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# Vanishing theorems for associative submanifolds

Damien Gayet

September 11, 2009

## Abstract

Let  $M$  a 7-dimensional manifold with holonomy in  $G_2$ , and containing a compact associative 3-dimensional submanifold  $Y$ . It is known since McLean [9] that the problem of the associative deformations of  $Y$  is related to a Dirac-like operator on  $Y$ , hence is elliptic and of vanishing index. In this paper we adapt the Bochner method to give sufficient metric conditions which implies that  $Y$  is isolated. In the case where  $Y$  has boundary in a coassociative 4-dimensional submanifold  $X$ , it has been proved in [5] that the related deformation problem is still elliptic, and moreover its index is in general non vanishing. In this case, we give a vanishing theorem that mixes the ambient curvature and the geometry of the boundary, and which insures the smoothness of the moduli space of associative deformations of  $Y$ . Thanks, we are able to describe explicit smooth moduli spaces for some families of examples in  $\mathbb{R}^7$  and in Calabi-Yau extensions. In particular, we prove that the moduli space of associative deformations of product of a special lagrangian by  $S^1$  in a Calabi-Yau manifold times  $S^1$  is always smooth and of dimension  $b^1(L) + 1$ .

MSC 2000: 53C38 (35J55, 53C21, 58J32).

KEYWORDS:  $G_2$  holonomy; calibrated submanifolds; elliptic boundary problems on manifolds ; Bochner method

## 1 Introduction

Let  $M$  a 7-dimensional riemannian manifold with holonomy included in  $G_2$ . In this case  $M$  supports a non degenerate harmonic 3-form  $\phi$  (and such that  $\nabla\phi = 0$ ). Moreover,  $M$  benefits a vector product  $\times$  defined by

$$\langle u \times v, w \rangle = \phi(u, v, w),$$

even if to be spin is enough for the existence of this product. Now, recall that a 3-dimensional submanifold  $Y$  is said *associative* if its tangent bundle is stable under the vector product. In other terms,  $\phi$  restricted to  $Y$  is a volume form. Likely, a 4-dimensional submanifold  $X$  is *coassociative* if all the fibers of its normal bundle are associative, or equivalently,  $\phi|_{TX}$  vanishes. We refer to the abundant literature on this subject, see [6] or [5] for a summary with our notations.

**The closed case.** It is known from [9] that the deformation of an associative submanifold  $Y$  without boundary is an elliptic problem, and hence is of vanishing index. In general, the situation is obstructed. For instance, consider the flat torus  $\mathbb{T}^3 \times \{pt\}$  in the flat space  $\mathbb{T}^7 = \mathbb{T}^3 \times \mathbb{T}^4$ . This is an associative submanifold, and its moduli space  $\mathcal{M}_{\mathbb{T}^3 \times \{pt\}}$  of associative deformations contains at least the 4-dimensional  $\mathbb{T}^4$ .

A natural question is to find conditions which force the moduli space to be smooth, or in other terms, which force the cokernel of the problem to vanish. For the closed case, Abkolut and Salur [2] allow a certain freedom for the connection on the normal bundle, the definition of associativity and genericity. But examples are often non generic, and we would like to get a condition that is not a perturbative one. For holomorphic curves in dimension 4, there are topological conditions on the degree of the normal bundle which imply smoothness of the moduli space of complex deformations, see for example [7]. The main point for this is that holomorphic curves intersect positively. In our case, there is no such phenomenon.

In [9], page 30, McLean gives an example of an isolated associative submanifold. Since this was the start point of our work, we recall it. In [4], Bryant and Salamon constructed on the spin bundle of the round 3-sphere  $S^3 \times \mathbb{R}^4$  a metric of holonomy  $G_2$ . In this case, the basis  $S^3 \times \{0\}$  is associative, the normal bundle is just the spin bundle of  $S^3$ , and the operator related to the associative deformations of  $S^3$  is just the Dirac operator. By the famous theorem of Lichnerowicz [8], there are no harmonic spinors on  $S^3$  for metric reasons (precisely, the riemaniann scalare curvature is non negative), so the sphere is isolated as an associative submanifold.

In this paper, we follow the way showed by McLean and the theorem of Lichnerowicz. That means that we use the Bochner method to get a geometric condition which allows the smoothness of  $\mathcal{M}_Y$ . For this, we define the following geometric object :

**Definition 1.1** Choose  $\{e_1, e_2, e_3 = e_1 \times e_2\}$  a local orthonormal frame field of  $TY$ . We define the 0-order operator

$$\begin{aligned} \mathcal{R}_\nu : \Gamma(Y, \nu) &\longrightarrow \Gamma(Y, \nu) \\ \psi &\longmapsto -\frac{1}{2} \sum_{i,j} (e_i \times e_j) \times R^\perp(e_i, e_j)\psi, \end{aligned}$$

where  $R^\perp$  is the curvature of the normal bundle  $\nu$  over  $Y$ , and  $\psi \in \Gamma(Y, \nu)$ .

*Fact.* The definition is independant of the choosen oriented orthonormal frame.

Now we can express the first theorem à la Lichnerowicz :

**Theorem 1.1** Let  $Y$  an associative submanifold of a  $G_2$ -manifold  $M$ . If the spectrum of  $\mathcal{R}_\nu$  is non negative, then  $Y$  is isolated.

**The case with boundary.** In [5], the authors proved that the deformation of an associative submanifold with boundary in a coassociative submanifold is an elliptic problem of finite index and given by the index of a certain Cauchy-Riemann operator. We sum up in the following the principal results of the paper :

**Theorem 1.2 ([5])** Let  $\nu_X$  the normal complementary of  $T\partial Y$  in  $TX|_{\partial Y}$ , and  $n$  the inner unit vector normal to  $\partial Y$  in  $Y$ . Then the bundle  $\nu_X$  is a subbundle of  $\nu|_{\partial Y}$  and is stable under  $n \times$ , as long as the orthogonal complement  $\mu_X$  of  $\nu_X$  in  $\nu$ . Viewing  $T\partial Y$ ,  $\nu_X$  and  $\mu_X$  as  $n \times$ -complex line bundles, we have  $\mu_X^* \cong \nu_X \otimes_{\mathbb{C}} T\partial Y$ . Besides, the problem of the associative deformations of  $Y$  is elliptic and of index

$$\text{index}(Y, X) = \text{index} \bar{\partial}_{\nu_X} = c_1(\nu_X) + 1 - g.$$

In this context, we introduce a new geometric object that is related to the geometry on the boundary :

**Proposition 1.3** *Let  $n$  the inner normal vector field on  $\partial Y$ , and  $\{v, w = n \times v\}$  a local orthonormal frame for  $T\partial Y$ . Let  $L$  a real plane subbundle of  $\nu$  invariant by the action of  $n \times$ . We define*

$$\begin{aligned} \mathcal{D}_L : \Gamma(\partial Y, L) &\longrightarrow \Gamma(\partial Y, L) \\ \phi &\longmapsto \pi_L(v \times \nabla_w^\perp \phi - w \times \nabla_v^\perp \phi), \end{aligned}$$

where  $\pi_L$  is the orthogonal projection on  $L$  and  $\nabla^\perp$  the normal connection on  $\nu$  induced by the Levi-Civita connection  $\nabla$  on  $M$ . Then  $\mathcal{D}_L$  is independant of the choosen oriented frame, of order 0 and symmetric.

Now, we can express our main theorem :

**Theorem 1.4** *Let  $Y$  an associative submanifold of a  $G_2$ -manifold  $M$  with boundary in a coassociative  $X$ . If  $\mathcal{D}_{\mu_X}$  and  $\mathcal{R}_\nu$  are non negative, the moduli space  $\mathcal{M}_{Y,X}$  is locally smooth and of dimension given by the virtual one.*

When  $M = \mathbb{R}^7$ , we get the following very explicit example considered in [5]. Take a ball  $Y$  in  $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^7$ , with real analytic boundary, and choose  $e$  any constant vector field in  $\nu = Y \times \{0\} \oplus \mathbb{R}^4$ . By [6], there is a unique local coassociative  $X_e$  containing  $\partial Y \times \mathbb{R}e$ , such that

$$TX|_{\partial Y} = T\partial Y \oplus \nu_X = T\partial Y \oplus \text{Vect}(e, n \times e).$$

Of course, the translations in the  $e$ -direction gives associative deformations of  $Y$  with boundary in  $X_e$ . The next corollary shows that under a simple metric condition, this is the only way to deform  $Y$  :

**Corollary 1.5** *If the  $\partial Y$  is strictly convex in  $\mathbb{R}^3$ , then  $\mathcal{M}_{Y,X_e} = \mathbb{R}$ .*

**The Calabi-Yau extension.** Let  $(N, J, \Omega, \omega)$  a Calabi-Yau 6-dimensional manifold (we allow holonomies in  $SU(3)$ ), where  $J$  is an integrable complex structure,  $\Omega$  a non vanishing holomorphic 3-form and  $\omega$  a Kähler form. Then  $M = N \times S^1$  (or  $N \times \mathbb{R}$ ) is a manifold with holonomy in  $G_2$ , with calibration the 3-form given by

$$\phi = \omega \wedge dt + \text{Re} \Omega.$$

Take a special lagrangian 3-dimensional submanifold  $L$  in  $N$ . We know from [9] that  $\mathcal{M}_L$  the moduli space of special lagrangian deformations of  $L$  is smooth and of dimension  $b^1(L)$ . Now every product  $Y = L \times \{pt\}$  of a special lagrangian and a point is an associative submanifold of  $M$ . Since the translation along  $S^1$  preserves the associativity of  $Y$ , we hence have  $\mathcal{M}_L \times S^1 \subset \mathcal{M}_Y$ . In the last part of this paper, we prove that in fact, there is equality, without any condition :

**Theorem 1.6** *The moduli space  $\mathcal{M}_{L \times \{pt\}}$  of associative deformations of  $L \times \{pt\}$  is always smooth, and can be identified with the product  $\mathcal{M}_L \times S^1$ .*

As an immediate example, we get that

$$\mathcal{M}_{\mathbb{T}^3 \times \{pt\}} = \mathbb{T}^4.$$

When  $Y$  has boundary, the metric conditions reappear :

**Corollary 1.7** *Let  $L$  a special lagrangian submanifold in a 6-dimensional Calabi-Yau  $N$ , such that  $L$  has boundary in a complex surface  $\Sigma$ . Let  $Y = L \times \{t_0\}$  in  $N \times S^1$  and  $X = \Sigma \times \{t_0\}$ . If the Ricci curvature of  $L$  is non negative, and the boundary of  $L$  has non negative mean curvature in  $L$ , then  $\mathcal{M}_{Y,X}$  is locally smooth and has dimension  $g$ , where  $g$  is the genus of  $\partial L$ .*

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## 2 Closed associative submanifold

### 2.1 A Weitzenbck formula

The strategy followed for rigidity is the Bochner technique. For this, we have to get some Weitzenbck formula, and integrate. Firstly, recall the version of McLean's theorem proposed by Akbulut and Salur :

**Theorem 2.1** ([9],[1]) *Let  $Y$  an associative submanifold of a riemannian manifold  $M$  with  $G2$ -holonomy. Then the tangent space of its associative deformations is the kernel of the operator*

$$\begin{aligned} D : \Gamma(Y, \nu) &\longrightarrow \Gamma(Y, \nu), \\ \psi &\longmapsto \sum_{i=1}^3 e_i \times \nabla_{e_i}^\perp \psi, \end{aligned}$$

where  $\nabla^\perp$  is the connection on  $\nu$  induced by the Levi-Civita connection  $\nabla$ .

For the next proposition, we introduce the normal equivalent of the invariant second derivative. More precisely, let

$$\nabla_{v,w}^{\perp 2} = \nabla_v^\perp \nabla_w^\perp - \nabla_{\nabla_v^\perp w}^\perp,$$

which is tensorial in the fields  $v$  and  $w$  and acting on  $\Gamma(Y, \nu)$ . Moreover, define the equivalent of the connection laplacian :

$$\nabla^{\perp*} \nabla^\perp = -\text{trace} (\nabla^{\perp 2}) = -\sum_i \nabla_{e_i, e_i}^{\perp 2}.$$

**Proposition 2.2** *For the Dirac-like  $D$  we have*

$$D^2 = \nabla^{\perp*} \nabla^\perp + \mathcal{R}_\nu.$$

Before computing  $D^2$ , we need the following trivial lemma

**Lemma 2.3** *Let  $\nabla$  the Levi-Civita connection on  $M$  and  $R$  its curvature tensor. For any vector fields  $w, z, u$  and  $v$  on  $M$ , we have*

$$\begin{aligned} \nabla(u \times v) &= \nabla u \times v + u \times \nabla v \text{ and} \\ R(w, z)(u \times v) &= R(w, z)u \times v + u \times R(w, z)v. \end{aligned}$$

If  $u \in \Gamma(Y, TY)$ ,  $v \in \Gamma(Y, TY)$  and  $\eta \in \Gamma(Y, \nu)$ , then

$$\begin{aligned} \nabla^\top(u \times v) &= \nabla^\top u \times v + u \times \nabla^\top v \text{ and} \\ \nabla^\perp(u \times \eta) &= \nabla^\top u \times v + u \times \nabla^\perp v, \end{aligned}$$

where  $\nabla^\top = \nabla - \nabla^\perp$  is the orthogonal projection of  $\nabla$  on  $TY$ .

**Proof.** Let  $x_1, \dots, x_7$  normal coordinates on  $M$  near  $x$ , and  $e_i = \frac{\partial}{\partial x_i}$  their derivatives orthonormal at  $x$ . We have

$$u \times v = \sum_i \langle u \times v, e_i \rangle e_i = \sum_i \phi(u, v, e_i) e_i,$$

so that

$$\begin{aligned} \nabla(u \times v) &= \sum_i (\nabla \phi(u, v, e_i) + \phi(\nabla u, v, e_i) + \phi(u, \nabla v, e_i) + \phi(\nabla u, v, \nabla e_i)) e_i \\ &= \sum_i (\phi(\nabla u, v, e_i) + \phi(u, \nabla v, e_i)) e_i = \nabla u \times v + u \times \nabla v, \end{aligned}$$

because  $\nabla \phi = 0$  and  $\nabla e_i = 0$ . Now if  $u$  and  $v$  are in  $TY$ , then we get the result after remarking that  $(\nabla u \times v)^\top = \nabla^\top u \times v$ , because  $TY$  is invariant under  $\times$ . The last relation is implied by  $TY \times \nu \subset \nu$  and  $\nu \times \nu \subset TY$ . The curvature relation is easily derived from the definition  $R(w, z) = \nabla_w \nabla_z - \nabla_z \nabla_w - \nabla_{[w, z]}$  and the derivation of the vector product. ■

**Proof of proposition 2.2.** We compute  $D^2$  at a point  $x \in Y$ . For this, we choose normal coordinates on  $Y$  and  $e_i \in \Gamma(Y, TY)$  their associated derivatives, orthonormal at  $x$ . To be explicit,  $\nabla^\top e_i = 0$  at  $x$ . Let  $\psi \in \Gamma(Y, \nu)$ .

$$\begin{aligned} D^2 \psi &= \sum_{i,j} e_i \times \nabla_i^\perp (e_j \times \nabla_j^\perp \psi) \\ &= \sum_{i,j} e_i \times (e_j \times \nabla_i^\perp \nabla_j^\perp \psi) + \sum_{i,j} e_i \times (\nabla_i^\top e_j \times \nabla_j^\perp \psi) \\ &= - \sum_i \nabla_i^\perp \nabla_i^\perp \psi - \sum_{i \neq j} (e_i \times e_j) \times \nabla_i^\perp \nabla_j^\perp \psi \\ &= \nabla^{\perp*} \nabla^\perp \psi - \sum_{i < j} (e_i \times e_j) \times (\nabla_i^\perp \nabla_j^\perp - \nabla_j^\perp \nabla_i^\perp) \psi \\ &= \nabla^{\perp*} \nabla^\perp \psi - \sum_{i < j} (e_i \times e_j) \times R^\perp(e_i, e_j) \psi \\ &= \nabla^{\perp*} \nabla^\perp \psi - \frac{1}{2} \sum_{i,j} (e_i \times e_j) \times R^\perp(e_i, e_j) \psi = \nabla^{\perp*} \nabla^\perp \psi + \mathcal{R}_\nu \psi, \end{aligned}$$

since  $(e_i \times e_j) \times R^\perp(e_i, e_j)$  is symmetric in  $i, j$ . It is easy to check that  $\mathcal{R}_\nu$  is independant of the choosen orthonormal frame. ■

We can give a separation of the uneasy operator  $\mathcal{R}_\nu$ , in terms of the ambient curvature, and the Weingarten operators :

**Proposition 2.4** Let  $\mathcal{R}$  the 0-order operator :  $\mathcal{R} : \Gamma(Y, \nu) \rightarrow \Gamma(Y, \nu)$  defined by

$$\mathcal{R}(\psi) = -\frac{1}{2} \pi_\nu \sum_{i,j} (e_i \times e_j) \times R(e_i, e_j) \psi,$$

where  $R$  is the curvature tensor of the ambient space  $M$  and  $\pi_\nu$  the orthogonal projection on  $\nu$ . Then  $\mathcal{R}$  is a symmetric operator, and there is a positive symmetric 0-order operator  $\mathcal{A} : \Gamma(Y, \nu) \rightarrow \Gamma(Y, \nu)$  which vanishes if  $Y$  is totally geodesic in  $M$ , such that  $\mathcal{R}_\nu = \mathcal{R} - \mathcal{A}$ . As a consequence, the operator  $\mathcal{R}_\nu$  is symmetric.

The operator seems as complicated as  $\mathcal{R}_\nu$ , but it vanishes for flat spaces, like  $\mathbb{R}^7$  or  $\mathbb{T}^7$ .

**Proof.** We refer to the lemma 2.5 below for the proof of the symmetry of  $\mathcal{R}$ . The main tool for the second assertion of our proposition is the Ricci equation. Let  $u, v$  in  $\Gamma(Y, TY)$  and  $\phi, \psi$  in  $\Gamma(Y, \nu)$ .

$$\langle R^\perp(u, v)\psi, \phi \rangle = \langle R(u, v)\psi, \phi \rangle + \langle (A_\psi A_\phi - A_\phi A_\psi)u, v \rangle,$$

where  $A_\phi : TY \rightarrow TY$  is the Weingarten endomorphism in the normal direction  $\phi$  defined by

$$A_\phi(u) = -\nabla_u^\top \phi.$$

Choose  $\eta_1, \dots, \eta_4$  an orthonormal basis of  $\nu$  at the point  $x$ , we get

$$\begin{aligned} \mathcal{R}_\nu(\psi) &= -\frac{1}{2} \sum_{i,j,k} \langle (e_i \times e_j) \times R^\perp(e_i, e_j)\psi, \eta_k \rangle \eta_k \\ &= \frac{1}{2} \sum_{i,j,k} \langle R^\perp(e_i, e_j)\psi, (e_i \times e_j) \times \eta_k \rangle \eta_k \\ &= \frac{1}{2} \pi_\nu \sum_{i,j,k} (e_i \times e_j) \times R(e_i, e_j)\psi \\ &\quad + \frac{1}{2} \sum_{i,j,k} \langle (A_\psi A_{(e_i \times e_j) \times \eta_k} - A_{(e_i \times e_j) \times \eta_k} A_\psi) e_i, e_j \rangle \eta_k. \end{aligned}$$

The Weingarten endomorphisms are symmetric, so that the last term is

$$\frac{1}{2} \sum_{i,j,k} \langle A_{(e_i \times e_j) \times \eta_k} e_i, A_\psi e_j \rangle \eta_k - \frac{1}{2} \sum_{i,j,k} \langle A_\psi e_i, A_{(e_i \times e_j) \times \eta_k} e_j \rangle \eta_k.$$

It is easy to see that the second sum is the opposite of the first one. We compute

$$A_{(e_i \times e_j) \times \eta_k} e_i = -(\nabla_i^\perp e_i \times e_j) \times \eta_k - (e_i \times \nabla_i^\perp e_j) \times \eta_k + (e_i \times e_j) \times A_{\eta_k} e_i.$$

But we know that an associative submanifold is minimal, so that

$$\sum_i \nabla_i^\perp e_i = 0.$$

Moreover, deriving the relation  $e_3 = \pm e_1 \times e_2$ , one easily check that

$$\sum_i e_i \times \nabla_j^\perp e_i = 0.$$

Summing, the only resting term is

$$\sum_{i,j,k} \langle (e_i \times e_j) \times A_{\eta_k} e_i, A_\psi e_j \rangle \eta_k.$$

We now use the classical formula for vectors  $u, v$  and  $w$  in  $TY$  :

$$(v \times w) \times u = \langle u, v \rangle w - \langle u, w \rangle v,$$

hence

$$(e_i \times e_j) \times A_{\eta_k} e_i = \langle A_{\eta_k} e_i, e_i \rangle e_j - \langle A_{\eta_k} e_i, e_j \rangle e_i.$$

One more simplification comes from  $\sum_i \langle A_{\eta_k} e_i, e_i \rangle = 0$  for all  $k$  because since  $Y$  is minimal, so our sum is now equal to

$$-\sum_{i,j,k} \langle A_{\eta_k} e_i, e_j \rangle \langle e_i, A_{\psi} e_j \rangle \eta_k = -\mathcal{A}\psi.$$

This is clear that this operator is of vanishing order, and is symmetric, because

$$\begin{aligned} \langle \mathcal{A}\psi, \phi \rangle &= \sum_{i,j,k} \langle A_{\eta_k} e_i, e_j \rangle \langle e_i, A_{\psi} e_j \rangle \phi_k \\ &= \sum_{i,j,k} \langle A_{\phi} e_i, e_j \rangle \langle e_i, A_{\psi} e_j \rangle. \end{aligned}$$

At last,  $\mathcal{A}$  is positive, since

$$\langle \mathcal{A}\psi, \psi \rangle = \sum_{i,j,k} \langle A_{\psi} e_i, e_j \rangle^2.$$

Remark that it vanishes if and only if  $A_{\psi} = 0$ , which is the case in particular if  $Y$  is totally geodesic.  $\blacksquare$

We end this section with the following lemma :

**Lemma 2.5** *The operator  $\mathcal{R}$  is symmetric.*

**Proof.** Let  $\psi$  and  $\phi$  in  $\Gamma(Y, \nu)$ . Then

$$\begin{aligned} -2 \langle \mathcal{R}\psi, \phi \rangle &= \sum_{i,j} \langle (e_i \times e_j) \times R(e_i, e_j)\psi, \phi \rangle = -\sum_{i,j} \langle R(e_i, e_j)\psi, (e_i \times e_j) \times \phi \rangle \\ &= \sum_{i,j} \langle R(e_i, e_j)((e_i \times e_j) \times \phi), \psi \rangle \end{aligned}$$

because in general  $\langle R(u, v)x, y \rangle = -\langle R(u, v)y, x \rangle$ . Using lemma 2.3, this is equal to

$$\sum_{i,j} \langle (e_i \times e_j) \times R(e_i, e_j)\phi + (R(e_i, e_j)(e_i \times e_j)) \times \phi, \psi \rangle.$$

Now the first part gives  $-2 \langle \mathcal{R}\phi, \psi \rangle$  and  $\sum_{i,j} R(e_i, e_j)(e_i \times e_j)$  is equal to

$$R(e_1, e_2)e_3 + R(e_2, e_3)e_1 + R(e_3, e_1)e_2 = 0$$

because of Bianchi's first identity.  $\blacksquare$

## 2.2 A vanishing theorem

**Proof of theorem 1.1.** Suppose that we are given a fixed closed associative submanifold  $Y$ . The virtual dimension of its moduli space of deformation is vanishing. Consider a section  $\psi \in \Gamma(Y, \nu)$ . By classical calculations, using normal coordinates, we have

$$\begin{aligned} \frac{1}{2} \Delta \langle \psi, \psi \rangle &= \sum_i \langle \nabla_i^\perp \psi, \nabla_i^\perp \psi \rangle + \langle \psi, \nabla_i^\perp \nabla_i^\perp \psi \rangle \\ &= |\nabla^\perp \psi|^2 - \langle D^2 \psi, \psi \rangle + \langle \mathcal{R}_\nu \psi, \psi \rangle. \end{aligned}$$

Since the laplacian is equal to  $\text{div}(\vec{\nabla})$ , its integral over the closed  $Y$  vanishes. We get :

$$0 = \int_Y |\nabla^\perp \psi|^2 - \langle D^2 \psi, \psi \rangle + \langle \mathcal{R}_\nu \psi, \psi \rangle dy. \quad (1)$$

If we have a section  $\psi \in \text{coker } D = \ker D$  then  $D\psi = 0$ . Under the hypothesis that  $\mathcal{R}^\nu$  is non negative, the last equation implies  $\psi = 0$ . By the theorem of implicit functions,  $\mathcal{M}_Y$  is a smooth manifold of vanishing dimension, and  $Y$  is isolated.  $\blacksquare$



### 3 Associative submanifolds with boundary

In this section we extend our result for rigidity in the case of associative submanifolds with boundary in a coassociative submanifold. In this case the index is not zero, so rigidity transforms into smoothness of the moduli space. Recall the notations of proposition 1.3 : let  $n$  the inner normal vector to  $\partial Y$  in  $Y$ , and  $L$  a  $n \times$ -complex line bundle over  $\partial Y$ . Define

$$\ker(D, L) = \{\psi \in \Gamma(Y, \nu), D\psi = 0, \psi|_{\partial Y} \in L\}.$$

In [5] (see [3] too), the authors noted that :  $\text{coker}(D, \nu_X) = \ker(D, \mu_X)$ .

**Proof of theorem 1.4.** By the theorem of implicit functions, we have to prove that  $\text{coker}(D, \nu_X) = \ker(D, \mu_X) = \{0\}$ . So let  $\psi \in \ker(D, \mu_X)$ . The boundary changes the integration (1), because the divergence has to be considered :

$$\int_Y |\nabla^\perp \psi|^2 + \langle \mathcal{R}_\nu \psi, \psi \rangle dy = \frac{1}{2} \int_Y \text{div } \vec{\nabla} |\psi|^2 dy. \quad (2)$$

By Stokes, the last is equal to

$$\frac{1}{2} \int_{\partial Y} \langle \vec{\nabla} |\psi|^2, -n \rangle d\sigma = -\frac{1}{2} \int_{\partial Y} d|\psi|^2(n) d\sigma = - \int_{\partial Y} \langle \nabla_n^\perp \psi, \psi \rangle d\sigma,$$

where  $d\sigma$  is the volume induced by the restriction of  $g$  on the boundary. So using the fact that  $D\psi = 0$ , this is equal to

$$\int_{\partial Y} \langle w \times \nabla_v^\perp \psi - v \times \nabla_w^\perp \psi, \psi \rangle d\sigma = - \int_{\partial Y} \langle \mathcal{D}_{\mu_X} \psi, \psi \rangle d\sigma.$$

Summing up, we get the equation

$$\int_Y |\nabla^\perp \psi|^2 dy + \int_Y \langle \mathcal{R}_\nu \psi, \psi \rangle dy + \int_{\partial Y} \langle \mathcal{D}_{\mu_X} \psi, \psi \rangle d\sigma = 0. \quad (3)$$

Now we can prove the theorem 1.4. We see that if  $\mathcal{D}_{\mu_X}$  and  $\mathcal{R}$  are negatively definite, then  $\psi$  vanishes. This means that our deformation problem has no cokernel, and by the theorem of implicit functions, the moduli space is locally smooth.  $\blacksquare$

#### 3.1 Some properties of the operator $\mathcal{D}_L$

We sum up the main results about  $\mathcal{D}_L$  in the following

**Proposition 3.1** *Let  $Y$  an associative submanifold with boundary in a coassociative submanifold  $X$ ,  $L$  a subbundle of  $\nu$  over  $\partial Y$ , and  $\mathcal{D}_L$  as defined in the introduction. Then  $\mathcal{D}_L$  is of order 0, symmetric, and its trace is  $2H$ , where  $H$  is the mean curvature of  $\partial Y$  in  $Y$  with respect to the outside normal vector  $-n$ .*

**Proof.** Let  $L$  is a subbundle of  $\nu$  invariant under the action of  $n \times$ . It is straightforward to check that  $\mathcal{D}$  does not depend of the orthonormal frame  $\{v, w = n \times v\}$ . For every  $\psi \in \Gamma(\partial Y, L)$  and  $f$  a function,

$$\begin{aligned} \mathcal{D}_L(f\psi) &= \pi_L(v \times \nabla_w(f\psi) - w \times \nabla_v(f\psi)) \\ &= f\mathcal{D}_L\psi + \pi_L(d_w f v \times \psi - d_v f w \times \psi) = f\mathcal{D}_L\psi \end{aligned}$$

because  $w \times L$  as long as  $v \times L$  are orthogonal to  $L$ . Now, decompose

$$\nabla^\top = \nabla^{\top\partial} + \nabla^{\perp\partial}$$

into its two projections along  $T\partial Y$  and along the normal (in  $TY$ )  $n$ -direction. For the computations, choose  $v$  and  $w = n \times v$  the two orthogonal characteristic directions on  $T\partial Y$ , i.e  $\nabla_v^\top n = -k_v v$  and  $\nabla_w^\top n = -k_w w$ , where  $k_v$  and  $k_w$  are the two principal curvatures. We have  $\nabla_v^{\perp\partial} = k_v n$  and  $\langle \nabla^{\perp\partial} v, n \rangle = 0$ , and the same, *mutatis mutandis*, for  $w$ . Then, for  $\psi$  and  $\phi \in \Gamma(\partial Y, L)$ , using the fact that  $T\partial Y \times L$  is orthogonal to  $L$ ,

$$\begin{aligned} \langle \mathcal{D}_L \psi, \phi \rangle &= \langle \nabla_w^\perp(v \times \psi) - (\nabla_w^{\perp\partial} v) \times \psi - \nabla_v^\perp(w \times \psi) + (\nabla_v^{\perp\partial} w) \times \psi, \phi \rangle \\ &= \langle \nabla_w^\perp(v \times \psi) - \nabla_v^\perp(w \times \psi), \phi \rangle \\ &= -\langle v \times \psi, \nabla_w^\perp \phi \rangle + \langle w \times \psi, \nabla_v^\perp \phi \rangle \\ &= \langle \psi, v \times \nabla_w^\perp \phi - w \times \nabla_v^\perp \phi \rangle = \langle \psi, \mathcal{D}_L \phi \rangle. \end{aligned}$$

To prove that the trace of  $\mathcal{D}_L$  is  $2H$ , let  $e \in L$  a local unit section of  $L$ . We have  $n \times e \in L$  too, and

$$\begin{aligned} \langle \mathcal{D}_L(n \times e), n \times e \rangle &= \langle v \times ((\nabla_w^{\top\partial} n) \times e) + v \times (n \times \nabla_w^\perp e), n \times e \rangle \\ &\quad - \langle w \times (\nabla_v^{\top\partial} n) \times e - w \times (n \times \nabla_v^\perp e), n \times e \rangle \\ &= \langle v \times (-k_w w \times e) - w \times (-k_v v \times e), n \times e \rangle \\ &\quad + \langle v \times (n \times \nabla_w^\perp e) - w \times (n \times \nabla_v^\perp e), n \times e \rangle \\ &= k_w + k_v - \langle n \times (w \times (n \times \nabla_v^\perp e) - v \times (n \times \nabla_w^\perp e)), e \rangle \\ &= 2H - \langle \mathcal{D}_L e, e \rangle. \end{aligned}$$

This shows that trace  $\mathcal{D}_L = 2H$ . ■

### 3.2 Flatland

In flat spaces,  $\mathcal{R}$  vanishes, and so by proposition 2.4,  $\mathcal{R}_\nu = -\mathcal{A}$ . Hence a priori the theorem 1.4 does'nt apply. Nevertheless, we have the

**Corollary 3.2** *Let  $Y$  a totally geodesic associative submanifold in a flat  $M$ , with boundary in a coassociative  $X$ . If  $\mathcal{D}_{\mu_X}$  positive, then  $\mathcal{M}_{Y,X}$  is locally smooth and of expected dimension.*

**Proof.** By proposition 2.4,  $\mathcal{R}_\nu = 0$ . The formula (3) and the hypothesis show that  $\nabla^\perp \psi = 0$  and  $\psi|_{\partial Y} = 0$ . Using  $d|\psi|^2 = 2 \langle \nabla^\perp \psi, \psi \rangle = 0$ , we get that  $\psi = 0$ , and  $\text{coker}(Y, \nu_X) = \ker(Y, \mu_X) = 0$ . ■

**Proof of corollary 1.5.** Let  $Y$  in  $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^7$ , and  $e \in \{0\} \times \mathbb{R}^4$ . From [6] the boundary of  $Y$  lies in a local coassociatif submanifold  $X_e$  of  $\mathbb{R}^7$ , which contains  $\partial Y \times \mathbb{R}e$  and whose tangent space over  $\partial Y$  is  $T\partial Y \oplus \mathbb{R}e \oplus \mathbb{R}n \times e$ . Geometrically, we can immediatly see that  $Y$  has a direction of associative deformation along the fixed direction  $e$ , hence the dimension of the kernel of our problem is bigger than 1. On the other hand, the index is  $c_1(\nu_X) + 1 - g = 1$ . We want to show that  $\mathcal{D}_{\mu_X}$  is positive. To see that, we choose orthogonal characteristic directions  $v$  and  $w = n \times v$  in  $T\partial Y$  as before. From lemma 1.2, we know that  $v \times e$  is a non vanishing section of  $\mu_X$ . We compute :

$$\begin{aligned} \mathcal{D}_{\mu_X}(v \times e) &= v \times (\nabla_w^{\perp\partial} v \times e) - w \times (\nabla_v^{\perp\partial} v \times e) \\ &= -k_v w \times (n \times e) = k_v v \times e. \end{aligned}$$

This shows that  $k_v$  is an eigenvalue of  $\mathcal{D}_{\mu_X}$ , and since we know that its trace is  $2H$ , we get that the other eigenvalue is  $k_w$ . Those eigenvalues are positive if the boundary of  $Y$  is strictly convex. By the last corollary, we get the corollary.  $\blacksquare$

*Remark.* In fact, we can give a better statement. Indeed, let  $\psi \in \ker(D, \nu_X)$ , and decompose  $\psi|_{\partial Y}$  as  $\psi = \psi_1 e + \psi_2 n \times e$ . Of course,  $e$  is in the kernel of  $\mathcal{D}_{\nu_X}$ , and hence by proposition 1.3, the second term is an eigenvector of  $\mathcal{D}_{\nu_X}$  for the eigenvalue  $2H$ . So formula (3) gives

$$\int_Y |\nabla^\perp \psi|^2 + \int_{\partial Y} 2H |\psi_2|^2 = 0.$$

If  $H > 0$ , this imply immediatly that  $\psi_2 = 0$ , and  $\psi_1$  is constant, so  $\psi$  is proportional to  $e$ . This proves that  $\dim \ker(D, \nu_X) = 1$  under the weaker condition that  $H > 0$ .

## 4 Extensions from the Calabi-Yau world

**Closed extension.** Let  $(N, J, \Omega, \omega)$  a 6-dimensional manifold with holonomy in  $SU(3)$ . Then  $M = N \times S^1$  (or  $N \times \mathbb{R}$ ) is a manifold with holonomy in  $G_2$ , with the calibration the 3-form given by  $\phi = \omega \wedge dt + \text{Re} \Omega$ . Let  $L$  a special lagrangian 3-dimensional submanifold in  $N$ . Recall that since  $L$  is lagrangian, its normal bundle is simply  $JTL$ . Then  $Y = L \times \{pt\}$  is an associative submanifold of  $N \times S^1$ , and its normal bundle  $\nu$  is isomorphic to  $JTL \times \mathbb{R} \partial_t$ , where  $\partial_t$  is the dual vector field of  $dt$ .

**Proof of theorem 1.1.** We won't use the tricky formula with  $\mathcal{R}_\nu$ . In this situation, we give another formulation for the Dirac-like operator. If  $s = J\sigma \oplus \tau \partial_t$  is a section of  $\nu$ , with  $\sigma \in \Gamma(L, TL)$  and  $\tau \in \Gamma(L, \mathbb{R}) = \Omega^0(L)$ , we call  $\sigma^\vee \in \Omega^1(L, \mathbb{R})$  the 1-form dual to  $\sigma$ , and we use the same symbol for its inverse. Moreover, we use the classical notation  $*$  :  $\Omega^k(L) \rightarrow \Omega^{3-k}(L)$  for the Hodge star. Lastly, we define :

$$\begin{aligned} D^\vee : \Omega^1(L) \times \Omega^0(L) &\longrightarrow \Omega^1(L) \times \Omega^0(L) \\ (\alpha, \tau) &\longmapsto ((-J\pi_{NL}D(J\alpha^\vee))^\vee, \tau), \end{aligned}$$

where  $\pi_{NL}$  is the orthogonal projection  $\nu = NL \oplus \mathbb{R}$  on the first component. This is just a way to use forms on  $L$  instead of its vector fields.

**Proposition 4.1** For every  $(\alpha, \tau) \in \Omega^1(L) \times \Omega^0(L)$ ,

$$\begin{aligned} D^\vee(\alpha, \tau) &= (*d\alpha - d\tau, *d*\alpha) \text{ and} \\ D^{\vee 2}(\alpha, \tau) &= (\Delta\alpha, \Delta\tau), \end{aligned}$$

where  $\Delta = d^*d + dd^*$  (note that it is  $d^*d$  on  $\tau$ ).

Assuming for a while this proposition, we see that for an infinitesimal associative deformation of  $L \times \{pt\}$ , then  $\alpha$  and  $\tau$  are harmonic over the compact  $L$ . In particular,  $\tau$  is constant and  $\alpha$  describes an infinitesimal special lagrangian deformation of  $L$  (see [9]). In other words, the only way to displace  $Y$  is to perturb  $L$  as special Lagrangian in  $N$  and translate it along the  $S^1$ -direction. Lastly,  $\dim \text{coker} D = \dim \ker D = b^1(L) + 1$  and by the constant rank theorem,  $\mathcal{M}_Y$  is smooth and of dimension  $b^1(L) + 1$ .  $\blacksquare$

**Proof of proposition 4.1.** We will use the simple formula  $\nabla^\perp J s = J \nabla^\top s$  for all sections  $s \in \Gamma(L, NL)$ . For  $(s, \tau) \in \Gamma(L, NL) \times \mathbb{R}$ , and  $e_i$  local orthonormal frame on  $L$ ,

$$\begin{aligned}
D(s, \tau) &= \sum_{i,j} \langle e_i \times \nabla_i^\perp s, J e_j \rangle J e_j + \sum_i \langle e_i \times \nabla_i^\perp s, \partial_t \rangle \partial_t + \sum_i \partial_i \tau e_i \times \partial_t \\
&= \sum_{i,j} \phi(e_i, \nabla_i^\perp s, J e_j) J e_j + \sum_i \phi(e_i, \nabla_i^\perp s, \partial_t) \partial_t + \sum_{i,j} \partial_i \tau \langle e_i \times \partial_t, J e_j \rangle J e_j \\
&= \sum_{i,j} \operatorname{Re} \Omega(e_i, \nabla_i^\perp s, J e_j) J e_j + \sum_i \omega(e_i, \nabla_i^\perp s) \partial_t + \sum_{i,j} \partial_i \tau \phi(e_i, \partial_t, J e_j) J e_j \\
&= \sum_{i,j} -\operatorname{Re} \Omega(e_i, J \nabla_i^\top \sigma, J e_j) J e_j + \sum_i \omega(e_i, \nabla_i^\perp s) \partial_t + \sum_{i,j} \partial_i \tau \omega(J e_j, e_i) J e_j \\
&= \sum_{i,j} \operatorname{Re} \Omega(e_i, \nabla_i^\top \sigma, e_j) J e_j + \sum_i \omega(e_i, J \nabla_i^\top \sigma) \partial_t - \sum_{i,j} \partial_i \tau \langle e_j, e_i \rangle J e_j \\
&= \sum_{i,j} \operatorname{Vol}(e_i, \nabla_i^\top \sigma, e_j) J e_j + \sum_i \langle e_i, \nabla_i^\top \sigma \rangle \partial_t - J \sum_i \partial_i \tau e_i
\end{aligned}$$

It is easy to find that this is equivalent to  $D\psi = J(*d\sigma^\vee)^\vee + (*d*\sigma^\vee)\partial_t - J(d\tau)^\vee$ , and so

$$D^\vee(\sigma^\vee, \tau) = (*d\sigma^\vee - d\tau, *d*\sigma^\vee).$$

Now, since  $d^* = (-1)^{3p+1} * d*$  on the  $p$ -forms, one easily checks the formula for  $D^2$ .  $\blacksquare$

**Proof of corollary 1.7.** We prove now the corollary 1.7. We begin by computing the index of the boundary problem. This is very easy, because the direction in  $TS^1$  is in  $\mu_X$ . Hence  $\mu_X$  is trivial, and by theorem 1.2, we have  $\nu_X = T\partial L^*$ . Hence the index is

$$-c_1(T\partial L) + 1 - g = -(2 - g) + 1 - g = g - 1,$$

where  $g$  is the genus of  $\partial Y$ . Moreover, the constant vector  $\frac{\partial}{\partial t}$  over  $\partial Y$  lies clearly in the kernel of  $\mathcal{D}_{\mu_X}$ . By proposition 3.1, the other eigenvalue is  $2H$ . Now let

$$\psi = \tau \frac{\partial}{\partial t} + s$$

belonging to  $\operatorname{coker}(D, \nu_X) = \ker(D, \mu_X)$ , where  $s$  a section of  $NL$  and  $\tau \in \Gamma(L, \mathbb{R})$ . Over  $\partial Y$ ,  $s$  is proportional to  $n \times \frac{\partial}{\partial t}$ , and  $\mathcal{D}_{\mu_X} \psi = 2Hs$ . Let  $\alpha = -Js^\vee$ . By classical results for harmonic 1-forms, we have :

$$-\frac{1}{2} \Delta |\psi|^2 = -\frac{1}{2} \Delta (|\alpha|^2 + |\tau|^2) = |\nabla_L \alpha|^2 + |d\tau|^2 + \frac{1}{2} \operatorname{Ric}(\alpha, \alpha).$$

Integrating on  $L \times \{pt\}$ , we obtain the equivalence of formula (3) :

$$-\int_{\partial Y} 2H|\alpha|^2 d\sigma = \int_Y |\nabla_L \alpha|^2 + |d\tau|^2 + \frac{1}{2} \operatorname{Ric}(\alpha, \alpha) dy.$$

This equation, the positivity of the Ricci curvature and the positivity of  $H$  show that  $\alpha$  vanishes and  $\tau$  is constant. So we see that  $\dim \operatorname{coker}(Y, X) = 1$ , and by the constant rank theorem,  $\mathcal{M}_{Y,X}$  is locally smooth and of dimension  $\dim \ker(Y, X) = g$ .  $\blacksquare$

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