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Multiple Polylogarithms: An Introduction

by

M. WALDSCHMIDT(*)

Abstract

Multiple polylogarithms in a single variable are defined by

$$\mathrm{Li}_{(s_1, \dots, s_k)}(z) = \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} \dots n_k^{s_k}},$$

when s_1, \dots, s_k are positive integers and z a complex number in the unit disk. For $k = 1$, this is the classical polylogarithm $\mathrm{Li}_s(z)$. These multiple polylogarithms can be defined also in terms of iterated Chen integrals and satisfy *shuffle relations*. Multiple polylogarithms in several variables are defined for $s_i \geq 1$ and $|z_i| < 1$ ($1 \leq i \leq k$) by

$$\mathrm{Li}_{(s_1, \dots, s_k)}(z_1, \dots, z_k) = \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{z_1^{n_1} \dots z_k^{n_k}}{n_1^{s_1} \dots n_k^{s_k}},$$

and they satisfy not only shuffle relations, but also *stuffle relations*. When one specializes the shuffle relations in one variable at $z = 1$ and the stuffle relations in several variables at $z_1 = \dots = z_k = 1$, one gets linear or quadratic dependence relations between the Multiple Zeta Values

$$\zeta(s_1, \dots, s_k) = \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{1}{n_1^{s_1} \dots n_k^{s_k}}$$

which are defined for k, s_1, \dots, s_k positive integers with $s_1 \geq 2$. The *Main Diophantine Conjecture* states that one obtains in this way all algebraic relations between these MZV.

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0. Introduction

A long term project is to determine all algebraic relations among the values

$$\pi, \zeta(3), \zeta(5), \dots, \zeta(2n+1), \dots$$

of the Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

So far, one only knows that the first number in this list, π , is transcendental, that the second one, $\zeta(3)$, is irrational, and that the other ones span a \mathbb{Q} -vector space of infinite dimension [R], [BR].

The expected answer is disappointingly simple: it is widely believed that there are no relations, which means that these numbers should be algebraically independent:

(?) For any $n \geq 0$ and any nonzero polynomial $P \in \mathbb{Z}[X_0, \dots, X_n]$,

$$P(\pi, \zeta(3), \zeta(5), \dots, \zeta(2n+1)) \neq 0.$$

If true, this property would mean that there is no interesting algebraic structure.

The situation changes drastically if we enlarge our set so as to include the so-called Multiple Zeta Values (MZV, also called Euler-Zagier numbers – see [Eu] and [Z]):

$$\zeta(s_1, \dots, s_k) = \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{1}{n_1^{s_1} \dots n_k^{s_k}},$$

which are defined for k, s_1, \dots, s_k positive integers with $s_1 \geq 2$. It may be hoped that the initial goal could be reached if one could determine all algebraic relations between the MZV. Now there are plenty of relations between them, providing a rich algebraic structure. One type of such relations arises when one multiplies two such series: it is easy to see that one gets a linear combination of MZV. There is another type of algebraic relations between MZV, coming from their expressions as integrals. Again the product of two such integrals is a linear combination of MZV. Following [B³], we will use the name *stuffle* for the relations arising from the series, and *shuffle* for those arising from the integrals.

The *Main Diophantine Conjecture* (Conjecture 5.3 below) states that these relations are sufficient to describe all algebraic relations between MZV. One should be careful when stating such a conjecture: it is necessary to include some relations which are deduced from the stuffle and shuffle applied to divergent series (i.e. with $s_1 = 1$).

There are several ways of dealing with the divergent case. Here, we use the *multiple polylogarithms*

$$\text{Li}_{(s_1, \dots, s_k)}(z) = \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} \dots n_k^{s_k}},$$

which are defined for $|z| < 1$ when s_1, \dots, s_k are all ≥ 1 , and which are also defined for $|z| = 1$ if $s_1 \geq 2$.

These multiple polylogarithms can be expressed as iterated Chen integrals, and from this representation one deduces shuffle relations. There is no stuffle relations for multiple polylogarithms in a single variable, but one recovers them by introducing the multivariable functions

$$\text{Li}_{(s_1, \dots, s_k)}(z_1, \dots, z_k) = \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{z_1^{n_1} \dots z_k^{n_k}}{n_1^{s_1} \dots n_k^{s_k}} \quad (s_i \geq 1, 1 \leq i \leq k)$$

which are defined not only for $|z_i| < 1$ ($1 \leq i \leq k$), but also for $|z_i| \leq 1$ ($1 \leq i \leq k$) if $s_1 \geq 2$.

Notation. Given a string a_1, \dots, a_k of integers, the notation $\{a_1, \dots, a_k\}_n$ stands for the kn -tuple

$$(a_1, \dots, a_k, \dots, a_1, \dots, a_k),$$

where the string a_1, \dots, a_k is repeated n times.

1. Multiple Polylogarithms in One Variable and Multiple Zeta Values

Let k, s_1, \dots, s_k be positive integers. Write \underline{s} in place of (s_1, \dots, s_k) . One defines a complex function of one variable by

$$\text{Li}_{\underline{s}}(z) = \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} \dots n_k^{s_k}}.$$

This function is analytic in the open unit disk, and, in the case $s_1 \geq 2$, it is also continuous on the closed unit disk. In the latter case we have

$$\zeta(\underline{s}) = \text{Li}_{\underline{s}}(1).$$

One can also define in an equivalent way these functions by induction on the number $p = s_1 + \dots + s_k$ (the *weight* of \underline{s}) as follows. Plainly we have

$$(1.1) \quad z \frac{d}{dz} \text{Li}_{(s_1, \dots, s_k)}(z) = \text{Li}_{(s_1-1, s_2, \dots, s_k)}(z) \quad \text{if} \quad s_1 \geq 2$$

and

$$(1.2) \quad (1-z) \frac{d}{dz} \text{Li}_{(1, s_2, \dots, s_k)}(z) = \text{Li}_{(s_2, \dots, s_k)}(z).$$

Together with the initial conditions

$$(1.3) \quad \text{Li}_{\underline{s}}(0) = 0,$$

the differential equations (1.1) and (1.2) determine all the $\text{Li}_{\underline{s}}$.

Therefore, as observed by M. Kontsevich (cf. [Z]; see also [K] Chap. XIX, § 11 for an early reference to H. Poincaré, 1884), an equivalent definition for $\text{Li}_{\underline{s}}$ is given by integral formulae as follows. Starting^(*) with $k = s = 1$, we write

$$\text{Li}_1(z) = -\log(1-z) = \int_0^z \frac{dt}{1-t},$$

where the complex integral is over any path from 0 to z inside the unit circle. From the differential equations (1.1) one deduces, by induction, for $s \geq 2$,

$$\text{Li}_s(z) = \int_0^z \text{Li}_{s-1}(t) \frac{dt}{t} = \int_0^z \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \dots \int_0^{t_{s-2}} \frac{dt_{s-1}}{t_{s-1}} \int_0^{t_{s-1}} \frac{dt_s}{1-t_s}.$$

In the last formula, the complex integral which is written on the left (and which is the last to be computed) is over any path inside the unit circle from 0 to z , the second one is from 0 to t_1, \dots and the last one on the right, which is the first to be computed, is from 0 to t_{s-1} .

Chen iterated integrals (see [K] Chap. XIX, § 11) provide a compact form for such expressions as follows. For $\varphi_1, \dots, \varphi_p$ differential forms and x, y complex numbers, define inductively

$$\int_x^y \varphi_1 \cdots \varphi_p = \int_x^y \varphi_1(t) \int_x^t \varphi_2 \cdots \varphi_p.$$

For $\underline{s} = (s_1, \dots, s_k)$, set

$$\omega_{\underline{s}} = \omega_0^{s_1-1} \omega_1 \cdots \omega_0^{s_k-1} \omega_1,$$

where

$$\omega_0(t) = \frac{dt}{t} \quad \text{and} \quad \omega_1(t) = \frac{dt}{1-t}.$$

(*) This induction could as well be started from $k = 0$, provided that we set $\text{Li}_{\emptyset}(z) = 1$.

Then the differential equations (1.1) and (1.2) with initial conditions (1.3) can be written

$$(1.4) \quad \text{Li}_{\underline{s}}(z) = \int_0^z \omega_{\underline{s}}.$$

Example. For any $n \geq 1$ and $|z| < 1$ we have

$$(1.5) \quad \text{Li}_{\{1\}_n}(z) = \frac{1}{n!} \left(\log(1/(1-z)) \right)^n,$$

which can be written in terms of generating series as

$$\sum_{n=0}^{\infty} \text{Li}_{\{1\}_n}(z) x^n = (1-z)^{-x}.$$

The constant term $\text{Li}_{\{1\}_0}(z)$ is 1.

2. Shuffle Product and the First Standard Relations

Denote by $X = \{\omega_0, \omega_1\}$ the alphabet with two letters and by X^* the set of words on X . A word is nothing else than a non-commutative monomial in the two letters ω_0 and ω_1 . The linear combinations of such words with rational coefficients

$$\sum_u c_u u,$$

where $\{c_u ; u \in X^*\}$ is a set of rational numbers with finite support, is the non-commutative ring $\mathbb{Q}\langle\omega_0, \omega_1\rangle$. We are interested with the set $X^*\omega_1$ of words which end with ω_1 , together with the empty word \emptyset . The linear combinations of such words is a left ideal of $\mathbb{Q}\langle\omega_0, \omega_1\rangle$ which we denote by $\mathbb{Q}\langle\omega_0, \omega_1\rangle\omega_1$.

The set $X^*\omega_1$ is also the set of words $\omega_{\underline{s}}$, with $\underline{s} = (s_1, \dots, s_k)$. We define $\text{Li}_u(z)$ for $u \in X^*\omega_1$ by $\text{Li}_u(z) = \text{Li}_{\underline{s}}(z)$ when $u = \omega_{\underline{s}}$. By linearity we extend the definition of $\text{Li}_u(z)$ to the ideal $\mathbb{Q}\langle\omega_0, \omega_1\rangle\omega_1$:

$$\text{Li}_v(z) = \sum_u c_u \text{Li}_u(z) \quad \text{for} \quad v = \sum_u c_u u.$$

The set of *convergent words* is the set, denoted by $\omega_0 X^* \omega_1$, of words which start with ω_0 and end with ω_1 (including the empty word). The \mathbb{Q} -vector subspace they span in $\mathbb{Q}\langle\omega_0, \omega_1\rangle$ is denoted by $\omega_0 \mathbb{Q}\langle\omega_0, \omega_1\rangle \omega_1$, and for v in this space we set

$$\zeta(v) = \text{Li}_v(1).$$

Definition. The shuffle product of two words in X^* is the element in $\mathbb{Q}\langle\omega_0, \omega_1\rangle$ which is defined inductively as follows:

$$\emptyset \sqcup u = u \sqcup \emptyset = u$$

for any u in $X^*\omega_1$, and

$$(\omega_i u) \sqcup (\omega_j v) = \omega_i (u \sqcup \omega_j v) + \omega_j (\omega_i u \sqcup v)$$

for u, v in X^* and i, j equal to 0 or 1.

This product is extended bilinearly to $\mathbb{Q}\langle\omega_0, \omega_1\rangle$ and defines a commutative and associative law. Moreover $\mathbb{Q}\langle\omega_0, \omega_1\rangle\omega_1$ is stable under \sqcup .

Computing the product $\text{Li}_u(z)\text{Li}_{u'}(z)$ of the two associated Chen iterated integrals yields (see [MPH], Th. 2):

Proposition 2.1. For u and u' in $X^*\omega_1$,

$$\text{Li}_u(z)\text{Li}_{u'}(z) = \text{Li}_{u \sqcup u'}(z).$$

For instance from

$$\omega_1 \sqcup \omega_0 \omega_1 = \omega_1 \omega_0 \omega_1 + 2\omega_0 \omega_1^2$$

we deduce

$$(2.2) \quad \text{Li}_1(z)\text{Li}_2(z) = \text{Li}_{1,2}(z) + 2\text{Li}_{2,1}(z).$$

Setting $z = 1$, we deduce from Proposition 2.1:

$$(2.3) \quad \zeta(u)\zeta(u') = \zeta(u \sqcup u')$$

for u and u' in $\omega_0 X^*\omega_1$.

These are the *first standard relations* between multiple zeta values.

3. Shuffle Product for Multiple Polylogarithms in Several Variables

The functions of k complex variables(*)

$$\text{Li}_{\underline{s}}(z_1, \dots, z_k) = \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{z_1^{n_1} \dots z_k^{n_k}}{n_1^{s_1} \dots n_k^{s_k}}$$

(*) Our notation for

$$\text{Li}_{(s_1, \dots, s_k)}(z_1, \dots, z_k)$$

is the same as in [W], but it corresponds to Goncharov's notation [G2] for

$$\text{Li}_{(s_k, \dots, s_1)}(z_k, \dots, z_1).$$

have been considered as early as 1904 by N. Nielsen, and rediscovered later by A.B. Goncharov [G1,G2]. Recently, J. Écalle [É] used them for z_i roots of unity (in case $s_1 \geq 2$): these are the *decorated multiple polylogarithms*. Of course one recovers the one variable functions $\text{Li}_{\underline{s}}(z)$ by specializing $z_2 = \dots = z_k = 1$. For simplicity we write $\text{Li}_{\underline{s}}(\underline{z})$, where \underline{z} stands for (z_1, \dots, z_k) . There is an integral formula which extends (1.4). Define

$$\omega_z(t) = \begin{cases} \frac{zdt}{1-zt} & \text{if } z \neq 0, \\ \frac{dt}{t} & \text{if } z = 0. \end{cases}$$

From the differential equations

$$z_1 \frac{\partial}{\partial z_1} \text{Li}_{\underline{s}}(\underline{z}) = \text{Li}_{(s_1-1, s_2, \dots, s_k)}(\underline{z}) \quad \text{if } s_1 \geq 2$$

and

$$(1-z_1) \frac{\partial}{\partial z_1} \text{Li}_{(1, s_2, \dots, s_k)}(\underline{z}) = \text{Li}_{(s_2, \dots, s_k)}(z_1 z_2, z_3, \dots, z_k),$$

generalizing (1.1) and (1.2), we deduce

$$\text{Li}_{\underline{s}}(\underline{z}) = \int_0^1 \omega_0^{s_1-1} \omega_{z_1} \omega_0^{s_2-1} \omega_{z_1 z_2} \dots \omega_0^{s_k-1} \omega_{z_1 \dots z_k}.$$

Because of the occurrence of the products $z_1 \dots z_j$ ($1 \leq j \leq k$), the authors of [G1] and [B³L] perform the change of variables

$$y_j = z_1^{-1} \dots z_j^{-1} \quad (1 \leq j \leq k) \quad \text{and} \quad z_j = \frac{y_{j-1}}{y_j} \quad (1 \leq j \leq k)$$

with $y_0 = 1$, and introduce the differential forms

$$\omega'_y(t) = -\omega_{y^{-1}}(t) = \frac{dt}{t-y},$$

so that $\omega'_0 = \omega_0$ and $\omega'_1 = -\omega_1$. Also they define

$$\begin{aligned} \lambda \left(\begin{matrix} s_1, \dots, s_k \\ y_1, \dots, y_k \end{matrix} \right) &= \text{Li}_{\underline{s}}(1/y_1, y_1/y_2, \dots, y_{k-1}/y_k) \\ &= \sum_{\nu_1 \geq 1} \dots \sum_{\nu_k \geq 1} \prod_{j=1}^k y_j^{-\nu_j} \left(\sum_{i=j}^k \nu_i \right)^{-s_j} \\ &= (-1)^p \int_{\Delta_p} \omega_0^{s_1-1} \omega'_{y_1} \dots \omega_0^{s_k-1} \omega'_{y_k}. \end{aligned}$$

With this notation some formulae are simpler. For instance the shuffle relation is easier to write with λ : the shuffle is defined on words on the alphabet $\{\omega'_y; y \in \mathbb{C}\}$, (including $y = 0$), inductively by

$$(\omega'_y u) \sqcup (\omega'_{y'} v) = \omega'_y (u \sqcup \omega'_{y'} v) + \omega'_{y'} (\omega'_y u \sqcup v).$$

4. Stuffle Product and the Second Standard Relations

The functions $\text{Li}_{\underline{s}}(z)$ satisfy not only shuffle relations, but also *stuffle relations* arising from the product of two series:

$$(4.1) \quad \text{Li}_{\underline{s}}(z)\text{Li}_{\underline{s}'}(z') = \sum_{\underline{s}''} \text{Li}_{\underline{s}''}(z''),$$

where the notation is as follows: \underline{s}'' runs over the tuples $(s''_1, \dots, s''_{k''})$ obtained from $\underline{s} = (s_1, \dots, s_k)$ and $\underline{s}' = (s'_1, \dots, s'_{k'})$ by inserting, in all possible ways, some 0 in the string (s_1, \dots, s_k) as well as in the string $(s'_1, \dots, s'_{k'})$ (including in front and at the end), so that the new strings have the same length k'' , with $\max\{k, k'\} \leq k'' \leq k + k'$, and by adding the two sequences term by term. For each such \underline{s}'' , the component z''_i of z'' is z_j if the corresponding s''_i is just s_j (corresponding to a 0 in \underline{s}'), it is z'_ℓ if the corresponding s''_i is s'_ℓ (corresponding to a 0 in \underline{s}), and finally it is $z_j z'_\ell$ if the corresponding s''_i is $s_j + s'_\ell$. For instance

$$\begin{array}{cccccccc} \underline{s} & s_1 & s_2 & 0 & s_3 & s_4 & \cdots & 0 \\ \underline{s}' & 0 & s'_1 & s'_2 & 0 & s'_3 & \cdots & s'_{k'} \\ \underline{s}'' & s_1 & s_2 + s'_1 & s'_2 & s_3 & s_4 + s'_3 & \cdots & s'_{k'} \\ \underline{z}'' & z_1 & z_2 z'_1 & z'_2 & z_3 & z_4 z'_3 & \cdots & z'_{k'}. \end{array}$$

Of course the 0's are inserted so that no s''_i is zero.

Examples. For $k = k' = 1$ the stuffle relation (4.1) yields

$$(4.2) \quad \text{Li}_s(z)\text{Li}_{s'}(z') = \text{Li}_{(s,s')}(z, z') + \text{Li}_{(s',s)}(z', z) + \text{Li}_{s+s'}(zz'),$$

while for $k = 1$ and $k' = 2$ we have

$$(4.3) \quad \begin{aligned} \text{Li}_s(z)\text{Li}_{(s'_1, s'_2)}(z'_1, z'_2) &= \text{Li}_{(s, s'_1, s'_2)}(z, z'_1, z'_2) + \text{Li}_{(s'_1, s, s'_2)}(z'_1, z, z'_2) \\ &+ \text{Li}_{(s'_1, s'_2, s)}(z'_1, z'_2, z) + \text{Li}_{(s+s'_1, s'_2)}(zz'_1, z'_2) + \text{Li}_{(s'_1, s+s'_2)}(z'_1, zz'_2). \end{aligned}$$

The *stuffle product* is defined on $X^*\omega_1$ inductively by

$$\emptyset * u = u * \emptyset = u$$

for $u \in X^*\omega_1$ and

$$\begin{aligned} (\omega_0^{s-1}\omega_1 u) * (\omega_0^{t-1}\omega_1 u') &= \\ \omega_0^{s-1}\omega_1(u * \omega_0^{t-1}\omega_1 u') &+ \omega_0^{t-1}\omega_1(\omega_0^{s-1}\omega_1 u * u') + \omega_0^{s+t-1}\omega_1(u * u') \end{aligned}$$

for u and u' in $X^*\omega_1$, $s \geq 1$, $t \geq 1$.

Specializing (4.1) at $z_1 = \cdots = z_k = z'_1 = \cdots = z'_{k'} = 1$, we deduce

$$(4.4) \quad \zeta(u)\zeta(u') = \zeta(u * u')$$

for u and u' in $\omega_0 X^* \omega_1$.

These are the *second standard relations* between multiple zeta values. For instance (4.3) with $z = z'_1 = z'_2 = 1$ gives

$$\begin{aligned} \zeta(s)\zeta(s'_1, s'_2) &= \zeta(s, s'_1, s'_2) + \zeta(s'_1, s, s'_2) + \zeta(s'_1, s'_2, s) \\ &\quad + \zeta(s + s'_1, s'_2) + \zeta(s'_1, s + s'_2) \end{aligned}$$

for $s \geq 2$, $s'_1 \geq 2$ and $s'_2 \geq 1$.

5. The Third Standard Relations and the Main Diophantine Conjectures

We start with an example. Combining the stuffle relation (4.2) for $s = s' = 1$ with the shuffle relation (2.2) for $z' = z$, we deduce

$$(5.1) \quad \text{Li}_{1,2}(z, 1) + 2\text{Li}_{2,1}(z, 1) = \text{Li}_{1,2}(z, z) + \text{Li}_{2,1}(z, z) + \text{Li}_3(z^2).$$

The two sides are analytic inside the unit circle, but not convergent at $z = 1$. We claim that

$$F(z) = \text{Li}_{1,2}(z, 1) - \text{Li}_{1,2}(z, z) = \sum_{n_1 > n_2 \geq 1} \frac{z^{n_1}(1 - z^{n_2})}{n_1 n_2^2}$$

tends to 0 as z tends to 1 inside the unit circle. Indeed for $|z| < 1$ we have

$$|1 - z^{n_2}| = |(1 - z)(1 + z + \cdots + z^{n_2-1})| < n_2 |1 - z|,$$

hence

$$\sum_{n_2=1}^{n_1-1} \frac{|1 - z^{n_2}|}{n_2^2} < |1 - z| \sum_{n_2=1}^{n_1-1} \frac{1}{n_2}.$$

From (1.5) with $n = 2$ we deduce

$$|F(z)| \leq |1 - z| \text{Li}_{1,1}(|z|) = \frac{1}{2} |1 - z| \left(\log(1/(1 - |z|)) \right)^2.$$

Therefore, taking the limit of the relation (5.1) as $z \rightarrow 1$ yields Euler's formula

$$\zeta(2, 1) = \zeta(3).$$

This argument works in a quite general setting and yields the relations

$$(5.2) \quad \zeta(\omega_1 * u - \omega_1 \sqcup u) = 0$$

for each $u \in \omega_0 X^* \omega_1$.

These are the *third standard relations* between multiple zeta values.

Zagier, Goncharov, Kontsevich, . . . (see [Z]) conjecture that the three standard relations (2.3), (4.4) and (5.2) constitute a basis of the ideal of algebraic relations between all numbers $\zeta(\underline{s})$. Here are precise statements.

We introduce independent variables Z_u , where u ranges over the set $X^* \omega_1$. For $v = \sum_u c_u u$ in $\mathbb{Q}\langle \omega_0, \omega_1 \rangle \omega_1$, we set

$$Z_v = \sum_u c_u Z_u.$$

In particular for u_1 and u_2 in $\omega_0 X^* \omega_1$, $Z_{u_1 \sqcup u_2}$ and $Z_{u_1 * u_2}$ are linear forms in Z_u , $u \in \omega_0 X^* \omega_1$. Also, for $v \in \omega_0 \mathbb{Q}\langle \omega_0, \omega_1 \rangle \omega_1$, $Z_{\omega_1 \sqcup v - \omega_1 * v}$ is a linear form in Z_u , $u \in \omega_0 X^* \omega_1$.

Denote by R the ring of polynomials with coefficients in \mathbb{Q} in the variables Z_u where u ranges over the set $\omega_0 X^* \omega_1$, and by \mathfrak{J} the ideal of R consisting of all polynomials which vanish under the specialization map

$$Z_u \mapsto \zeta(u) \quad (u \in \omega_0 X^* \omega_1).$$

Conjecture 5.3. *The polynomials*

$$Z_u Z_v - Z_{u \sqcup v}, \quad Z_u Z_v - Z_{u * v} \quad \text{and} \quad Z_{\omega_1 \sqcup u - \omega_1 * u},$$

where u and v range over the set of elements in $\omega_0 X^* \omega_1$, generate the ideal \mathfrak{J} .

Denote by \mathfrak{Z}_p the \mathbb{Q} -vector subspace of \mathbb{R} spanned by the real numbers $\zeta(\underline{s})$ with \underline{s} of weight p , with $\mathfrak{Z}_0 = \mathbb{Q}$ and $\mathfrak{Z}_1 = \{0\}$. Using any of the first two standard relations (2.3) or (4.4), one deduces $\mathfrak{Z}_p \cdot \mathfrak{Z}_{p'} \subset \mathfrak{Z}_{p+p'}$. This means that the \mathbb{Q} -vector subspace \mathfrak{Z} of \mathbb{R} spanned by all \mathfrak{Z}_p , $p \geq 0$, is a subalgebra of \mathbb{R} over \mathbb{Q} which is graded by the weight. From Conjecture 5.3 one deduces the following conjecture of Goncharov [G1]:

Conjecture 5.4. *As a \mathbb{Q} -algebra, \mathfrak{Z} is the direct sum of \mathfrak{Z}_p for $p \geq 0$.*

The dimension d_p of \mathfrak{Z}_p satisfies $d_0 = 1$, $d_1 = 0$, $d_2 = d_3 = 1$. The expected value for d_p is given by a conjecture of Zagier [Z]:

Conjecture 5.5. *For $p \geq 3$ we have*

$$d_p = d_{p-2} + d_{p-3}.$$

An interesting question is whether Conjecture 5.3 implies Conjecture 5.5. For this question as well as other related problems, see [É].

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