



HAL
open science

A convergent scheme for a non-local coupled system modelling dislocations densities dynamics

Ahmad El Hajj, Nicolas Forcadel

► **To cite this version:**

Ahmad El Hajj, Nicolas Forcadel. A convergent scheme for a non-local coupled system modelling dislocations densities dynamics. *Mathematics of Computation*, 2008, 77, pp.789–812. hal-00415944

HAL Id: hal-00415944

<https://hal.science/hal-00415944>

Submitted on 11 Sep 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A convergent scheme for a non-local coupled system modelling dislocations densities dynamics

A. El Hajj¹ & N. Forcadel¹

¹*CERMICS, Ecole des Ponts, ParisTech 6 et 8 avenue Blaise Pascal, Cité Descartes,
Champs-sur-Marne, 77455 Marne-la-Vallée Cedex 2*

*CERMICS — ENPC
6 et 8 avenue Blaise Pascal
Cité Descartes - Champs sur Marne
77455 Marne la Vallée Cedex 2*

<http://cermics.enpc.fr>

Abstract

In this paper, we study a non-local coupled system that arises in the theory of dislocations densities dynamics. Within the framework of viscosity solutions, we prove a long time existence and uniqueness result for the solution of this model. We also propose a convergent numerical scheme and we prove a Crandall-Lions type error estimate between the continuous solution and the numerical one. As far as we know, this is the first error estimate of Crandall-Lions type for Hamilton-Jacobi systems. We also provide some numerical simulations.

AMS Classification: 35Q72, 49L25, 35F25, 35L40, 65M06, 65M12, 65M15, 74H20, 74H25.

Keywords: Hamilton Jacobi equations, viscosity solutions, dislocations densities dynamics, numerical scheme, error estimate, system.

1 Introduction

1.1 Presentation and physical motivations

A dislocation is a crystal defect which corresponds to a discontinuity in the crystalline structure organisation. This concept has been introduced by Polanyi, Taylor and Orowan in 1934 as the main explanation at the microscopic scale of plastic deformation. A dislocation creates around it a perturbation that can be seen as an elastic field. Under an exterior strain, a dislocation moves according to its Burgers vector which characterize the intensity and the direction of the defect displacement (see Hirth and Lothe [17] for an introduction to dislocations).

Here, we are interested in dislocations densities dynamics. More precisely, we consider edge dislocations, *i.e* the Burgers vectors and dislocations are in the same plane. These dislocations are moving with the Burgers vectors $\pm \vec{b}$ (see figure 1). This model has been introduced by Groma, Balogh as a coupled system, namely a transport problem where the velocity is given by the elasticity equations in the 2-D case (see [16]).

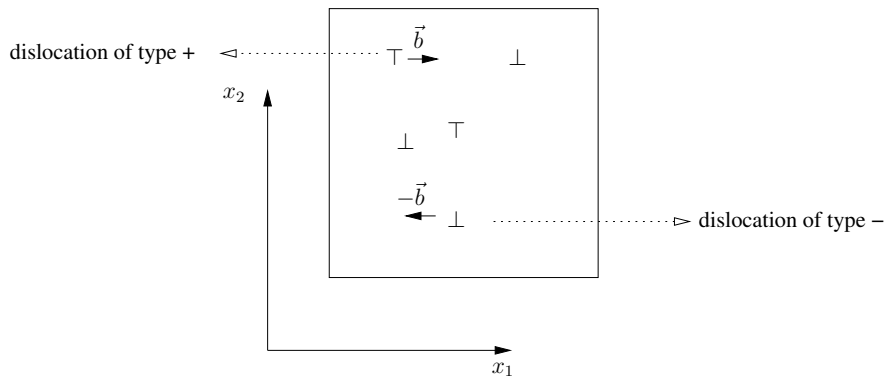


Figure 1: The cross-section of the dislocations lines.

If the 2-D domain is 1-periodic in x_1 and x_2 , and if the dislocations densities depend only on the variable $x = x_1 + x_2$ (where (x_1, x_2) are the coordinates of a point in \mathbb{R}^2), when $\vec{b} = (1, 0)$, the 2-D

model of [16] reduces to the system of coupled 1-D non local Hamilton-Jacobi equations (see Section 3)

$$\begin{cases} (\rho_+)_t = - \left(\rho_+ - \rho_- + \int_0^1 (\rho_+(x,t) - \rho_-(x,t)) dx + L(t) \right) |D\rho_+| & \text{on } \mathbb{R} \times (0, T) \\ (\rho_-)_t = \left(\rho_+ - \rho_- + \int_0^1 (\rho_+(x,t) - \rho_-(x,t)) dx + L(t) \right) |D\rho_-| & \text{on } \mathbb{R} \times (0, T) \end{cases} \quad (1.1)$$

where ρ_+, ρ_- are the unknown scalars such that $(\rho_+ - \rho_-)$ represents the plastic deformation, their space derivatives $D\rho_{\pm} := \frac{\partial \rho_{\pm}}{\partial x}$ are the dislocations densities and $L(t)$ represents the exterior shear stress field. From a physical viewpoint, $D\rho_{\pm} \geq 0$, however, here we do not make this assumption to remain on a more general framework. The initial conditions for the system (1.1) are defined as follows:

$$\rho_{\pm}(x, 0) = \rho_{\pm}^0(x) = P_{\pm}^0(x) + L_0 x \text{ on } \mathbb{R} \quad (1.2)$$

where P_{\pm}^0 are periodic of period 1 and Lipschitz continuous. In particular, $\rho_+^0 - \rho_-^0$ is a 1-periodic function. L_0 is a given constant which is the total densities of type \pm , *i.e.* we suppose that initially, we have the same total density of type $+$ and $-$.

1.2 Main Results

The first goal of our paper is to prove the existence and uniqueness for the solution of the non-local system (1.1)-(1.2). A natural framework for our study is the viscosity solution theory. We refer to Barles [7], Bardi, Capuzzo-dolcetta [6] and Crandall, Ishii, Lions [10] for a good introduction to this theory in the scalar case. We also refer to Ishii, Koike [19] and Ishii [18] for the vectorial case and to Engler, Lenhart [12], Ishii, Koike [20], Lenhart [23], Lenhart, Belbas [24], Lenhart, Yamada [25] and Yamada [28] for some applications.

We have the following existence and uniqueness result for the non local system:

Theorem 1.1 (Existence and uniqueness for the non-local problem) *For all $T > 0$, for all $L_0 \in \mathbb{R}$, suppose that $\rho_{\pm}^0 \in \text{Lip}(\mathbb{R})$ satisfy (1.2) and $L \in W^{1,\infty}(\mathbb{R}^+)$. Then, the system (1.1)-(1.2) admits a unique viscosity solution $\rho = (\rho_+, \rho_-)$. Moreover, this solution is uniformly Lipschitz continuous in space and time.*

Remark 1.2 *If at initial time, we have $D\rho_{\pm}^0 \geq 0$, then this remains true for $t \geq 0$, *i.e.*, $D\rho_{\pm}(x, t) \geq 0$ for all $(x, t) \in \mathbb{R} \times [0, T]$. This allows to treat the physical case where $D\rho_{\pm} \geq 0$.*

The main difficulty comes from the fact that the comparison principle does not hold because of the non-local term. In order to overcome this problem, we classically use a fixed point method by freezing the non-local term. In a first time, we give an existence and uniqueness result for the local problem (this is a simple adaptation of [19]). Then, we use Lipschitz estimates on the solution to prove the short time existence and uniqueness for the non-local system. In the third step, we obtained the result for all time by iterating the process.

Here, we are interested in the dislocations densities dynamics. Some others models have been proposed to describe the dynamics of dislocations lines. We recall some recent results. A non-local Hamilton-Jacobi equation have been proposed by Alvarez, Hoch, Le Bouar and Monneau [5] [4] for modelling dislocation dynamics. They also proved a short time existence and uniqueness result for this model. We also refer to Alvarez, Cardaliaguet, Monneau [1] and Barles, Ley [8] for a long time result under certain monotony assumptions and to Forcadel [13] for a short time result for dislocations dynamics with a mean curvature term.

The second result is a numerical analysis of the non-local system (1.1). We propose a numerical scheme for our non-local system. Then, we give an error estimate between the continuous solution and the numerical one.

We want to approximate the solution of (1.1)-(1.2). Given a mesh size Δx , Δt , we define

$$\Xi = \{i\Delta x, i \in \mathbb{Z}\} \quad \Xi_T = \Xi \times \{0, \dots, (\Delta t)N_T\}$$

where N_T is the integer part of $T/\Delta t$. We refer generically to the lattice by Δ in the sequel. The discrete running point is (x_i, t_n) with $x_i = i(\Delta x)$, $t_n = n(\Delta t)$. We assume that $\Delta x + \Delta t \leq 1$. The approximation of the solution ρ_k at the node (x_i, t_n) is written indifferently as $v_k(x_i, t_n)$ or $v_{k,i}^n$ according to whether we view it as a function defined on the lattice or as a sequence.

Now, we will introduce the numerical scheme. The main difficulty is due to the non-local term, which requires the availability of the solution we are intending to approximate. To solve this problem, we fix the solution $v_i^n = (v_{+,i}^n, v_{-,i}^n)$ at each time step on the interval $[t_n, t_{n+1}]$ and we apply the following monotone scheme,

$$v_i^0 = (v_{+,i}^0, v_{-,i}^0) = \tilde{\rho}^0(x_i) = (\tilde{\rho}_+^0, \tilde{\rho}_-^0), \quad (1.3)$$

where $\tilde{\rho}_\pm^0(x_i)$ is an approximation of $\rho_\pm^0(x_i)$;

$$v_{k,i}^{n+1} = v_{k,i}^n + \Delta t C_k^\Delta[v](x_i, t_n) \begin{cases} E^+ \left(D^+ v_{k,i}^n, D^- v_{k,i}^n \right) & \text{if } C_k^\Delta[v](x_i, t_n) \geq 0 \\ E^- \left(D^+ v_{k,i}^n, D^- v_{k,i}^n \right) & \text{if not} \end{cases} \quad \forall k \in \{+, -\} \quad (1.4)$$

where

$$C_k^\Delta[v](x_i, t_n) = -k (v_{+,i}^n - v_{-,i}^n + a^\Delta[v](t_n))$$

and the non-local term $a^\Delta[v](t_n)$ is given by

$$a^\Delta[v](t_n) = \sum_{i=0}^{N_x-1} \Delta x (v_+(x_i, t_n) - v_-(x_i, t_n)) + L(t_n) \quad (1.5)$$

where N_x is the integer part of $1/\Delta x$. E^\pm are the approximation of the Euclidean norm proposed by Osher and Sethian [26]:

$$E^+(P, Q) = (\max(P, 0)^2 + \min(Q, 0)^2)^{\frac{1}{2}}, \quad (1.6)$$

$$E^-(P, Q) = (\min(P, 0)^2 + \max(Q, 0)^2)^{\frac{1}{2}}$$

and $D^+ v_{k,i}^n$, $D^- v_{k,i}^n$ are the discrete gradient for all $n \in \{0, \dots, N_T\}$, $i \in \mathbb{Z}$ and $k \in \{+, -\}$:

$$D^+ v_{k,i}^n = \frac{v_{k,i+1}^n - v_{k,i}^n}{\Delta x}, \quad (1.7)$$

$$D^- v_{k,i}^n = \frac{v_{k,i}^n - v_{k,i-1}^n}{\Delta x}.$$

Finally, we assume the following uniform CFL condition (see the beginning of Section 5.2 for more details)

$$\Delta t \leq \frac{1}{2L_2} \Delta x \quad (1.8)$$

where

$$L_2 = 2M + 2$$

with $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$.

We then have the following error estimate:

Theorem 1.3 (Discrete-continuous error estimate) *Let $T \geq 0$. Assume that $\Delta x + \Delta t \leq 1$, $L \in W^{1,\infty}(\mathbb{R} \times [0, T])$ and that the CFL condition (1.8) holds.*

Then there exists a constant $K > 0$ depending only on $\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$, $\max_{k \in \{+, -\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ and $\|L\|_{W^{1,\infty}(0,T)}$ such that the error estimate between the solution ρ of the continuous system (1.1)-(1.2) and the discrete solution v of the finite difference scheme (1.3)-(1.4) is given by

$$\max_{k \in \{+, -\}} \sup_{\Xi_T} |\rho_k - v_k| \leq K \left((T + \sqrt{T})(\Delta x + \Delta t)^{1/2} + \max_{k \in \{+, -\}} \sup_{\Xi} |\rho_k^0 - v_k^0| \right)$$

provided $K \left((T + \sqrt{T})(\Delta x + \Delta t)^{1/2} + \max_{k \in \{+, -\}} \sup_{\Xi} (\rho_k^0 - v_k^0) \right) \leq 1$.

Remark 1.4 *In the condition $K \left((T + \sqrt{T})(\Delta x + \Delta t)^{1/2} + \max_{k \in \{+, -\}} \sup_{\Xi} (\rho_k^0 - v_k^0) \right) \leq 1$, we can replace the right hand side by any positive constant.*

In fact in the proof of this theorem, we mimic the continuous problem by considering the approximate solution of (1.1) as a fixed point of a local system. We are inspired by [3] to prove a Crandall-Lions rate of convergence [11], between the continuous solution of (1.1) and the numerical one. As far as we know, this is the first error estimate of Crandall-Lions type for Hamilton-Jacobi systems. We also refer to Jakobsen, Karlsen [21] and Jakobsen, Karlsen, Risebro [22] where they proved an error estimate for a weakly coupled system of the form

$$(u_i)_t + H_i(t, x, u_i, Du_i) = G_i(t, x, u) \quad \text{in } \mathbb{R}^N \times (0, T) \quad (1.9)$$

for $i = 1, \dots, M$. Their error estimate is in $O(\Delta t)$ for a semi-discrete splitting algorithm that they propose to approach the solution of (1.9). However, we obtain an error estimate in $O(\sqrt{\Delta t + \Delta x})$ because we also discretize in space.

In the dynamics of dislocations lines case, the model have also been numerically studied by Alvarez, Carlini, Monneau and Rouy [2, 3]. In their paper, they proposed a numerical scheme for the non-local Hamilton-Jacobi equation and they proved a Crandall-Lions type rate of convergence. Let us now explain how the paper is organised. In Section 2, we fix some notations. We present the formal derivation of the model in Section 3. Then, in Section 4, we study the continuous problem. First in Subsection 4.1, we give an existence and uniqueness result for a local system. Then, in Subsection 4.2, we prove Theorem 1.1 by using a fixed point method. In Section 5, we prove a Crandall-Lions type error estimate for the local problem and then we prove Theorem 1.3 on the non-local problem. Some numerical examples are displayed in Section 6 where we show some tests illustrating our error estimate and then an evolution approximation of dislocation densities.

2 Notation

For simplicity of presentation, we fix some notations:

1. *Order relation:* for $r = (r_1, r_2)$, $s = (s_1, s_2) \in \mathbb{R}^2$, we say that $r \leq s$ if $r_k \leq s_k$ for $k \in \{1, 2\}$.
2. *Addition vector-scalar:* for $r = (r_1, r_2) \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$, we denote by $r + \lambda$ the vector $(r_1 + \lambda, r_2 + \lambda)$.
3. *P -periodic plus L_0 -linear function:* we say that ρ is P -periodic plus L_0 -linear if there exists a vectorial periodic in space function $P^\rho = (P_+^\rho, P_-^\rho)$ of period P and a constant L_0 such that $\rho(x, t) = P^\rho(x, t) + L_0 x = (P_+^\rho(x, t) + L_0 x, P_-^\rho(x, t) + L_0 x)$.

3 Modelling

We denote by \mathbf{X} the vector $\mathbf{X} = (x_1, x_2)$. We consider a crystal with periodic deformation, namely the case where the total displacement of the crystal $U = (U_1, U_2) : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$ can be decomposed in a 1-periodic displacement $u = (u_1, u_2)$ and a linear displacement $A(t)^t\mathbf{X}$ with $A(t)$ a given 2×2 matrix which represents the shear stress

$$A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix}.$$

The displacement U is then given by

$$U(\mathbf{X}, t) = u(\mathbf{X}, t) + A(t)^t\mathbf{X}$$

and we define the total strain by

$$\varepsilon(U) = \frac{1}{2}(\nabla U + {}^t\nabla U) = \frac{1}{2}(\nabla u + {}^t\nabla u + A(t) + {}^tA(t)).$$

where the coefficients of ∇u are $(\nabla u)_{ij} = \frac{\partial u_i}{\partial x_j}$, $i, j \in \{1, 2\}$.

This total strain is decomposed in the form

$$\varepsilon(U) = \varepsilon^e(U) + \varepsilon^p,$$

where $\varepsilon^e(U)$ is the elastic deformation and ε^p the plastic deformation which is connected to the densities of dislocations by

$$\varepsilon^p = \varepsilon^0(\rho_+ - \rho_-), \quad (3.10)$$

where ρ_{\pm} represent the edge dislocation of type \pm , such that $\vec{b} \cdot \nabla \rho_{\pm} \geq 0$ is the density of dislocation of type \pm , $\vec{b} = (b_1, b_2)$ is the Burgers vector and

$$\varepsilon^0 = \frac{1}{2}(\vec{b} \otimes \vec{b}^{\perp} + \vec{b}^{\perp} \otimes \vec{b}) \quad (3.11)$$

where \vec{b}^{\perp} is a vector orthogonal to \vec{b} and $(\vec{b} \otimes \vec{b}^{\perp})_{ij} = b_i b_j^{\perp}$.

The stress is then given by

$$\sigma = \Lambda : \varepsilon^e(U), \quad (3.12)$$

i.e. the coefficients of the matrix σ are:

$$\sigma_{ij} = \sum_{k,l \in \{1,2\}} \Lambda_{ijkl} \varepsilon_{kl}^e(U) \quad i, j \in \{1, 2\}$$

with $\Lambda = (\Lambda_{ijkl})_{ijkl}$, $i, j, k, l = 1, 2$, Λ_{ijkl} are the elastic constant coefficients of the material, satisfying for $m > 0$:

$$\sum_{ijkl=1,2} \Lambda_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \geq m \sum_{ij=1,2} \varepsilon_{ij}^2 \quad (3.13)$$

for all symmetric matrix $\varepsilon = (\varepsilon_{ij})_{ij}$ *i.e.* such that $\varepsilon_{ij} = \varepsilon_{ji}$.

The functions ρ_{\pm} and u are then solutions of the coupled system (see Groma, Balogh [16], [15] and Groma [14]):

$$\left\{ \begin{array}{ll} \operatorname{div} \sigma = 0 & \text{in } \mathbb{R}^2 \times (0, T), \\ \sigma = \Lambda : (\varepsilon(U) - \varepsilon^p) & \text{in } \mathbb{R}^2 \times (0, T), \\ \varepsilon(U) = \frac{1}{2}(\nabla u + {}^t\nabla u + A(t) + {}^tA(t)) & \text{in } \mathbb{R}^2 \times (0, T), \\ \varepsilon^p = \varepsilon^0(\rho_+ - \rho_-) & \text{in } \mathbb{R}^2 \times (0, T), \\ (\rho_{\pm})_t = \pm(\sigma : \varepsilon^0)\vec{b} \cdot \nabla \rho_{\pm} & \text{in } \mathbb{R}^2 \times (0, T), \end{array} \right. \quad (3.14)$$

i.e. in the coordinates

$$\left\{ \begin{array}{ll} \sum_{j=1,2} \frac{\partial \sigma_{ij}}{\partial x_j} = 0 & \text{in } \mathbb{R}^2 \times (0, T), \\ \sigma_{ij} = \sum_{k,l \in \{1,2\}} \Lambda_{ijkl} (\varepsilon_{kl}(U) - \varepsilon_{kl}^p) & \text{in } \mathbb{R}^2 \times (0, T), \\ \varepsilon_{ij}(U) = \frac{1}{2} \left((\nabla u)_{ij} + (\nabla u)_{ij} + A_{ij}(t) + A_{ji}(t) \right) & \text{in } \mathbb{R}^2 \times (0, T), \\ \varepsilon_{ij}^p = \varepsilon_{ij}^0(\rho_+ - \rho_-) & \text{in } \mathbb{R}^2 \times (0, T), \\ (\rho_{\pm})_t = \pm \left(\sum_{i,j \in \{1,2\}} \sigma_{ij} \varepsilon_{ij}^0 \right) \vec{b} \cdot \nabla \rho_{\pm} & \text{in } \mathbb{R}^2 \times (0, T), \end{array} \right. \quad (3.15)$$

where the unknowns of the system are ρ_{\pm} and the displacement $u = (u_1, u_2)$ and with ε^0 defined by (3.11). The sign \pm comes from \pm in $\pm \vec{b}$.

To simplify, we consider the homogeneous case. The coefficients Λ_{ijkl} are such that

$$\sigma = 2\mu \varepsilon^e(U) + \lambda \operatorname{tr}(\varepsilon^e(U)) I_d, \quad (3.16)$$

where $\mu > 0$ and $\lambda + \mu > 0$ (consequence of (3.13)) are the Lamé coefficients and I_d the identity matrix. Then, the following *lemma* holds:

Lemma 3.1 (Equivalence between 2-D and 1-D models) *If we assume that the Burger vector is $\vec{b} = (1, 0)$, and that the densities of dislocations and u only depend on one variable $x = x_1 + x_2$ (as shown in Figure 2), the 2-D problem (3.14), with Λ defined by (3.16) is equivalent to the 1-D problem*

$$\left\{ \begin{array}{ll} (\rho_+)_t = -C_1 \left((\rho_+ - \rho_-) + C_2 \int_0^1 (\rho_+ - \rho_-) + L(t) \right) D\rho_+ & \text{in } \mathbb{R} \times (0, T) \\ (\rho_-)_t = C_1 \left((\rho_+ - \rho_-) + C_2 \int_0^1 (\rho_+ - \rho_-) + L(t) \right) D\rho_- & \text{in } \mathbb{R} \times (0, T) \end{array} \right. \quad (3.17)$$

where $L(t) = -\frac{(\lambda+2\mu)}{(\lambda+\mu)}(A_{12}(t) + A_{21}(t))$, $C_1 = \frac{(\lambda+\mu)\mu}{\lambda+2\mu}$, and $C_2 = \frac{\mu}{(\lambda+\mu)}$.

Proof of lemma 3.1

We can rewrite the first equation of (3.14) and (3.16) as

$$\operatorname{div} (2\mu \varepsilon(U) + \lambda \operatorname{tr}(\varepsilon(U)) I_d) = \operatorname{div} (2\mu \varepsilon^p + \lambda \operatorname{tr}(\varepsilon^p) I_d).$$

This implies by (3.10)

$$\mu \Delta u + (\lambda + \mu) \nabla (\operatorname{div} u) = \mu \begin{pmatrix} \frac{\partial(\rho_+ - \rho_-)}{\partial x_2} \\ \frac{\partial(\rho_+ - \rho_-)}{\partial x_1} \end{pmatrix}.$$

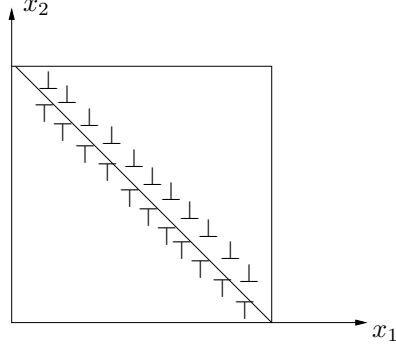


Figure 2: 1D sub-model for invariance by translation in the direction $(-1,1)$

Using the fact that $x = x_1 + x_2$, yields

$$2\mu \begin{pmatrix} \frac{\partial^2 u_1}{\partial x^2} \\ \frac{\partial^2 u_2}{\partial x^2} \end{pmatrix} + (\lambda + \mu) \begin{pmatrix} \frac{\partial^2 (u_1 + u_2)}{\partial x^2} \\ \frac{\partial^2 (u_1 + u_2)}{\partial x^2} \end{pmatrix} = \mu \begin{pmatrix} \frac{\partial(\rho_+ - \rho_-)}{\partial x} \\ \frac{\partial(\rho_+ - \rho_-)}{\partial x} \end{pmatrix}.$$

Now, by adding the two above equations, we obtain

$$\frac{\partial^2 (u_1 + u_2)}{\partial x^2} = \frac{\mu}{\lambda + 2\mu} \left(\frac{\partial(\rho_+ - \rho_-)}{\partial x} \right).$$

Integrating the above equation yields, since u is 1-periodic:

$$\frac{\partial(u_1 + u_2)}{\partial x} = \frac{\mu}{\lambda + 2\mu} \left((\rho_+ - \rho_-) - \int_0^1 (\rho_+ - \rho_-) \right). \quad (3.18)$$

Using the fact that

$$(\sigma : \varepsilon^0) = \sigma_{12} = 2\mu(\varepsilon^e(U))_{12} = \mu \left(\frac{\partial(u_1 + u_2)}{\partial x} + A_{12}(t) + A_{21}(t) - (\rho_+ - \rho_-) \right)$$

and (3.18) yields

$$(\sigma : \varepsilon^0) = -\frac{(\lambda + \mu)\mu}{\lambda + 2\mu} \left((\rho_+ - \rho_-) + \frac{\mu}{2(\lambda + \mu)} \int_0^1 (\rho_+ - \rho_-) + L(t) \right)$$

where $L(t) = -\frac{(\lambda + 2\mu)}{(\lambda + \mu)}(A_{12}(t) + A_{21}(t))$. We then deduce, if $\vec{b} = (1, 0)$, that the system (3.14) can be rewritten as (3.17). As the constant C_1, C_2 are positive, to simplify the notations, we can put them to 1 in the following without loss of generality on the system (1.1).

4 The continuous problem

To prove the existence and uniqueness result for the non-local problem, we use a fixed point method. In order to do that we freeze the non-local term and we study the following local problem

$$\begin{cases} (\rho_+)_t = -(\rho_+ - \rho_- + a(t)) |D\rho_+| & \text{on } \mathbb{R} \times (0, T), \\ (\rho_-)_t = (\rho_+ - \rho_- + a(t)) |D\rho_-| & \text{on } \mathbb{R} \times (0, T), \\ \rho_+(\cdot, 0) = \rho_+^0 & \text{on } \mathbb{R}, \\ \rho_-(\cdot, 0) = \rho_-^0 & \text{on } \mathbb{R}. \end{cases} \quad (4.19)$$

The assumptions are the following:

(H1) $a \in W^{1,\infty}(\mathbb{R}^+)$,

(H2) $\rho^0 = (\rho_+^0, \rho_-^0)$ is 1-periodic plus L_0 -linear, i.e., $\rho_\pm^0(x) = P_\pm^0(x) + L_0x$ where P_\pm^0 are periodic of period 1 and L_0 is a constant.

(H3) $P_\pm^0 \in \text{Lip}(\mathbb{R})$.

4.1 The local problem

Let $T > 0$, we denote by $USC(\mathbb{R} \times (0, T))$ (resp. $LSC(\mathbb{R} \times (0, T))$) the set of locally bounded upper (resp. lower) semi-continuous functions. The key point is that our system is quasi monotone in the sense of Ishii, Koike [19, (A.1)], (see *Lemma 4.1* below) and so we can extend their results to our system in unbounded domain and with unbounded initial condition using the well-known arguments of the scalar case.

Lemma 4.1 (Quasi-monotony of the Hamiltonian) *The Hamiltonian $H_k(t, \rho, p) = k(\rho_+ - \rho_- + a(t))|p|$ is quasi-monotone, i.e., for all vectors r and s such that*

$$r_j - s_j = \max_{k \in \{+, -\}} (r_k - s_k) \geq 0$$

then

$$H_j(t, r, p) - H_j(t, s, p) \geq 0. \quad (4.20)$$

We then have the following theorem:

Theorem 4.2 (The local problem) *Let $T \geq 0$. Assume (H1)-(H2)-(H3). We set $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$ and $B_0 = \max_{k \in \{+, -\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$. Then, the following holds:*

- (i) **Comparison principle.** *Let $\rho \in USC(\mathbb{R} \times (0, T))$ and $v \in LSC(\mathbb{R} \times (0, T))$ be respectively sub and super-solution of (1.1)-(1.2). We assume that there exists a constant $C > 0$ such that*

$$\rho^0(x) - Ct \leq \rho, v \leq \rho^0(x) + Ct. \quad (4.21)$$

If $\rho(\cdot, 0) \leq v(\cdot, 0)$ in \mathbb{R} then $\rho \leq v$ in $\mathbb{R} \times [0, T]$.

- (ii) **Existence.** *There exists a unique viscosity solution ρ of problem (4.19) satisfying*

$$\rho^0(x) - (M + \|a\|_{L^\infty(0, T)})B_0t \leq \rho(x, t) \leq \rho^0(x) + (M + \|a\|_{L^\infty(0, T)})B_0t. \quad (4.22)$$

Moreover, the solution is 1-periodic plus L_0 -linear.

- (iii) **Regularity.** *The solution ρ of (4.19) is Lipschitz continuous in space and time and satisfies*

$$\begin{aligned} \max_{k \in \{+, -\}} \|D\rho_k\|_{L^\infty(\mathbb{R} \times (0, T))} &\leq B_0, \\ \max_{k \in \{+, -\}} \|(\rho_k)_t\|_{L^\infty(\mathbb{R} \times (0, T))} &\leq B_0(M + \|a\|_{L^\infty(0, T)}). \end{aligned}$$

- (iv) **Estimate on the solution.** *The solution ρ satisfies*

$$\|\rho_+ - \rho_-\|_{L^\infty(\mathbb{R} \times (0, T))} \leq \|\rho_+^0 - \rho_-^0\|_{L^\infty(\mathbb{R})}.$$

Proof of Theorem 4.2

The comparison principle is just an extension of the one of Ishii, Koike [19, Th 4.7] for quasi-monotone Hamiltonians. For the existence, it suffices to use Perron's method by remarking that $\rho \pm (M + \|a\|_{L^\infty(0,T)}B_0)t$ are resp. super and sub-solution of (4.19). The fact that ρ is 1-periodic plus L_0 -linear comes from the fact that $\rho(x+1, t) + L_0$ is also solution of (iv).

The Lipschitz estimate in space comes from the fact that Problem (4.19) is invariant by space translation. To obtain the Lipschitz estimate in time, it is sufficient to bound the velocity using (4.22).

We now prove (iv). We set

$$m_+(t) = \sup_{x \in (0,1)} \rho_+(x, t) \quad \text{and} \quad m_-(t) = \inf_{x \in (0,1)} \rho_-(x, t).$$

It is easy to check that m_+ (resp. m_-) is subsolution (resp. supersolution) of $u_t = 0$ which implies the upper bound of (iv). The lower bound is proved similarly. This ends the proof of the theorem.

4.2 The non-local problem

Before to prove Theorem 1.1, we need the following *lemma*:

Lemma 4.3 (Stability of the solution with respect to the velocity) *Let $T \geq 0$. We consider for $i = 1, 2$ two different equations*

$$\begin{cases} (\rho_k^i)_t = -k(\rho_+^i - \rho_-^i + a_i(t)) |D\rho_k^i| & \text{for } k \in \{+, -\} \\ \rho_k^i(\cdot, 0) = \rho_k^0 & \text{for } k \in \{+, -\} \end{cases} \quad (4.23)$$

where the coefficients a_i satisfy (H1) and the initial conditions $\rho^0 = (\rho_+^0, \rho_-^0)$ satisfy (H2)-(H3). Then, we have

$$\max_{k \in \{+, -\}} \|\rho_k^2 - \rho_k^1\|_{L^\infty(\mathbb{R} \times (0, T))} \leq B_0 T \|a_2 - a_1\|_{L^\infty(0, T)},$$

where ρ^i for $i = 1, 2$ are the solutions of (4.23) given by Theorem 4.2.

Proof of Lemma 4.3

We set $K = \|a_2 - a_1\|_{L^\infty(0, T)}$. We remark that ρ^2 is a sub-solution of

$$(\rho_k)_t + k(\rho_+ - \rho_- + a_1(t)) |D\rho_k| - KB_0 = 0.$$

Moreover $\rho^1 + KB_0 t$ is solution of the same problem. By comparison principle, we then deduce

$$\max_{k \in \{+, -\}} \|\rho_k^2 - \rho_k^1\|_{L^\infty(\mathbb{R} \times (0, T))} \leq KB_0 T.$$

This is the estimate we want.

We have the following *lemma* which proof is trivial:

Lemma 4.4 (Stability of the velocity a) *Let ρ^1, ρ^2 be 1-periodic plus L_0 -linear. We set $a[\rho^i](t) = \int_0^1 \rho_+^i(x, t) - \rho_-^i(x, t) dx + L(t)$. Then the following holds*

$$\|a[\rho^2] - a[\rho^1]\|_{L^\infty(0, T)} \leq 2 \max_{k \in \{+, -\}} \|\rho_k^2 - \rho_k^1\|_{L^\infty(\mathbb{R} \times (0, T))}.$$

We now prove Theorem 1.1.

Proof of Theorem 1.1

We define the set:

$$U_T = \left\{ \rho = \begin{pmatrix} \rho_+ \\ \rho_- \end{pmatrix} \in (L_{\text{Loc}}^\infty)^2, s.t. \begin{cases} \|\rho_+ - \rho_-\|_{L^\infty} \leq M \\ \rho \text{ is 1-periodic plus } L_0\text{-linear} \\ \max_{k \in \{+, -\}} \|D\rho_k\|_{L^\infty} \leq B_0 \\ \max_{k \in \{+, -\}} \|(\rho_k)_t\|_{L^\infty} \leq B_0(2M + \|L\|_{L^\infty(0,T)}) \end{cases} \right\},$$

where L_0 is defined in (H2), $B_0 = \max_{k \in \{+, -\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ and $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$. For $\rho \in U_T$, we set

$$a[\rho](t) = \int_0^1 \rho_+(x, t) - \rho_-(x, t) dx + L(t). \quad (4.24)$$

We see that for any $\rho \in U_T$, $a[\rho]$ satisfies (H1) with $\|a[\rho]\|_{L^\infty(0,T)} \leq M + \|L\|_{L^\infty(0,T)}$.

For $\rho \in U_T$, we then define $v = G(\rho) = (G_+(\rho), G_-(\rho))$ as the unique viscosity solution for $k = 1, 2$ (see Theorem 4.2) of

$$\begin{cases} (v_k)_t = -k(v_+ - v_- + a[\rho](t)) |Dv_k| & \text{on } (0, T) \times \mathbb{R}, \\ v_k(\cdot, 0) = \rho_k^0 & \text{on } \mathbb{R}. \end{cases} \quad (4.25)$$

We will show that $G : U_T \rightarrow U_T$ is a strict contraction for T small enough. First, we will prove that G is well defined. By Theorem 4.2, we know that v is 1-periodic plus L_0 -linear. Moreover, we have

$$\max_{k \in \{+, -\}} \|Dv_k\|_{L^\infty(\mathbb{R} \times (0, T))} \leq B_0,$$

$$\max_{k \in \{+, -\}} \|(v_\pm)_t\|_{L^\infty(\mathbb{R} \times (0, T))} \leq B_0(M + \|a\|_{L^\infty(0, T)}) \leq B_0(2M + \|L\|_{L^\infty(0, T)})$$

and

$$\|v_+ - v_-\|_{L^\infty(\mathbb{R} \times (0, T))} \leq M$$

and so $v \in U_T$.

It thus remains to show that G is a contraction. For $v^i = G(\rho^i)$, according to Lemma 4.3 and Lemma 4.4, we have

$$\begin{aligned} \|v^2 - v^1\|_{L^\infty(\mathbb{R} \times (0, T))} &= \sup_{\{k \in \{+, -\}\}} \|v_k^2 - v_k^1\|_{L^\infty} \leq B_0 T \|a[\rho^2] - a[\rho^1]\|_{L^\infty(0, T)} \\ &\leq 2B_0 T \|\rho^1 - \rho^2\|_{L^\infty(\mathbb{R} \times (0, T))} \leq \frac{1}{2} \|\rho^1 - \rho^2\|_{L^\infty(\mathbb{R} \times (0, T))} \end{aligned}$$

for $T \leq T^* = \frac{1}{4B_0}$. And so G is a contraction on U_T which is a closed set. So, there exists a unique viscosity solution of (1.1)-(1.2) in U_T on $(0, T^*)$ where $T^* = \frac{1}{4B_0}$. By iterating this process, one can construct a solution for all $T > 0$. Indeed, T^* depends only on B_0 which does not change with time.

Proposition 4.5 (Estimate for the non-local solution) *Let $T \geq 0$. The solution ρ of (1.1)-(1.2) satisfies*

$$\|\rho_+ - \rho_-\|_{L^\infty(\mathbb{R} \times (0, T))} \leq M$$

where $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$.

The proof is the same of the one of the local case, see Theorem 4.2 (iv).

5 Numerical scheme

5.1 Approximation of the local system

In this subsection, we propose a finite difference scheme for the local system (4.19). Given a discrete velocity a^Δ , we consider the discrete solution v that approximates the solution of (4.19), given by the following explicit scheme

$$v_{k,i}^0 = \tilde{\rho}_k^0(x_i), \quad (5.26)$$

$$v_{k,i}^{n+1} = v_{k,i}^n + \Delta t \left(C_k^{\Delta, \text{Loc}}[v](x_i, t_n) \right) \begin{cases} E^+ \left(D^+ v_{k,i}^n, D^- v_{k,i}^n \right) & \text{if } C_k^{\Delta, \text{Loc}}[v](x_i, t_n) \geq 0 \\ E^- \left(D^+ v_{k,i}^n, D^- v_{k,i}^n \right) & \text{if not} \end{cases} \quad \forall k \in \{+, -\} \quad (5.27)$$

where $\tilde{\rho}^0(x_i)$ are defined in (1.3), E^\pm are the approximation of the Euclidean norm proposed by Osher and Sethian [26] defined in (1.6) (we also can use the one proposed by Rouy, Tourin [27]), $D^+ v_k^n$, $D^- v_k^n$ are the discrete gradient defined in (1.7) and

$$C_k^{\Delta, \text{Loc}}[w](x_i, t_n) = -k(w_+(x_i, t_n) - w_-(x_i, t_n) + a^\Delta(t_n)) \quad (5.28)$$

where a^Δ is an approximation of a satisfying

$$a^\Delta(t_n) = a(t_n). \quad (5.29)$$

In particular, the functions E^\pm are Lipschitz continuous with respect to the discrete gradients, *i.e.*

$$|E^\pm(P, Q) - E^\pm(P', Q')| \leq (|P - P'| + |Q - Q'|). \quad (5.30)$$

They are consistent with the Euclidean norm

$$E^\pm(P, P) = |P| \quad (5.31)$$

and enjoy suitable monotonicity with respect to each variable

$$\frac{\partial E^+}{\partial P^+} \geq 0, \quad \frac{\partial E^+}{\partial P^-} \leq 0, \quad \frac{\partial E^-}{\partial P^+} \geq 0, \quad \frac{\partial E^-}{\partial P^-} \leq 0. \quad (5.32)$$

Denoting by S^k the operator on the right-hand side of (5.27), we can rewrite the scheme more compactly as

$$v_{k,i}^0 = \tilde{\rho}_k^0(x_i), \quad v_k^{n+1} = S^k v^n.$$

Finally, we also assume that the mesh satisfies the following CFL condition (cf Remark 5.2)

$$\Delta t \leq \frac{1}{2L_1} \Delta x \quad (5.33)$$

where

$$L_1 = \|a\|_{L^\infty(0,T)} + M + 2.$$

Theorem 5.1 (Crandall-Lions rate of convergence) *Let $T \leq 1$. Assume that $\Delta x + \Delta t \leq 1$. Assume that $a \in W^{1,\infty}(\mathbb{R} \times [0, T])$ and that the CFL condition (5.33) holds.*

Then there exists a constant $K > 0$ depending only on $\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$, $\max_{k \in \{+, -\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ and $\|a\|_{W^{1,\infty}(0,T)}$ such that the error estimate between the solution ρ of the continuous system (4.19) and the discrete solution v of the finite difference scheme (5.26)-(5.27) is given by

$$\max_{k \in \{+, -\}} \sup_{\Xi_T} |\rho_k(x_i, t_n) - v_{k,i}^n| \leq K \sqrt{T} (\Delta x + \Delta t)^{1/2} + \max_{k \in \{+, -\}} \sup_{\Xi} |\rho_k^0(x_i) - v_{k,i}^0|$$

provided $K(\Delta x + \Delta t)^{\frac{1}{2}} + \max_{k \in \{+, -\}} \sup_{\Xi} (\rho_k^0(x_i) - v_{k,i}^0) \leq 1$.

Remark 5.2 (Monotony of the scheme) *Under the assumptions of Theorem 5.1, we have*

$$\begin{aligned} |v_{+,i}^n - v_{-,i}^n| &\leq |v_{+,i}^n - \rho_+(x_i, t_n)| + |\rho_+(x_i, t_n) - \rho_-(x_i, t_n)| + |\rho_-(x_i, t_n) - v_{-,i}^n| \\ &\leq 2 + M \end{aligned}$$

where we have used Theorem 4.2 (iv) for the second term. We then deduce that the discrete velocity is uniformly bounded:

$$C_k^{\Delta, \text{Loc}}[v] \leq \|a\|_{L^\infty(0,T)} + M + 2 = L_1.$$

Then, one can show that the scheme is monotone in the following sense: let v and w be two discrete functions such that $v_i^n \leq w_i^n$; then

$$S^k(v^n)(x_i) \leq S^k(w^n)(x_i), \quad \text{for } k \in \{+, -\}.$$

For the proof of Theorem 5.1, we need the following lemma:

Lemma 5.3 *If v_i^n is the numerical solution of (5.26)-(5.27), then*

$$-Kt_n - \mu^0 \leq \rho^0(x_i) - v(x_i, t_n) \leq Kt_n + \mu^0 \quad (5.34)$$

where $K = 2(\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})} + \|a\|_{L^\infty(0,T)}) \max_{k \in \{+, -\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ and

$$\mu^0 = \max_{k \in \{+, -\}} \sup_{\Xi} |\rho_k^0(x_i) - v_{k,i}^0| \geq 0. \quad (5.35)$$

Proof of Lemma 5.3

To prove this, we set $w_\pm(x_i, t_n) = \rho_\pm^0(x_i) - Kt_n - \mu^0$ and we show that for K large enough w is a discrete sub-solution. Indeed, we have

$$\begin{aligned} &w_{\pm,i}^{n+1} - (S^\pm w^n)_i \\ &= -K\Delta t - \Delta t C_\pm^{\Delta, \text{Loc}}[\rho^0](x_i, t_n) E^{\text{sgn}(C_\pm^{\Delta, \text{Loc}}[\rho^0](x_i, t_n))} (D^+ \rho_\pm^0(x_i), D^- \rho_\pm^0(x_i)) \\ &= -\Delta t \left(K \mp (\rho_+^0(x_i) - \rho_-^0(x_i) + a^\Delta(t_n)) E^{\text{sgn}(C_\pm^{\Delta, \text{Loc}}[\rho^0](x_i, t_n))} (D^+ \rho_\pm^0(x_i), D^- \rho_\pm^0(x_i)) \right) \\ &\leq -\Delta t \left(K - 2(\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})} + \|a\|_{L^\infty(0,T)}) \max_{k \in \{+, -\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})} \right) \end{aligned}$$

where $C_k^{\Delta, \text{Loc}}[w](x_i, t_n)$ is defined in (5.28) and $\text{sgn}(f)$ is the sign of f .

So, for every $K \geq 2(\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})} + \|a\|_{L^\infty(0,T)}) \max_{k \in \{+, -\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$, w is a discrete sub-solution. Moreover

$$w_{\pm,i}^0(x_i) = \rho_\pm^0(x_i) - \mu^0 \leq v_\pm^0(x_i).$$

Using the monotony of the scheme, we deduce $w_i^n \leq v_i^n$ and so

$$\rho^0(x_i) - v_i^n \leq Kt_n + \mu_0.$$

The lower bound is proved similarly.

We now give the proof of Theorem 5.1

Proof of Theorem 5.1

The proof is an adaptation for systems of the one of Crandall Lions [11], revisited by Alvarez *et al.* [2]. The proof splits into three steps. We denote throughout by K various constant depending only on $\|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$, $\max_{k \in \{+, -\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ and $\|a\|_{W^{1,\infty}(0,T)}$.

We first assume that

$$\rho^0(x_i) \geq v_i^0 \quad (5.36)$$

and we set

$$\mu^0 = \max_{k \in \{+, -\}} \sup_{\Xi} |\rho_k^0(x_i) - v_{k,i}^0| \geq 0. \quad (5.37)$$

We set a few notations. We put

$$\mu = \max_{k \in \{+, -\}} \sup_{\Xi_T} (\rho_k(x_i, t_n) - v_{k,i}^n).$$

For every $0 < \alpha \leq 1$, $0 < \varepsilon \leq 1$ and $\sigma > 0$, we set

$$M_\sigma^{\alpha, \varepsilon} = \sup_{\mathbb{R} \times [0, T] \times \Xi_T \times \{+, -\}} \Psi_\sigma^{\alpha, \varepsilon}(x, t, x_i, t_n, k),$$

with

$$\Psi_\sigma^{\alpha, \varepsilon}(x, t, x_i, t_n, k) = \rho_k(x, t) - v_k(x_i, t_n) - \frac{|x - x_i|^2}{2\varepsilon} - \frac{|t - t_n|^2}{2\varepsilon} - \sigma t - \alpha|x|^2 - \alpha|x_i|^2.$$

We shall drop the super and subscripts on Ψ when no ambiguity arises as concerning the value of the parameter.

Since ρ^0 is Lipschitz continuous and $T \leq 1$, we have by (4.22)

$$|\rho_\pm(x, t)| \leq K(1 + |x|). \quad (5.38)$$

Moreover by *Lemma 5.3* we have

$$\begin{aligned} |v_\pm(x_i, t_n)| &\leq |v_\pm(x_i, t_n) - \rho_\pm^0(x_i)| + |\rho_\pm^0(x_i)| \\ &\leq Kt_n + K(1 + |x_i|) \\ &\leq K(1 + |x_i|). \end{aligned}$$

We then deduce that Ψ achieves its maximum at some point that we denote by $(x^*, t^*, x_i^*, t_n^*, k^*)$.

Step 1: Estimates for the maximum point of Ψ

The maximum point of Ψ enjoys the following estimates

$$\alpha|x^*| + \alpha|x_i^*| \leq K, \quad (5.39)$$

and

$$|x^* - x_i^*| \leq K\varepsilon, \quad |t^* - t_n^*| \leq (K + 2\sigma)\varepsilon. \quad (5.40)$$

Indeed, by inequality $\Psi(x^*, t^*, x_i^*, t_n^*, k^*) \geq \Psi(0, 0, 0, 0, k^*) \geq 0$, we obtain

$$\alpha|x^*|^2 + \alpha|x_i^*|^2 \leq \rho_{k^*}(x^*, t^*) - v_{k^*}(x_i^*, t_n^*) \leq K(1 + |x^*| + |x_i^*|) \leq K + \frac{K^2}{\alpha} + \frac{\alpha}{2}|x^*|^2 + \frac{\alpha}{2}|x_i^*|^2.$$

This implies (5.39), since $\alpha \leq 1$.

The first bound of (5.40) follows from the Lipschitz in space regularity of ρ (see Theorem 4.2 (iii)), from the inequality $\Psi(x^*, t^*, x_i^*, t_n^*, k^*) \geq \Psi(x_i^*, t^*, x_i^*, t_n^*, k^*)$ and from (5.39). Indeed, this implies

$$\begin{aligned} \frac{|x^* - x_i^*|^2}{2\varepsilon} &\leq \rho_{k^*}(x^*, t^*) - \rho_{k^*}(x_i^*, t^*) - \alpha|x^*|^2 + \alpha|x_i^*|^2 \\ &\leq K|x^* - x_i^*| + \alpha|x^* - x_i^*|(|x^*| + |x_i^*|) \leq K|x^* - x_i^*|. \end{aligned}$$

The second bound of (5.40) is obtained in the same way, using the inequality $\Psi(x^*, t^*, x_i^*, t_n^*, k^*) \geq \Psi(x^*, t_n^*, x_i^*, t_n^*, k^*)$ and Theorem 4.2 (iii).

Step 2 : A better estimate for the maximum point of Ψ

Inequality (5.39) can be strengthened to

$$\alpha|x^*|^2 + \alpha|x_i^*|^2 \leq K. \quad (5.41)$$

Indeed, using the Lipschitz regularity of ρ , the inequality $\Psi(x^*, t^*, x_i^*, t_n^*, k^*) \geq \Psi(0, 0, 0, 0, k^*)$ and equations (4.22), (5.34) and (5.40), yields

$$\begin{aligned} \alpha|x^*|^2 + \alpha|x_i^*|^2 &\leq \rho_{k^*}(x^*, t^*) - v_{k^*}(x_i^*, t_n^*) + \rho(x_i^*, 0) - \rho(x_i^*, 0) \\ &\leq K(|x^* - x_i^*| + t^*) + Kt_n^* + \mu^0 \leq K. \end{aligned}$$

Step 3 : Upper bound of μ

We have the bound $\mu \leq K\sqrt{T}(\Delta x + \Delta t)^{\frac{1}{2}} + \mu^0$ if $\Delta x + \Delta t \leq \frac{1}{K^2}$.

First, we claim that for σ large enough, we have either $t^* = 0$ or $t_n^* = 0$. Suppose the contrary. Then the function $(x, t) \mapsto \Psi(x, t, x_i^*, t_n^*, k^*)$ achieves its maximum at a point of $\mathbb{R} \times (0, T]$. Using the fact that ρ is a sub-solution of the continuous problem, we obtain the inequality

$$\sigma + p_t^* \leq -k^*(\rho_+ - \rho_- + a(t^*))|p_x^* + 2\alpha x^*| \quad (5.42)$$

with $p_t^* = \frac{t^* - t_n^*}{\varepsilon}$, $p_x^* = \frac{x^* - x_i^*}{\varepsilon}$.

Since $t_n^* > 0$, we also have $\Psi(x^*, t^*, x_i^*, t_n^*, k^*) \geq \Psi(x^*, t^*, x_i, t_n^* - \Delta t, k^*)$. This implies

$$v_{k^*}(\cdot, t_n^* - \Delta t) \geq \varphi(\cdot, t_n^* - \Delta t) + v_{k^*}(x_i^*, t_n^*) - \varphi(x_i^*, t_n^*)$$

for $\varphi(x_i, t_n) = -\frac{|x^* - x_i|^2}{2\varepsilon} - \frac{|t^* - t_n|^2}{2\varepsilon} - \alpha|x_i|^2$. Using the fact that the scheme is monotone and commutes with the addition of constants, yields

$$\begin{aligned} &v_{k^*}(x_i^*, t_n^*) \\ &= S^{k^*}(v_{k^*}(\cdot, t_n^* - \Delta t))(x_i^*) \\ &\geq \varphi(x_i^*, t_n^* - \Delta t) + v_{k^*}(x_i^*, t_n^*) - \varphi(x_i^*, t_n^*) \\ &\quad + \Delta t \left(c_{k^*}^{\Delta, \text{Loc}}[v](x_i^*, t_n^*) \right) E^{l^*}(D^+ \varphi(x_i^*, t_n^* - \Delta t), D^- \varphi(x_i^*, t_n^* - \Delta t)) \end{aligned}$$

where $l^* = \text{sgn} \left(c_{k^*}^{\Delta, \text{Loc}}[v](x_i^*, t_n^*) \right)$. We set

$$c[v] = -c_{k^*}^{\Delta, \text{Loc}}[v](x_i^*, t_n^*), \quad c[\rho] = k^*(\rho_+(x^*, t^*) - \rho_-(x^*, t^*) + a(t^*)).$$

We then obtain the super-solution inequality:

$$\frac{\varphi(x_i^*, t_n^*) - \varphi(x_i^*, t_n^* - \Delta t)}{\Delta t} \geq -c[v]E^{l^*}(D^+ \varphi(x_i^*, t_n^* - \Delta t), D^- \varphi(x_i^*, t_n^* - \Delta t)).$$

Straightforward computations of the discrete derivative of φ yield

$$p_t^* + \frac{\Delta t}{2\varepsilon} \geq -c[v]E^{l^*} \left(p_x^* - \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^* + \Delta x), p_x^* + \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^* - \Delta x) \right).$$

Subtracting the above inequality to (5.42), we deduce

$$\begin{aligned} \sigma &\leq \frac{\Delta t}{2\varepsilon} - c[\rho]|p_x^* + 2\alpha x^*| + c[v]E^{l^*} \left(p_x^* - \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^* + \Delta x), p_x^* + \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^* - \Delta x) \right) \\ &\leq \frac{\Delta t}{2\varepsilon} - (c[\rho] - c[v])|p_x^*| + \alpha K|x^*| \\ &\quad + |c[v]| \left| E^{l^*} \left(p_x^* - \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^* + \Delta x), p_x^* + \frac{\Delta x}{2\varepsilon} - \alpha(2x_i^* - \Delta x) \right) - E^{l^*}(p_x^*, p_x^*) \right| \\ &\leq \frac{\Delta t}{2\varepsilon} - (c[\rho] - c[v])|p_x^*| + K\alpha|x^*| + K\frac{\Delta x}{\varepsilon} + 2\alpha K|x_i^*| + 2\alpha K\Delta x \end{aligned}$$

where we have used, for the second line, the fact that

$$c[\rho] \leq M + 2B_0(M + \|a\|_{L^\infty(0,T)})T \leq K$$

with $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$ and $B_0 = \max_{k \in \{+, -\}} \|D\rho_k^0\|_{L^\infty(\mathbb{R})}$ (see Theorem 4.2). Now, since $\rho_{k^*}(x^*, t^*) - v_{k^*}(x_i^*, t_n^*) = \max_{k \in \{+, -\}} (\rho_k(x^*, t^*) - v_k(x_i^*, t_n^*)) \geq 0$, by Lemma 4.1, we obtain

$$\begin{aligned} -(c[\rho] - c[v])|p_x^*| &= -k^* (\rho_+(x^*, t^*) - \rho_-(x^*, t^*) + a(t^*)) |p_x^*| \\ &\quad + k^* (v_+(x_i^*, t_n^*) - v_-(x_i^*, t_n^*) + a(t^*)) |p_x^*| + k^* (a^\Delta(t_n^*) - a(t^*)) |p_x^*| \\ &\leq |a^\Delta(t_n^*) - a(t^*)| |p_x^*| \leq K|t_n^* - t^*| |p_x^*| \end{aligned}$$

where we have used (5.29). This implies

$$\begin{aligned} \sigma &\leq \frac{\Delta t}{2\varepsilon} + K|t^* - t_n^*| |p_x^*| + K\alpha|x^*| + K\frac{\Delta x}{\varepsilon} + 2\alpha K|x_i^*| + 2\alpha K\Delta x \\ &\leq K\frac{\Delta x + \Delta t}{\varepsilon} + K\alpha^{1/2} + K\varepsilon. \end{aligned}$$

Putting

$$\sigma^* = \sigma^*(\Delta x + \Delta t, \varepsilon, \alpha) = K\frac{\Delta x + \Delta t}{\varepsilon} + K(\alpha^{1/2} + \varepsilon),$$

we therefore conclude that we must have $t^* = 0$ or $t_n^* = 0$ provided $\sigma \geq \sigma^*$. Whenever $t^* = 0$, we deduce from Lemma 5.3 and from (5.40) that

$$\begin{aligned} M_\sigma^{\alpha, \varepsilon} &= \Psi(x^*, 0, x_i^*, t_n^*, k^*) \leq \rho_{k^*}^0(x^*) - v_{k^*}(x_i^*, t_n^*) \\ &\leq \rho_{k^*}^0(x^*) - \rho_{k^*}^0(x_i^*) + Kt_n^* + \mu^0 \\ &\leq K(|x^* - x_i^*| + t_n^*) + \mu^0 \leq K(1 + \sigma)\varepsilon + \mu^0. \end{aligned}$$

Similarly, whenever $t_n^* = 0$, we deduce from the Lipschitz regularity of ρ and from (5.40) that

$$\begin{aligned} M_\sigma^{\alpha, \varepsilon} &= \Psi(x^*, t^*, x_i^*, 0, k^*) \leq \rho_{k^*}(x^*, t^*) - v_{k^*}(x_i^*, 0) \\ &\leq K(|x^* - x_i^*| + t^*) + \mu^0 \leq K(1 + \sigma)\varepsilon + \mu^0. \end{aligned}$$

To sum up, we have shown that

$$M_\sigma^{\alpha, \varepsilon} \leq K(1 + \sigma)\varepsilon + \mu^0 \leq K\varepsilon + \mu^0$$

provided $\sigma^* = K\frac{\Delta x + \Delta t}{\varepsilon} + K(\alpha^{1/2} + \varepsilon) \leq \sigma \leq 1$. We then deduce that, for every (x_i, t_n) and for every k , we have

$$\rho_k(x_i, t_n) - v_k(x_i, t_n) - \left(K\frac{\Delta x + \Delta t}{\varepsilon} + K(\alpha^{1/2} + \varepsilon) \right) T - 2\alpha|x_i|^2 \leq M_\sigma^{\alpha, \varepsilon} \leq K\varepsilon + \mu^0.$$

Sending $\alpha \rightarrow 0$, taking the supremum over (x_i, t_n) , the maximum over k and choosing $\varepsilon = T^{1/2}(\Delta x + \Delta t)^{1/2}$, we conclude that

$$\max_{k \in \{+, -\}} \sup_{\Xi_T} (\rho_k(x_i, t_n) - v_{k,i}^n) = \mu \leq K(\Delta x + \Delta t)^{1/2} \sqrt{T} + \max_{k \in \{+, -\}} \sup_{\Xi} (\rho_k^0(x_i) - v_{k,i}^0), \quad (5.43)$$

provided that $\Delta x, \Delta t$ are small enough $T \leq 1, \mu_0 \leq 1$ and (5.36) is assumed.

In the general case, we consider $\bar{\rho} = \rho + \mu^1$ with $\mu^1 = \max_{k \in \{+, -\}} \sup_{\Xi} (v_{k,i}^0 - \rho_k^0(x_i))$. We remark that $\bar{\rho}$ is solution of (4.19) and satisfies $\bar{\rho}^0(x_i) \geq v_i^0$. Then (5.43) is true with $\bar{\rho}$ in place of ρ , *i.e.*

$$\max_{k \in \{+, -\}} \sup_{\Xi_T} (\rho_k(x_i, t_n) + \mu^1 - v_{k,i}^n) \leq K (\Delta x + \Delta t)^{1/2} \sqrt{\bar{T}} + \max_{k \in \{+, -\}} \sup_{\Xi} (\rho_k^0(x_i) + \mu^1 - v_{k,i}^0),$$

which still implies (5.43) with $\max_{k \in \{+, -\}} \sup_{\Xi} |\rho_k^0(x_i) - v_{k,i}^0|$.

The lower bound for the error estimate is obtained by exchanging ρ and v . As the proof is similar to the above, we omit it.

5.2 Approximation of the non-local system

To solve numerically the non-local system (1.1)-(1.2), we use the finite difference scheme (1.3)-(1.4)-(1.5). We also assume the CFL condition (1.8). In particular, using Proposition 4.5, we deduce that the CFL condition (5.33) is satisfied uniformly for all a defined by (1.5) because

$$\|a[\rho]\|_{L^\infty(0,T)} \leq M + \|L\|_{L^\infty(0,T)}$$

and so $L_1 \leq L_2$.

Let $\bar{T} \geq 0$ which will be chosen later. To prove our convergence result, we mimic the continuous case. Before proving Theorem 1.3 we need to introduce some notations and *lemmata*. Defining $X_T^{1,\Delta} = \mathbb{R}^{\{0, \dots, N_T\}}$ and $X_T^{2,\Delta} = (\mathbb{R}^2)^{\mathbb{Z} \times \{0, \dots, N_T\}}$, the set of discrete functions defined on $\{0, \dots, N_T\}$ and on the mesh Ξ_T respectively, we denote by $G^\Delta : X_{\bar{T}}^{1,\Delta} \rightarrow X_{\bar{T}}^{2,\Delta}$ the operator that gives the discrete solution v of the local Problem (5.27) for a given velocity $a^\Delta \in X_{\bar{T}}^{1,\Delta}$, *i.e.*

$$(G_+^\Delta(a^\Delta), G_-^\Delta(a^\Delta)) = G^\Delta(a^\Delta) = v.$$

In particular, the scheme can be rewritten

$$v = G^\Delta(a^\Delta[v])$$

with $a^\Delta[\cdot]$ defined in (1.5). We set, for all $T \leq \bar{T}$:

$$U_T^\Delta = \left\{ w \in X_T^{2,\Delta} : \begin{array}{l} \sup_{\Xi_T} |D_x^+ w_\pm| \leq B_0, \\ \sup_{\Xi_T} |D_t^+ w_\pm| \leq 2B_0(2M + \|L\|_{L^\infty(0,T)} + 4), \\ \sup_{\Xi_T} |w_+ - w_-| \leq M + 2 \end{array} \right\}$$

and

$$V_T^\Delta = \left\{ a^\Delta \in X_T^{1,\Delta} : \sup_{\{0, \dots, N_T \Delta t\}} |a^\Delta| \leq M + \|L\|_{L^\infty} + 2 \right\}$$

where $M = \|P_+^0 - P_-^0\|_{L^\infty(\mathbb{R})}$. One can easily check that

$$\{(\rho)^\Delta \mid \rho \in U_T\} \subset U_T^\Delta$$

and

$$\{(a)^\Delta : \|a\|_{L^\infty(0,T)} \leq M + \|L\|_{L^\infty(0,T)}\} \subset V_T^\Delta$$

where $(f)^\Delta$ is the restriction to Ξ_T of the continuous function f . We have the following *Lemma*:

Lemma 5.4 *Assume that (1.8) holds. Then for all $T \leq \bar{T}$, the following inclusion holds*

- (i) $a^\Delta[U_T^\Delta] \subset V_T^\Delta$,
- (ii) $G^\Delta(V_T^\Delta) \subset U_T^\Delta$.

Proof of Lemma 5.4

The proof of (i) is just a simple computation. We prove (ii).

Let $a^\Delta \in V_T^\Delta$ and $v = G^\Delta(a^\Delta)$. We set $w(x_i, t_n) = v(x_{i+1}, t_n) - \Delta x B_0$. Then w is still solution of the discrete scheme (5.27) and satisfies $w^0 \leq v^0$. Using the monotony of the scheme yields

$$\frac{v_\pm(x_{i+1}, t_n) - v_\pm(x_i, t_n)}{\Delta x} \leq B_0.$$

Using Theorem 4.2, we deduce

$$|v_+ - v_-| \leq M + 2. \quad (5.44)$$

For the estimate in time, we have, using (5.44),

$$\left| \frac{v_i^{n+1} - v_i^n}{\Delta t} \right| \leq 2B_0 |C_k^{\Delta, \text{Loc}}[v](x_i, t_n)| \leq 2B_0(M + 2 + \sup_{\{0, \dots, N_T \Delta t\}} |a^\Delta|) \leq 2B_0(2M + \|L\|_{L^\infty(0, T)} + 4).$$

So $G^\Delta(V_T^\Delta) \subset U_T^\Delta$. This ends the proof of the *lemma*.

We now have to prove some consistency and stability results for the velocity a^Δ and for the operator G^Δ .

Lemma 5.5 (Consistency for the discrete velocity a^Δ) *There is a constant $K = 2B_0 + M$ such that, for every mesh Δ , for every $0 \leq T \leq \bar{T}$ and for $\rho \in U_T$, we have*

$$\sup_{\{0, \dots, N_T \Delta t\}} |(a[\rho])^\Delta - a^\Delta[(\rho)^\Delta]| \leq K \Delta x$$

where $(\rho)^\Delta$ is the restriction to Ξ_T of the continuous function ρ and $a[\cdot]$ is defined in (4.24).

Proof of Lemma 5.5

We set $\tilde{\rho}(x, t) = \rho_+(x, t) - \rho_-(x, t)$. The following holds:

$$\begin{aligned} |a[\rho](t_n) - a^\Delta[(\rho)^\Delta](t_n)| &= \left| \int_0^1 \tilde{\rho}(x, t_n) dx - \sum_{i=0}^{N_x-1} \Delta x \tilde{\rho}(x_i, t_n) \right| \\ &\leq \sum_{i=0}^{N_x-1} \left| \int_{i\Delta x}^{(i+1)\Delta x} \tilde{\rho}(x, t_n) dx - \Delta x \tilde{\rho}(x_i, t_n) \right| + \int_{N_x \Delta x}^1 \tilde{\rho}(x, t_n) dx \\ &\leq \Delta x \sum_{i=0}^{N_x-1} \sup_{[i\Delta x, (i+1)\Delta x]} |\tilde{\rho}(\cdot, t_n) - \tilde{\rho}(x_i, t_n)| + M \Delta x \\ &\leq \Delta x (2B_0 + M). \end{aligned}$$

We have the following *lemma* which proof is just a simple computation

Lemma 5.6 (Stability property of the velocity a^Δ) *For every mesh Δ , for every $0 \leq T \leq \bar{T}$ and every $v_1, v_2 \in U_T^\Delta$, the following holds*

$$\sup_{\{0, \dots, N_T \Delta t\}} |a^\Delta[v_2] - a^\Delta[v_1]| \leq 2 \max_{k \in \{+, -\}} \sup_{\Xi_T} |v_2 - v_1|.$$

Lemma 5.7 (Stability property of the operator G^Δ) *There is a constant $K = 2B_0$ so that, for every mesh Δ satisfying the uniform CFL condition (1.8), for all $0 \leq T \leq \bar{T}$ and all $a_1^\Delta, a_2^\Delta \in V_T^\Delta$*

$$\max_{k \in \{+, -\}} \sup_{\Xi_T} |G_k^\Delta(a_2^\Delta) - G_k^\Delta(a_1^\Delta)| \leq KT \sup_{\{0, \dots, N_T \Delta t\}} |a_2^\Delta - a_1^\Delta|.$$

Proof of Lemma 5.7

We set $v_i = G^\Delta(a_i^\Delta)$. Using the fact that

$$c_1 E^{\text{sgn}(c_1)} - c_2 E^{\text{sgn}(c_2)} \leq |c_1 - c_2| \max(E^+, E^-)$$

yields

$$\begin{aligned} & v_{2,k}^{n+1} - v_{2,k}^n + k\Delta t (v_{2,+}^n - v_{2,-}^n + a_1^\Delta(t_n)) E^{\text{sgn}(v_{2,+}^n - v_{2,-}^n + a_1^\Delta(t_n))} (D^+ v_2^n, D^- v_2^n) \\ & \leq \Delta t |a_2^\Delta(t_n) - a_1^\Delta(t_n)| \max(E^+(D^+ v_2^n, D^- v_2^n), E^-(D^+ v_2^n, D^- v_2^n)) \\ & \leq 2B_0 \Delta t \sup_{\{0, \dots, N_T \Delta t\}} |a_1^\Delta - a_2^\Delta|. \end{aligned}$$

Moreover $\tilde{v}_1(x_i, t_n) = v_1(x_i, t_n) + 2B_0 \sup_{\{0, \dots, N_T \Delta t\}} |a_1^\Delta - a_2^\Delta| t_n$ is solution of the same discrete equation.

Since the scheme is monotone, one deduces that

$$\max_{k \in \{+, -\}} \sup_{\Xi_T} |G_k^\Delta(a_2^\Delta) - G_k^\Delta(a_1^\Delta)| \leq 2B_0 T \sup_{\{0, \dots, N_T \Delta t\}} |a_2^\Delta - a_1^\Delta|.$$

This achieves the proof.

We now prove Theorem 1.3.

Proof of Theorem 1.3

We use the main idea of Alvarez *et al.* [2].

We first assume that $T \geq \bar{T}$ and we set, for every $l \geq 1$:

$$Q_l^\Delta = \Delta x \mathbb{Z} \times \{\Delta t N_l, \dots, \Delta t N_{l+1}\}$$

where N_l is the integer part of $\frac{l\bar{T}}{\Delta t}$. As in the continuous case, on each interval $(l\bar{T}, (l+1)\bar{T})$, we can iterate the process (since \bar{T} depends only on B_0 which does not change with time) and construct, using a fix point method (denoting by G and G^Δ), ρ and v respectively solution of (1.1)-(1.2) and (1.3)-(1.4)-(1.5). We then have the inequality

$$\begin{aligned} \max_{k \in \{+, -\}} \sup_{Q_l^\Delta} |\rho_k - v_k| & \leq \max_{k \in \{+, -\}} \sup_{Q_l^\Delta} |G_{k,l}(a[\rho]) - G_{k,l}^\Delta(a^\Delta[v])| \\ & \leq \max_{k \in \{+, -\}} \sup_{Q_l^\Delta} |G_{k,l}(a[\rho]) - G_{k,l}^\Delta((a[\rho])^\Delta)| \\ & \quad + \max_{k \in \{+, -\}} \sup_{Q_l^\Delta} |G_{k,l}^\Delta((a[\rho])^\Delta) - G_{k,l}^\Delta(a^\Delta[v])|, \end{aligned}$$

where the function $G_l^\Delta((a[\rho])^\Delta) = (G_{+,l}^\Delta((a[\rho])^\Delta), G_{-,l}^\Delta((a[\rho])^\Delta))$ (resp. $G_l(a[\rho])$) is simply the discrete solution of (5.26) (resp. the continuous solution of (1.1)) with the velocity $a[\rho]$ and initial condition v^{N_l} (resp. $\rho(\cdot, N_l \Delta t)$). From Theorem 5.1, we then deduce

$$\max_{k \in \{+, -\}} \sup_{Q_l^\Delta} |G_{k,l}(a[\rho]) - G_{k,l}^\Delta((a[\rho])^\Delta)| \leq K \sqrt{\bar{T} \Delta x} + \max_{k \in \{+, -\}} \sup_{\Delta x \mathbb{Z} \times N_l \Delta t} |\rho_k - v_k| \quad (5.45)$$

$$\leq lK \sqrt{\bar{T} \Delta x} + \max_{k \in \{+, -\}} \sup_{\Xi} |\rho_k^0 - v_k^0|.$$

For the second term, we use *Lemmata* 5.5, 5.6 and 5.7 to obtain

$$\begin{aligned}
& \max_{k \in \{+, -\}} \sup_{Q_t^\Delta} |G_{k,l}^\Delta((a[\rho])^\Delta) - G_{k,l}^\Delta(a^\Delta[v])| \\
& \leq K\bar{T} \sup_{\{N_l\Delta t, \dots, N_{l+1}\Delta t\}} |(a[\rho])^\Delta - a^\Delta[v]| \\
& \leq K\bar{T} \sup_{\{N_l\Delta t, \dots, N_{l+1}\Delta t\}} (|(a[\rho])^\Delta - a^\Delta[(\rho)^\Delta]| + |a^\Delta[(\rho)^\Delta] - a^\Delta[v]|) \\
& \leq K\bar{T} \left(\Delta x + \max_{k \in \{+, -\}} \sup_{Q_t^\Delta} |\rho_k - v_k| \right).
\end{aligned}$$

This implies, for $\bar{T}\Delta x \leq 1$ and $K\bar{T} < 1$,

$$\max_{k \in \{+, -\}} \sup_{Q_t^\Delta} |\rho_k - v_k| \leq \frac{LK}{1 - K\bar{T}} \sqrt{\bar{T}\Delta x} + \left(\max_{k \in \{+, -\}} \sup_{\Xi} |\rho_k^0 - v_k^0| \right) \frac{1}{1 - K\bar{T}}.$$

We now take $\bar{l} \geq 1$ such that

$$\bar{l}\bar{T} \leq T \leq (\bar{l} + 1)\bar{T}.$$

Then the following holds:

$$\begin{aligned}
\max_{k \in \{+, -\}} \sup_{\Xi_T} |\rho_k - v_k| & \leq \frac{\bar{l}K}{1 - K\bar{T}} \sqrt{\bar{T}\Delta x} + \left(\max_{k \in \{+, -\}} \sup_{\Xi} |\rho_k^0 - v_k^0| \right) \frac{1}{1 - K\bar{T}} \\
& \leq K\bar{T}\sqrt{\Delta x} + K \left(\max_{k \in \{+, -\}} \sup_{\Xi} |\rho_k^0 - v_k^0| \right), \text{ if } T \geq \bar{T}.
\end{aligned}$$

where we have used the fact that \bar{T} depends only on B_0 .

Notice that, in the case where $T \leq \bar{T}$, from Theorem 5.1, (5.45) is replaced by

$$\max_{k \in \{+, -\}} \sup_{\Xi_T} |G_k(a[\rho]) - G_k^\Delta((a[\rho])^\Delta)| \leq K\sqrt{T\Delta x} + \max_{k \in \{+, -\}} \sup_{\Xi} |\rho_k^0 - v_k^0|$$

and so we obtain

$$\max_{k \in \{+, -\}} \sup_{\Xi_T} |\rho_k - v_k| \leq K\sqrt{T\Delta x} + K \left(\max_{k \in \{+, -\}} \sup_{\Xi} |\rho_k^0 - v_k^0| \right), \text{ if } T \leq \bar{T}.$$

This ends the proof of the theorem.

6 Numerical results

In this Section, we present some numerical simulations of the 1-D Groma-Balogh problem (1.1)-(1.2) discretized by the numerical scheme (1.3)-(1.4)-(1.5).

6.1 Numerical error estimate

Here, we show a numerical test in order to confirm our error estimate for local system. Let us fix $L(t) = 0$ even if it is not physically relevant, let us choose the following initial conditions: $\rho_+^0(x) = -|x - 1/2| + 1/2$, and $\rho_-^0(x) = -|2x - 1| + 1$ on $[0, 1[$ (and extend it by periodicity on \mathbb{R}).

Figure (3) show the behaviour of the L^∞ -error versus the discretization parameter Δx . The regression slope is close to 0.7 and the ideal regression is $\frac{1}{2}$. Hence, the behaviour of this errors confirms that our error seems optimal.

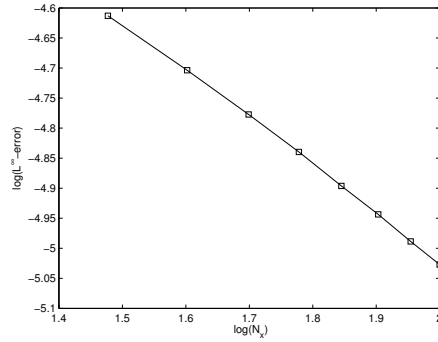


Figure 3: $\log(L^\infty - \text{error})$ of $|u_{N_T} - u_{N_T-1}|$ versus $\log(N_x)$ at $T = \frac{1}{2}$

6.2 dislocations density dynamics

In this paragraph, we are interested by the evolution of dislocations densities for the 1-D Groma-Balogh model (1.1)-(1.2) under the uniformly applied shear stress $L(t) = 3t$.

In this simulations, we choose an example of concentrated dislocations densities, *i.e.* where dislocations densities are initially periodic, and equal to zero on some sub-intervals of $[0, 1[$ (see Figure 4).

This initial condition means that there exists some regions without dislocations, and others with concentrated dislocations.

Intuitively, dislocations are intended to be uniformly distributed in the whole crystal as shown in (Figure 6) where finally a uniform distribution in all the crystal is observed, *i.e.* the density of dislocations becomes a constant.

We remark that when $L(t)$ is non-stationary, our system behaves as a diffusion equation (see [9] for further details). But evidently when $L(t) = 0$, with the same initial condition, the system does not evolve.

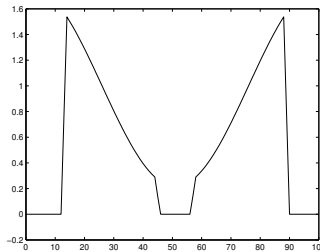


Figure 4: dislocations density ($D\rho_+^0(\cdot) = D\rho_-^0(\cdot)$)

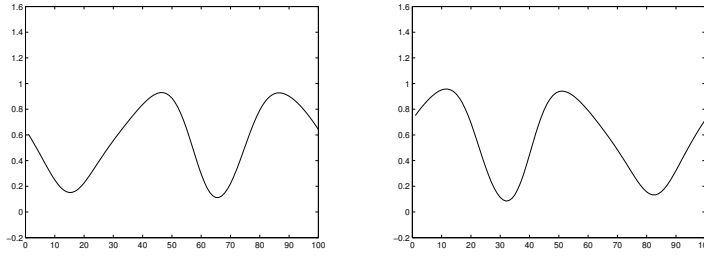


Figure 5: On the left: density ($D\rho_+(\cdot, \frac{1}{2})$); on the right: dislocations density ($D\rho_-(\cdot, \frac{1}{2})$)

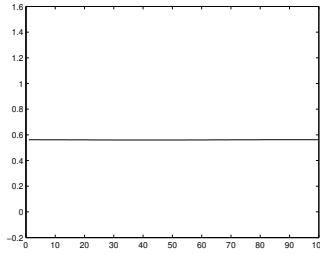


Figure 6: dislocations density ($D\rho_+(\cdot, 3) = D\rho_-(\cdot, 3)$)

Acknowledgements

The authors would like to thanks O. Alvarez, M. El Rhabi, C. Imbert, and R. Monneau for fruitful discussions in the preparation of this paper. This work was supported by the contract JC 1025 called “ACI jeunes chercheuses et jeunes chercheurs” of the French Ministry of Research (2003-2005).

References

- [1] OLIVIER ALVAREZ, PIERRE CARDALIAGUET, AND RÉGIS MONNEAU, *Existence and uniqueness for dislocation dynamics with nonnegative velocity*, Interfaces and Free Boundaries, 7 (2005), pp. 415–434.
- [2] OLIVIER ALVAREZ, ELISABETTA CARLINI, RÉGIS MONNEAU, AND ELISABETH ROUY, *Convergence of a first order scheme for a non local eikonal equation*, To appear in IMACS Journal "Applied Numerical Mathematics", (2005).
- [3] ———, *A convergent scheme for a nonlocal Hamilton-Jacobi equation, modeling dislocation dynamics*, To appear in Numerische Mathematik, (2006).
- [4] OLIVIER ALVAREZ, PHILIPPE HOCH, YANN LE BOUAR, AND RÉGIS MONNEAU, *Résolution en temps court d’une équation de Hamilton-Jacobi non locale décrivant la dynamique d’une dislocation*, C. R. Math. Acad. Sci. Paris, 338 (2004), pp. 679–684.
- [5] ———, *Dislocation dynamics: short time existence and uniqueness of the solution*, To appear in Archive for Rational Mechanics and Analysis, (2005).
- [6] MARTINO BARDI AND ITALO CAPUZZO-DOLCETTA, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Systems & Control: Foundations & Applications, Birkhäuser Boston Inc., Boston, MA, 1997. With appendices by Maurizio Falcone and Pierpaolo Soravia.

- [7] GUY BARLES, *Solutions de viscosité des équations de Hamilton-Jacobi*, vol. 17 of *Mathématiques & Applications* (Berlin) [Mathematics & Applications], Springer-Verlag, Paris, 1994.
- [8] GUY BARLES AND OLIVIER LEY, *Nonlocal first-order hamilton-jacobi equations modelling dislocations dynamics*, To appear in *Comm. Partial Differential Equations*, (2005).
- [9] ARIELA BRIANI, *et al. Homogenization in time for a model with dislocations densities*, In preparation, (2006).
- [10] MICHAEL G. CRANDALL, HITOSHI ISHII, AND PIERRE-LOUIS LIONS, *User's guide to viscosity solutions of second order partial differential equations*, *Bull. Amer. Math. Soc. (N.S.)*, 27 (1992), pp. 1–67.
- [11] M. G. CRANDALL AND P.-L. LIONS, *Two approximations of solutions of Hamilton-Jacobi equations*, *Math. Comp.*, 43 (1984), pp. 1–19.
- [12] HANS ENGLER AND SUZANNE M. LENHART, *Viscosity solutions for weakly coupled systems of Hamilton-Jacobi equations*, *Proc. London Math. Soc. (3)*, 63 (1991), pp. 212–240.
- [13] NICOLAS FORCADEL, *Dislocations dynamics with a mean curvature term: short time existence and uniqueness*, Preprint, (2005).
- [14] I. GROMA, *Link between the microscopic and mesoscopic length-scale description of the collective behaviour of dislocations*, *Phys.Rev.B*, 56 (1997), p. 5807.
- [15] I. GROMA AND P. BALOGH, *Link between the individual and continuum approaches of the description of the collective behavior of dislocations*, *Mat. Sci. Eng*, A234-236 (1997), pp. 249–252.
- [16] ———, *Investigation of dislocation pattern formation in a two-dimensional self-consistent field approximation*, *Acta mater*, 47 (1999), pp. 3647–3654.
- [17] JOHN PRICE HIRTH AND JENS LOTHE, *Theory of dislocations, Second edition*, Krieger, Malabar, Florida, 1992.
- [18] HITOSHI ISHII, *Perron's method for monotone systems of second-order elliptic partial differential equations*, *Differential Integral Equations*, 5 (1992), pp. 1–24.
- [19] HITOSHI ISHII AND SHIGEAKI KOIKE, *Viscosity solutions for monotone systems of second-order elliptic PDEs*, *Comm. Partial Differential Equations*, 16 (1991), pp. 1095–1128.
- [20] ———, *Viscosity solutions of a system of nonlinear second-order elliptic PDEs arising in switching games*, *Funkcial. Ekvac.*, 34 (1991), pp. 143–155.
- [21] E. R. JAKOBSEN AND K. H. KARLSEN, *Convergence rates for semi-discrete splitting approximations for degenerate parabolic equations with source terms*, *BIT*, 45 (2005), pp. 37–67.
- [22] ESPEN ROBSTAD JAKOBSEN, KENNETH HVISTENDAHL KARLSEN, AND NILS HENRIK RISEBRO, *On the convergence rate of operator splitting for Hamilton-Jacobi equations with source terms*, *SIAM J. Numer. Anal.*, 39 (2001), pp. 499–518 (electronic).
- [23] SUZANNE M. LENHART, *Viscosity solutions for weakly coupled systems of first-order partial differential equations*, *J. Math. Anal. Appl.*, 131 (1988), pp. 180–193.
- [24] SUZANNE M. LENHART AND STAVROS A. BELBAS, *A system of nonlinear partial differential equations arising in the optimal control of stochastic systems with switching costs*, *SIAM J. Appl. Math.*, 43 (1983), pp. 465–475.

- [25] SUZANNE M. LENHART AND NAOKI YAMADA, *Viscosity solutions associated with switching game for piecewise-deterministic processes*, Stochastics Stochastics Rep., 38 (1992), pp. 27–47.
- [26] STANLEY OSHER AND JAMES A. SETHIAN, *Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations*, J. Comput. Phys., 79 (1988), pp. 12–49.
- [27] ELISABETH ROUY AND AGNÈS TOURIN, *A viscosity solutions approach to shape-from-shading*, SIAM J. Numer. Anal., 29 (1992), pp. 867–884.
- [28] NAOKI YAMADA, *Viscosity solutions for a system of elliptic inequalities with bilateral obstacles*, Funkcial. Ekvac., 30 (1987), pp. 417–425.