



# Vanishing theorems for associative submanifolds

Damien Gayet

## ► To cite this version:

| Damien Gayet. Vanishing theorems for associative submanifolds. 2009. hal-00416114v3

**HAL Id: hal-00416114**

**<https://hal.science/hal-00416114v3>**

Preprint submitted on 9 Oct 2009 (v3), last revised 19 Nov 2009 (v4)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Vanishing theorems for associative submanifolds

Damien Gayet

October 9, 2009

## Abstract

Let  $Y$  is an associative submanifold in a manifold with holonomy in  $G_2$ , and  $\mathcal{M}_Y$  the moduli space of its associative deformations. We give some vanishing theorems that force  $\mathcal{M}_Y$  to be locally smooth, and analogous results when  $Y$  has a boundary in a fixed coassociative submanifold.

MSC 2000: 53C38 (35J55, 53C21, 58J32).

KEYWORDS:  $G_2$  holonomy; calibrated submanifolds; elliptic boundary problems ; Bochner technique

## 1 Introduction

Let  $M$  a 7-dimensional riemannian manifold with holonomy included in  $G_2$ . In this case by parallel transport,  $M$  supports a non degenerate harmonic 3-form  $\phi$  (and such that  $\nabla\phi = 0$ ). Moreover,  $M$  benefits a vector product  $\times$  defined by

$$\langle u \times v, w \rangle = \phi(u, v, w),$$

even if to be spin is enough for the existence of this product. Now, recall that a 3-dimensional submanifold  $Y$  is said *associative* if its tangent bundle is stable under the vector product. In other terms,  $\phi$  restricted to  $Y$  is a volume form. Likely, a 4-dimensional submanifold  $X$  is *coassociative* if all the fibers of its normal bundle are associative, or equivalently,  $\phi|_{TX}$  vanishes. We refer to the abundant literature on this subject, see [6] or [5] for a summary with our notations.

**The closed case.** It is known from [9] that the deformation of an associative submanifold  $Y$  without boundary is an elliptic problem, and hence is of vanishing index. In general, the situation is obstructed. For instance, consider the flat torus  $\mathbb{T}^3 \times \{pt\}$  in the flat space  $\mathbb{T}^7 = \mathbb{T}^3 \times \mathbb{T}^4$ . This is an associative submanifold, and its moduli space  $\mathcal{M}_{\mathbb{T}^3 \times \{pt\}}$  of associative deformations contains at least the 4-dimensional  $\mathbb{T}^4$ .

A natural question is to find conditions which force the moduli space to be smooth, or in other terms, which force the cokernel of the problem to vanish. For the closed case, Abkolut and Salur [2] allow a certain freedom for the connection on the normal bundle, the definition of associativity and genericity. But examples are often non generic, and we would like to get a condition that is not a perturbative one. For holomorphic curves in dimension 4, there are topological conditions on the degree of the normal bundle which imply smoothness of the moduli space of complex deformations, see for example [7]. The main point for this is

that holomorphic curves intersect positively. In our case, there is no such phenomenon.

In [9], page 30, McLean gives an example of an isolated associative submanifold. Since this was the start point of our work, we recall it. In [4], Bryant and Salamon constructed a metric of holonomy  $G_2$  on the spin bundle  $S^3 \times \mathbb{R}^4$  of the round 3-sphere. In this case, the basis  $S^3 \times \{0\}$  is associative, the normal bundle is just the spin bundle of  $S^3$ , and the operator related to the associative deformations of  $S^3$  is just the Dirac operator. By the famous theorem of Lichnerowicz [8], there are no non trivial harmonic spinors on  $S^3$  for metric reasons (precisely, the riemaniann scalar curvature is non negative), so the sphere is isolated as an associative submanifold.

In this paper, we follow the way showed by McLean and the theorem of Lichnerowicz. That means that we use the Bochner method to get a geometric condition which allows the smoothness of  $\mathcal{M}_Y$ . For this, we define the following geometric object :

**Definition 1.1** Choose  $\{e_1, e_2, e_3 = e_1 \times e_2\}$  a local orthonormal frame field of  $TY$ . We define the 0-order operator

$$\begin{aligned} \mathcal{R}_\nu : \Gamma(Y, \nu) &\longrightarrow \Gamma(Y, \nu) \\ \psi &\longmapsto -\frac{1}{2} \sum_{i,j} (e_i \times e_j) \times R^\perp(e_i, e_j) \psi, \end{aligned}$$

where  $R^\perp$  is the curvature of the normal bundle  $\nu$  over  $Y$ , and  $\psi \in \Gamma(Y, \nu)$ .

*Fact.* The definition is independant of the choosen oriented orthonormal frame.

Now we can express the first theorem à la Lichnerowicz :

**Theorem 1.1** Let  $Y$  an associative submanifold of a  $G_2$ -manifold  $M$ . If the spectrum of  $\mathcal{R}_\nu$  is non negative, then  $Y$  is isolated as an associative submanifold.

**The case with boundary.** In [5], the authors proved that the deformation of an associative submanifold  $Y$  with boundary in a coassociative submanifold  $X$  is an elliptic problem of finite index. Moreover, they gave the value of this index in terms of a certain Cauchy-Riemann operator related to the complex geometry of the boundary. We sum up in the following the principal results of the paper :

**Theorem 1.2 ([5])** Let  $\nu_X$  the normal complementary of  $T\partial Y$  in  $TX|_{\partial Y}$ , and  $n$  the inner unit vector normal to  $\partial Y$  in  $Y$ . Then the bundle  $\nu_X$  is a subbundle of  $\nu|_{\partial Y}$  and is stable under the left action by  $n$  under  $\times$ , as well as the orthogonal complement  $\mu_X$  of  $\nu_X$  in  $\nu$ . Viewing  $T\partial Y$ ,  $\nu_X$  and  $\mu_X$  as  $n \times$ -complex line bundles, we have  $\mu_X^* \cong \nu_X \otimes_{\mathbb{C}} T\partial Y$ . Besides, the problem of the associative deformations of  $Y$  is elliptic and of index

$$\text{index}(Y, X) = \text{index} \bar{\partial}_{\nu_X} = c_1(\nu_X) + 1 - g.$$

In this context, we introduce a new geometric object that is related to the geometry on the boundary :

**Proposition 1.3** Choose  $\{v, w = n \times v\}$  a local orthonormal frame for  $T\partial Y$ . Let  $L$  a real plane subbundle of  $\nu$  invariant by the action of  $n \times$ . We define

$$\begin{aligned} \mathcal{D}_L : \Gamma(\partial Y, L) &\longrightarrow \Gamma(\partial Y, L) \\ \phi &\longmapsto \pi_L(v \times \nabla_w^\perp \phi - w \times \nabla_v^\perp \phi), \end{aligned}$$

where  $\pi_L$  is the orthogonal projection on  $L$  and  $\nabla^\perp$  the normal connection on  $\nu$  induced by the Levi-Civita connection  $\nabla$  on  $M$ . Then  $\mathcal{D}_L$  is independant of the choosen oriented frame, of order 0 and symmetric.

Now, we can express our main theorem :

**Theorem 1.4** *Let  $Y$  an associative submanifold of a  $G_2$ -manifold  $M$  with boundary in a coassociative  $X$ . If  $\mathcal{D}_{\mu_X}$  and  $\mathcal{R}_\nu$  are non negative, the moduli space  $\mathcal{M}_{Y,X}$  is locally smooth and of dimension given by the virtual one index  $(Y, X)$ .*

When  $M = \mathbb{R}^7$ , we get the following very explicit example considered in [5]. Take a ball  $Y$  in  $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^7$ , with real analytic boundary, and choose  $e$  any constant vector field in  $\nu = Y \times \{0\} \oplus \mathbb{R}^4$ . By [6], there is a unique local coassociative  $X_e$  containing  $\partial Y \times \mathbb{R}$ , such that

$$TX|_{\partial Y} = T\partial Y \oplus \nu_X = T\partial Y \oplus \text{Vect}(e, n \times e).$$

Of course, the translation in the  $e$ -direction gives associative deformations of  $Y$  with boundary in  $X_e$ . The next corollary shows that under a simple metric condition, this is the only way to deform  $Y$  :

**Corollary 1.5** *If  $Y$  is a strictly convex ball in  $\mathbb{R}^3$ , then  $\mathcal{M}_{Y,X_e} = \mathbb{R}$ .*

**The Calabi-Yau extension.** Let  $(N, J, \Omega, \omega)$  a Calabi-Yau 6-dimensional manifold, where  $J$  is an integrable complex structure,  $\Omega$  a non vanishing holomorphic 3-form and  $\omega$  a Kähler form. Here we allow holonomies which are only subgroups of  $SU(3)$ . Then  $M = N \times S^1$  is a manifold with holonomy in  $SU(3) \subset G_2$ . The associated calibration 3-form is given by

$$\phi = \omega \wedge dt + \text{Re } \Omega.$$

Recall that a special lagrangian in  $N$  is 3-dimensional submanifold  $L$  in  $N$  satisfying both conditions  $\omega|_{TL} = 0$  and  $\text{Im } \Omega|_{TL} = 0$ . We know from [9] that  $\mathcal{M}_L$  the moduli space of special lagrangian deformations of  $L$  is smooth and of dimension  $b^1(L)$ . Now every product  $Y = L \times \{pt\}$  of a special lagrangian and a point is an associative submanifold of  $M$ . Since the translation along  $S^1$  preserves the associativity of  $Y$ , we hence have  $\mathcal{M}_L \times S^1 \subset \mathcal{M}_Y$ . In the last part of this paper, we prove that in fact, there is equality, without any condition :

**Theorem 1.6** *The moduli space  $\mathcal{M}_{L \times \{pt\}}$  of associative deformations of  $L \times \{pt\}$  is always smooth, and can be identified with the product  $\mathcal{M}_L \times S^1$ .*

As an immediate example, we get that  $\mathcal{M}_{\mathbb{T}^3 \times \{pt\}} = \mathbb{T}^4$ .

Recall that if  $\Sigma$  is a complex surface of  $N$ , then  $X = \Sigma \times \{pt\}$  is a coassociative submanifold of  $M$ . Consider the problem of associative deformations of  $Y = L \times \{pt\}$  with boundary in  $X$ . Then we must reintroduce the metric conditions :

**Corollary 1.7** *Let  $L$  a special lagrangian submanifold in a 6-dimensional Calabi-Yau  $N$ , such that  $L$  has boundary in a complex surface  $\Sigma$ . Let  $Y = L \times \{t_0\}$  in  $N \times S^1$  and  $X = \Sigma \times \{t_0\}$ . If the Ricci curvature of  $L$  is non negative, and the boundary of  $L$  has non negative mean curvature in  $L$ , then  $\mathcal{M}_{Y,X}$  is locally smooth and has dimension  $g$ , where  $g$  is the genus of  $\partial L$ .*

*Aknowledgements.* I would like to thank the French Agence nationale pour la Recherche for its support, and Vincent Borrelli (resp. Jean-Yves Welschinger) who convinced me that there is a life after curvature tensors (resp. Sobolev spaces).

## 2 Closed associative submanifolds

### 2.1 A Weitzenböck formula

The general strategy for our results is the Bochner technique, which gives some topological restrictions for manifolds with particular metric properties. For this, one computes the square of a 1-order operator acting (a Dirac operator) on some space of sections of a certain vector bundle. This is called a Weitzenböck formula, giving the square as the sum of some simple operator like a laplacian, and a 0th-order operator related to the curvatures of the ambient space and the bundle. Taking the scalar of this with the section and integrating gives, under some conditions on the curvatures, the triviality of the section.

So, we begin by giving the operator we are concerned in, and recall the version of McLean's theorem proposed by Akbulut and Salur. We will give below a new proof of this result.

**Theorem 2.1** ([9],[1]) *Let  $Y$  an associative submanifold of a riemannian manifold  $M$  with  $G_2$ -holonomy, and  $\nu$  its normal bundle. Then the tangent space of its associative deformations is the kernel of the operator*

$$\begin{aligned} D : \Gamma(Y, \nu) &\longrightarrow \Gamma(Y, \nu), \\ \psi &\longmapsto \sum_{i=1}^3 e_i \times \nabla_{e_i}^\perp \psi, \end{aligned} \tag{1}$$

where  $\nabla^\perp$  is the connection on  $\nu$  induced by the Levi-Civita connection  $\nabla$  on  $M$ .

For the next proposition, we introduce the normal equivalent of the invariant second derivative. More precisely, for every local vector fields  $v$  and  $w$  in  $\Gamma(Y, TY)$ , let

$$\nabla_{v,w}^{\perp 2} = \nabla_v^\perp \nabla_w^\perp - \nabla_{\nabla_v^\top w}^\perp,$$

acting on  $\Gamma(Y, \nu)$ . It is straightforward to see that it is tensorial in  $v$  and  $w$ . Moreover, define the equivalent of the connection laplacian :

$$\nabla^{\perp *} \nabla^\perp = -\text{trace} (\nabla^{\perp 2}) = -\sum_i \nabla_{e_i, e_i}^{\perp 2}.$$

**Proposition 2.2** *For the Dirac-like  $D$  we have  $D^2 = \nabla^{\perp *} \nabla^\perp + \mathcal{R}_\nu$ .*

Before computing  $D^2$ , we need the following trivial lemma

**Lemma 2.3** *Let  $\nabla$  the Levi-Civita connection on  $M$  and  $R$  its curvature tensor. For any vector fields  $w, z, u$  and  $v$  on  $M$ , we have*

$$\begin{aligned} \nabla(u \times v) &= \nabla u \times v + u \times \nabla v \text{ and} \\ R(w, z)(u \times v) &= R(w, z)u \times v + u \times R(w, z)v. \end{aligned}$$

*If  $Y$  is an associative submanifold of  $M$  with normal bundle  $\nu$ ,  $u \in \Gamma(Y, TY)$ ,  $v \in \Gamma(Y, TY)$  and  $\eta \in \Gamma(Y, \nu)$ , then*

$$\begin{aligned} \nabla^\top(u \times v) &= \nabla^\top u \times v + u \times \nabla^\top v \text{ and} \\ \nabla^\perp(u \times \eta) &= \nabla^\top u \times v + u \times \nabla^\perp v, \end{aligned}$$

where  $\nabla^\top = \nabla - \nabla^\perp$  is the orthogonal projection of  $\nabla$  on  $TY$ .

**Proof.** Let  $x_1, \dots, x_7$  normal coordinates on  $M$  near  $x$ , and  $e_i = \frac{\partial}{\partial x_i}$  their derivatives, orthonormal at  $x$ . We have

$$u \times v = \sum_i \langle u \times v, e_i \rangle e_i = \sum_i \phi(u, v, e_i) e_i,$$

so that at  $x$ , where  $\nabla_{e_j} e_i = 0$ ,

$$\begin{aligned} \nabla(u \times v) &= \sum_i (\nabla \phi(u, v, e_i) + \phi(\nabla u, v, e_i) + \phi(u, \nabla v, e_i) + \phi(\nabla u, v, \nabla e_i)) e_i \\ &= \sum_i (\phi(\nabla u, v, e_i) + \phi(u, \nabla v, e_i)) e_i = \nabla u \times v + u \times \nabla v, \end{aligned}$$

because  $\nabla \phi = 0$ . Now if  $u$  and  $v$  are in  $TY$ , then we get the result after remarking that  $(\nabla u \times v)^\top = \nabla^\top u \times v$ , because  $TY$  is invariant under  $\times$ . The last relation is implied by  $TY \times \nu \subset \nu$  and  $\nu \times \nu \subset TY$ . The curvature relation is easily derived from the definition  $R(w, z) = \nabla_w \nabla_z - \nabla_z \nabla_w - \nabla_{[w, z]}$  and the derivation of the vector product.  $\blacksquare$

**Proof of proposition 2.2.** We compute  $D^2$  at a point  $x \in Y$ . For this, we choose normal coordinates on  $Y$  and  $e_i \in \Gamma(Y, TY)$  their associated derivatives, orthonormal at  $x$ . To be explicit,  $\nabla^\top e_i = 0$  at  $x$ . Let  $\psi \in \Gamma(Y, \nu)$ .

$$\begin{aligned} D^2 \psi &= \sum_{i,j} e_i \times \nabla_i^\perp (e_j \times \nabla_j^\perp \psi) \\ &= \sum_{i,j} e_i \times (e_j \times \nabla_i^\perp \nabla_j^\perp \psi) + \sum_{i,j} e_i \times (\nabla_i^\top e_j \times \nabla_j^\perp \psi) \\ &= - \sum_i \nabla_i^\perp \nabla_i^\perp \psi - \sum_{i \neq j} (e_i \times e_j) \times \nabla_i^\perp \nabla_j^\perp \psi \\ &= \nabla^{\perp*} \nabla^\perp \psi - \sum_{i < j} (e_i \times e_j) \times (\nabla_i^\perp \nabla_j^\perp - \nabla_j^\perp \nabla_i^\perp) \psi \\ &= \nabla^{\perp*} \nabla^\perp \psi - \sum_{i < j} (e_i \times e_j) \times R^\perp(e_i, e_j) \psi. \end{aligned}$$

Since  $(e_i \times e_j) \times R^\perp(e_i, e_j)$  is symmetric in  $i, j$ , this is equal to

$$\nabla^{\perp*} \nabla^\perp \psi - \frac{1}{2} \sum_{i,j} (e_i \times e_j) \times R^\perp(e_i, e_j) \psi = \nabla^{\perp*} \nabla^\perp \psi + \mathcal{R}_\nu \psi.$$

It is easy to check that  $\mathcal{R}_\nu$  is independant of the choosen orthonormal frame.  $\blacksquare$

We can split the uneasy operator  $\mathcal{R}_\nu$ , in terms of the ambient curvature, and the Weingarten operators :

**Proposition 2.4** *Let  $\mathcal{R}$  the 0-order operator :  $\mathcal{R} : \Gamma(Y, \nu) \rightarrow \Gamma(Y, \nu)$  defined by*

$$\mathcal{R}(\psi) = -\frac{1}{2} \pi_\nu \sum_{i,j} (e_i \times e_j) \times R(e_i, e_j) \psi,$$

where  $R$  is the curvature tensor of the ambient space  $M$  and  $\pi_\nu$  the orthogonal projection on  $\nu$ . Then  $\mathcal{R}$  is a symmetric operator, and there is a positive symmetric 0-order operator  $\mathcal{A} : \Gamma(Y, \nu) \rightarrow \Gamma(Y, \nu)$  which vanishes if  $Y$  is totally geodesic in  $M$ , such that  $\mathcal{R}_\nu = \mathcal{R} - \mathcal{A}$ . As a consequence, the operator  $\mathcal{R}_\nu$  is symmetric.

The operator seems as complicated as  $\mathcal{R}_\nu$ , but it vanishes for flat spaces, like  $\mathbb{R}^7$  or  $\mathbb{T}^7$ .

**Proof.** We refer to the lemma 2.5 below for the proof of the symmetry of  $\mathcal{R}$ . The main tool for the second assertion of our proposition is the Ricci equation. Let  $u, v$  in  $\Gamma(Y, TY)$  and  $\phi, \psi$  in  $\Gamma(Y, \nu)$ .

$$\langle R^\perp(u, v)\psi, \phi \rangle = \langle R(u, v)\psi, \phi \rangle + \langle (A_\psi A_\phi - A_\phi A_\psi)u, v \rangle,$$

where  $A_\phi : TY \rightarrow TY$  is the Weingarten endomorphism in the normal direction  $\phi$  defined by

$$A_\phi(u) = -\nabla_u^\top \phi.$$

Choosing  $\eta_1, \dots, \eta_4$  an orthonormal basis of  $\nu$  at the point  $x$ , we get

$$\begin{aligned} \mathcal{R}_\nu(\psi) &= -\frac{1}{2} \sum_{i,j,k} \langle (e_i \times e_j) \times R^\perp(e_i, e_j)\psi, \eta_k \rangle \eta_k \\ &= \frac{1}{2} \sum_{i,j,k} \langle R^\perp(e_i, e_j)\psi, (e_i \times e_j) \times \eta_k \rangle \eta_k \\ &= \frac{1}{2} \pi_\nu \sum_{i,j,k} (e_i \times e_j) \times R(e_i, e_j)\psi \\ &\quad + \frac{1}{2} \sum_{i,j,k} \langle (A_\psi A_{(e_i \times e_j) \times \eta_k} - A_{(e_i \times e_j) \times \eta_k} A_\psi) e_i, e_j \rangle \eta_k. \end{aligned}$$

The Weingarten endomorphisms are symmetric, so that the last sum is

$$\frac{1}{2} \sum_{i,j,k} \langle A_{(e_i \times e_j) \times \eta_k} e_i, A_\psi e_j \rangle \eta_k - \frac{1}{2} \sum_{i,j,k} \langle A_\psi e_i, A_{(e_i \times e_j) \times \eta_k} e_j \rangle \eta_k.$$

It is easy to see that the second sum is the opposite of the first one. We compute

$$A_{(e_i \times e_j) \times \eta_k} e_i = -(\nabla_i^\perp e_i \times e_j) \times \eta_k - (e_i \times \nabla_i^\perp e_j) \times \eta_k + (e_i \times e_j) \times A_{\eta_k} e_i.$$

But we know that an associative submanifold is minimal, so that

$$\sum_i \nabla_i^\perp e_i = 0.$$

Moreover, deriving the relation  $e_3 = \pm e_1 \times e_2$ , one easily check that

$$\sum_i e_i \times \nabla_j^\perp e_i = 0.$$

Summing, the only resting term is

$$\sum_{i,j,k} \langle (e_i \times e_j) \times A_{\eta_k} e_i, A_\psi e_j \rangle \eta_k.$$

We now use the classical formula for vectors  $u, v$  and  $w$  in  $TY$  :

$$(v \times w) \times u = \langle u, v \rangle w - \langle u, w \rangle v,$$

hence

$$(e_i \times e_j) \times A_{\eta_k} e_i = \langle A_{\eta_k} e_i, e_i \rangle e_j - \langle A_{\eta_k} e_i, e_j \rangle e_i.$$

One more simplification comes from  $\sum_i \langle A_{\eta_k} e_i, e_i \rangle = 0$  for all  $k$  because since  $Y$  is minimal, so our sum is now equal to

$$-\sum_{i,j,k} \langle A_{\eta_k} e_i, e_j \rangle \langle e_i, A_{\psi} e_j \rangle \eta_k = -\mathcal{A}\psi.$$

This is clear that this operator is of vanishing order, and is symmetric, because

$$\begin{aligned} \langle \mathcal{A}\psi, \phi \rangle &= \sum_{i,j,k} \langle A_{\eta_k} e_i, e_j \rangle \langle e_i, A_{\psi} e_j \rangle \phi_k \\ &= \sum_{i,j,k} \langle A_{\phi} e_i, e_j \rangle \langle e_i, A_{\psi} e_j \rangle. \end{aligned}$$

At last,  $\mathcal{A}$  is positive, since

$$\langle \mathcal{A}\psi, \psi \rangle = \sum_{i,j,k} \langle A_{\psi} e_i, e_j \rangle^2.$$

Remark that it vanishes if and only if  $A_{\psi} = 0$ , which is the case in particular if  $Y$  is totally geodesic. ■

We end this section with the following lemma :

**Lemma 2.5** *The operator  $\mathcal{R}$  is symmetric.*

**Proof.** Let  $\psi$  and  $\phi$  in  $\Gamma(Y, \nu)$ . Then

$$\begin{aligned} -2 \langle \mathcal{R}\psi, \phi \rangle &= \sum_{i,j} \langle (e_i \times e_j) \times R(e_i, e_j)\psi, \phi \rangle = - \sum_{i,j} \langle R(e_i, e_j)\psi, (e_i \times e_j) \times \phi \rangle \\ &= \sum_{i,j} \langle R(e_i, e_j)((e_i \times e_j) \times \phi), \psi \rangle \end{aligned}$$

because in general  $\langle R(u, v)x, y \rangle = - \langle R(u, v)y, x \rangle$ . Using lemma 2.3, this is equal to

$$\sum_{i,j} \langle (e_i \times e_j) \times R(e_i, e_j)\phi + (R(e_i, e_j)(e_i \times e_j)) \times \phi, \psi \rangle.$$

Now the first part gives  $-2 \langle \mathcal{R}\phi, \psi \rangle$  and  $\sum_{i,j} R(e_i, e_j)(e_i \times e_j)$  is equal to

$$R(e_1, e_2)e_3 + R(e_2, e_3)e_1 + R(e_3, e_1)e_2 = 0$$

because of Bianchi's first identity. ■

## 2.2 The implicit function stuff

We give now the analytic elements needed for the clear definition of our moduli spaces and their smoothness. The general steps are classical, at least since Gromov-Witten theory. We define the Banach space  $\mathcal{E}$  of (not necessary associative) embedding of a 3-manifold in  $M$ , and a function  $F$  that measure the lack of associativity. Then we linearise  $F$ , and identify the tangent space of the moduli space of associative, a priori weak, deformations with its kernel. Since the derivative of  $F$  is elliptic, those deformations are in fact smooth, and by the implicit theorem, this moduli space is smooth if the cokernel vanishes.



Firstly, recall the existence on  $(M, \phi)$  of a important object, the 3-form with values in  $TM$  defined, for  $u, v, w \in TM$  by :

$$\chi(u, v, w) = -u \times (v \times w) - \langle u, v \rangle w + \langle v, w \rangle u. \quad (2)$$

It is easy to check [1] that  $\chi(u, v, w)$  is orthogonal to the 3-plane  $u \wedge v \wedge w$ . Besides, if  $Y$  is a 3-dimensional submanifold in  $(M, \phi)$ , then  $\chi|_{TY} = 0$  if and only if  $Y$  is associative.

For the future computations, we will use the the following usefull formula [6] :

$$\langle \chi(u, v, w), \eta \rangle = * \phi(u, v, w, \eta),$$

where  $*$  is the Hodge star, and  $\eta \in TM$ . Then

$$\chi = \sum_k \eta_k \lrcorner * \phi \otimes \eta_k, \quad (3)$$

where  $(\eta_k)_{k=1,2,\dots,7}$  is an orthonormal basis of the tangent space of  $M$ .

Now, as in [9], we use this characterization to study the moduli space of associative deformations of an associative  $Y$ . To be honnest, we would like to clarify some obscure points of [9] and simplify the proof of [1] in the case of holonomy in  $G_2$ .

**Proof of theorem 2.1.** Let  $(Y, g)$  any riemanian 3-manifod. For every embedding  $f : Y \rightarrow M$ , define

$$F(f) = f^* \chi(\omega) \in \Gamma(Y, f_* TM),$$

where  $\omega$  is the volume form on  $Y$ . Then  $f(Y)$  is associative if and only if  $F(f)$  vanishes. Consider a path of embeddings  $(f_t)_{t \in [0,1]}$ . After a reparametrization of  $Y$ , we can suppose that

$$s = \frac{df_t}{dt} \Big|_{t=0} \in \Gamma(Y, \nu_{f_0}),$$

where  $\nu_f$  is the normal bundle over  $f(Y)$ .

Suppose that  $f_0(Y)$  is an associative submanifold of  $M$ , and that  $f_0$  is the injection of  $Y \subset M$ . For derivating the vector-valued form  $F$  at  $f_0$ , we use the Levi-Civita connection :

$$\nabla_{\frac{\partial}{\partial t}} F(f_t) \Big|_{t=0} = \sum_k \mathcal{L}_s(\eta_k \lrcorner * \phi)(\omega) \otimes \eta_k + (\eta_k \lrcorner * \phi)(\omega) \otimes \nabla_s \eta_k,$$

where  $\mathcal{L}_s$  is the Lie derivative in the direction  $s$ . The second member vanishes because  $Y$  is associative. Thanks to classical riemannian formulas, we compute the first term :

$$\begin{aligned} \mathcal{L}_s(\eta_k \lrcorner * \phi)(\omega) &= (\eta_k \wedge \omega) \lrcorner \mathcal{L}_s(*\phi) + ([\eta_k, s] \wedge \omega) \lrcorner * \phi \\ &= \mathcal{L}_s(*\phi)(\eta_k, \omega), \end{aligned}$$

since  $(\eta_k \lrcorner * \phi)(\omega) = \langle \chi(\omega), \eta_k \rangle = 0$ . Writing  $\omega = e_1 \wedge e_2 \wedge e_3$ , where  $(e_i)_{i=1,2,3}$  is a local orthonormal frame of  $TY$  with  $e_3 = e_1 \times e_2$ , this is equal to

$$\nabla_s * \phi(\eta_k, \omega) + * \phi(\nabla_{\eta_k} s, \omega) + * \phi(\eta_k, \nabla_{e_1} s, e_2, e_3) + * \phi(\eta_k, e_1, \nabla_{e_2} s, e_3) + * \phi(\eta_k, e_1, e_2, \nabla_{e_3} s).$$

The first term vanishes since  $*\phi$  is covariantly constant, the second one vanishes too for the former reason, and the third is

$$* \phi(\eta_k, \nabla_{e_1} s, e_2, e_3) = * \phi(\eta_k, \nabla_{e_1}^\perp s, e_2, e_3) = - \langle \nabla_{e_1}^\perp s \times (e_2 \times e_3), \eta_k \rangle.$$

Using equation (3) and the relation  $e_2 \times e_3 = e_1$  and summing up the two others similar terms, we have :

$$Ds = \nabla_s F = \sum_i e_i \times \nabla_i^\perp s. \quad (4)$$

■

After this linearisation, we come back to the problem of the moduli space.

**Proposition 2.6** *Let  $Y$  an associative submanifold in  $M$ . If the kernel of the operator  $D$  given by (1) vanishes, then  $\mathcal{M}_Y$  is locally smooth and of dimension  $\dim \ker D$ . If the moduli space  $\mathcal{M}_Y$  contains a smooth submanifold  $\mathcal{M}_k$  of dimension  $k$ , and if for every  $Y'$  in  $\mathcal{M}_k$  the kernel of  $D$  at  $Y'$  is of dimension  $k$ , then  $\mathcal{M}_Y = \mathcal{M}_k$ .*

**Proof.** For  $kp > 3$ , it has a sense to consider the Banach space

$$\mathcal{E} = W^{k,p}(Y, M),$$

with tangent space at  $f$  equal to  $T_f \mathcal{E} = W^{k,p}(Y, f_* TM)$ . Moreover for  $(k - r)/3 > 1/p$ , then

$$W^{k,p}(Y, M) \subset C^r(Y, M),$$

and so  $f \in \mathcal{E}$  is  $C^1$  if  $k > 1 + 3/p$ . In particular, one can define  $\mathcal{F}$  the Banach bundle over  $\mathcal{E}$  with fiber

$$\mathcal{F}_f = W^{k-1,p}(Y, \nu_f).$$

It is clear that the operator  $F$  extends to a section  $F_{k,p}$  of  $\mathcal{F}$  over  $\mathcal{E}$ . We just proved before that  $F$  is differentiable, and if  $f_0(Y)$  is associative, its derivative at  $f_0(Y)$  along a vector field  $s \in T_{f_0} \mathcal{E}$  is given by (4).

Now, the operator  $D$  has symbol

$$\sigma(\xi) : s \mapsto \sum_i \xi_i s \times e_i = s \times \xi,$$

which is always invertible for  $\xi \in TY \setminus \{0\}$ . This proves that  $D$  is elliptic. Remark that  $\sigma(\xi)^2 s = -|\xi|^2 s$ , which is the symbol of the laplacian. Hence  $\dim \ker D$  and  $\dim \operatorname{coker} D$  are of finite dimension. By the implicit function theorem for Banach bundles, if  $\operatorname{coker} D$  vanishes, then  $F^{-1}(0)$  is a locally a smooth Banach submanifold of  $\mathcal{E}$  of finite dimension equal to  $\dim \ker D = \operatorname{index} D$ , which vanishes because  $Y$  is odd-dimensional. More generally, if  $\dim \operatorname{coker} D$  is constant on the component of  $\mathcal{M}_X$  containing  $Y$ , then the moduli space is still smooth of dimension  $\dim \ker D$ . Lastly, still because of ellipticity, all elements of  $\mathcal{M}_X$  are smooth. ■

### 2.3 A vanishing theorem

We can now give a

**Proof of theorem 1.1.** Suppose that we are given a fixed closed associative submanifold  $Y$ . The virtual dimension of its moduli space of deformation is vanishing. Consider a section  $\psi \in \Gamma(Y, \nu)$ . By classical calculations, using normal coordinates, we have

$$\begin{aligned} -\frac{1}{2} \Delta |\psi|^2 &= \sum_i \langle \nabla_i^\perp \psi, \nabla_i^\perp \psi \rangle + \langle \psi, \nabla_i^\perp \nabla_i^\perp \psi \rangle \\ &= |\nabla^\perp \psi|^2 - \langle D^2 \psi, \psi \rangle + \langle \mathcal{R}_\nu \psi, \psi \rangle. \end{aligned}$$

Since the laplacian is equal to  $-\text{div}(\vec{\nabla})$ , its integral over the closed  $Y$  vanishes. We get :

$$0 = \int_Y |\nabla^\perp \psi|^2 - \langle D^2 \psi, \psi \rangle + \langle \mathcal{R}_\nu \psi, \psi \rangle dy. \quad (5)$$

Suppose that we have a section  $\psi \in \ker D$ . Under the hypothesis that  $\mathcal{R}^\nu$  is non negative, the last equation implies  $\psi = 0$ . Hence  $\dim \text{coker } D = \dim \ker D = 0$ , and by proposition 2.6,  $\mathcal{M}_Y$  is locally a smooth manifold of vanishing dimension, and  $Y$  is isolated. ■

### 3 Associative submanifolds with boundary

In this section we extend our result for rigidity in the case of associative submanifolds with boundary in a coassociative submanifold. In this case the index may be not zero, so rigidity transforms into smoothness of the moduli space.

#### 3.1 Implicit function stuff

As before, define the adapted  $\mathcal{E}$ , for  $kp > 3$  and  $(k-r)/3 > 1/p$  :

$$\mathcal{E}_X = \{f \in W^{k,p}(Y, M), f(\partial Y) \subset X\}.$$

This has the following tangent space :

$$T\mathcal{E}_{X,f} = \{s \in W^{k,p}(Y, f^*TM), s_{T\partial Y} \in f^*TX\}.$$

As before, we have a map :

$$F : \mathcal{E}_X \rightarrow W^{k-1,p}(Y, \nu_{f(Y)}).$$

It is enough to compute the derivative of  $F_X$  at an application  $f_0$  where  $f_0(Y)$  is an associative submanifold. We suppose as in the closed case that  $f_0$  is an injection  $Y \hookrightarrow M$ . In this case, lemma 1.2 showed that  $TX$  is orthogonal to  $TY$  at  $\partial Y$ , hence the derivative of  $F$  at  $f_0$  is :

$$D : \{s \in W^{k,p}(Y, \nu), s|_{\partial Y} \in \nu_X\} \rightarrow W^{k-1,p}(Y, \nu).$$

Now, to get some trace properties and use the results of [3], we need to restrict to the Sobolev space  $p = 2$ . In particular, if  $f \in H^s(Y, \nu) = W^{s,2}(Y, \nu)$ , then  $f|_{\partial Y} \in H^{s-\frac{1}{2}}(Y, \nu)$ . By theorem 20.8 of [3], the operator  $D$  is Fredholm. In [5], the authors computed its index, given by theorem 1.2. Moreover, we will need the usefull

**Proposition 3.1** *The operator  $D$  is formally self-adjoint, i.e for  $s$  and  $s' \in \Gamma(Y, \nu)$ ,*

$$\int_Y \langle Ds, s' \rangle - \langle s, Ds' \rangle dy = - \int_{\partial Y} \langle n \times s, s' \rangle d\sigma, \quad (6)$$

$d\sigma$  is the volume induced by the restriction of  $g$  on the boundary, and  $n$  is the normal inner unit vector of  $\partial Y$ . Moreover,  $\text{coker}(D, \nu_X) = \ker(D, \mu_X)$ .

**Proof.** The proof of the first assertion is *mutatis mutandis* the classical one for the classical Dirac operator, see proposition 3.4 in [3] for example. For the reader's convenience we give

a proof of this.

$$\begin{aligned}
\langle Ds, s' \rangle &= \langle \sum_i e_i \times \nabla_i^\perp s, s' \rangle = - \sum_i \langle \nabla_i^\perp s, e_i \times s' \rangle \\
&= - \sum_i d_{e_i} \langle s, e_i \times s' \rangle + \langle s, \nabla_i^\perp (e_i \times s') \rangle \\
&= - \sum_i d_{e_i} \langle s, e_i \times s' \rangle + \langle s, \nabla_i^\top e_i \times s' + e_i \times \nabla_i^\perp s' \rangle.
\end{aligned}$$

By a classical trick, define the vector field  $X \in \Gamma(Y, TY)$  by

$$\langle X, w \rangle = - \langle s, w \times s' \rangle \quad \forall w \in TY.$$

Note that the first product the one of  $TY$ , and the second one the one of  $\nu$ . Now

$$\begin{aligned}
- \sum_i d_{e_i} \langle s, e_i \times s' \rangle &= \sum_i d_{e_i} \langle X, e_i \rangle \\
&= \sum_i \langle \nabla_i^\top X, e_i \rangle + \langle X, \nabla_i^\top e_i \rangle \\
&= \sum_i \operatorname{div} X - \langle s, \nabla_i^\top e_i \times s' \rangle.
\end{aligned}$$

By Stokes we get

$$\begin{aligned}
\int_Y \langle Ds, s' \rangle dy &= \int_{\partial Y} \langle X, -n \rangle d\sigma + \int_Y \langle s, Ds' \rangle dy \\
&= \int_{\partial Y} \langle s, n \times s' \rangle d\sigma + \int_Y \langle s, Ds' \rangle dy,
\end{aligned}$$

which is what we wanted. Now, let  $s' \in \Gamma(Y, \nu)$  lying in  $\operatorname{coker}(D, \nu_X)$ . This is equivalent to say that for every  $s \in \Gamma(Y, \nu_X)$ , we have  $\int_Y \langle Ds, s' \rangle dy = 0$ . By the former result, we see that this equivalent to

$$\int_Y \langle s, Ds' \rangle + \int_{\partial Y} \langle n \times s, s' \rangle = 0.$$

This clearly implies that  $Ds' = 0$ , and  $s'_{|\partial Y}(\nu) \perp \nu_X$ , because  $\nu_X$  is invariant under the action of  $n \times$ . So  $s'_{|\partial Y} \in \mu_X$ , and  $s' \in \ker(D, \mu_X)$ . The inverse inclusion holds too by similar reasons.  $\blacksquare$

### 3.2 Vanishing theorem

**Proof of theorem 1.4.** In order to get some smooth moduli spaces in the case with boundary, we want to prove that  $\operatorname{coker}(D, \nu_X) = \ker(D, \mu_X)$  is trivial or has constant rank. So let  $\psi \in \ker(D, \mu_X)$ . The boundary changes the integration (5), because the divergence has to be considered :

$$\int_Y |\nabla^\perp \psi|^2 + \langle \mathcal{R}_\nu \psi, \psi \rangle dy = \frac{1}{2} \int_Y \operatorname{div} \vec{\nabla} |\psi|^2 dy. \quad (7)$$

By Stokes, the last is equal to

$$-\frac{1}{2} \int_{\partial Y} d|\psi|^2(n) d\sigma = - \int_{\partial Y} \langle \nabla_n^\perp \psi, \psi \rangle d\sigma,$$

where  $n$  is the normal inner unit vector of  $\partial Y$ . Choosing a local orthonormal frame  $\{v, w = n \times v\}$  of  $T\partial Y$ , and using the fact that  $D\psi = 0$ , this is equal to

$$\int_{\partial Y} \langle w \times \nabla_v^\perp \psi - v \times \nabla_w^\perp \psi, \psi \rangle d\sigma = - \int_{\partial Y} \langle \mathcal{D}_{\mu_X} \psi, \psi \rangle d\sigma.$$

Summing up, we get the equation

$$\int_Y |\nabla^\perp \psi|^2 dy + \int_Y \langle \mathcal{R}_\nu \psi, \psi \rangle dy + \int_{\partial Y} \langle \mathcal{D}_{\mu_X} \psi, \psi \rangle d\sigma = 0. \quad (8)$$

Now we can prove the theorem 1.4. We see that if  $\mathcal{D}_{\mu_X}$  and  $\mathcal{R}$  are negatively definite, then  $\psi$  vanishes. This means that our deformation problem has no cokernel, and by a straightforward generalization of proposition 2.6, the moduli space is locally smooth.  $\blacksquare$

### 3.3 Some properties of the operator $\mathcal{D}_L$

We sum up the main results about  $\mathcal{D}_L$  in the following

**Proposition 3.2** *Let  $Y$  an associative submanifold with boundary in a coassociative submanifold  $X$ ,  $L$  a subbundle of  $\nu$  over  $\partial Y$ , and  $\mathcal{D}_L$  as defined in the introduction. Then  $\mathcal{D}_L$  is of order 0, symmetric, and its trace is  $2H$ , where  $H$  is the mean curvature of  $\partial Y$  in  $Y$  with respect to the outside normal vector  $-n$ .*

**Proof.** Let  $L$  is a subbundle of  $\nu$  invariant under the action of  $n \times$ . It is straightforward to check that  $\mathcal{D}$  does not depend of the orthonormal frame  $\{v, w = n \times v\}$ . For every  $\psi \in \Gamma(\partial Y, L)$  and  $f$  a function,

$$\begin{aligned} \mathcal{D}_L(f\psi) &= \pi_L(v \times \nabla_w(f\psi) - w \times \nabla_v(f\psi)) \\ &= f\mathcal{D}_L\psi + \pi_L(d_w f v \times \psi - d_v f w \times \psi) = f\mathcal{D}_L\psi \end{aligned}$$

because  $w \times L$  and  $v \times L$  are orthogonal to  $L$ . Now, decompose

$$\nabla^\top = \nabla^\top \partial + \nabla^\perp \partial$$

into its two projections along  $T\partial Y$  and along the normal (in  $TY$ )  $n$ -direction. For the computations, choose  $v$  and  $w = n \times v$  the two orthogonal characteristic directions on  $T\partial Y$ , i.e  $\nabla_v^\top \partial n = -k_v v$  and  $\nabla_w^\top \partial n = -k_w w$ , where  $k_v$  and  $k_w$  are the two principal curvatures. We have  $\nabla_v^\perp \partial = k_v n$  and  $\langle \nabla^\perp \partial w, n \rangle = 0$ , and the same, *mutatis mutandis*, for  $w$ . Then, for  $\psi$  and  $\phi \in \Gamma(\partial Y, L)$ , using the fact that  $T\partial Y \times L$  is orthogonal to  $L$ ,

$$\begin{aligned} \langle \mathcal{D}_L \psi, \phi \rangle &= \langle \nabla_w^\perp (v \times \psi) - (\nabla_w^\perp \partial v) \times \psi - \nabla_v^\perp (w \times \psi) + (\nabla_v^\perp \partial w) \times \psi, \phi \rangle \\ &= \langle \nabla_w^\perp (v \times \psi) - \nabla_v^\perp (w \times \psi), \phi \rangle \\ &= -\langle v \times \psi, \nabla_w^\perp \phi \rangle + \langle w \times \psi, \nabla_v^\perp \phi \rangle \\ &= \langle \psi, v \times \nabla_w^\perp \phi - w \times \nabla_v^\perp \phi \rangle = \langle \psi, \mathcal{D}_L \phi \rangle. \end{aligned}$$

To prove that the trace of  $\mathcal{D}_L$  is  $2H$ , let  $e \in L$  a local unit section of  $L$ . We have  $n \times e \in L$  too, and

$$\begin{aligned} \langle \mathcal{D}_L(n \times e), n \times e \rangle &= \langle v \times ((\nabla_w^\top \partial n) \times e) + v \times (n \times \nabla_w^\perp e), n \times e \rangle \\ &\quad - \langle w \times (\nabla_v^\top \partial n) \times e - w \times (n \times \nabla_v^\perp e), n \times e \rangle \\ &= \langle v \times (-k_w w \times e) - w \times (-k_v v \times e), n \times e \rangle \\ &\quad + \langle v \times (n \times \nabla_w^\perp e) - w \times (n \times \nabla_v^\perp e), n \times e \rangle \\ &= k_w + k_v - \langle n \times (w \times (n \times \nabla_v^\perp e) - v \times (n \times \nabla_w^\perp e)), e \rangle \\ &= 2H - \langle \mathcal{D}_L e, e \rangle. \end{aligned}$$

This shows that  $\text{trace } \mathcal{D}_L = 2H$ . ■

### 3.4 Flatland

In flat spaces,  $\mathcal{R}$  vanishes, and so by proposition 2.4,  $\mathcal{R}_\nu = -\mathcal{A}$ . Hence a priori the theorem 1.4 does'nt apply. Nevertheless, we have the

**Corollary 3.3** *Let  $Y$  a totally geodesic associative submanifold in a flat  $M$ , with boundary in a coassociative  $X$ . If  $\mathcal{D}_{\mu_X}$  positive, then  $\mathcal{M}_{Y,X}$  is locally smooth and of expected dimension.*

**Proof.** By proposition 2.4, the hypothesis on  $Y$  implies that  $\mathcal{R}_\nu = 0$ . The formula (8) show that  $\nabla^\perp \psi = 0$  and  $\psi|_{\partial Y} = 0$ . Using  $d|\psi|^2 = 2 \langle \nabla^\perp \psi, \psi \rangle = 0$ , we get that  $\psi = 0$ , and  $\text{coker}(Y, \nu_X) = \ker(Y, \mu_X) = 0$ . ■

**Proof of corollary 1.5.** Let  $Y$  in  $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^7$ , and  $e \in \{0\} \times \mathbb{R}^4$ . From [6] the boundary of  $Y$  lies in a local coassociatif submanifold  $X_e$  of  $\mathbb{R}^7$ , which contains  $\partial Y \times \mathbb{R}e$  and whose tangent space over  $\partial Y$  is  $T\partial Y \oplus \mathbb{R}e \oplus \mathbb{R}n \times e$ . We see that  $Y$  has a direction of associative deformation along the fixed direction  $e$ , hence the dimension of the kernel of our problem is bigger than 1. On the other hand, the index is  $c_1(\nu_X) + 1 - g = 1$ . We want to show that  $\mathcal{D}_{\mu_X}$  is positive. To see that, we choose orthogonal characteristic directions  $v$  and  $w = n \times v$  in  $T\partial Y$  as before. From theorem 1.2, we know that  $v \times e$  is a non vanishing section of  $\mu_X$ . We compute :

$$\begin{aligned} \mathcal{D}_{\mu_X}(v \times e) &= v \times (\nabla_w^\perp v \times e) - w \times (\nabla_v^\perp v \times e) \\ &= -k_v w \times (n \times e) = k_v v \times e. \end{aligned}$$

This shows that  $k_v$  is an eigenvalue of  $\mathcal{D}_{\mu_X}$ , and since we know that its trace is  $2H$ , we get that the other eigenvalue is  $k_w$ . Those eigenvalues are positive if the boundary of  $Y$  is strictly convex. By the last corollary, we get the result. ■

*Remark.* In fact, we can give a better statement. Indeed, let  $\psi \in \ker(D, \nu_X)$ , and decompose  $\psi|_{\partial Y}$  as  $\psi = \psi_1 e + \psi_2 n \times e$ . Of course,  $e$  is in the kernel of  $\mathcal{D}_{\nu_X}$ , and hence by proposition 1.3, the second term is an eigenvector of  $\mathcal{D}_{\nu_X}$  for the eigenvalue  $2H$ . So formula (8) gives

$$\int_Y |\nabla^\perp \psi|^2 + \int_{\partial Y} 2H |\psi_2|^2 = 0.$$

If  $H > 0$ , this imply immediatly that  $\psi_2 = 0$ , and  $\psi_1$  is constant, so  $\psi$  is proportional to  $e$ . This proves that  $\dim \ker(D, \nu_X) = 1$  under the weaker condition that  $H > 0$ .

## 4 Extensions from the Calabi-Yau world

**Closed extension.** Let  $(N, J, \Omega, \omega)$  a 6-dimensional manifold with holonomy in  $SU(3)$ . Then  $M = N \times S^1$  is a manifold with holonomy in  $G_2$ , with the calibration the 3-form given by  $\phi = \omega \wedge dt + \text{Re } \Omega$ . Let  $L$  a special lagrangian 3-dimensional submanifold in  $N$ . Recall that since  $L$  is lagrangian, its normal bundle is simply  $JTL$ . Then  $Y = L \times \{pt\}$  is an associative submanifold of  $N \times S^1$ , and its normal bundle  $\nu$  is isomorphic to  $JTL \times \mathbb{R}\partial_t$ , where  $\partial_t$  is the dual vector field of  $dt$ .

**Proof of theorem ??.** We won't use the tricky formula with  $\mathcal{R}_\nu$ . In this situation, we give another formulation for the Dirac-like operator. If  $s = J\sigma \oplus \tau \partial_t$  is a section of  $\nu$ , with  $\sigma \in \Gamma(L, TL)$  and  $\tau \in \Gamma(L, \mathbb{R}) = \Omega^0(L)$ , we call  $\sigma^\vee \in \Omega^1(L, \mathbb{R})$  the 1-form dual to  $\sigma$ , and we use the same symbol for its inverse. Moreover, we use the classical notation  $*$  :  $\Omega^k(L) \rightarrow \Omega^{3-k}(L)$  for the Hodge star. Lastly, we define :

$$\begin{aligned} D^\vee : \Omega^1(L) \times \Omega^0(L) &\longrightarrow \Omega^1(L) \times \Omega^0(L) \\ (\alpha, \tau) &\longmapsto ((-J\pi_L D(J\alpha^\vee, \tau))^\vee, \pi_t D(J\alpha^\vee, \tau)), \end{aligned}$$

where  $\pi_L$  (resp.  $\pi_t$ ) is the orthogonal projection  $\nu = NL \oplus \mathbb{R}$  on the first (resp. the second) component. This is just a way to use forms on  $L$  instead of normal ambient vector fields.

**Proposition 4.1** *For every  $(\alpha, \tau) \in \Omega^1(L) \times \Omega^0(L)$ ,*

$$\begin{aligned} D^\vee(\alpha, \tau) &= (- * d\alpha - d\tau, *d * \alpha) \text{ and} \\ D^{\vee 2}(\alpha, \tau) &= -\Delta(\alpha, \tau), \end{aligned}$$

where  $\Delta = d^*d + dd^*$  (note that it is  $d^*d$  on  $\tau$ ).

Assuming for a while this proposition, we see that for an infinitesimal associative deformation of  $L \times \{pt\}$ , then  $\alpha$  and  $\tau$  are harmonic over the compact  $L$ . In particular,  $\tau$  is constant and  $\alpha$  describes an infinitesimal special lagrangian deformation of  $L$  (see [9]). In other words, the only way to displace  $Y$  is to perturb  $L$  as special Lagrangian in  $N$  and translate it along the  $S^1$ -direction. Lastly,  $\dim \text{coker } D = \dim \ker D = b^1(L) + 1$  and by the constant rank theorem 2.6,  $\mathcal{M}_Y$  is smooth and of dimension  $b^1(L) + 1$ . ■

**Proof of proposition 4.1.** We will use the simple formula  $\nabla^\perp Js = J\nabla^\top s$  for all sections  $s \in \Gamma(L, NL)$ . For  $(s, \tau) \in \Gamma(L, NL) \times \mathbb{R}$ , and  $e_i$  local orthonormal frame on  $L$ ,

$$\begin{aligned} D(s, \tau) &= \sum_{i,j} \langle e_i \times \nabla_i^\perp s, J e_j \rangle J e_j + \sum_i \langle e_i \times \nabla_i^\perp s, \partial_t \rangle \partial_t + \sum_i \partial_i \tau e_i \times \partial_t \\ &= J \sum_{i,j} \phi(e_i, \nabla_i^\perp s, J e_j) e_j + \sum_i \phi(e_i, \nabla_i^\perp s, \partial_t) \partial_t + J \sum_{i,j} \partial_i \tau \langle e_i \times \partial_t, J e_j \rangle e_j, \end{aligned}$$

where we used that  $e_i \times \partial_t \perp \partial_t$ .

$$\begin{aligned} &= J \sum_{i,j} \text{Re } \Omega(e_i, \nabla_i^\perp s, J e_j) e_j + \sum_i \omega(e_i, \nabla_i^\perp s) \partial_t + J \sum_{i,j} \partial_i \tau \phi(e_i, \partial_t, J e_j) e_j \\ &= J \sum_{i,j} \text{Re } \Omega(e_i, J \nabla_i^\top \sigma, J e_j) e_j + \sum_i \omega(e_i, J \nabla_i^\top \sigma) \partial_t + J \sum_{i,j} \partial_i \tau \omega(J e_j, e_i) e_j, \end{aligned}$$

where  $\sigma = -Js \in \Gamma(L, TL)$ .

$$\begin{aligned} &= -J \sum_{i,j} \text{Re } \Omega(e_i, \nabla_i^\top \sigma, e_j) e_j + \sum_i \langle e_i, \nabla_i^\top \sigma \rangle \partial_t - J \sum_{i,j} \partial_i \tau \langle e_j, e_i \rangle e_j \\ &= -J \sum_{i,j} \text{Vol}(e_i, \nabla_i^\top \sigma, e_j) e_j + \sum_i \langle e_i, \nabla_i^\top \sigma \rangle \partial_t - J \sum_i \partial_i \tau e_i, \end{aligned}$$

since  $\text{Re } \Omega$  is the volume form on  $TL$ . It is easy to find that this is equivalent to

$$D(s, \tau) = -J(*d\sigma^\vee)^\vee + (*d * \sigma^\vee) \partial_t - J(d\tau)^\vee,$$

and so

$$D^\vee(\sigma^\vee, \tau) = (- * d\sigma^\vee - d\tau, *d * \sigma^\vee).$$

Now, since  $d^* = (-1)^{3p+1} * d *$  on the  $p$ -forms, one easily checks the formula for  $D^2$ . ■

**Proof of corollary 1.7.** Consider  $L$  a special lagrangian with boundary in a complex surface  $\Sigma$ , and  $Y = L \times \{pt\}$  (resp.  $X = \Sigma \times \{pt\}$ ) its associative (resp. coassociative) extension. It is clear that  $\nu_X$  is equal (as a real bundle) to  $JT\partial L$ , and  $\mu_X$  is the trivial  $n \times$ -bundle generated by  $\partial_t$ . We begin by computing the index of the boundary problem. This is very easy, since  $\mu_X$  is trivial, and by theorem 1.2, we have  $\nu_X \cong T\partial L^*$  (as  $n \times$ -bundles). Hence the index is

$$-c_1(T\partial L) + 1 - g = -(2 - g) + 1 - g = g - 1,$$

where  $g$  is the genus of  $\partial Y$ . Now let  $\psi = s + \tau \frac{\partial}{\partial t}$  belonging to  $\text{coker}(D, \nu_X) = \ker(D, \mu_X)$ , where  $s$  a section of  $NL$  and  $\tau \in \Gamma(L, \mathbb{R})$ . Let  $\alpha = -Js^\vee$ . By proposition 4.1,  $\alpha$  is a harmonic 1-form, and  $\tau$  is harmonic (note that  $Y$  is not closed, so  $\tau$  may be not constant). By classical results for harmonic 1-forms, we have :

$$\frac{1}{2}\Delta|\psi|^2 = \frac{1}{2}\Delta(|\alpha|^2 + |\tau|^2) = |\nabla_L \alpha|^2 + |d\tau|^2 + \frac{1}{2}\text{Ric}(\alpha, \alpha).$$

Integrating on  $L \times \{pt\}$ , we obtain the equivalence of formula (8) :

$$-\int_{\partial Y} \langle \mathcal{D}_{\mu_X} \psi, \psi \rangle d\sigma = \int_Y |\nabla_L \alpha|^2 + |d\tau|^2 + \frac{1}{2}\text{Ric}(\alpha, \alpha) dy.$$

Lastly, let us compute the eigenvalues of  $\mathcal{D}_{\mu_X}$ . The constant vector  $\frac{\partial}{\partial t}$  over  $\partial Y$  lies clearly in the kernel of  $\mathcal{D}_{\mu_X}$ . By proposition 3.2, the other eigenvalue of  $\mathcal{D}_{\mu_X}$  is  $2H$ , with eigenspace generated by  $n \times \frac{\partial}{\partial t}$ . Over  $\partial Y$ ,  $s$  lies in  $JTL \cap \mu_X$ , hence is proportional to  $n \times \frac{\partial}{\partial t}$ . Consequently,  $\mathcal{D}_{\mu_X} \psi = 2Hs$  and

$$-\int_{\partial Y} 2H|s|^2 d\sigma = \int_Y |\nabla_L \alpha|^2 + |d\tau|^2 + \frac{1}{2}\text{Ric}(\alpha, \alpha) dy.$$

This equation, the positivity of the Ricci curvature and the positivity of  $H$  show that  $\alpha$  vanishes and  $\tau$  is constant. So we see that  $\dim \text{coker}(Y, X) = 1$ , and by the constant rank theorem,  $\mathcal{M}_{Y,X}$  is locally smooth and of dimension  $\dim \ker(Y, X) = g$ .  $\blacksquare$

## References

- [1] S. AKBULUT, S. SALUR, *Deformations in  $G_2$  manifolds*, Adv. Math. **217** no. 5 (2008), 2130–2140.
- [2] S. AKBULUT, S. SALUR, *Calibrated manifolds and gauge theory*, J. Reine Angew. Math. **625** (2008), 187–214.
- [3] B. BOOSS, K. WOJCIECHOWSKI, *Elliptic Boundary Problems for Dirac Operators*, Birkhäuser Verlag, Boston, 1993.
- [4] R. BRYANT, S. SALAMON, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58** no. 3 (1989), 829–850.
- [5] D. GAYET, F. WITT, *Deformations of associative submanifolds with boundary*, arXiv:0802.1283v2.
- [6] R. HARVEY, H. LAWSON, *Calibrated geometries*, Acta Math. **148**, (1982), 47–157.



- [7] H. HOFER, V. LIZAN, J-C SIKORAV, *On genericity for holomorphic curves in four-dimensional almost-complex manifolds*, J. Geom. Anal., **7**, no. 1 (1997), 149–157.
- [8] A. LICHNEROWICZ, *Spineurs harmoniques*, C. R. Acad. Sci. Paris Ser A-B **257**, (1963), 7–9.
- [9] R. MCLEAN, *Deformations of calibrated submanifolds*, Comm. Anal. Geom. **6** no. 4 (1998), 705–747.

D. GAYET

Université de Lyon, CNRS, Université Lyon 1, Institut Camille Jordan,  
F-69622 Villeurbanne Cedex, France  
e-mail: `gayet@math.univ-lyon1.fr`