb{P}}^\uparrow$,
- the law of the process $(V(x), m_c^+ \leq x \leq \underline{\tau}_c^+)$ under \mathbb{P} is absolutely continuous with respect to the law of $(V^\uparrow(x), 0 \leq x \leq \tau_c(V^\uparrow))$ under \mathbb{P}^\uparrow ,

where the times \hat{m}_c^\uparrow and $\tau_c(V^\uparrow)$ will be defined in Section 2.

This result has already been obtained by H. Tanaka [32] in the special case of a Brownian environment ; in this case, the law of the process $(V(x), m_c^+ \leq x \leq \underline{\tau}_c^+)$ is \mathbb{P}^\uparrow and moreover $\hat{\mathbb{P}}^\uparrow$ and \mathbb{P}^\uparrow are both the law of a 3-dimensional Bessel process.

This paper is organized as follows: in Section 2, we define the environment V and we give the asymptotic law of the standard valley V on \mathbb{R}_+ (Proposition 2.14) and in Section 3, we prove the convergence theorem of the local time of the diffusion (Theorem 1.1).

2. STANDARD VALLEY OF A LÉVY PROCESS.

2.1. Lévy process on \mathbb{R} . We use a definition of Carmona (see [6]). A \mathcal{V} -valued random process V is a Lévy process on \mathbb{R} if it is a process with independent and stationary increments:

- (1) $V(0) = 0$,
- (2) if x_0, x_1, \dots, x_n are real numbers, then the random variables $(V(x_{i+1}) - V(x_i), 0 \leq i \leq n-1)$ are independent,
- (3) for all x and y , the law of $V(x+y) - V(x)$ only depends on y .

We denote by \mathbb{P} the law of the Lévy process V .

Remark 2.1. An equivalent definition of a \mathbb{R} -valued Lévy process with indexed by \mathbb{R} is given by Cheliotis in [7]: let $(V^+(t), t \geq 0)$ be a \mathbb{R} -valued Lévy process starting from 0 and $(V^-(t), t \geq 0)$ a process independent of V^+ and with the same law as $-V^+$, for all $t \in \mathbb{R}$, we define a process $(V(t), t \geq 0)$ by

$$(3) \quad V(t) = \mathbf{1}_{t \geq 0} V^+(t) + \mathbf{1}_{t \leq 0} V^-((-t)^-).$$

We can check that this process has càd-làg paths and check the three assumptions of the previous definition. Conversely, a process that checks the three assumptions can be decomposed in two Lévy processes as in Formula (3).

Let us define \underline{V}^+ the infimum process of the Lévy process V^+ and \overline{V}^+ the supremum process: for all $t \geq 0$, $\underline{V}^+(t) = \inf_{0 \leq s \leq t} V^+(s)$ and $\overline{V}^+(t) = \sup_{0 \leq s \leq t} V^+(s)$. Let \mathbb{N} be the excursion measure of the process $V^+ - \underline{V}^+$ away from 0. We denote by L the local time of $V^+ - \underline{V}^+$ in 0 and $L^{-1}(t) = \inf\{s \geq 0; L(s) > t\}$ its right continuous inverse.

We denote by \mathcal{V}_+ the set of càd-làg functions $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}$. For $c > 0$ fixed, we define the first passage time above a level c of ω and the last previous time in which ω reaches 0: for all $\omega \in \mathcal{V}_+$,

$$\tau_c(\omega) = \inf\{t \geq 0; \omega(t) \geq c\} \text{ and } n_c(\omega) = \sup\{t \leq \tau_c(\omega); \omega(t) = 0 \text{ or } \omega(t-) = 0\}.$$

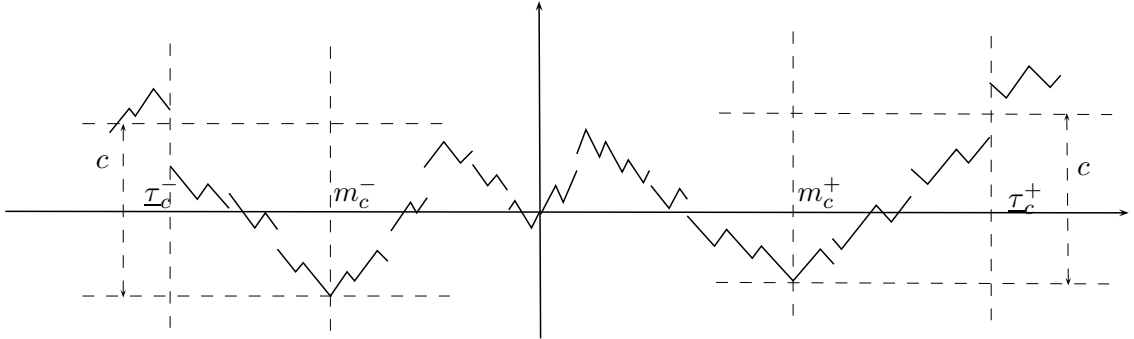
We then define

$$\underline{\tau}_c^+ = \tau_c(V^+ - \underline{V}^+) \text{ , } m_c^+ = n_c(V^+ - \underline{V}^+) \text{ and}$$

$$J_c^+ = (V^+(m_c^+) + c) \vee \overline{V}^+(m_c^+).$$

Similarly, we define analogue variables $\underline{\tau}_c^-$, m_c^- and J_c^- using the process V^- .

FIGURE 1. Standard valley with height c



Remark 2.2. We show that the minimum of the standard valley with height c defined in the introduction checks $m_c = m_c^+$ if $J_c^+ < J_c^-$ and $m_c = m_c^-$ otherwise (see [20] for a proof in the Brownian case which is also true in our case). Thus to study the process V in a neighbourhood of m_c , it is sufficient to study the processes V^+ in a neighbourhood of m_c^+ and V^- in a neighbourhood of m_c^- .

In this section, we will study the Lévy process V^+ defined on \mathbb{R}_+ , we still denote \mathbb{P} its law. Until the end of this section, all the probability measures are measures on \mathcal{V}^+ . In order to simplify the notations, we write V , m_c and $\underline{\tau}_c$ for V^+ , m_c^+ and $\underline{\tau}_c^+$.

2.2. Valley of the Lévy process on \mathbb{R}_+ . We say that $(V(t), 0 \leq t \leq \underline{\tau}_c)$ is the standard valley with height c of the Lévy process on \mathbb{R}_+ .

Furthermore we say that 0 is regular for $(0, +\infty)$ (resp. for $(-\infty, 0)$) if $\mathbb{P}(\inf\{t > 0; V(t) > 0\} = 0) = 1$ (resp. $\mathbb{P}(\inf\{t > 0; V(t) < 0\} = 0) = 1$). We only work with Lévy process for which 0 is regular for $(-\infty, 0)$ and $(0, +\infty)$. Thereby, we begin by recalling some properties of such processes stated in [7], which, in particular, gives us the existence of the triplet $(\mathfrak{p}_c, m_c, \mathfrak{q}_c)$ defined in the introduction.

For $\omega \in \mathcal{V}^+$, the point x_0 is called a left local maximum (resp. minimum) if there exists $\varepsilon > 0$ such that $\omega(x) \leq \omega(x_0-)$ (resp. $\omega(x) \geq \omega(x_0-)$) for all $x \in (x_0 - \varepsilon, x_0)$. Similarly, we define the right local maximum and minimum.

Lemma 2.3. ([7], Lemma 3)

Let V be a Lévy process such that 0 is regular for $(-\infty, 0)$ and $(0, +\infty)$, then \mathbb{P} -a.s.:

- (1) V is continuous at its left local extrema and at its right local extrema,
- (2) In no two local extrema V has the same value.

The point (2) of the previous Lemma gives us the uniqueness of the minimum m_c such that $V(m_c) = \inf_{[0, \tau_c]} V$. And m_c is a local minimum, thus according to point (1) the process V is continuous in m_c , that means $V(m_c) = V(m_c-)$.

Remark 2.4. If $\omega \in \mathcal{V}^+$ is càd-làg on a bounded interval I , then there exists $x \in \bar{I}$ such that $\sup_I \omega = \omega(x) \vee \omega(x-)$ and $y \in \bar{I}$ such that $\inf_I \omega = \omega(y) \wedge \omega(y-)$. So if ω is continuous at its local extrema, we get:

$$\exists x \in \bar{I}, \text{ s.t. } \sup_I \omega = \omega(x) \text{ and } \exists y \in \bar{I}, \text{ s.t. } \inf_I \omega = \omega(y).$$

2.2.1. *The pre-infimum and post-infimum processes.* We now define the pre-infimum and post-infimum processes $(\underline{V}_{m_c}(t), 0 \leq t \leq m_c)$ and $(\overline{V}_{m_c}(t), 0 \leq t \leq \tau_c - m_c)$ by

$$\underline{V}_{m_c}(t) = V((m_c - t)^-) - V(m_c),$$

$$\overline{V}_{m_c}(t) = V(m_c + t) - V(m_c).$$

Unless there is an ambiguity, we write \underline{V} instead of \underline{V}_{m_c} and \overline{V} instead of \overline{V}_{m_c} . We also denote respectively \mathbb{P}^c and \mathbb{P}^c the laws of the processes \underline{V} stopped at time m_c and \overline{V} stopped at time $\tau_c(V)$.

Let us study the law of these two processes, we need the notion of Lévy process conditioned to stay positive.

2.2.2. *Lévy process conditioned to stay positive.* We use Bertoin's construction of a Lévy process conditioned to stay positive (see Section 3, [3]) by concatenation of the excursions of the Lévy process above 0.

The process V is a semi-martingale, its local continuous martingale part is null or proportional to a standard Brownian motion. We denote by ℓ the local time, in a semi-martingale sense, of V at 0, see for example [23]. We consider the time passed by V in $(0, \infty)$ and its right continuous inverse:

$$A_t^+ = \int_0^t ds \mathbf{1}_{V(s) > 0} \text{ and } \alpha_t^+ = \inf\{s \geq 0; A_s^+ > t\}.$$

We define the process V^\uparrow by: for all $t \geq 0$,

$$V^\uparrow(t) = V(\alpha_t^+) + \frac{1}{2} \ell_{\alpha_t^+} + \sum_{0 < s \leq \alpha_t^+} (0 \vee V(s-)) \mathbf{1}_{V(s) \leq 0} - (0 \wedge V(s-)) \mathbf{1}_{V(s) > 0} \text{ if } t < A_\infty^+$$

and $V^\uparrow(t) = +\infty$ otherwise.

Remark 2.5. We will use this expression mainly to prove that when the process V is stable, V^\uparrow is stable too.

Remark 2.6.

- \mathbb{P} -a.s. for all $t > 0$, $V^\uparrow(t) > 0$.
- in the case where V has no Brownian part, $\ell = 0$.
- the process V^\uparrow tends to $+\infty$ as t goes to $+\infty$.

We denote by \mathbb{P}^\uparrow the law of the process V^\uparrow and $\hat{\mathbb{P}}^\uparrow$ the law of the dual process $\hat{V} = -V$ conditioned to stay positive. For every process Y that tends to $+\infty$ at $+\infty$, we define the future infimum: for all $t \geq 0$, $\underline{Y}(t) = \inf_{s \geq t} Y(s)$. For $c > 0$, we introduce the time

$$\hat{m}_c^\uparrow = n_c(\hat{V}^\uparrow - \underline{\hat{V}}^\uparrow).$$

We write $\mathbb{P}^{\uparrow,c}$ (resp. $\hat{\mathbb{P}}^{\uparrow,c}$) for the law of the process V^\uparrow (resp. \hat{V}^\uparrow) stopped at time $\tau_c(V^\uparrow)$ (resp. \hat{m}_c^\uparrow).

2.2.3. Valley with height c . Unlike the Brownian case where the law of the post-infimum process is equal to the law of a 3 dimensional Bessel process, in the case of a Lévy process, this law is only absolutely continuous with respect to the law of a Lévy process conditioned to stay positive. In order to have the law of the standard valley, we use Proposition 4.7 of Duquesne [12].

For $t \geq 0$, we define $U(t) = -V(L^{-1}(t))$ if $L(\infty) > t$ and $U(t) = +\infty$ otherwise. The process (L^{-1}, U) is called the ladder process, it is a subordinator. We then define the potential measure \mathcal{U} associated with U : for all measurable positive function F ,

$$\int_{\mathbb{R}} \mathcal{U}(dx) F(x) = \mathbb{E} \left[\int_0^{L(\infty)} dv F(U(v)) \right].$$

For the excursion measure \mathbb{N} and for all measurable non-negative function f on \mathcal{V} , we write

$$\mathbb{N}(f(v)) = \int f(v) \mathbb{N}(dv).$$

Proposition 2.7. *We consider a Lévy process V , which does not go to $+\infty$ neither to $-\infty$ and such that 0 is regular for $(0, +\infty)$ and for $(-\infty, 0)$.*

Then the pre-infimum and post-infimum processes $(\underline{V}(t), 0 \leq t \leq m_c)$ and $(\overline{V}(t), 0 \leq t \leq \tau_c(\overline{V}))$ are independent. Moreover

- (1) *The law $\mathbb{P}^c_{\leftarrow}$ of the pre-infimum process $(\underline{V}(t), 0 \leq t \leq m_c)$ is equal to the law $\hat{\mathbb{P}}^{\uparrow,c}$ of the process $(\hat{V}^\uparrow(t), 0 \leq t \leq \hat{m}_c^\uparrow)$.*
- (2) *The law $\mathbb{P}^c_{\rightarrow}$ of the post-infimum process $(\overline{V}(t), 0 \leq t \leq \tau_c(\overline{V}))$ is absolutely continuous with respect to the law $\mathbb{P}^{\uparrow,c}$ of the process $(V^\uparrow(t), 0 \leq t \leq \tau_c(V^\uparrow))$.*

More precisely : $\mathbb{P}^c_{\rightarrow}(d\omega) = f_c(\omega(\tau_c(\omega))) \mathbb{P}^{\uparrow,c}(d\omega)$

where for all $x \in \mathbb{R}_+$, $\frac{1}{f_c(x)} = \mathcal{U}([0, x]) \mathbb{N}(\tau_c(v) < \infty)$.

Remark 2.8.

- (1) Let us remark that in the case where V is a Lévy process with no positive jumps, $\omega(\tau_c(\omega)) = c$, $\mathbb{P}^{\uparrow,c}$ - a.s. Thus $\omega \mapsto f_c(\omega(\tau_c(\omega)))$ is constant and equal to 1 and the post-infimum process is equal in law to the process V^\uparrow conditioned to stay positive.
- (2) \mathbb{P} -a.s., \underline{V} is continuous in its local extrema (see Lemma 2.3), then if c tends to $+\infty$, using point (1) of Proposition 2.7, \hat{V}^\uparrow and V^\uparrow are continuous at their local extrema too.

Proof of the Proposition 2.7. (1) First, let us prove the first part of Proposition 2.7 and the independence of the two processes for $c = 1$. To simplify the notations, we will not write the index 1.

We want to express the pre-infimum process \underline{V} in terms of the excursions of V above its minimum and of the associated jump of \underline{V} . We consider the couple of processes $(V - \underline{V}, -\underline{V})$ and we give an expression using excursions. We use the proof of Lemma 4 of [4], remark that the result is still true for every Lévy process that does not converge to $-\infty$, see Theorem 27 of [10].

In order to use excursion theory, we denote by $\bar{\mathcal{V}}$ the set of the càd-làg paths $v : [0, \infty) \rightarrow \mathbb{R}_+$ such that

- (i) $v(t) > 0$ for $0 < t < \sigma(v) = \inf\{s > 0, v(s) = 0\}$,
- (ii) $v(0) = v(t) = 0$ for $t \geq \sigma(v)$.

The process

$$t \mapsto ((V - \underline{V})(L^{-1}(t^-) + r), \underline{V}(L^{-1}(t^-)) - \underline{V}(L^{-1}(t)) \ ; \ 0 \leq r \leq L^{-1}(t) - L^{-1}(t^-))$$

is a $\bar{\mathcal{V}} \times \mathbb{R}_+$ -valued Poisson point process (see for example [13]). We denote it $p(t) = (p_1(t), p_2(t))$. Let us remark that $p_1(L(\underline{\tau}))$ is the excursion starting from m , that is to say that $L(\underline{\tau})$ is the first time t where the excursion $p_1(t)$ has a height bigger than 1. Thus, the processes $(p(t); 0 \leq t < L(\underline{\tau}))$ and $(p(t); t \geq L(\underline{\tau}))$ are independent. The pre-infimum process depends only on the first and the post-infimum process depends only on the second, then these two processes are independent too.

In order to obtain the process \underline{V} , we first inverse the excursions of $V - \underline{V}$ then we inverse the time in each of these excursions.

Let us denote by $\tilde{p}_i = (p_i(L(\underline{\tau}) - t)\mathbf{1}_{t < L(\underline{\tau})} + p_i(t)\mathbf{1}_{t \geq L(\underline{\tau})}; t \geq 0)$ the process p_i where the order of the excursions before $L(\underline{\tau})$ is changed for $i = 1$ and $i = 2$. The law of the Poisson point process p is stable by inversion of the order of its excursions before $L(\underline{\tau})$, we deduce the equality in law of the two processes $(\tilde{p}_1(t), \tilde{p}_2(t))$ and $(p_1(t), p_2(t))$.

Now we use a notation in order to inverse the time in the excursions. We define for $v \in \bar{\mathcal{V}}$,

$$[v](s) = \begin{cases} v((\sigma(v) - s)^-) & \text{for } 0 \leq s \leq \sigma(v) \\ 0 & \text{for } s > \sigma(v). \end{cases}$$

Using this transformation in the previous law, we obtain that the processes $([\tilde{p}_1(t)], \tilde{p}_2(t); t \geq 0)$ and $([p_1(t)], p_2(t); t \geq 0)$ have same law. With the first one we get the process \underline{V} . Indeed, $([\tilde{p}_1(t)]; t \geq 0)$ is the excursion process of \underline{V} above its future infimum and the process $(\tilde{p}_2(t); t \geq 0)$ represents the jumps of the future infimum of \underline{V} . We now use the couple $([p_1(t)], p_2(t); t \geq 0)$ to get a Lévy process conditioned to stay positive.

We recall the notations of [4], for $t \geq 0$, we use $g(t) = \sup\{s < t; V(s) = \underline{V}(s)\}$ and $d(t) = \inf\{s > t; V(s) = \underline{V}(s)\}$, the left and right extremities of the excursion interval of $V - \underline{V}$ away from 0 that contains t and

$$\mathcal{R}_{V-\underline{V}}(t) = \begin{cases} (V - \underline{V})((d(t) + g(t) - t)^-) & \text{if } g(t) < d(t) \\ 0 & \text{otherwise.} \end{cases}$$

See Figure 2 and 3.

The excursion process of $\mathcal{R}_{V-\underline{V}}$ above 0 is the Poisson point process $([p_1(t)], t \geq 0)$ and, similarly, the excursion process of $((V - \underline{V})((m - t)^-)\mathbf{1}_{0 \leq t < m} + \mathcal{R}_{V-\underline{V}}(t)\mathbf{1}_{t \geq m}, t \geq 0)$ above 0 is the Poisson point process $([\tilde{p}_1(t)], t \geq 0)$. We have an analogue equality for the jumps associated with every excursion. Then we get the equality in law of the processes

$$((V - \underline{V})((m - t)^-) \mathbf{1}_{0 \leq t < m} + \mathcal{R}_{V - \underline{V}}(t) \mathbf{1}_{t \geq m},$$

$$(\underline{V}((m - t)^-) - \underline{V}(m)) \mathbf{1}_{0 \leq t < m} - \underline{V}(d(t)) \mathbf{1}_{t \geq m}; t \geq 0)$$

and

$$(\mathcal{R}_{V - \underline{V}}(t), -\underline{V}(d(t)); t \geq 0).$$

Adding the two parts for $t < m$ we get:

$$\left(\underline{V}(t); 0 \leq t < m \right) \stackrel{\mathcal{L}}{=} \left(\mathcal{R}_{V - \underline{V}}(t) - \underline{V}(d(t)); 0 \leq t < m \right),$$

See Figure 3 and ??.

FIGURE 2. The Lévy process $(V(t); t \geq 0)$

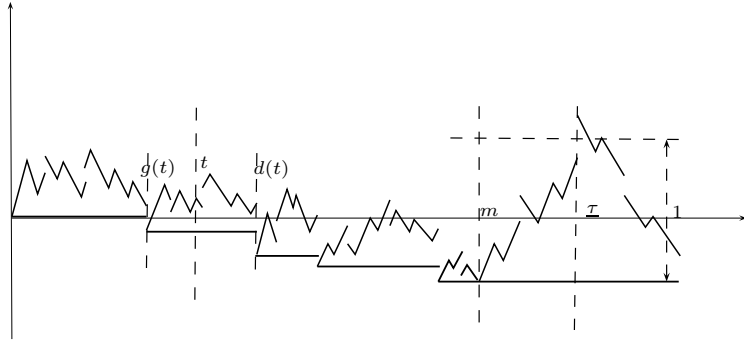
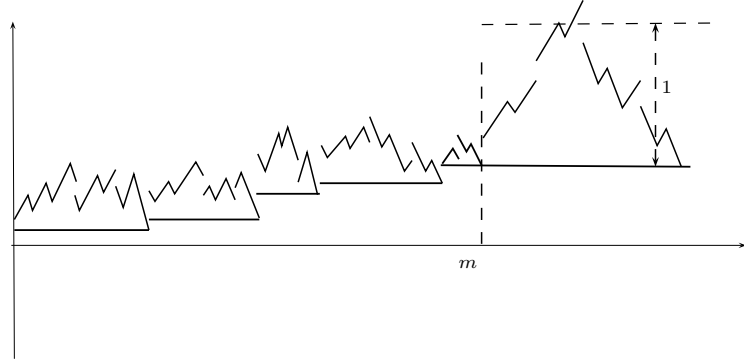


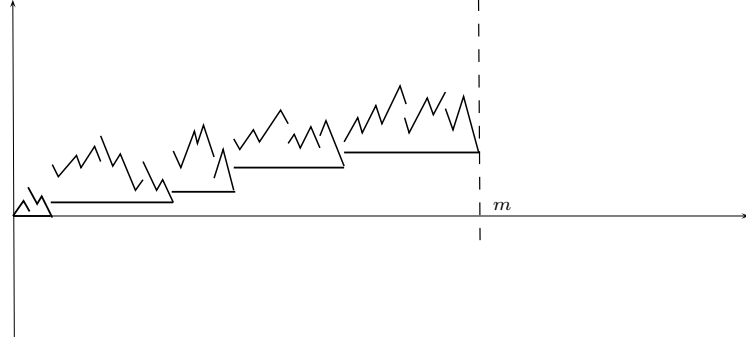
FIGURE 3. The process $(\mathcal{R}_{V - \underline{V}}(t) - \underline{V}(d(t)); t \geq 0)$



We now use Theorem 28 of [10] : under \mathbb{P} , the law of the process $(\mathcal{R}_{V - \underline{V}}(t) - \underline{V}(d(t)), t \geq 0)$ is $\hat{\mathbb{P}}^\uparrow$.

Now we express the final time m in terms of the process $(\mathcal{R}_{V - \underline{V}}(t) - \underline{V}(d(t)), 0 \leq t \leq m)$.

Recall that $\underline{\tau} = \inf\{t \geq 0; V(t) - \underline{V}(t) > 1\}$ and that $m = \sup\{t \leq \underline{\tau}; V(t) - \underline{V}(t) = 0\}$. We define the variable $\underline{\tau}' = \tau(\mathcal{R}_{V - \underline{V}})$. Notice that $\underline{\tau}$ and $\underline{\tau}'$ belong to the same excursion



interval of $V - \underline{V}$. The variable m remains stable by time inversion in every excursions of $V - \underline{V}$ and by the inversion of the excursions before m , then $m = \sup\{t \leq \underline{\tau}'; \mathcal{R}_{V-\underline{V}}(t) = 0\}$. We write

$$Y(t) := \mathcal{R}_{V-\underline{V}}(t) - \underline{V}(d(t)).$$

Let us prove that

$$\mathcal{R}_{V-\underline{V}}(t) = Y(t) - \underline{\underline{Y}}(t),$$

it is enough to prove that, for $t \geq 0$ fixed, the post-infimum process $\underline{\underline{Y}}(t) = \inf_{s \geq t} Y(s)$ is equal to $-\underline{V}(d(t))$.

On the interval $[t, d(t))$, the process $(\underline{V}(d(s)), t \leq s < d(t))$ is constant and $\inf_{t \leq s < d(t)} \mathcal{R}_{V-\underline{V}}(s) = \mathcal{R}_{V-\underline{V}}(d(t)^-) = 0$ then $\inf_{t \leq s < d(t)} Y(s) = -\underline{V}(d(t))$. The process $\mathcal{R}_{V-\underline{V}}$ is non-negative and vanishes at times $\{d(t)^-, t \geq 0\}$ (see Corollary 1, [24]), and the process $-\underline{V}(d(\cdot))$ increases by jumps at times $\{d(t), t \geq 0\}$, then $\underline{\underline{Y}}(t) = \inf_{t \leq s < d(t)} Y(s)$ and so $\underline{\underline{Y}}(t) = -\underline{V}(d(t))$. We deduce that

$$m = \sup\{t \leq \underline{\tau}'; Y(t) - \underline{\underline{Y}}(t) = 0\}.$$

Finally, under \mathbb{P} , the law of the process $(\mathcal{R}_{V-\underline{V}}(t) - \underline{V}(d(t)))_{0 \leq t \leq m}$ is the law of the process \hat{V}^\uparrow stopped at time \hat{m}^\uparrow and the point (1) is proved.

(2) Now we consider the post-infimum process. Let F be a bounded measurable function on Ω . We can write

$$\mathbb{E}^c [F(\omega)] = \mathbb{N}(F(v(\cdot \wedge \tau_c)) | \tau_c(v) < \infty).$$

We use Proposition 4.7 of [12]: for all bounded measurable function G on \mathcal{V}^+ ,

$$\begin{aligned} \mathbb{E}^{\uparrow, c} [G(\omega)] &= \mathbb{N}(G(v(\cdot \wedge \tau_c)) \mathcal{U}([0, v(\tau_c)[) | \tau_c < \infty) \mathbb{N}(\tau_c(v) < \infty) \\ &= \mathbb{N}\left(\frac{G(v(\cdot \wedge \tau_c))}{f_c(v(\tau_c))} \middle| \tau_c(v) < \infty\right) \end{aligned}$$

where we recall that for all $x \in \mathbb{R}_+$, $\frac{1}{f_c(x)} = \mathcal{U}([0, x]) \mathbb{N}(\tau_c(v) < \infty)$. Using the function $G(v(\cdot \wedge \tau_c)) = F(v(\cdot \wedge \tau_c)) f_c(v(\tau_c))$, we obtain the second point of the proposition. \square

2.3. Stable Lévy process.

Definition 2.9. ([2], Section VIII)

The Lévy process V is stable with index $\alpha \in (0, 2]$ if, for all $c > 0$

$$(cV(c^\alpha t), t \geq 0) \stackrel{\mathcal{L}}{=} (V(t), t \geq 0).$$

For $\alpha \in (0, 1) \cup (1, 2)$, the Fourier transform of an α -stable Lévy process V is : for all $t \geq 0$ and for all $\lambda \in \mathbb{R}$, $\mathbb{E}[e^{i\lambda V(t)}] = e^{-t\psi(\lambda)}$ with

$$\psi(\lambda) = k|\lambda|^\alpha \left(1 - i\beta \operatorname{sgn}(\lambda) \tan\left(\frac{\pi\alpha}{2}\right)\right)$$

where $k > 0$ and $\beta \in [-1, 1]$.

The Lévy measure of the process V is absolutely continuous with respect to the Lebesgue measure, it can be written as follows:

$$\pi(dx) = c^+ x^{-\alpha-1} \mathbf{1}_{x>0} dx - c^- x^{-\alpha-1} \mathbf{1}_{x<0} dx$$

where c^+ and c^- are positive numbers non equal to 0 together and such that $\beta = \frac{c^+ - c^-}{c^+ + c^-}$.

In the quadratic case ($\alpha = 2$), V is a Brownian motion and its characteristic exponent can be written $\psi(\lambda) = k\lambda^2$ where k is a positive number. The Lévy measure is then trivial.

In the case $\alpha = 1$, V is a Cauchy process and $\psi(\lambda) = k|\lambda| + id\lambda$ with $k > 0$ and $d \in \mathbb{R}$. The Lévy measure is then proportional to $x^{-2}dx$.

Remark 2.10. If the process V is α -stable, then the pre-infimum process $(c^{-1}\underline{V}_{m_c}(c^\alpha t), t \geq 0)$ and \underline{V}_{m_1} (stopped at correct times) have the same law and the same for the post-infimum process $(c^{-1}\underline{V}_{m_c}(c^\alpha t), t \geq 0)$ and the process \underline{V}_{m_1} .

Lemma 2.11. *Let V be an α -stable Lévy process which is not a pure drift. Then V is recurrent if and only if $\alpha \geq 1$.*

Proof. We use a condition of recurrence given in [2], Theorem I.17: V is recurrent if and only if $\int_{-r}^r \Re \left[\frac{1}{\psi(\lambda)} \right] d\lambda = +\infty$ where $\Re[z]$ is the real part of the complex number z .

For $\alpha = 1$, $\int_{-r}^r \frac{k|\lambda|}{(k^2 + d^2)\lambda^2} d\lambda = +\infty$ because $k \neq 0$.

For $\alpha \neq 1$, $\int_{-r}^r \frac{1}{k|\lambda|^\alpha (1 + \beta^2 \tan^2(\frac{\pi\alpha}{2}))} d\lambda < +\infty$ if and only if $\alpha < 1$. □

Then we use a stable Lévy process with index $\alpha \in [1, 2]$ and which is not a pure drift. We need others properties for V , mainly to use the results of Section 2.2.

Lemma 2.12. *Let V be a stable Lévy process with index $\alpha \in [1, 2]$ and which is not a pure drift. Then,*

- 0 is regular for $(0, +\infty)$ and for $(-\infty, 0)$,
- \mathbb{P} -a.s., $\limsup_{t \rightarrow +\infty} V(t) = +\infty$ and $\liminf_{t \rightarrow +\infty} V(t) = -\infty$.

Proof.

• Rogozin criterion (see Prop. VI.11 of [2]) gives a necessary and sufficient condition to have the regularity in 0:

0 is regular for $(0, +\infty)$ (resp. for $(-\infty, 0)$) if and only if $\int_0^1 t^{-1} \mathbb{P}(V(t) \geq 0) dt = +\infty$ (resp. $\int_0^1 t^{-1} \mathbb{P}(V(t) \leq 0) dt = +\infty$).

Remark, using the stability property of V , that $\mathbb{P}(V(t) \geq 0)$ does not depend on t . Moreover this probability is positive if V is not a pure drift, see [2], VIII.1.

With the previous criterion, 0 is regular for $(0, +\infty)$ and for $(-\infty, 0)$.

• We have $\int_1^\infty t^{-1} \mathbb{P}(V(t) \geq 0) dt = \int_1^\infty t^{-1} \mathbb{P}(V(t) \leq 0) dt = +\infty$, this implies that \mathbb{P} -a.s., $\limsup_{t \rightarrow +\infty} V(t) = +\infty$ and $\liminf_{t \rightarrow +\infty} V(t) = -\infty$, see [2], Theo. VI.12. □

Lemma 2.13. *We consider a stable Lévy process V with index $\alpha \in (0, 2]$. Then the process V^\uparrow is an α -stable process.*

Proof. In the case where $\alpha = 2$, the stable Lévy process V is a standard Brownian motion, V^\uparrow is a 3-dimensional Bessel process (see [21]) and then is stable with index 2.

If $\alpha \in (0, 2)$, the stable Lévy process has no Brownian part. Recall that α_t^+ is the right continuous inverse of $A_t^+ = \int_0^t \mathbf{1}_{V(s) > 0} ds$. These two processes are stable with index 1. We use the definition given in Section 2.2.2:

$$V^\uparrow(t) = V(\alpha_t^+) + \sum_{0 < s \leq \alpha_t^+} (0 \vee V(s^-)) \mathbf{1}_{V(s) \leq 0} - (0 \wedge V(s^-)) \mathbf{1}_{V(s) > 0},$$

We deduce that the process V^\uparrow is stable with index α . □

Proposition 2.14. *We consider a stable Lévy process with index $\alpha \in [1, 2]$ and which is not a pure drift. When c converges to $+\infty$:*

- (1) *the law $\mathbb{P}^c \xleftarrow{c}$ converges weakly (in the Skorohod sense) to the law $\hat{\mathbb{P}}^\uparrow$.*
- (2) *the law $\mathbb{P}^c \xrightarrow{c}$ converges weakly (in the Skorohod sense) to the law \mathbb{P}^\uparrow .*

To prove part (2), we need the following lemma.

Lemma 2.15. *We fix $\varepsilon > 0$ and we define*

$$\sigma_\varepsilon = \sigma_\varepsilon(V^\uparrow) = \inf \left\{ t > 0 / V^\uparrow(t) - \underline{V}^\uparrow(t) = 0 \text{ and } \exists s < t, V^\uparrow(s) - \underline{V}^\uparrow(s) > \varepsilon \right\}.$$

Then the process $(V_{\sigma_\varepsilon}^\uparrow(t), t \geq 0) = (V^\uparrow(\sigma_\varepsilon + t) - V^\uparrow(\sigma_\varepsilon), t \geq 0)$ is independent of the process $(V^\uparrow(t), 0 \leq t \leq \sigma_\varepsilon)$ and its law is given by \mathbb{P}^\uparrow .

Proof. Thanks to Theorem 28 of [10], the law of the process $\mathcal{R}_{\overline{V}-V} + V(d(\cdot))$ is \mathbb{P}^\uparrow (where \mathcal{R} and d are defined like page 7 but using the process $\overline{V} - V$). It is enough to prove the result in the case where $V^\uparrow = \mathcal{R}_{\overline{V}-V} + V(d(\cdot))$. But in this case, σ_ε can be written as

$$\sigma_\varepsilon = \inf \left\{ t > 0 / \overline{V}(t) - V(t) = 0 \text{ and } \exists s < t, \overline{V}(s) - V(s) > \varepsilon \right\}$$

then σ_ε is a stopping time for V .

Thus the process $(V_{\sigma_\varepsilon}(t), t \geq 0) = (V(\sigma_\varepsilon + t) - V(\sigma_\varepsilon), t \geq 0)$ is a Lévy process with law \mathbb{P} independent of $(V(t), 0 \leq t \leq \sigma_\varepsilon)$. We can check that $V_{\sigma_\varepsilon}^\uparrow(t) = \mathcal{R}_{\overline{V}_{\sigma_\varepsilon} - V_{\sigma_\varepsilon}}(t) + V_{\sigma_\varepsilon}(d(t))$, this proves the Lemma. □

Proof of Proposition 2.14. (1) Thanks to Proposition 2.7, the laws of \mathbb{P}^c and $\hat{\mathbb{P}}^{\uparrow, c}$ are equal. When c converges to infinity, $\hat{\mathbb{P}}^\uparrow$ -a.s., \hat{m}_c^\uparrow converges to infinity, then $\mathbb{P}^c \xleftarrow{c} \hat{\mathbb{P}}^{\uparrow, c}$ converges to $\hat{\mathbb{P}}^\uparrow$.

Now we prove that for a continuous bounded function F in the Skorohod topology such that $F(\omega) = F(\omega \mathbf{1}_{[-K, K]})$, we have

$$\lim_{c \rightarrow \infty} \mathbb{E}^c [F(\omega)] = \mathbb{E}^\dagger [F(\omega)].$$

As \mathbb{P} -a.s., $\lim_{c \rightarrow \infty} \tau_c(V_{m_c}) = \infty$, so $\lim_{c \rightarrow \infty} \mathbb{P}^c(K \leq \tau_c(\omega)) = 1$ and thus

$$\lim_{c \rightarrow \infty} \mathbb{E}^c [F(\omega)] = \lim_{c \rightarrow \infty} \mathbb{E}^c [F(\omega), K \leq \tau_c(\omega)].$$

We first use the stability of V . Thanks to Remark 2.10,

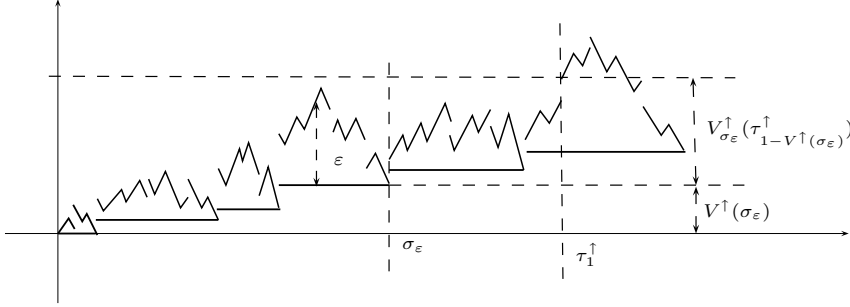
$$\begin{aligned} \mathbb{E}^c [F(\omega), K \leq \tau_c(\omega)] &= \mathbb{E}^\dagger [F(c\omega(c^{-\alpha} \cdot)), K \leq c^\alpha \tau_1(\omega)], \\ &= \mathbb{E}^\dagger [F(c\omega(c^{-\alpha} \cdot)) f_1(\omega(\tau_1)), K \leq c^\alpha \tau_1(\omega)] \end{aligned}$$

where we use the result (2) of Proposition 2.7, for the second equality.

We now prove the asymptotic independence of $F(c\omega(c^{-\alpha} \cdot))$ and of $f_1(\omega(\tau_1))$ under \mathbb{P}^\dagger . Then we express these two terms with respect to the processes $(\omega(t), 0 \leq t \leq \sigma_\varepsilon)$ and $(\omega(t), \sigma_\varepsilon \leq t)$ where, for $\varepsilon > 0$, we recall that:

$$\sigma_\varepsilon = \sigma_\varepsilon(\omega) = \inf \{t > 0 / \omega(t) - \underline{\omega}(t) = 0 \text{ and } \exists s < t, \omega(s) - \underline{\omega}(s) > \varepsilon\}.$$

FIGURE 4. The process V^\dagger



We fix $\varepsilon > 0$, we remark that $\lim_{c \rightarrow +\infty} \mathbb{P}^\dagger(K \leq c^\alpha \tau_1) = \lim_{c \rightarrow +\infty} \mathbb{P}^\dagger(K \leq c^\alpha \sigma_\varepsilon) = 1$. Then we can replace $K \leq c^\alpha \tau_1$ by $K \leq c^\alpha \sigma_\varepsilon$ in the expectation.

Moreover, \mathbb{P}^\dagger -a.s., $\lim_{\varepsilon \rightarrow 0} \sigma_\varepsilon = 0$. Indeed, if there exists $\delta > 0$ such that for all $\varepsilon > 0$, $\sigma_\varepsilon > \delta$, then the excursion of $\omega - \underline{\omega}$ containing time δ is the first one. And this is \mathbb{P}^\dagger -a.s. impossible because 0 is regular for the process V . Thus $\lim_{\varepsilon \rightarrow 0} \mathbb{P}^\dagger(\sigma_\varepsilon < \tau_1) = 1$. By conditioning with respect to $(\omega(t), 0 \leq t \leq \sigma_\varepsilon)$ and using that $\{\sigma_\varepsilon < \tau_1^\dagger\} = \{\forall t \leq \sigma_\varepsilon, V_t^\dagger < 1\}$ is measurable with respect to $(\omega(t), 0 \leq t \leq \sigma_\varepsilon)$, we get:

$$(4) \quad \mathbb{E}^\dagger [F(c\omega(c^{-\alpha} \cdot)) f_1(\omega(\tau_1)), c^{-\alpha} K \leq \sigma_\varepsilon < \tau_1] =$$

$$\mathbb{E}^\dagger \left[F(c\omega(c^{-\alpha} \cdot)) \mathbf{1}_{c^{-\alpha} K \leq \sigma_\varepsilon < \tau_1} \mathbb{E}^\dagger [f_1(\omega(\tau_1)) \mid (\omega(t), 0 \leq t \leq \sigma_\varepsilon)] \right].$$

We compute the limit of the conditional expectation when ε converges to 0. When $\sigma_\varepsilon < \tau_1$, we can write $\omega(\tau_1) = \omega(\sigma_\varepsilon) + \omega_{\sigma_\varepsilon}(\tau_1 - \omega(\sigma_\varepsilon))$ where $\omega_{\sigma_\varepsilon}$ is the path starting from σ_ε : $(\omega_{\sigma_\varepsilon}(t), t \geq 0) = (\omega(\sigma_\varepsilon + t) - \omega(\sigma_\varepsilon), t \geq 0)$, see Figure 4. Thanks to Lemma 2.15,

$$\mathbb{E}^\dagger \left[f_1(\omega(\tau_1)) \mid (\omega(t), 0 \leq t \leq \sigma_\varepsilon) \right] = \chi(\omega(\sigma_\varepsilon)) \text{ where } \chi(x) = \mathbb{E}^\dagger [f_1(x + \omega(\tau_1 - x))].$$

Remark that \mathbb{P}^\dagger -a.s., $\lim_{\varepsilon \rightarrow 0} \omega(\sigma_\varepsilon) = 0$. Let us prove that \mathbb{P}^\dagger -a.s., $\lim_{y \rightarrow 1^-} \omega(\tau_y) = \omega(\tau_1)$. \mathbb{P}^\dagger -a.s., ω is continuous at its local extrema (see (2) of Remark 2.8). Thus, thanks to Remark 2.4, \mathbb{P}^\dagger -a.s., $\sup_{[0, \tau_1]} \omega$ is attained in a point $x_0 \in [0, \tau_1]$ and $\omega(x_0) \leq 1$.

If $x_0 = \tau_1$, then $\omega(\tau_1) = 1$ and $\lim_{y \rightarrow 1^-} 1 - \omega(\tau_y) \leq \lim_{y \rightarrow 1^-} 1 - y = 0$.

If $x_0 < \tau_1$, then $\omega(x_0) < 1$ and then, for $1 - y$ small enough, $\tau_1 = \tau_y$, thereby $\lim_{y \rightarrow 1^-} \omega(\tau_y) = \omega(\tau_1)$.

Moreover, in the case where V is a stable process with index $\alpha \in [1, 2)$, an explicit expression of \mathcal{U} is given in Example 7 of [11]:

$$\mathcal{U}([0, x]) = \frac{x^{\alpha\rho}}{\Gamma(\alpha\rho + 1)} \text{ where } \rho = \mathbb{P}(V(t) \geq 0) \text{ does not depend on } t.$$

Thus using the expression of f_c given in Proposition 2.7, we get: $f_1(x) = \frac{\Gamma(\alpha\rho + 1)}{x^{\alpha\rho} \mathbb{N}(\tau_1(\omega) < \infty)}$, then f_1 is continuous on \mathbb{R}_+^* . Then

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \mathbb{E}^\dagger \left[f_1(\omega(\tau_1)) \mid (\omega(t), 0 \leq t \leq \sigma_\varepsilon) \right] &= \lim_{\varepsilon \rightarrow 0} \chi(\omega(\sigma_\varepsilon)) = \mathbb{E}^\dagger [f_1(\omega(\tau_1))], \\ (5) \qquad \qquad \qquad &= 1, \mathbb{P}^\dagger\text{-a.s.}, \end{aligned}$$

because $f_1(\omega(\tau_1))$ is the density of \mathbb{P}^\dagger with respect to \mathbb{P}^\dagger .

Using the stability of the process V^\dagger given in Lemma 2.13, we get that for all $c \geq 0$,

$$\mathbb{E}^\dagger [F(c\omega(c^{-\alpha}\cdot))] = \mathbb{E}^\dagger [F(\omega)] = \mathbb{E} [F(V^\dagger)].$$

So to prove the proposition, it is enough to show that

$$\lim_{\varepsilon \rightarrow 0} \lim_{c \rightarrow \infty} \left| \mathbb{E}^\dagger [F(c\omega(c^{-\alpha}\cdot)) \mathbf{1}_{c^{-\alpha}K \leq \sigma_\varepsilon} \chi(\omega(\sigma_\varepsilon)) \mathbf{1}_{\sigma_\varepsilon < \tau_1}] - \mathbb{E}^\dagger [F(c\omega(c^{-\alpha}\cdot))] \right| = 0.$$

We introduce $\mathbb{E}^\dagger [F(c\omega(c^{-\alpha}\cdot)) \mathbf{1}_{c^{-\alpha}K \leq \sigma_\varepsilon}]$, and we get the following upper bound :

$$\begin{aligned} &\left| \mathbb{E}^\dagger [F(c\omega(c^{-\alpha}\cdot)) \mathbf{1}_{c^{-\alpha}K \leq \sigma_\varepsilon} \chi(\omega(\sigma_\varepsilon)) \mathbf{1}_{\sigma_\varepsilon < \tau_1}] - \mathbb{E}^\dagger [F(c\omega(c^{-\alpha}\cdot))] \right| \leq \\ &\mathbb{E}^\dagger [|F(c\omega(c^{-\alpha}\cdot)) \mathbf{1}_{c^{-\alpha}K \leq \sigma_\varepsilon} (\chi(\omega(\sigma_\varepsilon)) \mathbf{1}_{\sigma_\varepsilon < \tau_1} - 1)|] + \mathbb{E}^\dagger [|F(c\omega(c^{-\alpha}\cdot)) (\mathbf{1}_{c^{-\alpha}K \leq \sigma_\varepsilon} - 1)|]. \end{aligned}$$

For ε fixed, we have $\lim_{c \rightarrow +\infty} \mathbb{P}^\dagger(c^{-\alpha}K \leq \sigma_\varepsilon) = 1$. Then for the second part, we get :

$$\lim_{c \rightarrow \infty} \mathbb{E}^\dagger [|F(c\omega(c^{-\alpha}\cdot)) (\mathbf{1}_{c^{-\alpha}K \leq \sigma_\varepsilon} - 1)|] \leq \|F\|_\infty \lim_{c \rightarrow \infty} \mathbb{E}^\dagger [|\mathbf{1}_{c^{-\alpha}K \leq \sigma_\varepsilon} - 1|] = 0,$$

and for the first part, we have :

$$\overline{\lim}_{c \rightarrow \infty} \mathbb{E}^\dagger [|F(c\omega(c^{-\alpha}\cdot)) \mathbf{1}_{c^{-\alpha}K \leq \sigma_\varepsilon} (1 - \chi(\omega(\sigma_\varepsilon)) \mathbf{1}_{\sigma_\varepsilon < \tau_1})|] \leq \|F\|_\infty \mathbb{E}^\dagger [|1 - \chi(\omega(\sigma_\varepsilon)) \mathbf{1}_{\sigma_\varepsilon < \tau_1}|].$$

Let ε tends to 0, we have \mathbb{P}^\dagger -a.s., $\lim_{\varepsilon \rightarrow 0} \sigma_\varepsilon = 0$ and we use (5) to get the convergence :

$$\lim_{\varepsilon \rightarrow 0} \lim_{c \rightarrow \infty} \mathbb{E}^\dagger \left[\left| F(c\omega(c^{-\alpha}\cdot)) \mathbf{1}_{c^{-\alpha}K \leq \sigma_\varepsilon} (1 - \chi(\omega(\sigma_\varepsilon)) \mathbf{1}_{\sigma_\varepsilon < \tau_1}) \right| \right] = 0.$$

This completes the proof. □

3. DIFFUSION IN A RANDOM ENVIRONMENT

3.1. Asymptotic behaviour of a diffusion in a stable environment. We suppose that the diffusion X in the random environment V is defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and we write, as previously, \mathbb{P} the law of V . But we take a general case for the environment: *in all this section, we only suppose that V is an α -stable càd-làg process on \mathbb{R} ($\alpha > 0$) and not necessary a Lévy process.* We will prove that the asymptotic behaviour of L_X , the local time of the diffusion, depends only on the environment.

We first give some properties of X . We can prove (see for example [27]) that there exists a Brownian motion B independent of V such that \mathcal{P} -a.s.,

$$(6) \quad \forall t > 0, \quad X(V, t) = S_V^{-1}(B(T_V^{-1}(t)))$$

where

$$\forall x \in \mathbb{R}, \quad S_V(x) := \int_0^x e^{V(y)} dy$$

and

$$\forall t \geq 0, \quad T_V(t) := \int_0^t e^{-2V(S_V^{-1}(B(s)))} ds.$$

In order to simplify the notations, we write S and T respectively for S_V and T_V , when there is no confusion. From Formula (6) and the occupation time Formula (1), we deduce an expression of the local time process of X using the local time process L_B of the Brownian motion B . For all $x \in \mathbb{R}$ and all $t \geq 0$,

$$(7) \quad L_X(t, x) = e^{-V(x)} L_B(T^{-1}(t), S(x)).$$

To study the asymptotic behaviour of the diffusion, we first work in a quenched environment.

As Brox in the Brownian environment case (see [5]), we introduce the process $X_c(V, t) := X(cV, t)$. For all $x \in \mathbb{R}$, we write $V^c(x) := c^{-1}V(c^\alpha x)$. There is a direct link between the law of X_c and the law of X :

Lemma 3.1. *For every fixed c , conditionally on V , the processes $(c^{-\alpha}X(V, c^{2\alpha}t); t \geq 0)$ and $(X_c(V^c, t); t \geq 0)$ have the same law.*

Proof. We can easily check the result with Formula (6). □

With the definition of the local time (Formula (1)), we get an analogous relation for the local time processes:

Corollary 3.2. *For every fixed c , conditionally on V , the processes $(L_{X_c(V^c, \cdot)}(t, x); t \geq 0, x \in \mathbb{R})$ and $(c^{-\alpha}L_{X(V, \cdot)}(c^{2\alpha}t, c^\alpha x); t \geq 0, x \in \mathbb{R})$ have the same law.*

We will first give a convergence result for the process L_{X_c} and then we will deduce a convergence result for the process L_X . The asymptotic behaviour of L_{X_c} with a quenched environment can be only described with respect to the environment if this one is a "good" environment in the sense of the following definition:

Definition 3.3. We call a good environment every path $\omega \in \mathcal{V}$ that checks

- (1) $\liminf_{x \rightarrow \pm\infty} \omega(x) = -\infty$ and $\limsup_{x \rightarrow \pm\infty} \omega(x) = +\infty$,
- (2) ω is continuous at its local extrema and the values of ω in these points are pairwise distinct.

Let us denote by $\tilde{\mathcal{V}}$ the set of good environments.

Remark that if the paths of V are in $\tilde{\mathcal{V}}$, then the minimum $\mathbf{m}_c(V)$ of the standard valley with height c (see page 2) is well defined.

Then we can define, for every $c > 0$ and every $r > 0$,

$$a_{cr} = a_{cr}(V_{\mathbf{m}_c}) := \sup \{x \leq 0 / V_{\mathbf{m}_c}(x) > cr\} \text{ and}$$

$$b_{cr} = b_{cr}(V_{\mathbf{m}_c}) := \inf \{x \geq 0 / V_{\mathbf{m}_c}(x) > cr\}$$

where for $x \in \mathbb{R}$, $(V_x(y), y \in \mathbb{R})$ is the refocused environment in x : for all $y \in \mathbb{R}$,

$$V_x(y) := V(x+y) - V(x).$$

Now we can give the result with a quenched environment, we denote by \mathcal{P}_V the probability measure $\mathcal{P}(\cdot | V)$.

Proposition 3.4.

We fix two real numbers $K > 0$ and $r \in (0, 1)$ and we fix a \mathbb{R} -valued function h such that $\lim_{c \rightarrow +\infty} h(c) = 1$. If V is a good environment, then for all $\delta > 0$,

$$\lim_{c \rightarrow +\infty} \mathcal{P}_V \left(\sup_{-K \leq x \leq K} \left| \frac{L_{X_c}(e^{ch(c)}, \mathbf{m}_1 + c^{-\alpha}x) \int_{a_r}^{b_r} e^{-cV_{\mathbf{m}_1}(y)} dy}{e^{ch(c)} e^{-cV_{\mathbf{m}_1}(c^{-\alpha}x)}} - 1 \right| \leq \delta \right) = 1.$$

It is easy to adapt Proposition 31 of [1] to prove the previous result.

Now we can take the process L_X and prove the following Theorem, main result of this Section:

Theorem 3.5.

Let X be a diffusion in a random environment V . If the environment V is a good environment \mathbb{P} -a.s. and there exists $\alpha > 0$ such that V is α -stable then for all $K > 0$, $r \in (0, 1)$ and $\delta > 0$,

$$\lim_{c \rightarrow +\infty} \mathcal{P} \left(\sup_{-K \leq x \leq K} \left| \frac{L_X(e^c, \mathbf{m}_c + x) \int_{a_{cr}}^{b_{cr}} e^{-V_{\mathbf{m}_c}(y)} dy}{e^c e^{-V_{\mathbf{m}_c}(x)}} - 1 \right| \leq \delta \right) = 1.$$

Remark 3.6. So this Theorem is still true when the process V is fractional Brownian motion.

Proof. We fix $c > 0$ and we recall that $V^c(\cdot) := c^{-1}V(c^\alpha \cdot)$. By the stability of V , the processes V and V^c have same law. We first remark that, as the paths of V are in $\tilde{\mathcal{V}}$, \mathcal{P} -a.s., the standard valley is well defined and therefore

$$\begin{aligned} \mathbf{m}_1(V^c) &= c^{-\alpha} \mathbf{m}_c(V), \\ a_r(V_{\mathbf{m}_1(V^c)}^c) &= c^{-\alpha} a_{cr}(V_{\mathbf{m}_c(V)}), \\ b_r(V_{\mathbf{m}_1(V^c)}^c) &= c^{-\alpha} b_{cr}(V_{\mathbf{m}_c(V)}), \end{aligned}$$

and for all $x \in \mathbb{R}$,

$$(8) \quad V_{\mathbf{m}_1(V^c)}^c(x) = \frac{1}{c} V_{\mathbf{m}_c(V)}(c^\alpha x).$$

Thus, using the substitution $z = c^\alpha y$, we get that

$$\int_{a_r}^{b_r} e^{-cV_{m_1}(y)} dy = c^{-\alpha} \int_{a_{cr}}^{b_{cr}} e^{V_{m_c}(z)} dz.$$

Now we replace t by $c^{-2\alpha}e^c$ and x by $m_1(V^c) + c^{-\alpha}x$ in Corollary 3.2, then we get that, conditionally on V ,

$(L_{X_c(V^c, \cdot)}(c^{-2\alpha}e^c, m_1(V^c) + c^{-\alpha}x), x \in \mathbb{R})$ and $\left(\frac{1}{c^\alpha}L_{X(V, \cdot)}(e^c, m_c(V) + x), x \in \mathbb{R}\right)$ have the same law. Moreover the processes V and V^c have the same law, so we obtain that

$(L_{X_c(V, \cdot)}(c^{-2\alpha}e^c, m_1(V) + c^{-\alpha}x), x \in \mathbb{R})$ and $\left(\frac{1}{c^\alpha}L_{X(V, \cdot)}(e^c, m_c(V) + x), x \in \mathbb{R}\right)$ have the same law without conditioning. Thus for all $c > 0$, $K > 0$, $r \in (0, 1)$ and $\delta > 0$,

$$\begin{aligned} & \mathcal{P} \left(\sup_{-K \leq x \leq K} \left| \frac{L_{X(V, \cdot)}(e^c, m_c(V) + x) \int_{a_{cr}}^{b_{cr}} e^{-V_{m_c}(y)} dy}{e^c e^{-V_{m_c}(x)}} - 1 \right| < \delta \right) = \\ & \mathcal{P} \left(\sup_{-K \leq x \leq K} \left| \frac{L_{X_c(V, \cdot)}(c^{-2\alpha}e^c, m_1(V) + c^{-\alpha}x) \int_{a_r}^{b_r} e^{-cV_{m_1}(y)} dy}{c^{-2\alpha}e^c e^{-cV_{m_1}(c^{-\alpha}x)}} - 1 \right| < \delta \right). \end{aligned}$$

To finish, it is enough to remark that $c^{-2\alpha}e^c = e^{c(1 - \frac{2\alpha}{c} \log c)}$ and $\lim_{c \rightarrow +\infty} (1 - \frac{2\alpha}{c} \log c) = 1$. Then, using Proposition 3.4, we obtain Theorem 3.5. \square

3.2. Limit law of the renormalized local time. Now we suppose again that our environment is an α -stable Lévy process with $\alpha \in [1, 2]$ and is not a pure drift. As in Formula (2), we define, for all $\omega \in \mathcal{V}$, $\omega(t) = \omega^+(t)\mathbf{1}_{t \geq 0} + \omega^-((-t)^-)\mathbf{1}_{t \leq 0}$. We recall the notations of the first part:

$$\underline{\tau}_c^+ = \inf\{t \geq 0; V^+(t) - \underline{V}^+(t) \geq c\} \quad \text{and} \quad m_c^+ = \sup\{t \leq \underline{\tau}_c^+; V^+(t) - \underline{V}^+(t) = 0\}.$$

We recall,

$$J_c^+ = (V^+(m_c^+) + c) \vee \bar{V}^+(m_c^+)$$

and similarly, using V^- , we define the variables $\underline{\tau}_c^-$, m_c^- and J_c^- . As previously, we can prove that the minimum of the standard valley m_c is equal to m_c^+ if $J_c^+ < J_c^-$ and m_c^- otherwise.

Proposition 3.7. *Let V be a stable Lévy process with index $\alpha \in [1, 2]$ which is not a pure drift. We fix $r \in (0, 1)$.*

When c converges to $+\infty$, the process $\frac{e^{-V_{m_c}}}{\int_{a_{rc}}^{b_{rc}} e^{-V_{m_c}(y)} dy}$ converges in law (in the Skorohod sense) to $\frac{e^{-\tilde{V}}}{\int_{-\infty}^{+\infty} e^{-\tilde{V}(y)} dy}$ where the law of \tilde{V} is $\tilde{\mathbb{P}}$ and, under $\tilde{\mathbb{P}}(d\omega)$, ω^+ and ω^- are independent, and distributed respectively as \mathbb{P}^\uparrow , the law of the Lévy process V conditioned to stay positive, and $\hat{\mathbb{P}}^\uparrow$ the law of the dual process V^- conditioned to stay positive.

In order to prove Proposition 3.7, we need the following Lemmas. In the first one, we prove that the integral $\int_{-\infty}^{+\infty} e^{-\tilde{V}(y)} dy$ is finite.

Lemma 3.8.

$$\text{We have } \tilde{\mathbb{P}}\text{-a.s. } \int_{-\infty}^{+\infty} e^{-\tilde{V}(y)} dy < \infty.$$

Proof. It suffices to prove the integrability on \mathbb{R}_+ for the process $(\hat{V}^\uparrow(x), x \geq 0)$ whose law is $\hat{\mathbb{P}}^\uparrow$.

We fix $t \geq 0$. We prove the integrability of e^{-U_t} where we recall that $U(t) = -V(L^{-1}(t))$. The process U is a subordinator, then it is characterized by its Laplace exponent ψ_U (see Section III.1, [2]): for all $\lambda \geq 0$,

$$\mathbb{E} \left[e^{-\lambda U(t)} \right] = e^{-t\psi_U(\lambda)}.$$

For all $t > 0$, $U(t) > 0$ then $\psi_U(1) > 0$, we obtain that $\mathbb{E} \left[e^{-U(t)} \right] = e^{-t\psi_U(1)}$ is integrable on \mathbb{R}_+ . We recall the notation of the Proposition 2.7, we get

$$U(t) = -\underline{V}(d(t)) \leq \mathcal{R}_{V-\underline{V}}(t) - \underline{V}(d(t)).$$

Thanks to this Proposition, the process $(\mathcal{R}_{V-\underline{V}}(t) - \underline{V}(d(t)); t \geq 0)$ is equal in law to the process \hat{V}^\uparrow . Thus, $\mathbb{E} \left[e^{-\hat{V}^\uparrow(t)} \right]$ is also integrable on \mathbb{R}_+ , we deduce that $e^{-\hat{V}^\uparrow}$ is $\hat{\mathbb{P}}^\uparrow$ -a.s. integrable. \square

Lemma 3.9. *Let ω be a non-negative càd-làg function with $\omega(0) = 0$ which is continuous in its extrema. Then for all $a \leq \alpha < 0 < \beta \leq b$, we get*

$$\int_a^b e^{-c\omega(y)} dy \underset{c \rightarrow +\infty}{\sim} \int_\alpha^\beta e^{-c\omega(y)} dy.$$

Proof.

$$\begin{aligned} \int_a^b e^{-c\omega(y)} dy - \int_\alpha^\beta e^{-c\omega(y)} dy &= \int_a^\alpha e^{-c\omega(y)} dy + \int_\beta^b e^{-c\omega(y)} dy \\ &\leq |\alpha - a| e^{-c \inf_{y \in (a, \alpha)} \omega(y)} + |b - \beta| e^{-c \inf_{y \in (\beta, b)} \omega(y)}. \end{aligned}$$

As ω is positive and continuous at its extrema, for all $x \neq 0$, $\inf_{(a, \alpha)} \omega > 0$ and $\inf_{(\beta, b)} \omega > 0$ (see Remark 2.4), then $\lim_{c \rightarrow +\infty} \int_a^b e^{-c\omega(y)} dy - \int_\alpha^\beta e^{-c\omega(y)} dy = 0$.

By the Laplace method,

$$\lim_{c \rightarrow +\infty} \frac{1}{c} \log \int_a^b e^{-c\omega(y)} dy = \sup_{(a, b)}(-\omega).$$

As $a \leq 0 \leq b$ then $\sup_{(a, b)}(-\omega) = 0$, thus $\int_a^b e^{-c\omega(y)} dy = e^{o(c)}$.

$$\lim_{c \rightarrow +\infty} \frac{\int_a^b e^{-c\omega(y)} dy - \int_\alpha^\beta e^{-c\omega(y)} dy}{\int_a^b e^{-c\omega(y)} dy} \leq \lim_{c \rightarrow +\infty} e^{-c \inf_{(a, \alpha)} \omega - c \inf_{(\beta, b)} \omega - o(c)} = 0.$$

This implies the equivalence. \square

Proof of Proposition 3.7. We follow the same idea as in the proof of Proposition 2.14.

It is enough to prove that for a bounded continuous function F in the Skorohod topology and such that $F(\omega) = F(\omega \mathbf{1}_{[-K, K]})$,

$$\lim_{c \rightarrow +\infty} \mathbb{E} \left[F \left(\frac{e^{-V_{m_c}}}{\int_{a_{rc}}^{b_{rc}} e^{-V_{m_c}(y)} dy} \right) \right] = \mathbb{E} \left[F \left(\frac{e^{-\tilde{V}}}{\int_{-\infty}^{+\infty} e^{-\tilde{V}(y)} dy} \right) \right].$$

Using Formula (8) and the fact that $\mathbf{m}_1 \in \{m_1^-, m_1^+\}$,

$$\mathbb{E} \left[F \left(\frac{e^{-V_{\mathbf{m}_c}}}{\int_{a_{rc}}^{b_{rc}} e^{-V_{\mathbf{m}_c}(y)} dy} \right) \right] = \mathbb{E} \left[F \left(\frac{e^{-cV_{\mathbf{m}_1}(c^{-\alpha \cdot})}}{c^\alpha \int_{a_r}^{b_r} e^{-cV_{\mathbf{m}_1}(y)} dy} \right), \mathbf{m}_1 = m_1^+ \right] +$$

$$\mathbb{E} \left[F \left(\frac{e^{-cV_{\mathbf{m}_1}(c^{-\alpha \cdot})}}{c^\alpha \int_{a_r}^{b_r} e^{-cV_{\mathbf{m}_1}(y)} dy} \right), \mathbf{m}_1 = m_1^- \right].$$

We will compute the two parts of the sum by the same method, thus we only study the first part. Thanks to Lemma 3.9, it suffices to take the integral from $-m_1 \vee a_r$ to b_r . By definition, the law of the process $(\tilde{V}(t), t \geq 0)$ is \mathbb{P}^\uparrow and the law of the process $(\tilde{V}((-t)^-), t \geq 0)$ is $\hat{\mathbb{P}}^\uparrow$, moreover they are independent. Then we write them respectively V^\uparrow and \hat{V}^\uparrow . Using the fact that $\mathbf{m}_1 = m_1^+$ if and only if $J_1^+ < J_1^-$ and the result of Proposition 2.7, we get that

$$\mathbb{E} \left[F \left(\frac{e^{-cV_{\mathbf{m}_1}(c^{-\alpha \cdot})}}{c^\alpha \int_{-m_1 \vee a_r}^{b_r} e^{-cV_{\mathbf{m}_1}(y)} dy} \right), \mathbf{m}_1 = m_1^+, \frac{K}{c^\alpha} \leq \mathbf{m}_1 \wedge \tau_1(V) \right] =$$

$$(9) \quad \mathbb{E} \left[F \left(\frac{e^{-c\tilde{V}(c^{-\alpha \cdot})}}{c^\alpha \int_{-\tilde{a}_r}^{\tilde{b}_r} e^{-c\tilde{V}(y)} dy} \right) f_1(V^\uparrow(\tau_1)), \tilde{J}_1^+ \leq \tilde{J}_1^-, \frac{K}{c^\alpha} \leq \hat{m}_1^\uparrow \wedge \tau_1(V^\uparrow) \right]$$

where $\tilde{a}_r, \tilde{b}_r, \tilde{J}_1^+$ and \tilde{J}_1^- are defined similarly as a_r, b_r, J_1^+ and J_1^- but using the process \tilde{V} :

$$\begin{aligned} \tilde{a}_r &:= \hat{m}_1^\uparrow \vee \inf \left\{ x \geq 0 / \hat{V}^\uparrow(x) > r \right\}, \\ \tilde{b}_r &:= \inf \left\{ x \geq 0 / V^\uparrow(x) > r \right\}, \\ \tilde{J}_1^+ &:= \left(1 - \hat{V}^\uparrow(\hat{m}_1^\uparrow) \right) \vee \left(\sup_{(0, \hat{m}_1^\uparrow)} \hat{V}^\uparrow - \hat{V}^\uparrow(\hat{m}_1^\uparrow) \right), \\ \text{and } \tilde{J}_1^- &:= \left(1 - W^\uparrow(m_1^\uparrow) \right) \vee \left(\sup_{(0, m_1^\uparrow)} W^\uparrow - W^\uparrow(m_1^\uparrow) \right) \end{aligned}$$

where W^\uparrow is a process with law \mathbb{P}^\uparrow independent of \tilde{V} . To simplify the notation, we write $F_{\tilde{a}_r, \tilde{b}_r}$ for $F \left(\frac{e^{-c\tilde{V}(c^{-\alpha \cdot})}}{c^\alpha \int_{-\tilde{a}_r}^{\tilde{b}_r} e^{-c\tilde{V}(y)} dy} \right)$.

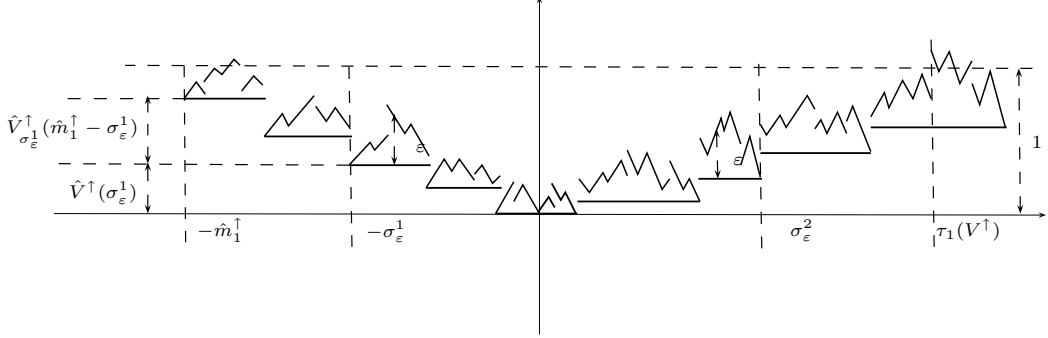
As in the proof of Proposition 2.14, we fix $\varepsilon > 0$ and we separate the several terms of the expectation with respect to $(\tilde{V}(t), 0 \leq t \leq \sigma_\varepsilon^1)$, $(\tilde{V}(t), \sigma_\varepsilon^1 \leq t \leq \sigma_\varepsilon^2)$ and $(\tilde{V}(t), \sigma_\varepsilon^2 \leq t)$, where:

$$\begin{aligned} \sigma_\varepsilon^1 &= \sigma_\varepsilon(\hat{V}^\uparrow) = \inf \left\{ t > 0 / \hat{V}^\uparrow(t) - \underline{\underline{\hat{V}^\uparrow}}(t) = 0 \text{ and } \exists s < t, \hat{V}^\uparrow(s) - \underline{\underline{\hat{V}^\uparrow}}(s) > \varepsilon \right\}, \\ \sigma_\varepsilon^2 &= \sigma_\varepsilon(V^\uparrow) = \inf \left\{ t > 0 / V^\uparrow(t) - \underline{\underline{V^\uparrow}}(t) = 0 \text{ and } \exists s < t, V^\uparrow(s) - \underline{\underline{V^\uparrow}}(s) > \varepsilon \right\}, \end{aligned}$$

to obtain the asymptotic independence of $F_{\tilde{a}_r, \tilde{b}_r}$, of $V^\uparrow(\tau_1)$ and of \tilde{J}_1^+ .

We study the hypothesis $\tilde{J}_1^+ < \tilde{J}_1^-$, let us denote

$$M_+ = \sup_{(0, \hat{m}_1^\uparrow)} \hat{V}^\uparrow - \hat{V}^\uparrow(\hat{m}_1^\uparrow) \text{ and } M_\varepsilon = \sup_{(0, \sigma_\varepsilon^1)} \hat{V}^\uparrow - \hat{V}^\uparrow(\hat{m}_1^\uparrow).$$

FIGURE 5. The process \tilde{V} .

For ε small enough, $M_\varepsilon < M_+$, in this case: $\sigma_\varepsilon^1 < \hat{m}_1^\dagger$, $\sup_{(0, \hat{m}_1^\dagger)} \hat{V}^\dagger = \sup_{(\sigma_\varepsilon^1, \hat{m}_1^\dagger)} \hat{V}^\dagger$ and $\hat{V}^\dagger(\hat{m}_1^\dagger) = \hat{V}^\dagger(\sigma_\varepsilon^1) + \hat{V}_{\sigma_\varepsilon^1}^\dagger(\hat{m}_1^\dagger - \sigma_\varepsilon^1)$. Moreover, for ε small enough, $\sigma_\varepsilon^2 < \tau_1(V^\dagger)$, thus

$$\tilde{J}_1^+ = \phi_\varepsilon \left(\hat{V}^\dagger(\sigma_\varepsilon^1) \right) \text{ with } \phi_\varepsilon(x) = \left(1 - x - \hat{V}_{\sigma_\varepsilon^1}^\dagger(\hat{m}_1^\dagger - \sigma_\varepsilon^1) \right) \vee \left(\sup_{(\sigma_\varepsilon^1, \hat{m}_1^\dagger)} \hat{V}_{\sigma_\varepsilon^1}^\dagger + \hat{V}_{\sigma_\varepsilon^1}^\dagger(\hat{m}_1^\dagger - \sigma_\varepsilon^1) \right).$$

We follow the idea of the proof of Proposition 2.14 with Lemma 2.15. We use Lemma 3.9 for the integral in the function F and we remark that

$$\lim_{c \rightarrow +\infty} \mathbb{P} \left(\frac{K}{c^\alpha} \leq \hat{m}_1^\dagger \wedge \tau_1(V^\dagger) \right) = \lim_{c \rightarrow +\infty} \mathbb{P} \left(\frac{K}{c^\alpha} \leq \sigma_\varepsilon^1 \wedge \sigma_\varepsilon^2 \right) = 1.$$

We recall (9) and we obtain

$$\begin{aligned} & \lim_{c \rightarrow +\infty} \mathbb{E} \left[F_{\tilde{a}_r, \tilde{b}_r} \left(f_1 \left(V^\dagger(\tau_1) \right), \tilde{J}_1^+ \leq \tilde{J}_1^-, \frac{K}{c^\alpha} \leq \hat{m}_1^\dagger \wedge \tau_1(V^\dagger) \right) \right] = \\ & \lim_{\varepsilon \rightarrow 0} \lim_{c \rightarrow +\infty} \mathbb{E} \left[F_{\sigma_\varepsilon^1, \sigma_\varepsilon^2} \chi_2 \left(V^\dagger(\sigma_\varepsilon^2) \right) \chi_1 \left(\hat{V}^\dagger(\sigma_\varepsilon^1) \right), \frac{K}{c^\alpha} \leq \sigma_\varepsilon^1 \wedge \sigma_\varepsilon^2, \sigma_\varepsilon^1 \leq \hat{m}_1^\dagger, \sigma_\varepsilon^2 \leq \tau_1(V^\dagger) \right] \end{aligned}$$

where

$$\begin{aligned} \chi_1(x) &= \mathbb{P} \left(\left(1 - x - \hat{V}^\dagger(\hat{m}_1^\dagger) \right) \vee M_+ < \tilde{J}_1^- \right) \text{ and} \\ \chi_2(x) &= \mathbb{E} \left[f_1 \left(x + V^\dagger(\tau_{1-x}) \right) \right]. \end{aligned}$$

Moreover we have, $\lim_{\varepsilon \rightarrow 0} \chi_1 \left(\hat{V}^\dagger(\sigma_\varepsilon^1) \right) = \mathbb{P}(\tilde{J}_1^+ \leq \tilde{J}_1^-) = \mathbb{P}(\mathbf{m}_1 > 0)$ and $\lim_{\varepsilon \rightarrow 0} \chi_2 \left(V^\dagger(\sigma_\varepsilon^2) \right) = 1$ thanks to (5). Then we obtain:

$$\lim_{\varepsilon \rightarrow 0} \lim_{c \rightarrow +\infty} \left| \mathbb{E} \left[F_{\tilde{a}_r, \tilde{b}_r} \left(\mathbb{P}(\mathbf{m}_1 > 0) - \chi_1 \left(\hat{V}^\dagger(\sigma_\varepsilon^1) \right) \chi_2 \left(V^\dagger(\sigma_\varepsilon^2) \right) \right) \right] \right| = 0.$$

We use the stability of V^\dagger and of \hat{V}^\dagger (see Lemma 2.13). Moreover, for all $r \in (0, 1)$, $\lim_{c \rightarrow +\infty} \tilde{a}_{cr} = -\infty$ and $\lim_{c \rightarrow +\infty} \tilde{b}_{cr} = +\infty$ a.s., then $\frac{e^{-\tilde{V}}}{\int_{a_{rc}}^{b_{rc}} e^{-\tilde{V}(y)} dy}$ converges to $\frac{e^{-\tilde{V}}}{\int_{-\infty}^{+\infty} e^{-\tilde{V}(y)} dy}$ in the Skorohod topology on $[-K, K]$. Thus

$$\lim_{c \rightarrow +\infty} \mathbb{E} \left[F \left(\frac{e^{-cV_{\mathbf{m}_1}(c^{-\alpha})}}{c^\alpha \int_{a_r}^{b_r} e^{-cV_{\mathbf{m}_1}(y)} dy} \right), \mathbf{m}_1 = m_1^\dagger \right] = \mathbb{E} \left[F \left(\frac{e^{-\tilde{V}}}{\int_{-\infty}^{+\infty} e^{-\tilde{V}(y)} dy} \right) \right] \mathbb{P}(\mathbf{m}_1 > 0).$$

We similarly study the case where $\mathbf{m}_1 = m_1^-$, we obtain:

$$\lim_{c \rightarrow +\infty} \mathbb{E} \left[F \left(\frac{e^{-cV_{\mathbf{m}_1}(c^{-\alpha \cdot})}}{c^\alpha \int_{a_r}^{b_r} e^{-cV_{\mathbf{m}_1}(y)} dy} \right), \mathbf{m}_1 = m_1^- \right] = \mathbb{E} \left[F \left(\frac{e^{-\tilde{V}}}{\int_{-\infty}^{+\infty} e^{-\tilde{V}(y)} dy} \right) \right] \mathbb{P}(\mathbf{m}_1 < 0).$$

Then we obtain the result. □

Using Theorem 3.5 and Proposition 3.7, we deduce Theorem 1.1 given in the introduction.

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