



**HAL**  
open science

# Large Time existence For 1D Green-Naghdi equations

Samer Israwi

► **To cite this version:**

| Samer Israwi. Large Time existence For 1D Green-Naghdi equations. 2009. hal-00415875

**HAL Id: hal-00415875**

**<https://hal.science/hal-00415875>**

Preprint submitted on 11 Sep 2009

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# LARGE TIME EXISTENCE FOR 1D GREEN-NAGHDI EQUATIONS

SAMER ISRAWI

ABSTRACT. We consider here the 1D Green-Naghdi equations that are commonly used in coastal oceanography to describe the propagation of large amplitude surface waves. We show that the solution of the Green-Naghdi equations can be constructed by a standard Picard iterative scheme so that there is no loss of regularity of the solution with respect to the initial condition.

## 1. INTRODUCTION

**1.1. Presentation of the problem.** The water-waves problem for an ideal liquid consists of describing the motion of the free surface and the evolution of the velocity field of a layer of perfect, incompressible, irrotational fluid under the influence of gravity. This motion is described by the free surface Euler equations that are known to be well-posed after the works of Nalimov [16], Yasihara [21], Craig [6], Wu [19, 20] and Lannes [11]. But, because of the complexity of these equations, they are often replaced for practical purposes by approximate asymptotic systems. The most prominent examples are the Green-Naghdi equations (GN) – which is a widely used model in coastal oceanography ([8, 4, 7] and, for instance, [18, 10])–, the Shallow-Water equations, and the Boussinesq systems; their range of validity depends on the physical characteristics of the flow under consideration. In other words, they depend on certain assumptions made on the dimensionless parameters  $\varepsilon$ ,  $\mu$  defined as:

$$\varepsilon = \frac{a}{h_0}, \quad \mu = \frac{h_0^2}{\lambda^2};$$

where  $a$  is the order of amplitude of the waves and the bottom variations;  $\lambda$  is the wave-length of the waves and the bottom variations;  $h_0$  is the reference depth. The parameter  $\varepsilon$  is often called nonlinearity parameter; while  $\mu$  is the shallowness parameter. In the shallow-water scaling ( $\mu \ll 1$ ), and without smallness assumption on  $\varepsilon$  one can derive the so-called Green-Naghdi equations (see [8, 13] for a derivation and [2] for a rigorous justification) also called Serre or fully nonlinear Boussinesq equations [15].

In nondimensionalized variables, denoting by  $\zeta(t, x)$  and  $u(t, x)$  the parameterization of the surface and the vertically averaged horizontal component of the velocity

---

Ce travail a bénéficié d'une aide de l'Agence Nationale de la Recherche portant la référence ANR-08-BLAN-0301-01.

at time  $t$ , and by  $b(x)$  the parameterization of the bottom, the equations read

$$(1) \quad \begin{cases} \partial_t \zeta + \nabla \cdot (hu) = 0, \\ (h + \mu h \mathcal{T}[h, \varepsilon b]) \partial_t u + h \nabla \zeta + \varepsilon h (u \cdot \nabla) u \\ \quad + \mu \varepsilon \left\{ -\frac{1}{3} \nabla [(h^3 ((u \cdot \nabla)(\nabla \cdot u) - (\nabla \cdot u)^2))] + h \mathfrak{R}[h, \varepsilon b] u \right\} = 0, \end{cases}$$

where  $h = 1 + \varepsilon(\zeta - b)$  and

$$\mathcal{T}[h, \varepsilon b] W = -\frac{1}{3h} \nabla (h^3 \nabla \cdot W) + \frac{\varepsilon}{2h} [\nabla (h^2 \nabla b \cdot W) - h^2 \nabla b \nabla \cdot W] + \varepsilon^2 \nabla b \nabla b \cdot W,$$

while the purely topographical term  $\mathfrak{R}[h, \varepsilon b] u$  is defined as:

$$\begin{aligned} \mathfrak{R}[h, \varepsilon b] u &= \frac{\varepsilon}{2h} [\nabla (h^2 (u \cdot \nabla)^2 b) - h^2 ((u \cdot \nabla)(\nabla \cdot u) - (\nabla \cdot u)^2) \nabla b] \\ &\quad + \varepsilon^2 ((u \cdot \nabla)^2 b) \nabla b. \end{aligned}$$

This model is often used in coastal oceanography because it takes into account the dispersive effects neglected by the shallow-water and it is more nonlinear than the Boussinesq equations. A recent rigorous justification of the GN model was given by Li [14] in 1D and for flat bottoms, and by B. Alvarez-Samaniego and D. Lannes [2] in 2008 in the general case. This latter reference relies on well-posedness results for these equations given in [3] and based on general well-posedness results for evolution equations using a Nash-Moser scheme. The result of [3] covers both the case of 1D and 2D surfaces, and allows for non flat bottoms. The reason why a Nash-Moser scheme is used there is because the estimates on the linearized equations exhibit losses of derivatives. However, in the 1D case with flat bottoms, such losses do not occur and it is possible to construct a solution with a standard Picard iterative scheme as in [14]. Our goal here is to show that it is also possible to use such a simple scheme in the 1D case with non flat bottoms, thanks to a careful analysis of the linearized equations.

**1.2. Organization of the paper.** We start by giving some preliminary results in Section 2.1; the main theorem is then stated in Section 2.2 and proved in Section 2.3. Finally, in Appendix A, we give the existence and uniqueness of a solution to the linear Cauchy problem associated to the Green-Naghdi equations. The proof of the energy conservation, stated in the main theorem, is given in Appendix B.

**1.3. Notation.** We denote by  $C(\lambda_1, \lambda_2, \dots)$  a constant depending on the parameters  $\lambda_1, \lambda_2, \dots$  and whose dependence on the  $\lambda_j$  is always assumed to be nondecreasing. The notation  $a \lesssim b$  means that  $a \leq Cb$ , for some nonnegative constant  $C$  whose exact expression is of no importance (*in particular, it is independent of the small parameters involved*).

Let  $p$  be any constant with  $1 \leq p < \infty$  and denote  $L^p = L^p(\mathbb{R})$  the space of all Lebesgue-measurable functions  $f$  with the standard norm

$$\|f\|_{L^p} = \left( \int_{\mathbb{R}} |f(x)| dx \right)^{1/p} < \infty.$$

When  $p = 2$ , we denote the norm  $\|\cdot\|_{L^2}$  simply by  $\|\cdot\|_2$ . The inner product of any functions  $f_1$  and  $f_2$  in the Hilbert space  $L^2(\mathbb{R})$  is denoted by

$$(f_1, f_2) = \int_{\mathbb{R}} f_1(x) f_2(x) dx.$$

The space  $L^\infty = L^\infty(\mathbb{R})$  consists of all essentially bounded, Lebesgue-measurable functions  $f$  with the norm

$$|f|_{L^\infty} = \text{ess sup } |f(x)| < \infty.$$

We denote by  $W^{1,\infty} = W^{1,\infty}(\mathbb{R}) = \{f \in L^\infty, \partial_x f \in L^\infty\}$  endowed with its canonical norm.

For any real constant  $s$ ,  $H^s = H^s(\mathbb{R})$  denotes the Sobolev space of all tempered distributions  $f$  with the norm  $|f|_{H^s} = |\Lambda^s f|_2 < \infty$ , where  $\Lambda$  is the pseudo-differential operator  $\Lambda = (1 - \partial_x^2)^{1/2}$ .

For any functions  $u = u(x, t)$  and  $v(x, t)$  defined on  $\mathbb{R} \times [0, T)$  with  $T > 0$ , we denote the inner product, the  $L^p$ -norm and especially the  $L^2$ -norm, as well as the Sobolev norm, with respect to the spatial variable  $x$ , by  $(u, v) = (u(\cdot, t), v(\cdot, t))$ ,  $|u|_{L^p} = |u(\cdot, t)|_{L^p}$ ,  $|u|_{L^2} = |u(\cdot, t)|_{L^2}$ , and  $|u|_{H^s} = |u(\cdot, t)|_{H^s}$ , respectively.

Let  $C^k(\mathbb{R})$  denote the space of  $k$ -times continuously differentiable functions and  $C_0^\infty(\mathbb{R})$  denote the space of infinitely differentiable functions, with compact support in  $\mathbb{R}$ ; we also denote by  $C_b^\infty(\mathbb{R})$  the space of infinitely differentiable functions that are bounded together with all their derivatives.

Let  $f$  be a function of the independent variables  $x_1, x_2, \dots, x_m$ ; its partial derivative with respect to  $x_k$  is denoted by  $\partial_{x_k} f = f_{x_k}$  for  $1 \leq k \leq m$ .

For any closed operator  $T$  defined on a Banach space  $X$  of functions, the commutator  $[T, f]$  is defined by  $[T, f]g = T(fg) - fT(g)$  with  $f, g$  and  $fg$  belonging to the domain of  $T$ .

## 2. WELL-POSEDNESS OF THE GREEN-NAGHDI EQUATIONS IN 1D

For one dimensional surfaces, the Green-Naghdi equations (1) can be simplified, after some computations, into

$$(2) \quad \begin{cases} \partial_t \zeta + \partial_x(hu) = 0, \\ (h + \mu h T[h, \varepsilon b])[\partial_t u + \varepsilon u \partial_x u] + h \partial_x \zeta + \varepsilon \mu h Q[h, \varepsilon b](u) = 0 \end{cases}$$

where  $h = 1 + \varepsilon(\zeta - b)$  and

$$\begin{aligned} \mathcal{T}[h, \varepsilon b]w &= -\frac{1}{3h} \partial_x(h^3 w_x) + \frac{\varepsilon}{2h} [\partial_x(h^2 b_x w) - h^2 b_x w_x] + \varepsilon^2 b_x^2 w, \\ Q[h, \varepsilon b](w) &= \frac{2}{3h} \partial_x(h^3 w_x^2) + \varepsilon h w_x^2 b_x + \varepsilon \frac{1}{2h} \partial_x(h^2 w^2 b_{xx}) + \varepsilon^2 w^2 b_{xx} b_x. \end{aligned}$$

**Remark 1.** *The interest of the formulation (2) of the Green-Naghdi equation is that all the third order derivatives of  $u$  have been factorized by  $(h + \mu h T[h, \varepsilon b])$ . Indeed,  $Q[h, \varepsilon b]$  is a second order differential operator. This was used in [14] in the case of flat bottoms ( $b = 0$ ).*

**2.1. Preliminary results.** For the sake of simplicity, we write

$$\mathfrak{T} = h + \mu h T[h, \varepsilon b].$$

We always assume that the nonzero depth condition

$$(3) \quad \exists h_0 > 0, \quad \inf_{x \in \mathbb{R}} h \geq h_0, \quad h = 1 + \varepsilon(\zeta - b)$$

is valid initially, which is a necessary condition for the GN system (2) to be physically valid. We shall demonstrate that the operator  $\mathfrak{T}$  plays an important role in

the energy estimate and the local well-posedness of the GN system (2). Therefore, we give here some of its properties.

The following lemma gives an important invertibility result on  $\mathfrak{T}$ .

**Lemma 1.** *Let  $b \in C_b^\infty(\mathbb{R})$  and  $\zeta \in W^{1,\infty}(\mathbb{R})$  be such that (3) is satisfied. Then the operator*

$$\mathfrak{T} : H^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

*is well defined, one-to-one and onto.*

**Remark 2.** *Here and throughout the rest of this paper, and for the sake of simplicity, we do not try to give some optimal regularity assumption on the bottom parameterization  $b$ . This could easily be done, but is of no interest for our present purpose. Consequently, we omit to write the dependance on  $b$  of the different quantities that appear in the proof.*

*Proof.* In order to prove the invertibility of  $\mathfrak{T}$ , let us first remark that the quantity  $|v|_*^2$  defined as

$$|v|_*^2 = |v|_2^2 + \mu |\partial_x v|_2^2,$$

is equivalent to the  $H^1(\mathbb{R})$ -norm but not uniformly with respect to  $\mu \in (0, 1)$ . We define by  $H_*^1(\mathbb{R})$  the space  $H^1(\mathbb{R})$  endowed with this norm. The bilinear form:

$$a(u, v) = (hu, v) + \mu \left( h \left( \frac{h}{\sqrt{3}} u_x - \frac{\sqrt{3}}{2} \varepsilon b_x u \right), \frac{h}{\sqrt{3}} v_x - \frac{\sqrt{3}}{2} \varepsilon b_x v \right) + \frac{\mu \varepsilon^2}{4} (hb_x u, b_x v).$$

is obviously continuous on  $H_*^1(\mathbb{R}) \times H_*^1(\mathbb{R})$ . Remarking that

$$\begin{aligned} a(v, v) = (\mathfrak{T}v, v) &= (hv, v) \\ &+ \mu \left( h \left( \frac{h}{\sqrt{3}} v_x - \frac{\sqrt{3}}{2} \varepsilon b_x v \right), \frac{h}{\sqrt{3}} v_x - \frac{\sqrt{3}}{2} \varepsilon b_x v \right) + \frac{\mu \varepsilon^2}{4} (hb_x v, b_x v), \end{aligned}$$

we have

$$\begin{aligned} |v|_*^2 &\leq |v|_2^2 + \frac{3\mu}{h_0^2} \left| \frac{h}{\sqrt{3}} v_x \right|_2^2 \\ &\leq |v|_2^2 + \frac{6\mu}{h_0^2} \left( \left| \frac{h}{\sqrt{3}} v_x - \frac{\sqrt{3}}{2} \varepsilon b_x v \right|_2^2 + \frac{3\varepsilon^2}{4} |b_x v|_2^2 \right). \end{aligned}$$

One deduces that

$$\max \left\{ 1, \frac{18}{h_0^2} \right\} \left( |v|_2^2 + \mu \left| \frac{h}{\sqrt{3}} v_x - \frac{\sqrt{3}}{2} \varepsilon b_x v \right|_2^2 + \frac{\mu \varepsilon^2}{4} |b_x v|_2^2 \right) \geq |v|_*^2.$$

Since from (3) we also get

$$a(v, v) \geq h_0 |v|_2^2 + \mu h_0 \left( \left| \frac{h}{\sqrt{3}} v_x - \frac{\sqrt{3}}{2} \varepsilon b_x v \right|_2^2 + \frac{\mu \varepsilon^2}{4} |b_x v|_2^2 \right),$$

it is easy to deduce that

$$(4) \quad a(v, v) \geq \frac{h_0}{\max \left\{ 1, \frac{18}{h_0^2} \right\}} |v|_*^2.$$

In particular,  $a$  is coercive on  $H_*^1$ . Using Lax-Milgram lemma, for all  $f \in L^2(\mathbb{R})$ , there exists unique  $u \in H_*^1(\mathbb{R})$  such that, for all  $v \in H_*^1(\mathbb{R})$

$$a(u, v) = (f, v);$$

equivalently, there is a unique variational solution to the equation

$$(5) \quad \mathfrak{T}u = f.$$

We then get from the definition of  $\mathfrak{T}$  that

$$\partial_x^2 u = \frac{hu + \frac{\varepsilon\mu}{2}\partial_x(h^2b_x)u + \varepsilon^2\mu hb_x^2u - \frac{\mu}{3}\partial_x h^3\partial_x u - f}{\frac{\mu h^3}{3}};$$

since  $u \in H^1(\mathbb{R})$  and  $f \in L^2(\mathbb{R})$ , we get  $\partial_x^2 u \in L^2(\mathbb{R})$  and thus  $u \in H^2(\mathbb{R})$ .  $\square$

The following lemma then gives some properties of the inverse operator  $\mathfrak{T}^{-1}$ .

**Lemma 2.** *Let  $b \in C_b^\infty(\mathbb{R})$ ,  $t_0 > 1/2$  and  $\zeta \in H^{t_0+1}(\mathbb{R})$  be such that (3) is satisfied. Then:*

$$(i) \quad \forall 0 \leq s \leq t_0 + 1, \quad |\mathfrak{T}^{-1}f|_{H^s} + \sqrt{\mu}|\partial_x \mathfrak{T}^{-1}f|_{H^s} \leq C\left(\frac{1}{h_0}, |h-1|_{H^{t_0+1}}\right)|f|_{H^s};$$

$$(ii) \quad \forall 0 \leq s \leq t_0 + 1, \quad \sqrt{\mu}|\mathfrak{T}^{-1}\partial_x g|_{H^s} \leq C\left(\frac{1}{h_0}, |h-1|_{H^{t_0+1}}\right)|g|_{H^s};$$

(iii) *If  $s \geq t_0 + 1$  and  $\zeta \in H^s(\mathbb{R})$  then:*

$$\|\mathfrak{T}^{-1}\|_{H^s(\mathbb{R}) \rightarrow H^s(\mathbb{R})} + \sqrt{\mu}\|\mathfrak{T}^{-1}\partial_x\|_{H^s(\mathbb{R}) \rightarrow H^s(\mathbb{R})} \leq c_s,$$

where  $c_s$  is a constant depending on  $\frac{1}{h_0}$ ,  $|h-1|_{H^s}$  and independent of  $(\mu, \varepsilon) \in (0, 1)^2$ .

*Proof. Step 1.* We prove that if  $u \in H_*^1(\mathbb{R})$  solves

$$\mathfrak{T}u = f + \sqrt{\mu}\partial_x g$$

for  $f, g \in L^2(\mathbb{R})$ , then one has

$$|u|_{H_*^1} \leq C\left(\frac{1}{h_0}\right)(|f|_2 + |g|_2).$$

Indeed, multiplying the equation by  $u$  and integrating by parts, one gets, with the notations used in the proof of lemma 1

$$a(u, u) \leq (f, u) - (g, \sqrt{\mu}\partial_x u).$$

We thus get from the proof of Lemma 1 and Cauchy-Schwarz inequality that

$$\frac{h_0}{\max\{1, \frac{18}{h_0^2}\}}|u|_{H_*^1}^2 \leq |f|_2|u|_2 + |g|_2|u|_{H_*^1},$$

and the result follows easily.

**Step 2.** We prove here that  $|\mathfrak{T}^{-1}f|_{H^s} + \sqrt{\mu}|\partial_x \mathfrak{T}^{-1}f|_{H^s} \leq C\left(\frac{1}{h_0}, |h-1|_{H^{t_0+1}}\right)|f|_{H^s}$ . Indeed, if  $f \in H^s$  and  $u = \mathfrak{T}^{-1}f$  then  $\mathfrak{T}u = f$ . Applying  $\Lambda^s$  to this identity, we get

$$\begin{aligned} \mathfrak{T}(\Lambda^s u) &= \Lambda^s f + [\mathfrak{T}, \Lambda^s]u \\ &= \tilde{f} + \sqrt{\mu}\partial_x \tilde{g}, \end{aligned}$$

with,

$$\tilde{f} = \Lambda^s f - [\Lambda^s, h]u + \frac{\varepsilon\mu}{2}[\Lambda^s, h^2b_x]u_x - \varepsilon^2\mu[\Lambda^s, h^2b_x]u,$$

and

$$\tilde{g} = \frac{\sqrt{\mu}}{3}[\Lambda^s, h^3]u_x - \frac{\varepsilon\sqrt{\mu}}{2}[\Lambda^s, h^2b_x]u.$$

Now, one can deduce from the commutator estimate (see e.g Lemma 4.6 of [3])

$$(6) \quad |[\Lambda^s, F]G|_2 \lesssim |\nabla F|_{H^{t_0}} |G|_{H^{s-1}}$$

that

$$|\tilde{f}|_2 + |\tilde{g}|_2 \leq |f|_{H^s} + C\left(\frac{1}{h_0}, |h-1|_{H^{t_0+1}}\right) (|u|_{H^{s-1}} + \sqrt{\mu}|\partial_x u|_{H^{s-1}}).$$

One can use Step 1 and a continuous induction on  $s$  to show that the inequality (i) holds for  $0 \leq s \leq t_0 + 1$ .

**Step 3.** We prove here that  $\sqrt{\mu}|\mathfrak{T}^{-1}\partial_x g|_{H^s} \leq C\left(\frac{1}{h_0}, |h-1|_{H^{t_0+1}}\right)|g|_{H^s}$ . Indeed, if  $g \in H^s$  and  $u = \sqrt{\mu}\mathfrak{T}^{-1}\partial_x g$  then  $\mathfrak{T}u = \sqrt{\mu}\partial_x g$  and thus

$$\mathfrak{T}(\Lambda^s u) = \tilde{f} + \sqrt{\mu}\partial_x \tilde{g},$$

with,

$$\tilde{f} = -[\Lambda^s, h]u + \frac{\varepsilon\mu}{2}[\Lambda^s, h^2 b_x]u_x - \varepsilon^2\mu[\Lambda^s, h^2 b_x]u,$$

and

$$\tilde{g} = \Lambda^s g + \frac{\sqrt{\mu}}{3}[\Lambda^s, h^3]u_x - \frac{\varepsilon\sqrt{\mu}}{2}[\Lambda^s, h^2 b_x]u.$$

Proceeding now as for the Step 2, one can deduce (ii).

**Step 4.** If  $s \geq t_0 + 1$  then one can prove (iii) proceeding as in Step 2 and 3 above, but replacing the commutator estimate (6) by the following one

$$(7) \quad |[\Lambda^s, F]G|_2 \lesssim |\nabla F|_{H^{s-1}} |G|_{H^{s-1}}.$$

□

**2.2. Linear analysis.** In order to rewrite the GN equations (2) in a condensed form, let us decompose  $Q[h, \varepsilon b](u)$  as

$$\varepsilon\mu h Q[h, \varepsilon b](u) = Q_1[U]u_x + q_2(U)$$

where  $U = (\zeta, u)^T$  and

$$(8) \quad Q_1[U]f = \frac{2}{3}\varepsilon\mu\partial_x(h^3 u_x f) + \varepsilon^2\mu h^2 b_x u_x f + \varepsilon^2\mu h^2 b_{xx} u f$$

$$q_2(U) = \varepsilon^3\mu h b_{xx} b_x u^2 + \frac{1}{2}\varepsilon^2\mu\partial_x(h^2 b_{xx})u^2.$$

The Green-Naghdi equations (2) can be written after applying  $\mathfrak{T}^{-1}$  to both sides of the second equation in (2) as

$$(9) \quad \partial_t U + A[U]\partial_x U + B(U) = 0,$$

with  $U = (\zeta, u)^T$  and where

$$(10) \quad A[U] = \begin{pmatrix} \varepsilon u & h \\ \mathfrak{T}^{-1}(h \cdot) & \varepsilon u + \mathfrak{T}^{-1}Q_1[U] \end{pmatrix}$$

and

$$(11) \quad B(U) = \begin{pmatrix} \varepsilon b_x u \\ \mathfrak{T}^{-1} q_2(U) \end{pmatrix}.$$

This subsection is devoted to the proof of energy estimates for the following initial value problem around some reference state  $\underline{U} = (\underline{\zeta}, \underline{u})^T$ :

$$(12) \quad \begin{cases} \partial_t U + A[\underline{U}] \partial_x U + B(\underline{U}) = 0; \\ U|_{t=0} = U_0. \end{cases}$$

We define now the  $X^s$  spaces, which are the energy spaces for this problem.

**Definition 1.** For all  $s \geq 0$  and  $T > 0$ , we denote by  $X^s$  the vector space  $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$  endowed with the norm

$$\forall U = (\zeta, u) \in X^s, \quad |U|_{X^s}^2 := |\zeta|_{H^s}^2 + |u|_{H^s}^2 + \mu |\partial_x u|_{H^s}^2,$$

while  $X_T^s$  stands for  $C([0, \frac{T}{\varepsilon}]; X^s)$  endowed with its canonical norm.

First remark that a symmetrizer for  $A[\underline{U}]$  is given by

$$(13) \quad S = \begin{pmatrix} 1 & 0 \\ 0 & \underline{\mathfrak{T}} \end{pmatrix},$$

with  $\underline{h} = 1 + \varepsilon(\underline{\zeta} - b)$  and  $\underline{\mathfrak{T}} = \underline{h} + \mu \underline{h} \mathcal{T}[\underline{h}, \varepsilon b]$ . A natural energy for the IVP (12) is given by

$$(14) \quad E^s(U)^2 = (\Lambda^s U, S \Lambda^s U).$$

The link between  $E^s(U)$  and the  $X^s$ -norm is investigated in the following Lemma.

**Lemma 3.** Let  $b \in C_b^\infty(\mathbb{R})$ ,  $s \geq 0$  and  $\underline{\zeta} \in W^{1,\infty}(\mathbb{R})$ . Under the condition (3),  $E^s(U)$  is uniformly equivalent to the  $|\cdot|_{X^s}$ -norm with respect to  $(\mu, \varepsilon) \in (0, 1)^2$ :

$$E^s(U) \leq C(|\underline{h}|_{L^\infty}, |\underline{h}_x|_{L^\infty}) |U|_{X^s},$$

and

$$|U|_{X^s} \leq C\left(\frac{1}{h_0}\right) E^s(U).$$

*Proof.* Notice first that

$$E^s(U)^2 = |\Lambda^s \zeta|_2^2 + (\Lambda^s u, \underline{\mathfrak{T}} \Lambda^s u),$$

one gets the first estimate using the explicit expression of  $\underline{\mathfrak{T}}$ , integration by parts and Cauchy-Schwarz inequality.

The other inequality can be proved by using that  $\inf_{x \in \mathbb{R}} h \geq h_0 > 0$  and proceeding as in the proof of Lemma 1.  $\square$

We prove now the energy estimates in the following proposition:

**Proposition 1.** Let  $b \in C_b^\infty(\mathbb{R})$ ,  $t_0 > 1/2$ ,  $s \geq t_0 + 1$ . Let also  $\underline{U} = (\underline{\zeta}, \underline{u})^T \in X_T^s$  be such that  $\partial_t \underline{U} \in X_T^{s-1}$  and satisfying the condition (3) on  $[0, \frac{T}{\varepsilon}]$ . Then for



all  $U_0 \in X^s$  there exists a unique solution  $U = (\zeta, u)^T \in X_T^s$  to (12) and for all  $0 \leq t \leq \frac{T}{\varepsilon}$

$$E^s(U(t)) \leq e^{\varepsilon\lambda_T t} E^s(U_0) + \varepsilon \int_0^t e^{\varepsilon\lambda_T(t-t')} C(E^s(\underline{U})(t')) dt'.$$

For some  $\lambda_T = \lambda_T(\sup_{0 \leq t \leq T/\varepsilon} E^s(\underline{U}(t)), \sup_{0 \leq t \leq T/\varepsilon} |\partial_t \underline{h}(t)|_{L^\infty})$ .

*Proof.* Existence and uniqueness of a solution to the IVP (12) is achieved in appendix A and we thus focus our attention on the proof of the energy estimate. For any  $\lambda \in \mathbb{R}$ , we compute

$$e^{\varepsilon\lambda t} \partial_t (e^{-\varepsilon\lambda t} E^s(U)^2) = -\varepsilon\lambda E^s(U)^2 + \partial_t (E^s(U)^2).$$

Since

$$E^s(U)^2 = (\Lambda^s U, S\Lambda^s U),$$

we have

$$(15) \quad \partial_t (E^s(U)^2) = 2(\Lambda^s \zeta, \Lambda^s \zeta_t) + 2(\Lambda^s u, \underline{\mathfrak{I}}\Lambda^s u_t) + (\Lambda^s u, [\partial_t, \underline{\mathfrak{I}}]\Lambda^s u).$$

One gets using the equations (12) and integrating by parts,

$$(16) \quad \begin{aligned} \frac{1}{2} e^{\varepsilon\lambda t} \partial_t (e^{-\varepsilon\lambda t} E^s(U)^2) &= -\frac{\varepsilon\lambda}{2} E^s(U)^2 - (SA[\underline{U}]\Lambda^s \partial_x U, \Lambda^s U) \\ &\quad - ([\Lambda^s, A[\underline{U}]] \partial_x U, S\Lambda^s U) - (\Lambda^s B(\underline{U}), S\Lambda^s U) \\ &\quad + \frac{1}{2} (\Lambda^s u, [\partial_t, \underline{\mathfrak{I}}]\Lambda^s u). \end{aligned}$$

We now turn to bound from above the different components of the r.h.s of (16).

- Estimate of  $(SA[\underline{U}]\Lambda^s \partial_x U, \Lambda^s U)$ . Remarking that

$$SA[\underline{U}] = \begin{pmatrix} \varepsilon \underline{u} & \underline{h} \\ \underline{h} & \underline{\mathfrak{I}}(\varepsilon \underline{u}) + Q_1[\underline{U}] \end{pmatrix},$$

we get

$$\begin{aligned} (SA[\underline{U}]\Lambda^s \partial_x U, \Lambda^s U) &= (\varepsilon \underline{u} \Lambda^s \zeta_x, \Lambda^s \zeta) + (\underline{h} \Lambda^s u_x, \Lambda^s \zeta) \\ &\quad + (\underline{h} \Lambda^s \zeta_x, \Lambda^s u) + ((\underline{\mathfrak{I}}(\varepsilon \underline{u}) + Q_1[\underline{U}]) \Lambda^s u_x, \Lambda^s u) \\ &=: A_1 + A_2 + A_3 + A_4. \end{aligned}$$

We now focus to control  $(A_j)_{1 \leq j \leq 4}$ .

- Control of  $A_1$ . Integrating by parts, one obtains

$$A_1 = (\varepsilon \underline{u} \Lambda^s \zeta, \Lambda^s \zeta_x) = -\frac{1}{2} (\varepsilon \underline{u}_x \Lambda^s \zeta, \Lambda^s \zeta)$$

one can conclude by Cauchy-Schwarz inequality that

$$|A_1| \leq \varepsilon C(|\underline{u}_x|_{L^\infty}) E^s(U)^2.$$

- Control of  $A_2 + A_3$ . First remark that

$$|A_2 + A_3| = |(\underline{h}_x \Lambda^s u, \Lambda^s \zeta)| \leq |\underline{h}_x|_{L^\infty} E^s(U)^2;$$

we get,

$$|A_2 + A_3| \leq \varepsilon C(|\underline{h}_x|_{L^\infty}) E^s(U)^2.$$

– Control of  $A_4$ . One computes,

$$\begin{aligned} A_4 &= \varepsilon(\underline{\mathfrak{T}}(\underline{u}\Lambda^s u_x), \Lambda^s u) + (Q_1[\underline{U}]\Lambda^s u_x, \Lambda^s u) \\ &=: A_{41} + A_{42}. \end{aligned}$$

Note that

$$\begin{aligned} A_{41} &= \varepsilon(\underline{h}\underline{u}\Lambda^s u_x, \Lambda^s u) + \frac{\varepsilon\mu}{3}(\underline{h}^3(\underline{u}\Lambda^s u_x)_x, \Lambda^s u_x) \\ &\quad - \frac{\varepsilon^2\mu}{2}(\underline{h}^2 b_x(\underline{u}\Lambda^s u_x)_x, \Lambda^s u) - \frac{\varepsilon^2\mu}{2}(\underline{h}^2 b_x \underline{u}\Lambda^s u_x, \Lambda^s u_x) \\ &\quad + \varepsilon^3\mu(\underline{h}\underline{u}b_x^2 \Lambda^s u_x, \Lambda^s u); \end{aligned}$$

since

$$(\underline{h}^3(\underline{u}\Lambda^s u_x)_x, \Lambda^s u_x) = \frac{1}{2}(-(\underline{h}_x^3 \underline{u}\Lambda^s u_x, \Lambda^s u_x) + (\underline{h}^3 \underline{u}_x \Lambda^s u_x, \Lambda^s u_x)),$$

by using successively integration by parts and the Cauchy-Schwarz inequality, one obtains directly:

$$|A_{41}| \leq \varepsilon C(|\underline{u}|_{W^{1,\infty}}, |\underline{\zeta}|_{W^{1,\infty}}) E^s(U)^2.$$

For  $A_{42}$ , remark that

$$\begin{aligned} |A_{42}| &= |(Q_1[\underline{U}]\Lambda^s u_x, \Lambda^s u)| \\ &= \left| -\frac{2}{3}\varepsilon\mu(\underline{h}^3 \underline{u}_x \Lambda^s u_x, \Lambda^s u_x) + \varepsilon^2\mu(\underline{h}^2 \underline{u}_x b_x \Lambda^s u_x, \Lambda^s u) \right. \\ &\quad \left. + \varepsilon^2\mu(\underline{h}^2 \underline{u} b_{xx} \Lambda^s u_x, \Lambda^s u) \right| \end{aligned}$$

therefore

$$|A_{42}| \leq \varepsilon C(|\underline{u}|_{W^{1,\infty}}, |\underline{\zeta}|_{W^{1,\infty}}) E^s(U)^2.$$

This shows that

$$|A_4| \leq \varepsilon C(|\underline{u}|_{W^{1,\infty}}, |\underline{\zeta}|_{W^{1,\infty}}) E^s(U)^2.$$

• Estimate of  $([\Lambda^s, A[\underline{U}]]\partial_x U, S\Lambda^s U)$ . Remark first that

$$\begin{aligned} ([\Lambda^s, A[\underline{U}]]\partial_x U, S\Lambda^s U) &= ([\Lambda^s, \varepsilon\underline{u}]\zeta_x, \Lambda^s \zeta) + ([\Lambda^s, \underline{h}]u_x, \Lambda^s \zeta) \\ &\quad + ([\Lambda^s, \underline{\mathfrak{T}}^{-1} \underline{h}]\zeta_x, \underline{\mathfrak{T}}\Lambda^s u) + ([\Lambda^s, \varepsilon\underline{u}]u_x, \underline{\mathfrak{T}}\Lambda^s u) \\ &\quad + ([\Lambda^s, \underline{\mathfrak{T}}^{-1} Q_1[\underline{U}]]u_x, \underline{\mathfrak{T}}\Lambda^s u) \\ &=: B_1 + B_2 + B_3 + B_4 + B_5. \end{aligned}$$

– Control of  $B_1 + B_2 = ([\Lambda^s, \varepsilon\underline{u}]\zeta_x, \Lambda^s \zeta) + ([\Lambda^s, \underline{h}]u_x, \Lambda^s \zeta)$ . Since  $s \geq t_0 + 1$ , we can use the commutator estimate (7) to get

$$|B_1 + B_2| \leq \varepsilon C(E^s(\underline{U})) E^s(U)^2.$$

– Control of  $B_4 = ([\Lambda^s, \varepsilon \underline{u}]u_x, \underline{\mathfrak{T}}\Lambda^s u)$ . By using the explicit expression of  $\underline{\mathfrak{T}}$  we get

$$\begin{aligned} B_4 &= ([\Lambda^s, \varepsilon \underline{u}]u_x, \underline{h}\Lambda^s u) + \frac{\mu}{3}(\partial_x[\Lambda^s, \varepsilon \underline{u}]u_x, \underline{h}^3 \Lambda^s u_x) \\ &\quad - \frac{\varepsilon\mu}{2}([\Lambda^s, \varepsilon \underline{u}]u_x, \underline{h}^2 b_x \Lambda^s u_x) + \frac{\varepsilon\mu}{2}([\Lambda^s, \varepsilon \underline{u}]u_x, \partial_x(\underline{h}^2 b_x \Lambda^s u)) \\ &\quad + \varepsilon^2 \mu([\Lambda^s, \varepsilon \underline{u}]u_x, \underline{h} b_x^2 \Lambda^s u), \end{aligned}$$

using the Cauchy-Schwarz inequality and the fact that

$$\partial_x[\Lambda^s, f]g = [\Lambda^s, f_x]g + [\Lambda^s, f]g_x$$

one obtains directly:

$$|B_4| \leq \varepsilon C(E^s(\underline{U}))E^s(U)^2.$$

– Control of  $B_3 = ([\Lambda^s, \underline{\mathfrak{T}}^{-1} \underline{h}]\zeta_x, \underline{\mathfrak{T}}\Lambda^s u)$ . Remark first that

$$\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1} \underline{h}]\zeta_x = \underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1} \underline{h}]\zeta_x - [\Lambda^s, \underline{h}]\zeta_x;$$

moreover, since  $[\Lambda^s, \underline{\mathfrak{T}}^{-1}] = -\underline{\mathfrak{T}}^{-1}[\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}$ , one gets

$$\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1} \underline{h}]\zeta_x = -[\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1} \underline{h}\zeta_x + [\Lambda^s, \underline{h}]\zeta_x,$$

and one can check by using the explicit expression of  $\underline{\mathfrak{T}}$  that

$$\begin{aligned} \underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1} \underline{h}]\zeta_x &= -[\Lambda^s, \underline{h}]\underline{\mathfrak{T}}^{-1} \underline{h}\zeta_x + \frac{\mu}{3}\partial_x\{[\Lambda^s, \underline{h}^3]\partial_x(\underline{\mathfrak{T}}^{-1} \underline{h}\zeta_x)\} \\ &\quad - \frac{\varepsilon\mu}{2}\partial_x[\Lambda^s, \underline{h}^2 b_x]\underline{\mathfrak{T}}^{-1} \underline{h}\zeta_x + \frac{\varepsilon\mu}{2}[\Lambda^s, \underline{h}^2 b_x]\partial_x \underline{\mathfrak{T}}^{-1} \underline{h}\zeta_x \\ &\quad - \varepsilon^2 \mu[\Lambda^s, \underline{h} b_x^2]\underline{\mathfrak{T}}^{-1} \underline{h}\zeta_x + [\Lambda^s, \underline{h}]\zeta_x. \end{aligned}$$

One deduces directly from Lemma 2, an integration by parts, and Cauchy-Schwarz inequality that

$$\begin{aligned} |B_3| &\leq C\left(\frac{1}{h_0}, |\underline{h} - 1|_{H^s}\right) \left\{ \left( |\underline{h}_x|_{H^{s-1}} + \frac{\varepsilon\mu}{2} |\underline{h}^2 b_x|_{H^s} + \varepsilon^2 \mu |\underline{h} b_x^2|_{H^s} \right) |\underline{h}\zeta_x|_{H^{s-1}} \right. \\ &\quad \left. + \left( \frac{\sqrt{\mu}}{3} |\underline{h}_x^3|_{H^{s-1}} + \frac{\varepsilon\sqrt{\mu}}{2} |\underline{h}^2 b_x|_{H^s} \right) |\underline{h}\zeta_x|_{H^{s-1}} + |\underline{h}_x|_{H^{s-1}} |\zeta_x|_{H^{s-1}} \right\} |\Lambda^s u|_{H_*^1}. \end{aligned}$$

Finally, since

$$|\underline{h}\zeta_x|_{H^{s-1}} \leq C(E^s(\underline{U}))E^s(U) \quad \text{and} \quad |\underline{h} b_x^2|_{H^s} + |\underline{h}^2 b_x|_{H^s} \leq C(E^s(\underline{U})),$$

we deduce

$$|B_3| \leq \varepsilon C(E^s(\underline{U}))E^s(U)^2.$$

– Control of  $B_5 = ([\Lambda^s, \underline{\mathfrak{T}}^{-1} Q_1[\underline{U}]]u_x, \underline{\mathfrak{T}}\Lambda^s u)$ . Let us first write

$$\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1} Q_1[\underline{U}]]u_x = -[\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1} Q_1[\underline{U}]u_x + [\Lambda^s, Q_1[\underline{U}]]u_x$$

so, that

$$\begin{aligned}\underline{\mathfrak{I}}[\Lambda^s, \underline{\mathfrak{I}}^{-1}Q_1[\underline{U}]]u_x &= -[\Lambda^s, \underline{h}]\underline{\mathfrak{I}}^{-1}Q_1[\underline{U}]u_x + \frac{\mu}{3}\partial_x\{[\Lambda^s, \underline{h}^3]\partial_x(\underline{\mathfrak{I}}^{-1}Q_1[\underline{U}]u_x)\} \\ &\quad - \frac{\varepsilon\mu}{2}\partial_x\{[\Lambda^s, \underline{h}^2b_x]\underline{\mathfrak{I}}^{-1}Q_1[\underline{U}]u_x\} + \frac{\varepsilon\mu}{2}[\Lambda^s, \underline{h}^2b_x]\partial_x(\underline{\mathfrak{I}}^{-1}Q_1[\underline{U}]u_x) \\ &\quad - \varepsilon^2\mu[\Lambda^s, \underline{h}b_x^2]\underline{\mathfrak{I}}^{-1}Q_1[\underline{U}]u_x + [\Lambda^s, Q_1[\underline{U}]]u_x.\end{aligned}$$

To control the term  $([\Lambda^s, Q_1[\underline{U}]]u_x, \Lambda^s u)$  we use the explicit expression of  $Q_1[\underline{U}]$ :

$$Q_1[\underline{U}]f = \frac{2}{3}\varepsilon\mu\partial(\underline{h}^3\underline{u}_x f) + \varepsilon^2\mu\underline{h}^2b_x\underline{u}_x f + \varepsilon^2\mu\underline{h}^2b_{xx}\underline{u}f,$$

and the fact that

$$\partial_x[\Lambda^s, f]g = [\Lambda^s, \partial_x(f\cdot)]g.$$

Similarly to control the term  $(\partial_x\{[\Lambda^s, \underline{h}^3]\partial_x(\underline{\mathfrak{I}}^{-1}Q_1[\underline{U}]u_x)\}, \Lambda^s u)$  we use the explicit expression of  $Q_1[\underline{U}]$ , the commutator estimate (7) and Lemma 2. Indeed,

$$\begin{aligned}(\partial_x\{[\Lambda^s, \underline{h}^3]\partial_x(\underline{\mathfrak{I}}^{-1}Q_1[\underline{U}]u_x)\}, \Lambda^s u) &= -\frac{2}{3}\varepsilon\mu([\Lambda^s, \underline{h}^3]\partial_x(\underline{\mathfrak{I}}^{-1}\partial_x(\underline{h}^3\underline{u}_x u_x)), \Lambda^s u_x) \\ &\quad - \varepsilon^2\mu([\Lambda^s, \underline{h}^3]\partial_x(\underline{\mathfrak{I}}^{-1}(\underline{h}^2b_x\underline{u}_x u_x)), \Lambda^s u_x) \\ &\quad - \varepsilon^2\mu([\Lambda^s, \underline{h}^3]\partial_x(\underline{\mathfrak{I}}^{-1}(\underline{h}^2b_{xx}\underline{u}u_x)), \Lambda^s u_x).\end{aligned}$$

and thus, after remarking that

$$\begin{aligned}|\partial_x(\underline{\mathfrak{I}}^{-1}\partial_x(\underline{h}^3\underline{u}_x u_x))|_{H^{s-1}} &\leq |\underline{\mathfrak{I}}^{-1}\partial_x(\underline{h}^3\underline{u}_x u_x)|_{H^s} \\ &\leq \|\underline{\mathfrak{I}}^{-1}\partial_x\|_{H^s(\mathbb{R}) \rightarrow H^s(\mathbb{R})} |\underline{h}^3\underline{u}_x u_x|_{H^s}.\end{aligned}$$

we can proceed as for the control of  $B_3$  to get

$$|B_5| \leq \varepsilon C(E^s(\underline{U}))E^s(U)^2.$$

- Estimate of  $(\Lambda^s B(\underline{U}), S\Lambda^s U)$ . Note first that

$$B(\underline{U}) = \begin{pmatrix} \varepsilon b_x \underline{u} \\ \underline{\mathfrak{I}}^{-1} q_2(\underline{U}) \end{pmatrix}$$

where,  $q_2(\cdot)$  as in (8), so that

$$\begin{aligned}(\Lambda^s B(\underline{U}), S\Lambda^s U) &= (\Lambda^s(\varepsilon b_x \underline{u}), \Lambda^s \zeta) + (\Lambda^s(\underline{\mathfrak{I}}^{-1} q_2(\underline{U})), \underline{\mathfrak{I}}\Lambda^s u) \\ &= (\Lambda^s(\varepsilon b_x \underline{u}), \Lambda^s \zeta) - ([\Lambda^s, \underline{\mathfrak{I}}]\underline{\mathfrak{I}}^{-1} q_2(\underline{U}), \Lambda^s u) \\ &\quad + (\Lambda^s q_2(\underline{U}), \Lambda^s u).\end{aligned}$$

Using again here the explicit expressions of  $\underline{\mathfrak{I}}$ ,  $q_2(\underline{U})$  and Lemma 2, we get

$$(\Lambda^s B(\underline{U}), S\Lambda^s U) \leq \varepsilon C(E^s(\underline{U}))E^s(U).$$

- Estimate of  $(\Lambda^s u, [\partial_t, \underline{\mathfrak{I}}]\Lambda^s u)$ . We have that

$$\begin{aligned} (\Lambda^s u, [\partial_t, \underline{\mathfrak{I}}]\Lambda^s u) &= (\Lambda^s u, \partial_t \underline{h} \Lambda^s u) + \frac{\mu}{3} (\Lambda^s u_x, \partial_t \underline{h}^3 \Lambda^s u_x) \\ &\quad - \frac{\varepsilon \mu}{2} (\Lambda^s u, \partial_t \underline{h}^2 b_x \Lambda^s u_x) - \frac{\varepsilon \mu}{2} (\Lambda^s u_x, \partial_t \underline{h}^2 b_x \Lambda^s u) \\ &\quad + \varepsilon^2 \mu (\Lambda^s u, \partial_t \underline{h} b_x^2 \Lambda^s u). \end{aligned}$$

Controlling these terms by  $\varepsilon C(E^s(\underline{U}), |\partial_t \underline{h}|_{L^\infty}) E^s(U)^2$  follows directly from a Cauchy-Schwarz inequality and an integration by parts.

Gathering the informations provided by the above estimates and using the fact that  $H^s(\mathbb{R}) \subset W^{1,\infty}$ , we get

$$e^{\varepsilon \lambda t} \partial_t (e^{-\varepsilon \lambda t} E^s(U)^2) \leq \varepsilon (C(E^s(\underline{U}), |\partial_t \underline{h}|_{L^\infty}) - \lambda) E^s(U)^2 + \varepsilon C(E^s(\underline{U})) E^s(U).$$

Taking  $\lambda = \lambda_T$  large enough (how large depending on  $\sup_{t \in [0, \frac{T}{\varepsilon}]} C(E^s(\underline{U}), |\partial_t \underline{h}|_{L^\infty})$ ) to have the first term of the right hand side negative for all  $t \in [0, \frac{T}{\varepsilon}]$ , one deduces

$$\forall t \in [0, \frac{T}{\varepsilon}], \quad e^{\varepsilon \lambda t} \partial_t (e^{-\varepsilon \lambda t} E^s(U)^2) \leq \varepsilon C(E^s(\underline{U})) E^s(U).$$

Integrating this differential inequality yields therefore

$$\forall t \in [0, \frac{T}{\varepsilon}], \quad E^s(U) \leq e^{\varepsilon \lambda_T t} E^s(U_0) + \varepsilon \int_0^t e^{\varepsilon \lambda_T (t-t')} C(E^s(\underline{U})(t')) dt'.$$

□

**2.3. Main result.** In this subsection we prove the main result of this paper, which shows well-posedness of the Green-Naghdi equations over large times.

**Theorem 1.** *Let  $b \in C_b^\infty(\mathbb{R})$ ,  $t_0 > 1/2$ ,  $s \geq t_0 + 1$ . Let also the initial condition  $U_0 = (\zeta_0, u_0)^T \in X^s$ , and satisfy (3). Then there exists a maximal  $T_{max} > 0$ , uniformly bounded from below with respect to  $\varepsilon, \mu \in (0, 1)$ , such that the Green-Naghdi equations (2) admit a unique solution  $U = (\zeta, u)^T \in X_{T_{max}}^s$  with the initial condition  $(\zeta_0, u_0)^T$  and preserving the nonvanishing depth condition (3) for any  $t \in [0, \frac{T_{max}}{\varepsilon})$ . In particular if  $T_{max} < \infty$  one has*

$$|U(t, \cdot)|_{X^s} \longrightarrow \infty \quad \text{as } t \longrightarrow T_{max},$$

or

$$\inf_{\mathbb{R}} h(t, \cdot) = \inf_{\mathbb{R}} 1 + \varepsilon (\zeta(t, \cdot) - b(\cdot)) \longrightarrow 0 \quad \text{as } t \longrightarrow T_{max}.$$

Moreover, the following conservation of energy property holds

$$\partial_t \left( |\zeta|_2^2 + (hu, u) + \mu(h\mathcal{T}u, u) \right) = 0,$$

where  $\mathcal{T} = \mathcal{T}[h, \varepsilon b]$ .

**Remark 3.** *For 2D surface waves, non flat bottoms, B. A. Samaniego and D. Lannes [3] proved a well-posedness result to the Green-Naghdi using a Nash-Moser scheme. Our result only use a standard Picard iterative and there is therefore no loss of regularity of the solution with respect to the initial condition. In the one-dimensional case and for flat bottoms, our result coincides with the one proved by Li in [14].*

**Remark 4.** *Our approach does not admit a straightforward generalization to the 2D case. The main reason is that the natural energy norm  $X^s$  is then given by*

$$|U|_{X^s}^2 = |\zeta|_{H^s}^2 + |u|_{H^s}^2 + \mu |\nabla \cdot u|_{H^s}^2,$$

*which does not control the  $H^1(\mathbb{R}^2)$  norm of  $u$  (since  $u$  takes its values in  $\mathbb{R}^2$ , the information on the rotational of  $u$  is missing).*

**Remark 5.** *No smallness assumption on  $\varepsilon$  nor  $\mu$  is required in the theorem. The fact that  $T_{max}$  is uniformly bounded from below with respect to these parameters allows us to say that if some smallness assumption is made on  $\varepsilon$ , then the existence time becomes larger, namely of order  $O(1/\varepsilon)$ . This is consistent with the existence time obtained for the (simpler) physical models derived under some smallness assumption on  $\varepsilon$ , like the Boussinesq models. In fact, such models can be derived from the Green-Naghdi equations [13]. The present theorem also has some direct implication for the justification of variable-bottom Camassa-Holm equations [9].*

*Proof.* We want to construct a sequence of approximate solution  $(U^n = (\zeta^n, u^n))_{n \geq 0}$  by the induction relation

$$(17) \quad U^0 = U_0, \quad \text{and} \quad \forall n \in \mathbb{N}, \quad \begin{cases} \partial_t U^{n+1} + A[U^n] \partial_x U^{n+1} + B(U^n) = 0; \\ U^{n+1}|_{t=0} = U_0. \end{cases}$$

By Proposition 1, we know that there is a unique solution  $U^{n+1} \in C([0, \infty); X^s)$  to (17) if  $U^n \in C([0, \infty); X^s)$  and  $U^n$  satisfies (3) for all times. Let  $R > 0$  be such that  $E^s(U^0) \leq R/2$ , it follows from Proposition 1 that  $U^{n+1}$  satisfies the following inequality

$$E^s(U^{n+1}(t)) \leq e^{\varepsilon \lambda_T t} E^s(U^0) + \varepsilon \int_0^t e^{\varepsilon \lambda_T (t-t')} C(E^s(U^n(t'))) dt',$$

we suppose now that

$$\sup_{t \in [0, \frac{T}{\varepsilon}]} E^s(U^n(t)) \leq R,$$

therefore

$$E^s(U^{n+1}(t)) \leq R/2 + (e^{\varepsilon \lambda_T t} - 1)(R/2 + \frac{C(R)}{\lambda_T}).$$

Hence, there is  $T > 0$  small enough such that

$$\sup_{t \in [0, \frac{T}{\varepsilon}]} E^s(U^{n+1}(t)) \leq R.$$

Using now the link between  $E^s(U)$  and  $|U|_{X^s}$  given by Lemma 3 we get

$$\sup_{t \in [0, \frac{T}{\varepsilon}]} |U^{n+1}(t)|_{X^s} \leq C\left(\frac{1}{h_0}\right)R.$$

We also know from the equations that

$$\partial_t \zeta^{n+1} = -h^n u_x^{n+1} - \varepsilon \zeta_x^n u^{n+1} + \varepsilon b_x u^{n+1}.$$

Hence, one gets

$$(18) \quad |\partial_t h^{n+1}|_{L^\infty} = \varepsilon |\partial_t \zeta^{n+1}|_{L^\infty} \leq \varepsilon C\left(\frac{1}{h_0}\right)R.$$

Since moreover

$$h^{n+1} = h_{t=0}^{n+1} + \int_0^t \partial_t \zeta^{n+1},$$

we can deduce from (18) and the fact that  $h_{t=0}^{n+1} = 1 + \varepsilon(\zeta_0 - b) \geq h_0$  that it is possible to choose  $T$  small enough for  $U^{n+1}$  to satisfy (3) on  $[0, \frac{T}{\varepsilon}]$ , with  $h_0$  replaced by  $h_0/2$ .

Finally, we deduce that the Cauchy problem

$$\begin{cases} \partial_t U^{n+1} + A[U^n] \partial_x U^{n+1} + B(U^n) = 0; \\ U_{|t=0}^{n+1} = U_0 \end{cases}$$

has a unique solution  $U^{n+1}$  satisfying (3) and the inequality

$$E^s(U^{n+1}) \leq e^{\varepsilon \lambda_T t} E^s(U_0) + \varepsilon \int_0^t e^{\varepsilon \lambda_T (t-t')} C(E^s(U^n)(t')) dt',$$

when  $0 \leq t \leq \frac{T}{\varepsilon}$  and  $\lambda_T$  depending only on  $\sup_{t \in [0, \frac{T}{\varepsilon}]} E^s(U^n)$ . Thanks to this energy estimate, one can conclude classically (see e.g. [1]) to the existence of

$$T_{max} = T(E^s(U_0)) > 0,$$

and of a unique solution  $U \in X_{T_{max}}^s$  to (2) preserving the inequality (3) for any  $t \in [0, \frac{T_{max}}{\varepsilon}]$  as a limit of the iterative scheme

$$U^0 = U_0, \quad \text{and} \quad \forall n \in \mathbb{N}, \quad \begin{cases} \partial_t U^{n+1} + A[U^n] \partial_x U^{n+1} + B(U^n) = 0; \\ U_{|t=0}^{n+1} = U_0. \end{cases}$$

The fact that  $T_{max}$  is bounded from below by some  $T > 0$  independent of  $\varepsilon, \mu \in (0, 1)$  follows from the analysis above, while the behavior of the solution as  $t \rightarrow T_{max}$  if  $T_{max} < \infty$  follows from standard continuation arguments.

Though the conservation of the energy can be found in some references (e.g. [5]), we reproduce it in Appendix B for the sake of completeness.  $\square$

#### APPENDIX A. EXISTENCE OF SOLUTIONS FOR THE LINEARIZED EQUATIONS

In this section we examine existence, uniqueness, and regularity for solutions to the following system of equations:

$$(19) \quad \begin{cases} \partial_t U + A[\underline{U}] \partial_x U = f; \\ U_{|t=0} = U_0, \end{cases}$$

where  $\underline{U} = (\underline{\zeta}, \underline{u})^T \in X_T^s$  is such that  $\partial_t \underline{U} \in X_T^{s-1}$  and satisfy the condition (3) on  $[0, \frac{T}{\varepsilon}]$ . We begin the proof by the following lemma (see for instance [17]):

**Lemma 4.** *Let  $\varphi \in C_0^\infty(\mathbb{R})$ , such that  $\varphi(r) = 1$  for  $|r| \leq 1$ . Let also*

$$J^\delta = \varphi(\delta|D|), \quad \delta > 0.$$

*Then:*

- (i)  $\forall s, s' \in \mathbb{R}, J^\delta: H^s(\mathbb{R}) \mapsto H^{s'}(\mathbb{R})$  is a bounded linear operator.
- (ii)  $J^\delta$  commutes with  $\Lambda^s$  and is self-adjoint operator.
- (iii)  $\forall f \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), v \in L^2(\mathbb{R})$  there exists  $C$  independent of  $\delta$  such that

$$|[f, J^\delta]v|_{H^1} \leq C|f|_{C^1}|v|_{L^2}.$$

- (iv)  $\forall f \in H^s(\mathbb{R}), s > \frac{1}{2}, J^\delta f \in L^\infty(\mathbb{R})$  with

$$|J^\delta f|_{L^\infty} \leq C|f|_{L^\infty}$$

where  $C$  is a constant independent of  $\delta$ .

Our strategy will be to obtain a solution to (19) as a limit of solutions  $U_\delta$  to

$$(20) \quad \begin{cases} \partial_t U_\delta + J^\delta A[\underline{U}] J^\delta \partial_x U_\delta = f; \\ U_{\delta|_{t=0}} = U_0. \end{cases}$$

For any  $\delta > 0$ ,  $A^\delta = J^\delta A[\underline{U}] J^\delta$  is a bounded linear operator on each  $X^s$ , and  $F^\delta = A^\delta \partial_x + B(\underline{U}) \in C^1(X^s)$  so by Cauchy-Lipschitz the ODE (20) has a unique solution,  $U_\delta \in C([0, T/\varepsilon], X^s)$ . Our task will be to obtain estimates on  $U_\delta$ , independent of  $\delta \in (0, 1)$  and to show that the solution  $U_\delta$  has a limit as  $\delta \searrow 0$  solving (19). To do this, we remark that

$$(21) \quad \begin{aligned} \frac{1}{2} \partial_t E^s(U_\delta)^2 &= -(SA[\underline{U}] \Lambda^s \partial_x J^\delta U_\delta, \Lambda^s J^\delta U_\delta) - ([\Lambda^s, A[\underline{U}]] \partial_x J^\delta U_\delta, S \Lambda^s J^\delta U_\delta) \\ &\quad + (\Lambda^s f, S \Lambda^s U_\delta) + \frac{1}{2} (\Lambda^s u_\delta, [\partial_t, \underline{\mathfrak{I}}] \Lambda^s u_\delta) \\ &\quad + ([S, J^\delta] \Lambda^s U_\delta, A[\underline{U}] \Lambda^s \partial_x J^\delta U_\delta) + ([S, J^\delta] \Lambda^s U_\delta, [\Lambda^s, A[\underline{U}]] \partial_x J^\delta U_\delta). \end{aligned}$$

Note that we do not give any details for the control of the components of the r.h.s (21) other than the last two terms because the others can be handled exactly as in Proposition 1. To estimate the last two terms of the r.h.s (21), we have that

$$\begin{aligned} ([S, J^\delta] \Lambda^s U_\delta, A[\underline{U}] \Lambda^s \partial_x J^\delta U_\delta) &= ([\underline{\mathfrak{I}}, J^\delta] \Lambda^s u_\delta, \underline{\mathfrak{I}}^{-1} (\underline{h} \Lambda^s \partial_x J^\delta u_\delta)) \\ &\quad + ([\underline{\mathfrak{I}}, J^\delta] \Lambda^s u_\delta, \varepsilon \underline{u} \Lambda^s \partial_x J^\delta u_\delta) \\ &\quad + ([\underline{\mathfrak{I}}, J^\delta] \Lambda^s u_\delta, \underline{\mathfrak{I}}^{-1} Q_1[\underline{U}] \Lambda^s \partial_x J^\delta u_\delta). \end{aligned}$$

One can check by using the explicit expression of  $\underline{\mathfrak{I}}$  that

$$\begin{aligned} [\underline{\mathfrak{I}}, J^\delta] &= [\underline{h}, J^\delta] - \frac{\mu}{3} \partial_x [\underline{h}^3, J^\delta] \partial_x - \frac{\varepsilon \mu}{2} [\underline{h}^2 b_x, J^\delta] \partial_x \\ &\quad + \frac{\varepsilon \mu}{2} \partial_x [\underline{h}^2 b_x, J^\delta] + \varepsilon^2 \mu [\underline{h} b_x^2, J^\delta]. \end{aligned}$$

One deduces directly from Lemma 4, an integration by parts, the Cauchy-Schwarz inequality and the explicit expression of  $Q_1[\underline{U}]$  that

$$([S, J^\delta] \Lambda^s U_\delta, A[\underline{U}] \Lambda^s \partial_x J^\delta U_\delta) \leq C E^s(U_\delta)^2,$$

similarly, one can conclude

$$([S, J^\delta] \Lambda^s U_\delta, [\Lambda^s, A[\underline{U}]] \partial_x J^\delta U_\delta) \leq C E^s(U_\delta)^2,$$

where  $C$  is a constant independent of  $\delta$ . By the Proposition 1, we have

$$\begin{aligned} &-(SA[\underline{U}] \Lambda^s \partial_x J^\delta U_\delta, \Lambda^s J^\delta U_\delta) - ([\Lambda^s, A[\underline{U}]] \partial_x J^\delta U_\delta, S \Lambda^s J^\delta U_\delta) \\ &+ (\Lambda^s f, S \Lambda^s U_\delta) + \frac{1}{2} (\Lambda^s u_\delta, [\partial_t, \underline{\mathfrak{I}}] \Lambda^s u_\delta) \\ &\leq C E^s(U_\delta)^2 + C E^s(f)^2. \end{aligned}$$

Consequently, we obtain an estimate of the form

$$(22) \quad \frac{d}{dt} E^s(U_\delta)^2 \leq C E^s(U_\delta)^2 + C E^s(f)^2.$$



Thus Gronwall's inequality yields an estimate

$$(23) \quad E^s(U_\delta)^2 \leq C(t) \left[ E^s(U_0)^2 + \sup_{[0,t]} E^s(f)^2 \right],$$

independent of  $\delta \in (0, 1)$ . Thanks to this energy estimate, one can conclude classically (see e.g. [17]) to the existence of a unique solution  $U \in C([0, T], X^s)$  to (19).

#### APPENDIX B. CONSERVATION OF THE ENERGY

In order to prove that

$$\partial_t \left( |\zeta|_2^2 + (hu, u) + \mu(h\mathcal{T}u, u) \right) = 0,$$

we multiply the first equation of (2) by  $\zeta$  and the second by  $u$ , integrate on  $\mathbb{R}$ , and sum both equations to find

$$\frac{1}{2} \partial_t |\zeta|_2^2 + (\partial_x(hu), \zeta) + (\partial_t u, hu) + \mu(h\mathcal{T}\partial_t u, u) + (\partial_x \zeta, hu) + \varepsilon(u\partial_x u, hu) + \mu\varepsilon(\mathcal{Q}u, u) = 0.$$

Therefore

$$\frac{1}{2} \partial_t |\zeta|_2^2 + \frac{1}{2} \partial_t (hu, u) - \frac{1}{2} (\partial_t h, u^2) + \varepsilon(u\partial_x u, hu) + \mu(h\mathcal{T}\partial_t u, u) + \mu\varepsilon(\mathcal{Q}u, u) = 0,$$

where the term  $\mathcal{Q}u$  is defined as:

$$\begin{aligned} \mathcal{Q}u &= -\frac{1}{3} \partial_x [(h^3(u\partial_x^2 u - (\partial_x u)^2) + \frac{\varepsilon}{2} [\partial_x (h^2 u \partial_x (u\partial_x b) - h^2 \partial_x b (u\partial_x^2 u - (\partial_x u)^2))] \\ &\quad + \varepsilon^2 h \partial_x b (u\partial_x (u\partial_x b))]. \end{aligned}$$

Using now the fact that  $h = 1 + \varepsilon(\zeta - b)$  and the first equation of (2), we get

$$-\frac{1}{2} (\partial_t h, u^2) + \varepsilon(u\partial_x u, hu) = -\frac{\varepsilon}{2} (\partial_x(hu), u^2) + \varepsilon(u\partial_x u, hu) = 0.$$

Thus,

$$(24) \quad \frac{1}{2} \partial_t |\zeta|_2^2 + \frac{1}{2} \partial_t (hu, u) + \mu(h\mathcal{T}\partial_t u, u) + \mu\varepsilon(\mathcal{Q}u, u) = 0.$$

Regarding now the term  $\mu(h\mathcal{T}\partial_t u, u)$ , we remark as in [5] that

$$\begin{aligned} \mu(h\mathcal{T}\partial_t u, u) &= \mu(\mathcal{T}_1^* h \mathcal{T}_1 \partial_t u, u) + \mu(\mathcal{T}_2^* h \mathcal{T}_2 \partial_t u, u), \\ &= \mu(h\mathcal{T}_1 \partial_t u, \mathcal{T}_1 u) + \mu(h\mathcal{T}_2 \partial_t u, \mathcal{T}_2 u), \\ &= \mu(h(\partial_t(\mathcal{T}_1 u) - \partial_t \mathcal{T}_1 u), \mathcal{T}_1 u) + \mu(h\partial_t(\mathcal{T}_2 u), \mathcal{T}_2 u), \end{aligned}$$

with  $\mathcal{T}_j^*$  ( $j = 1, 2$ ) denoting the adjoint of the operators  $\mathcal{T}_j$  given by

$$\mathcal{T}_1 u = \frac{h}{\sqrt{3}} \partial_x u - \varepsilon \frac{\sqrt{3}}{2} \partial_x b u, \quad \text{and} \quad \mathcal{T}_2 u = \frac{\varepsilon}{2} \partial_x b u.$$

It comes:

$$\begin{aligned} \mu(h\mathcal{T}\partial_t u, u) &= \frac{\mu}{2} \partial_t (h\mathcal{T}_1 u, \mathcal{T}_1 u) - \frac{\mu}{2} (\partial_t h, (\mathcal{T}_1 u)^2) - \mu(h(\partial_t \mathcal{T}_1 u), \mathcal{T}_1 u) \\ &\quad + \frac{\mu}{2} \partial_t (h\mathcal{T}_2 u, \mathcal{T}_2 u) - \frac{\mu}{2} (\partial_t h, (\mathcal{T}_2 u)^2), \\ &= \frac{\mu}{2} \partial_t (h\mathcal{T}_1 u, \mathcal{T}_1 u) + \frac{\mu}{2} \partial_t (h\mathcal{T}_2 u, \mathcal{T}_2 u) - \frac{\mu}{2} (\partial_t h, (\mathcal{T}_1 u)^2 + (\mathcal{T}_2 u)^2) \\ &\quad - \mu(h\partial_t \mathcal{T}_1 u, \mathcal{T}_1 u). \end{aligned}$$

Inject this result in (24) to get:

$$\begin{aligned} \frac{1}{2}\partial_t\left(|\zeta|_2^2 + (hu, u) + \mu(h\mathcal{T}u, u)\right) &= \frac{\mu}{2}(\partial_t h, (\mathcal{T}_1 u)^2 + (\mathcal{T}_2 u)^2) \\ &\quad + \mu(h\partial_t \mathcal{T}_1 u, \mathcal{T}_1 u) - \mu\varepsilon(\mathcal{Q}u, u). \end{aligned}$$

Noting that  $h\partial_t \mathcal{T}_1 u = \partial_t h(\mathcal{T}_1 u + \sqrt{3}\mathcal{T}_2 u)$ , it comes:

$$\begin{aligned} \frac{1}{2}(\partial_t h, (\mathcal{T}_1 u)^2 + (\mathcal{T}_2 u)^2) + (h\partial_t \mathcal{T}_1 u, \mathcal{T}_1 u) &= \frac{1}{2}(\partial_t h, 3(\mathcal{T}_1 u)^2 + (\mathcal{T}_2 u)^2 + 2\sqrt{3}\mathcal{T}_1 u\mathcal{T}_2 u), \\ &= \frac{\mu}{2}(\partial_t h, (\sqrt{3}\mathcal{T}_1 u + \mathcal{T}_2 u)^2), \\ &= \varepsilon\left(u, \frac{h}{2}\partial_x(\sqrt{3}\mathcal{T}_1 u + \mathcal{T}_2 u)^2\right), \end{aligned}$$

where we used here the first equation of (2). Finally, we get:

$$\frac{1}{2}\partial_t\left(|\zeta|_2^2 + (hu, u) + \mu(h\mathcal{T}u, u)\right) = \mu\varepsilon\left(\frac{h}{2}\partial_x(\sqrt{3}\mathcal{T}_1 u + \mathcal{T}_2 u)^2 - \mathcal{Q}u, u\right).$$

One can easily show that  $\left(\frac{h}{2}\partial_x(\sqrt{3}\mathcal{T}_1 u + \mathcal{T}_2 u)^2 - \mathcal{Q}u, u\right) = 0$ , which implies easily the result.

**Acknowledgments.** The author is grateful to David Lannes for encouragement and many helpful discussions.

#### REFERENCES

- [1] S. Alinhac, P. Gérard, *Opérateurs pseudo-différentiels et théorème de Nash-Moser*, Savoires Actuels. InterEditions, Paris; Editions du Centre National de la Recherche Scientifique (CNRS), Meudon, 1991. 190 pp.
- [2] B. Alvarez-Samaniego, D. Lannes, Large time existence for 3D water-waves and asymptotics, *Inventiones Mathematicae* **171** (2008), 485–541.
- [3] B. Alvarez-Samaniego, D. Lannes, A Nash-Moser theorem for singular evolution equations. Application to the Serre and Green-Naghdi equations. *Indiana Univ. Math. J.* **57** (2008), 97–131.
- [4] S. V. Basenkova, N. N. Morozov, and O. P. Pogutse. *Dispersive effects in two-dimensional hydrodynamics*. Dokl. Akad. Nauk SSSR, 1985.
- [5] F. Chazel, *Influence de la topographie sur les ondes de surface*, Thèse Université Bordeaux I (2007), <http://tel.archives-ouvertes.fr/tel-00200419-v2/>
- [6] W. Craig, *An existence theory for water waves and the Boussinesq and the Korteweg-de Vries scaling limits*. Commun. Partial Differ. Equations **10**, 787–1003 (1985).
- [7] A. E. Green, N. Laws, and P. M. Naghdi. *On the theory of water waves*. Proc. Roy. Soc. (London) Ser. A, 338: 43–55, 1974.
- [8] A. E. Green and P. M. Naghdi, A derivation of equations for wave propagation in water of variable depth, *J. Fluid Mech.* **78** (1976), 237–246.
- [9] S. Israwi, *Variable depth KDV equations and generalizations to more nonlinear regimes*. (2009) arXiv: 0901.3201v1.
- [10] J. W. Kim, K. J. Bai, R. C. Ertekin, and W. C. Webster. *A strongly-nonlinear model for water waves in water of variable depth: the irrotational green-naghdi model*. Journal of Oshore Mechanics and Arctic Engineering, Trans. of ASME., 2003.
- [11] D. Lannes. *Well-posedness of the water waves equations*, J. Amer. Math. Soc. **18** (2005), 605–654.
- [12] D. Lannes *Sharp Estimates for pseudo-differential operators with symbols of limited smoothness and commutators*, J. Funct. Anal. , **232** (2006), 495–539.
- [13] D. Lannes, P. Bonneton, *Derivation of asymptotic two-dimensional time-dependent equations for surface water wave propagation*, Physics of fluids **21** (2009).
- [14] Y. A. Li, *A shallow-water approximation to the full water wave problem*, *Commun. Pure Appl. Math.* **59** (2006), 1225–1285.

- [15] Madsen, P.A. and Bingham, H.B. and Liu, H., 2002. *A new Boussinesq method for fully nonlinear waves from shallow to deep water*. J. Fluid Mech. **462**, 1-30.
- [16] V. I. Nalimov, *The Cauchy-Poisson problem*. (Russian) Dinamika Splošn. Sredy Vyp. 18 Dinamika Zidkost. so Svobod. Granicami, **254**, (1974) 104-210.
- [17] Michael E. Taylor, *Partial Differential Equations II*, Applied Mathematical Sciences Volume 116. Springer.
- [18] Ge Wei, James T. Kirby, Stephan T. Grilli, and Ravishankar Subramanya. *A fully nonlinear Boussinesq model for surface waves. w I*. Highly nonlinear unsteady waves. J. Fluid Mech., 294: 71-92, 1995.
- [19] S. Wu, *Well-posedness in sobolev spaces of the full water wave problem in 2-D*, Invent. Math. **130** (1997), no. 1, 39-72.
- [20] S. Wu, *Well-posedness in sobolev spaces of the full water wave problem in 3-D*, J. Amer. Math. Soc. **12** (1999), no. 2, 445-495.
- [21] H. Yosihara, *Gravity waves on the free surface of an incompressible perfect fluid of finite depth*. Publ. Res. Inst. Math. Sci. **18** (1982), no.1, 49-96.

UNIVERSITÉ BORDEAUX I; IMB, 351 COURS DE LA LIBÉRATION, 33405 TALENCE CEDEX, FRANCE  
*E-mail address:* `Samer.Israwi@math.u-bordeaux1.fr`