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OBSTRUCTED BUNDLES OF RANK TWO ON A QUINTIC SURFACE

NICOLE MESTRANO AND CARLOS SIMPSON

To the memory of Masaki Maruyama

ABSTRACT. In this note we consider the moduli space of stable bundles of rank two on a very general quintic surface. We study the obstructed points of the moduli space via the spectral covering of a twisted endomorphism. This analysis leads in some examples to a generically non-reduced component of the moduli space, and a component which is generically smooth of one bigger than the expected dimension. We obtain a sharp bound asked for by O'Grady saying when the moduli space is good.

1. INTRODUCTION

The moduli space of stable vector bundles on an algebraic surface was introduced by Maruyama [21] and Gieseker [11]. These moduli spaces played an important role in Donaldson's theory as applied to 4-manifolds which are complex surfaces. Many authors have closely studied the structure of the moduli space for big values of c_2 (for a full set of references see [17]), but it remains an interesting and largely open question to understand the structure for intermediate values of c_2 .

One of the objectives of this paper is to investigate how to use the following interpretation of singularities in the moduli space. The space of obstructions to deforming E is $H^2(\text{End}^0(E))$ where the superscript End^0 denotes the trace-free endomorphisms. We say that E is *obstructed* if this space is nonzero. Then the dimension of the Zariski tangent space to the moduli space is bigger than the expected dimension. If the moduli space is good, i.e. generically smooth of the expected dimension near E , it means that E must be a singular point. The moduli space might on the other hand be smooth but overdetermined, i.e. having dimension bigger than the expected dimension. And of course it could also be overdetermined and singular too.

As is well known (see for example [31, §1]), the dual of the space of obstructions is $H^0(\text{End}^0(E) \otimes K_X)$ by Serre duality. An element ϕ in this dual space is a trace-free morphism $\phi : E \rightarrow E \otimes K_X$. Such a ϕ corresponds, by Kuranishi theory, to an equation of the moduli space locally at E , and we call this a *co-obstruction*. A pair (E, ϕ) consisting of a bundle together with a nonzero co-obstruction, may be thought of as a K_X -valued *Hitchin pair* on X . These pairs are somewhat different from those considered in [33] for the surface X : the Higgs bundles corresponding to representations of π_1 are endomorphisms taking values in Ω_X^1 . Over a curve these two notions coincide and indeed Hitchin used the notation K_X in his original paper

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[15]. Generalizing his notation as written leads to the notion of a Higgs field $E \rightarrow E \otimes K_X$ which is exactly a co-obstruction, often called a “twisted endomorphism”.

The basic tool in the analysis of Hitchin pairs is the notion of *spectral cover* [6] [3]. A twisted endomorphism $\phi : E \rightarrow E \otimes K_X$ gives E the structure of a coherent sheaf on the total space of the vector bundle K_X , and the support of the coherent sheaf is the *spectral covering* associated to ϕ . It consists of the set of pairs (x, u) where $x \in X$ and $u \in K_{X,x}$ such that u is an eigenvalue of ϕ_x .

In our rank two case the spectral cover is particularly simple to describe: it is the divisor $Z \subset K_X$ determined by the equation $z^2 = \beta$ where $\beta = \det(\phi) \in H^0(K_X^{\otimes 2})$. The *Hitchin map* is $(E, \phi) \mapsto \beta$.

We investigate in a very basic way the possible classification of such spectral covers, and the implications for the locus of singularities of the moduli space.

Many authors have shown that the moduli spaces of bundles of odd degree on abelian and K3 surfaces are smooth, going back to [8] and Mukai [25], see the discussions and references in [34], [35]. O’Grady has observed an important example of symplectic singularities in the moduli of rank two bundles on a K3 surface [32], along the locus of reducible bundles. In view of these properties and examples, for understanding bundles on surfaces of general type it seems like a good idea to look at surfaces of general type which are as close as possible to K3 surfaces. This motivates our consideration of the example of a very general quintic surface in \mathbb{P}^3 , where $K_X = \mathcal{O}_X(1)$.

The main advantage of the quintic surface is that $\beta \in H^0(\mathcal{O}_X(2))$ is a quadric on X , and because of its low degree there are not too many possibilities for the spectral covering.

If $\beta = 0$ then the Higgs field is nilpotent and we get a presentation of E as an extension. Furthermore, there are no nontrivial line bundles between \mathcal{O}_X and $K_X = \mathcal{O}_X(1)$ so stability of E yields a particularly easy description of the extension.

If $\beta = \alpha^2$ for $\alpha \in H^0(\mathcal{O}_X(1))$ then the spectral cover is a union of two copies of X joined along the curve C defined by α , the intersection of X with a plane. In this case, E can be presented as an elementary transformation. Again using the smallness of the situation on the quintic surface, we can see that the space of co-obstructions contains a full copy of $\mathfrak{sl}_2 \cong \mathbb{C}^3$, in particular it must also contain a nilpotent element. This argument is the basis of the classification statement Theorem 5.1.

If β is not a square, it follows from the very general condition on X that it defines a reduced curve $D \subset X$ and Z is irreducible branched along D . In this case, E is the direct image of a line bundle on the desingularization \tilde{Z} of Z , so the dimension of the possible space of such obstructed bundles is bounded independently of $c_2(E)$: a rough estimate using Lefschetz theory shows the dimension is ≤ 13 (Corollary 5.4), a bound which could undoubtedly be improved.

After this general discussion, the main case to be treated is that of nilpotent Higgs field. A more precise classification is obtained in Theorem 6.2, allowing to estimate the dimensions of the various components in Proposition 6.3. This yields a statement about the singular locus of the moduli space when c_2 is big. We can identify the biggest irreducible component of the singular locus, and show by a general position argument that the quadratic term of the Kuranishi map has the biggest possible rank at a general point of this component, in other words the singularity is an ordinary double point in the transverse direction.

On the other hand, we consider the examples of $c_2(E) = 8$, in which case we find at least one irreducible component which is completely obstructed but which is smooth of dimension 13 bigger than the expected dimension 12; and construct another family of obstructed bundles of dimension 12 which we conjecture is a non-reduced component not in the closure of the previous family.

In the case $c_2 = 9$ these two examples come together in a certain sense, giving a single irreducible component of the moduli space which is completely obstructed but of the expected dimension, with the quadratic term of the Kuranishi map giving an equation of the form $x^2 = 0$ at a general point.

Theorem 6.4 says that if $c_2(E) \geq 11$ then the moduli space is good. This answers a question of O'Grady [30] who predicted a sharp bound, and in fact our bound improves upon his predicted bound by 1. It is likely that for $c_2 = 10$ the moduli space will also be good.

Our discussion is undoubtedly subsumed in some sense by the general theory of the structure of moduli spaces of stable bundles on surfaces, of Donaldson [7], Friedman [10], Gieseker, Li [13] [19] and others. Many of these works have concentrated on the range $c_2 \gg 0$, although Gieseker [12] and O'Grady [30] construct components for intermediate values of c_2 having more than the expected dimension. We hope that the relatively explicit considerations here can provide some insight into the complicated middle range where c_2 is neither too big nor too small, continuing in the direction of [24].

Here are some further remarks and questions.

We obtain some non-reduced components of the moduli space of rank 2 bundles. This theory should be somewhat related to the theory of generically non-reduced components of Hilbert schemes of curves as constructed following Mumford's original example by Kleppe [18], Ellia [9], Floystad [14], Martin-Deschamps and Perrin [20], Azziz [2], Mukai-Nasu [26] and others.

The investigation of quintic surfaces here looks somewhat similar to the examples discussed in [5]. It would be interesting to understand what happens for a quintic surface which is no longer very general, i.e. such that the Neron-Severi group has rank ≥ 2 . In a similar spirit, recall that smooth quintic surfaces in \mathbb{P}^3 are connected by deformation to the "Horikawa surfaces" which form a different irreducible component of moduli, the two intersecting along a locus of quintic surfaces with ordinary double points. It would be interesting to see how much of our discussion could be done for Horikawa surfaces.

The construction of some irreducible components of the moduli space $M_X(2, -1, d)$ raises the question of the existence of Poincaré bundles over open subsets of these components.

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2. OBSTRUCTIONS FOR VECTOR BUNDLES ON A SURFACE

Let X be a smooth projective algebraic surface with ample line bundle $\mathcal{O}_X(1)$ and corresponding hyperplane class $H := c_1(\mathcal{O}_X(1)) \in H^2(X, \mathbb{Q})$. We consider the moduli space of H -Gieseker-semistable vector bundles on X of rank r , and given c_1 and c_2 denoted $M = M(r, c_1, c_2)$. The open subset of stable points is denoted

$M^s(r, c_1, c_2)$, and the projective moduli space of torsion-free sheaves is denoted by $\overline{M}(r, c_1, c_2)$. If necessary, the underlying variety X and/or the polarization will be indicated by subscripts as in $M_{X,H}(r, c_1, c_2)$.

We will concentrate on the case of rank $r = 2$ and $c_1 = \mathcal{O}_X(-1)$ but many of the initial definitions are valid for any rank.

Consider a point $E \in M^s(r, c_1, c_2)$. The deformation theory of E is controlled by the space

$$\text{Def}(E) := H^1(\text{End}(E)),$$

while the obstruction theory is controlled by

$$\text{Obs}(E) := H^2(\text{End}^0(E)).$$

Here $\text{End}^0(E) := \ker(\text{tr} : \text{End}(E) \rightarrow \mathcal{O}_X)$ is the trace-free part of the endomorphism bundle of E . The map tr is split by the diagonal embedding so

$$\text{End}(E) = \text{End}^0(E) \oplus \mathcal{O}_X.$$

The trace-free part is self-dual: $\text{End}^0(E)^* \cong \text{End}^0(E)$ via the pairing

$$\text{End}^0(E) \otimes \text{End}^0(E) \xrightarrow{\langle \cdot, \cdot \rangle} \mathcal{O}_X,$$

with $\langle A, B \rangle := \text{tr}(AB)$.

Let $K_X := \Omega_X^2$ denote the dualizing sheaf. By Serre duality,

$$H^2(\text{End}^0(E)) \cong H^0(\text{End}^0(E) \otimes K_X)^*$$

using $\text{End}^0(E) \cong \text{End}^0(E)^*$. We obtain

$$\text{Obs}(E) = H^0(\text{End}^0(E) \otimes K_X)^*,$$

so $\text{Obs}(E) \neq \{0\}$ if and only if there exists a nonzero element $\phi \in \text{End}^0(E) \otimes K_X$. Such an element may be interpreted as a *Higgs field* [15] or “twisted endomorphism”

$$\phi : E \rightarrow E \otimes K_X$$

with $\text{tr}(\psi) = 0$. Notice that this is a Higgs field twisted by the canonical line bundle K_X rather than by Ω_X^1 as in [33]. Higgs bundles on higher dimensional varieties with twisting by a line bundle have been considered by a number of authors.

Lemma 2.1. *Suppose E is a vector bundle on a smooth projective surface. Then the obstruction space $\text{Obs}(E)$ is nonzero, if and only if there exists a nonzero Higgs field $\phi : E \rightarrow E \otimes K_X$ of trace zero.* \square

A pair (E, ϕ) corresponds to a coherent sheaf denoted $\mathcal{F} = \mathcal{F}_{E, \phi}$ on the total space of the line bundle K_X . We continue denoting this total space by K_X with its projection denoted by $p : K_X \rightarrow X$. The sheaf \mathcal{F} is of pure dimension 2, and indeed $p_*(\mathcal{F}) = E$ so \mathcal{F} should be flat over X if E is to be a vector bundle.

The *spectral surface* is the reduced subscheme $Z \subset K_X$ which is the support of \mathcal{F} . It can be viewed as the subset of eigenvalues of ψ . Note here that we define Z as a reduced subscheme, so \mathcal{F} may be a coherent sheaf not on Z but only on some infinitesimal neighborhood thereof.

We would like to classify pairs (E, ϕ) in the case when E has rank two.

In our case, the *Hitchin invariant* of the spectral surface is just the determinant $\beta := \det(\phi) \in H^0(X, K_X^{\otimes 2})$. The subscheme Z is defined by the equation $z^2 = \beta$.

Lemma 2.2. *Suppose $\beta \neq 0$. Then \mathcal{F} is a rank one torsion-free sheaf on Z , flat over X .* \square

2.1. **The obstruction corresponding to ϕ .** Recall that the local deformation theory of E is governed by the formal *Kuranishi map*

$$\kappa : H^1(\widehat{End}(E)) \rightarrow H^2(End^0(E)).$$

Recall that the Kuranishi map is a “formal function” i.e. a power series at the origin represented by a vector of $h^2(End^0(E))$ elements of the complete local ring of the vector space $H^1(End(E))$ at the origin. This is the reason for the hat in the notation.

The linear term of κ vanishes, and the quadratic term is a function

$$\kappa_2 : Sym^2 H^1(End(E)) \rightarrow H^2(End^0(E)).$$

For each “co-obstruction” or linear form $\phi \in H^2(End^0(E))^*$ we get a function denoted

$$obs(\phi) = \phi \cdot \kappa_2 : Sym^2 H^1(End(E)) \rightarrow \mathbb{C}.$$

By the general theory, we have the formula

$$(2.1) \quad \phi \cdot \kappa_2(\eta, \eta') = obs(\phi; \eta, \eta') = \int_X Tr(\phi \cdot [\eta, \eta']).$$

In this formula $\eta, \eta' \in H^1(End(E))$ and their commutator is

$$[\eta, \eta'] \in H^2(End^0(E)),$$

which is then multiplied by ϕ to get an element of $H^2(End(E) \otimes K_X)$. The trace on the factor $End(E)$ sends us to $H^2(K_X)$ and then we apply the duality isomorphism $\int_X : H^2(K_X) \xrightarrow{\cong} \mathbb{C}$.

The easiest way of interpreting (2.1) is to think of the cohomology classes as being given by their Dolbeault representatives, and with this formulation the obstruction can be calculated as

$$obs(\phi; \eta, \eta') = \int_X Tr(\phi \cdot \eta \cdot \eta' + \phi \cdot \eta' \cdot \eta)$$

where \cdot indicates matrix multiplication in $End(E)$ coupled with wedge product of Dolbeault $(0, q)$ -forms when necessary. The sign in the commutator is from the sign relations for differential forms. Using the matrix relation $Tr(AB - BA) = 0$ and again the sign conventions, the obstruction element is equal to

$$obs(\phi; \eta, \eta') = \int_X Tr(\phi \cdot \eta \cdot \eta' - \eta \cdot \phi \cdot \eta') = \int_X Tr(Ad(\phi)(\eta) \cdot \eta').$$

Here $Ad(\phi)(\eta) = \phi \cdot \eta - \eta \cdot \phi$. We have

$$Ad(\phi)(\eta) \in H^1(End^0(E) \otimes K_X) \cong H^1(End(E))^*$$

and the obstruction $obs(\phi; \eta, \eta')$ is just the duality pairing between $Ad(\phi)(\eta)$ and η' .

Proposition 2.3. *The rank of the quadratic form on $H^1(End^0(E))$ given by the obstruction $obs(\phi)$, is equal to the rank of the linear map*

$$Ad(\phi) : H^1(End(E)) \rightarrow H^1(End(E) \otimes K_X).$$

Proof. The duality pairing is a perfect pairing so the rank of the quadratic form is equal to the rank of the linear map. \square

3. THE CASE OF NILPOTENT CO-OBSTRUCTION

In this section we study the case when $\phi : E \rightarrow E \otimes K_X$ is a nonzero nilpotent matrix at the general point of X . In this case, $\ker(\phi)$ is a saturated subsheaf of rank 1 in E , thus it is a line bundle which we denote L . The quotient E/L is a torsion-free sheaf of rank 1 whose double dual is a line bundle which we denote by L' . Put $\mathcal{J}_P := (L')^* \otimes (E/L)$. It is the ideal in \mathcal{O}_X of a zero-dimensional subscheme which we denote by P . The bundle E fits into an exact sequence

$$(3.1) \quad 0 \rightarrow L \rightarrow E \rightarrow \mathcal{J}_P \otimes L' \rightarrow 0.$$

This kind of extension goes back to Serre's construction, and has been used systematically ever since [8] [30]

In order to insure the existence of a locally free extension of this form, one introduces the following condition: if U is a line bundle then say that P is *totally superfluous for sections of U* if for any subscheme $P' \subset P$ with $\text{length}(P') = \text{length}(P) - 1$, the conditions imposed by P and P' on sections of U are the same, in other words

$$\ker(H^0(U) \rightarrow H^0(P, U|_P)) = \ker(H^0(U) \rightarrow H^0(P', U|_{P'})).$$

Lemma 3.1. *Given line bundles L and L' and a zero-dimensional subscheme P of length d , there exists a rank 2 vector bundle fitting into an extension (3.1), if and only if P is a local complete intersection totally superfluous for sections of $L' \otimes L^* \otimes K_X$. Let c be the number of conditions imposed by P on $H^0(L' \otimes L \otimes K_X)$, then*

$$\dim \text{Ext}^1(\mathcal{J}_P \otimes L', L) = d - c.$$

Proof. Here is a brief account of this well-known fact (which was used implicitly in [30, (3.29)] for example). A locally free extension exists locally if and only if P is a local complete intersection. An extension corresponds to a linear form f on $H^0((L' \otimes L \otimes K_X)|_P)$ which vanishes on the image of $H^0(L' \otimes L \otimes K_X)$. It is quite classical that the condition for this to yield a locally free extension, is that $f(I) \neq 0$ for any subsheaf $I \subset \mathcal{O}_P \otimes L' \otimes L \otimes K_X$ which can be supposed of rank one. Such a subsheaf corresponds to the ideal of a subscheme $P' \subset P$ of colength 1. Saying that the general form f which vanishes on the image of $H^0(L' \otimes L \otimes K_X)$, is nonzero on I , is equivalent to saying that I is not in the image of $H^0(\quad)$, which in turn says that P' imposes the same number of conditions as P .

The dimension of the *Ext* group is given by the standard exact sequence. \square

Given a vector bundle E sitting in a sequence of the form (3.1), we have $\det(E) \cong L \otimes L'$,

$$\begin{aligned} E^* \otimes L' &\cong E \otimes L^*, \\ \text{End}^0(E) \otimes L \otimes L' &\cong \text{Sym}^2(E), \end{aligned}$$

and there is an exact sequence

$$(3.2) \quad 0 \rightarrow E \otimes (L')^* \rightarrow \text{End}^0(E) \rightarrow \mathcal{J}_P^2 \otimes L' \otimes L^* \rightarrow 0.$$

Lemma 3.2. *Taking the dual of (3.2) gives an exact sequence of the form*

$$0 \rightarrow L \otimes (L')^* \rightarrow \text{End}^0(E) \rightarrow \mathcal{G} \rightarrow 0$$

where \mathcal{G} fits into an exact sequence of the form

$$0 \rightarrow \mathcal{G} \rightarrow E^* \otimes L' \rightarrow \underline{\text{Ext}}^2(\mathcal{O}_{2P}, \mathcal{O}_X) \otimes L \otimes (L')^* \rightarrow 0,$$

and $2P$ denotes the subscheme defined by \mathcal{J}_P^2 .

Proof. Taking the dual gives a long exact sequence of the form

$$\dots \rightarrow \text{End}^0(E) \rightarrow E^* \otimes L' \rightarrow \underline{\text{Ext}}^1(\mathcal{J}_P^2 \otimes L' \otimes L^*, \mathcal{O}) \rightarrow 0.$$

But the long exact sequence for the standard sequence defining \mathcal{O}_{2P} gives

$$\underline{\text{Ext}}^1(\mathcal{J}_P^2, \mathcal{O}_X) \cong \underline{\text{Ext}}^2(\mathcal{O}_{2P}, \mathcal{O}_X).$$

□

The nilpotent Higgs field ϕ factors as

$$E \rightarrow \mathcal{J}_P \otimes L' \rightarrow L \otimes K_X \rightarrow E \otimes K_X$$

where the middle map comes from a map of line bundles denoted

$$\varphi : L' \rightarrow L \otimes K_X.$$

with transpose $\varphi^t : L^* \rightarrow (L')^* \otimes K_X$.

Proposition 3.3. *In the situation of an exact sequence (3.1) with \mathcal{G} defined as in Lemma 3.2, the map*

$$\text{Ad}(\phi) : H^1(\text{End}^0(E)) \rightarrow H^1(\text{End}^0(E) \otimes K_X)$$

factors as the map fitting into the following diagram:

$$\begin{array}{ccccccc} & & & & H^0((L' \otimes L^* \otimes K_X)|_{2P})^* & & \\ & & & & \downarrow & & \\ \dots & H^1(\text{End}^0(E)) & \longrightarrow & H^1(\mathcal{G}) & \longrightarrow & H^2(L \otimes (L')^*) & \dots \\ & & & \downarrow & & & \\ & & & H^1(E \otimes L^*) & \xrightarrow{\varphi^t} & H^1(E \otimes (L')^* \otimes K_X) & \\ & & & \downarrow & & \downarrow & \\ & & & 0 & & H^1(\text{End}^0(E) \otimes K_X) & \end{array}$$

where the main horizontal and vertical sequences are exact.

Proof. The exact sequences are just the long exact sequences associated to the exact sequences of Lemma 3.2. The element at the top is explained by

$$\begin{aligned} H^0(\underline{\text{Ext}}^2(\mathcal{O}_{2P}, \mathcal{O}_X) \otimes L \otimes (L')^*) &= \text{Ext}^2(\mathcal{O}_{2P} \otimes L^* \otimes L' \otimes K_X, K_X) \\ &\cong H^0(\mathcal{O}_{2P} \otimes L^* \otimes L' \otimes K_X)^*. \end{aligned}$$

The factorization is obtained by factoring the map $\text{Ad}(\phi)$ on the level of sheaves, then applying H^1 . □

Remark 3.4. *The composed map*

$$H^0(2P, \mathcal{O}_{2P} \otimes L' \otimes L^* \otimes K_X)^* \rightarrow H^2(L \otimes (L')^*)$$

is dual to a map

$$H^0(L' \otimes L^* \otimes K_X) \rightarrow H^0(2P, \mathcal{O}_{2P} \otimes L' \otimes L^* \otimes K_X),$$

which is just the evaluation map for sections over the subscheme $2P$ up to multiplying by a unit.

4. THE CASE OF REDUCIBLE SPECTRAL SURFACE

In this section we study the special case when the spectral surface decomposes into two irreducible components meeting along a smooth curve.

Hypothesis 4.1. *Suppose that E is a slope-stable bundle with co-obstruction ϕ such that $\det(\phi) = \alpha^2$ for a nonzero section $\alpha \in H^0(X, K_X)$. Assume that the curve $C \subset X$ defined by $\alpha = 0$ is reduced.*

In this case, $Z = Z^+ \cup Z^-$ where $Z^+ \subset K_X$ is the graph of α and Z^- is the graph of $-\alpha$. Note that $Z^\pm \cong X$ and $Z^+ \cap Z^- = C$ the latter being contained in the zero-section of K_X .

Elementary transformations are a classical method for constructing bundles introduced by Maruyama [22] [23], and recently for example Nakashima constructs stable vector bundles on CY threefolds using this technique [27]. The following proposition says that E is obtained by an elementary transformation along C , and determines the structure of ϕ .

Proposition 4.2. *Assume Hypothesis 4.1. With the previous notations, the restriction $\mathcal{F}|_{Z^+}$ determines a line bundle denoted L on $Z^+ \cong X$, and the restriction $\mathcal{F}|_{Z^-}$ determines a line bundle L' on $Z^- \cong X$. We have an exact sequence*

$$(4.1) \quad 0 \rightarrow E \rightarrow L \oplus L' \rightarrow i_*R \rightarrow 0$$

where R is a rank one torsion-free sheaf over C and $i : C \hookrightarrow X$ denotes the inclusion. The K_X -valued endomorphism ϕ is the operator on E induced by the K_X -valued endomorphism of $L \oplus L'$ whose matrix is $\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$.

Conversely, any pair of line bundles L, L' and such a surjection $L \oplus L' \rightarrow i_*R$ determines a pair (E, ϕ) .

Proof. The sheaf \mathcal{F} on the total space of K_X corresponding to (E, ϕ) is torsion-free and saturated in the sense that it admits no extension equal outside codimension 2. Take the restriction $\mathcal{F}|_{Z^+}$, divide by torsion, and look at the double dual. This is a line bundle L and we have a map $E \rightarrow L$. Similarly for $\mathcal{F}|_{Z^-}$ we get a map to a line bundle $E \rightarrow L'$. This gives an injection of sheaves $E \hookrightarrow L \oplus L'$, an isomorphism away from C . The quotient \mathcal{R} has to be nonzero, indeed E is assumed to be slope-stable so it is indecomposable. But, the quotient must also be pure of dimension 1, since E is reflexive. Along the smooth locus of C , the quotient has to be of rank one—if it were of rank 2 then E would be of the form $(L \oplus L')(-C)$, again decomposable. We get the structure result of the proposition, away from the finite singular set of C . In particular, \mathcal{R} is supported scheme-theoretically on C at the smooth locus, but this then has to hold at the singular points too because \mathcal{R} has pure dimension 1. We get $\mathcal{R} = i_*(R)$ for a torsion-free sheaf R of rank 1 on C . The statement identifying ϕ holds outside the singular set of C , and then extends. \square

In terms of the description of the previous proposition, we have

$$\begin{aligned} c_1(E) &= L + L' - C, \\ c_2(E) &= c_1(L)c_1(L') + \deg_C(R). \end{aligned}$$

On any surface X which admits a reduced canonical divisor, the above proposition gives a different construction of obstructed stable bundles E of arbitrarily high c_2 , by taking R to have very high degree.

We next consider the obstruction determined by ϕ . This discussion is just a start and is not used elsewhere; it is included here for eventual future reference. In what follows, we assume that the ramification curve C is smooth.

The first step is to calculate explicitly the deformation space $H^1(\text{End}(E))$. From (4.1) we get maps $L(-C) \rightarrow E$ and the same for L' , which gives an exact sequence

$$(4.2) \quad 0 \rightarrow (L \oplus L')(-C) \rightarrow E \rightarrow S \rightarrow 0.$$

Here S is again a line bundle supported on C , and we have two exact sequences,

$$(4.3) \quad 0 \rightarrow S \rightarrow (L \oplus L')|_C \rightarrow R \rightarrow 0$$

and

$$(4.4) \quad 0 \rightarrow R(-C) \rightarrow E|_C \rightarrow S \rightarrow 0.$$

The dual of (4.2) is

$$(4.5) \quad 0 \rightarrow E^* \rightarrow (L \oplus L')^*(C) \rightarrow S^*(C) \rightarrow 0.$$

Using the left arrows in (4.1) and (4.2) we get an injection of sheaves; and define Q to be the quotient, to give altogether an exact sequence

$$(4.6) \quad 0 \rightarrow \text{End}(E) \xrightarrow{u} \text{End}(L \oplus L')(C) \rightarrow Q \rightarrow 0.$$

This gives a long exact sequence of cohomology

$$(4.7) \quad \dots H^0(\text{End}(L \oplus L')(C)) \rightarrow H^0(Q) \xrightarrow{\delta} H^1(\text{End}(E)) \rightarrow H^1(\text{End}(L \oplus L')(C)) \rightarrow \dots$$

The main piece of $H^1(\text{End}(E))$ will come from the connecting map δ , so we would like to understand the structure of Q . The right maps of (4.1) and (4.5) give a map fitting into an exact sequence defining a sheaf G ,

$$0 \rightarrow G(-C) \rightarrow \text{End}(L \oplus L') \rightarrow (L \oplus L')^* \otimes R \oplus S^* \otimes (L \oplus L') \rightarrow S^* \otimes R \rightarrow 0.$$

This leads to an exact sequence

$$(4.8) \quad 0 \rightarrow Q|_C \rightarrow (L \oplus L')^* \otimes R(C) \oplus S^* \otimes (L \oplus L')(C) \rightarrow S^* \otimes R(C) \rightarrow 0.$$

However, Q is not supported on C , but only on the second infinitesimal neighborhood. This can be seen by looking at the contribution of $Q|_C$ to $c_1(\text{End}(E))$, one can see that we are missing a piece of rank one supported on C . There is an exact sequence

$$0 \rightarrow T \rightarrow Q \rightarrow Q|_C \rightarrow 0$$

where T is the image of the map $Q(-C) \xrightarrow{m} Q$. Note that T is a line bundle supported on C .

On the infinitesimal neighborhood $2C$ the sheaf $W = \underline{\text{Ext}}^1(Q, \mathcal{O}_X)$ fits as the kernel in the exact sequence

$$(4.9) \quad 0 \rightarrow W \rightarrow \text{End}(L \oplus L')(C)|_{2C} \rightarrow Q \rightarrow 0.$$

The above discussion passes to trace-free parts, denoted by a superscript $(\)^0$. For example W and Q split as

$$W \cong W^0 \oplus \mathcal{O}_C, \quad Q \cong Q^0 \oplus \mathcal{O}_C(C)$$

and these are compatible with the splitting of the exact sequence (4.9) into

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_{2C}(C) \rightarrow \mathcal{O}_C(C) \rightarrow 0$$

direct sum with

$$(4.10) \quad 0 \rightarrow W^0 \rightarrow \text{End}^0(L \oplus L')(C)|_{2C} \rightarrow Q^0 \rightarrow 0.$$

Furthermore, we have the exact sequences

$$(4.11) \quad 0 \rightarrow \text{End}(L \oplus L')(-C) \rightarrow \text{End}(E) \rightarrow W \rightarrow 0$$

and

$$(4.12) \quad 0 \rightarrow \text{End}^0(L \oplus L')(-C) \rightarrow \text{End}^0(E) \rightarrow W^0 \rightarrow 0$$

Lemma 4.3. *We have a perfect pairing*

$$H^0(Q^0) \otimes H^0(Q^0 \otimes K_X) \rightarrow \mathbb{C}$$

fitting with isomorphisms

$$H^0(Q^0 \otimes K_X) \cong H^1(W^0) \cong H^0(Q^0)^*,$$

$$H^0(Q^0) \cong H^1(W^0 \otimes K_X) \cong H^0(Q^0 \otimes K_X)^*.$$

The connecting map for the sequence (4.10) is the map

$$(4.13) \quad H^0(Q^0) \rightarrow H^1(W^0)$$

obtained by the composition of the connecting map for the trace-free version of (4.6) with the right map of (4.12),

$$H^0(Q^0) \rightarrow H^1(\text{End}^0(E)) \rightarrow H^1(W^0).$$

If, furthermore we assume that $H^1(\text{End}^0(L \oplus L')(C)) = 0$ and $H^1(\text{End}^0(L \oplus L')(-C)) = 0$ then $H^1(\text{End}^0(E))$ is the image of (4.13). \square

Apply Proposition 2.3 to the the present situation.

Corollary 4.4. *Suppose $H^1(\text{End}^0(L \oplus L')(C)) = 0$ and $H^1(\text{End}^0(L \oplus L')(-C)) = 0$. Then the rank of the quadratic form corresponding to $\text{obs}(\phi)$ is the same as the rank of the composed linear map*

$$H^0(Q^0) \xrightarrow{\text{Ad}(\phi)} H^0(Q^0 \otimes K_X) \rightarrow H^1(W^0 \otimes K_X).$$

Proof. By Lemma 4.3, $H^1(\text{End}^0(E))$ is the image of the map $H^0(Q^0) \rightarrow H^1(W^0)$. Similarly, $H^1(\text{End}^0(E) \otimes K_X)$ is the image of the map $H^0(Q^0 \otimes K_X) \rightarrow H^1(W^0 \otimes K_X)$. We have a commutative diagram

$$\begin{array}{ccccc} H^0(Q^0) & \rightarrow & H^1(\text{End}^0(E)) & \rightarrow & H^1(W^0) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(Q^0 \otimes K_X) & \rightarrow & H^1(\text{End}^0(E) \otimes K_X) & \rightarrow & H^1(W^0 \otimes K_X) \end{array}$$

where the vertical maps are given by $\text{Ad}(\phi)$. These compatibilities can be seen from the fact that $\text{Ad}(\phi)$ acts on the exact sequences of the form (4.1), (4.11) ... and the conclusion comes from a diagram chase. \square

Further work would be needed to obtain a full calculation of $\text{Ad}(\phi)$.

5. ON A VERY GENERAL QUINTIC SURFACE

From now on we specialize to the case when $X \subset \mathbb{P}^3$ is a very general quintic surface. By “very general” we mean smooth with

$$\text{Pic}(X) \cong \text{Pic}(\mathbb{P}^3) = \mathbb{Z}.$$

This condition holds on the complement of the *Noether-Lefschetz locus* which is a countable union of subvarieties. Notice that

$$H^1(\mathcal{O}_X) = 0, \quad H^2(\mathcal{O}_X) = \mathbb{C}^4, \quad K_X = \mathcal{O}_X(1),$$

and if $n \in \mathbb{Z}$ then

$$H^1(\mathcal{O}_X(n)) = 0.$$

This is seen by looking at the piece of long exact sequence

$$H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) \rightarrow H^1(\mathcal{O}_X(n)) \rightarrow H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n-5))$$

and using $H^i(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) = 0$ for $i = 1, 2$ and all m . Similarly, for $m = 0, 1, 2$ we have

$$H^0(\mathcal{O}_X(m)) = H^0(\mathcal{O}_{\mathbb{P}^3}(m)).$$

Since $\text{Pic}(X) \cong \mathbb{Z}$ all choices of hyperplane class give the same notion of stability and we don't need to include this choice in the notation. Also, for bundles of degree $c_1(E) = -1$ the four notions of Gieseker, slope / stability, semistability coincide.

The expected dimension of the moduli space $M = M_X(2, -1, d)$ of stable bundles of rank $\text{rk}(E) = 2$, with $c_1(E) = \mathcal{O}_X(-1)$ and $c_2(X) = d$, is

$$(5.1) \quad \dim_E^{\text{exp}}(M) = 4d - 20.$$

5.1. First classification of obstructed bundles. Recall that E is said to be *obstructed* if $h^2(\text{End}^0(E)) > 0$. This is equivalent to saying that E is either a singular point of the moduli space, or lies in an irreducible component whose dimension is strictly greater than the expected dimension (or both).

Theorem 5.1. *Suppose $X \subset \mathbb{P}^3$ is a general quintic surface. Suppose E is a stable bundle with $\det(E) \cong \mathcal{O}_X(-1)$, and E is obstructed. Then either:*

(i)—*there exists an exact sequence*

$$(5.2) \quad 0 \rightarrow \mathcal{O}_X(-1) \rightarrow E \rightarrow \mathcal{J}_P \rightarrow 0$$

where $\mathcal{J}_P \subset \mathcal{O}_X$ is the ideal of a zero-dimensional subscheme $P \subset X$; or else

(ii)—*there is a section $\beta \in H^0(\mathcal{O}_X(2))$ which is not a square, defining a double cover $r : Z \rightarrow X$ with $Z \subset K_X$ and r ramified along $\text{Zero}(\beta)$, together with a line bundle L over a desingularization $\epsilon : \tilde{Z} \rightarrow Z$ such that $E \cong r_*\epsilon_*(L)^{**}$.*

Proof. There is a nonzero twisted endomorphism $\phi : E \rightarrow E(1)$. Let $\beta := \det(\phi) \in H^0(\mathcal{O}_X(2))$. If $\beta = 0$ then ϕ is nilpotent at the general point, so we are in case (i) by the discussion of §3.

Assume from now on $\beta \neq 0$.

Let $D := \text{Zero}(\beta)$, which is a divisor in the linearly equivalence class corresponding to $\mathcal{O}_X(2)$. Since $\text{Pic}(X) = \mathbb{Z}$ with generator $\mathcal{O}_X(1)$, either D is irreducible and reduced, or else $D = 2C$ with C in the linear system of $\mathcal{O}_X(1)$.

If D is irreducible and reduced, we are in case (ii).

The remaining case is when $D = 2C$. Then, using the fact that the sections of $\mathcal{O}_X(1)$ all come from \mathbb{P}^3 and that the restriction on quadrics is injective, we get $\beta = \alpha^2$ for α a linear form. Notice that C itself has to be irreducible and

reduced since it is indecomposable in the positive cone of the Neron-Severi group, so the theory of §4 applies. Notice that Hypothesis 4.1 holds because C is reduced. Proposition 4.2 expresses E as an elementary transformation

$$0 \rightarrow E \rightarrow L \oplus L' \rightarrow i_*(R) \rightarrow 0$$

with L, L' line bundles and R a torsion free rank one sheaf on C . Stability shows that $\deg(L) > -1/2$, and for the inclusion $L(-C)$ which comes from the fact that $i_*(R)$ is supported on C , it says $\deg(L) - 1 < -1/2$. This gives $\deg(L) = 0$ so $L \cong \mathcal{O}_X$; similarly $L' \cong \mathcal{O}_X$. Our elementary transformation can thus be written

$$0 \rightarrow E \rightarrow \mathcal{O}_X \oplus \mathcal{O}_X \rightarrow i_*(R) \rightarrow 0.$$

But since $i_*(R)$ is supported on C we get

$$\mathcal{O}_X(-1) \oplus \mathcal{O}_X(-1) \rightarrow E.$$

In particular, for any 2×2 matrix m with complex entries we get a map

$$E \rightarrow \mathcal{O}_X \oplus \mathcal{O}_X \xrightarrow{m} (\mathcal{O}_X(-1) \oplus \mathcal{O}_X(-1))(1) \rightarrow E(1),$$

so on trace-free parts this gives a whole

$$\mathfrak{sl}_2(\mathbb{C}) = \text{End}^0(\mathbb{C}^2) \subset H^0(\text{End}^0(E) \otimes K_X).$$

In other words, the co-obstruction ϕ is not unique but lies in a space of dimension at least 3. The image of a nilpotent element of $\mathfrak{sl}_2(\mathbb{C})$ is a new co-obstruction ψ which is nilpotent. Looking at (E, ψ) we get back to case (i). \square

5.2. Case (ii). Let A be the maximum irregularity of a surface \tilde{Z} arising as the desingularization of a spectral cover Z associated to a non-square $\beta \in H^0(\mathcal{O}_X(2))$. Notice that the family of possible double covers $r : Z \rightarrow X$ arising in Case (ii) is bounded, i.e. parametrized by a single finite-type constructible set. Hence the same can be said for the possible desingularizations \tilde{Z} , and we get a global bound $q(\tilde{Z}) \leq A$. Recall that $\dim \text{Pic}^0(\tilde{Z}) = q(\tilde{Z})$.

Proposition 5.2. *The dimensions of components of the spaces of obstructed bundles falling into case (i) of Theorem 5.1, are bounded by a constant $A + 8$ which is independent of $d = c_2(E)$.*

Proof. The dimension of the space of β is bounded by 9, and for each choice of β the dimension of the space of rank one bundles on Z is bounded by A . However, at a general value of β the spectral cover Z is smooth itself and has irregularity 0, so the estimate of $A + 9$ can be reduced to $A + 8$. This kind of argument in the style of Ngo's dimension estimates for the Hitchin fibers in the case of curves [28], can probably be pushed further. \square

Using Lefschetz theory we can give a bound $A \leq 5$. Let $D \subset Z$ denote the ramification locus of r i.e. $D = \text{Zero}(\beta)$, and let $Q \subset \mathbb{P}^3$ be a general plane. For this discussion, denote by an asterisk $()^*$ the complement of the intersection of the variety in question with D or its preimage.

The Lefschetz theorem for the quasiprojective variety $X^* = X - D$ gives a surjection

$$\pi_1(X^* \cap Q) \twoheadrightarrow \pi_1(X^*).$$

Let Z_Q denote the preimage in Z of $X \cap Q$, which is the same as the preimage in \tilde{Z} ; thus $Z_Q^* = Z_Q^* - r^{-1}(D \cap Q)$. Note that $\tilde{Z}^* = Z^*$ is an étale covering of X^* with group $\mathbb{Z}/2$ so there is a diagram

$$\begin{array}{ccccc} \pi_1(Z_Q^*) & \hookrightarrow & \pi_1(X^* \cap Q) & \rightarrow & \mathbb{Z}/2 \\ \downarrow & & \downarrow & & \downarrow = \\ \pi_1(\tilde{Z}^*) & \hookrightarrow & \pi_1(X^*) & \rightarrow & \mathbb{Z}/2 \end{array}$$

where the middle vertical arrow is surjective. It follows that the left vertical arrow is surjective too (this result is a somewhat more advanced statement but which is still standard in Lefschetz theory). On the other hand the diagram

$$\begin{array}{ccc} \pi_1(Z_Q^*) & \rightarrow & \pi_1(Z_Q) \\ \downarrow & & \downarrow \\ \pi_1(\tilde{Z}^*) & \rightarrow & \pi_1(\tilde{Z}) \end{array}$$

with surjective map on the bottom, shows that the map $\pi_1(Z_Q) \rightarrow \pi_1(\tilde{Z})$ is surjective. It follows that the restriction map on Picard groups is injective:

$$(5.3) \quad \text{Pic}^0(\tilde{Z}) \hookrightarrow \text{Pic}^0(Z_Q).$$

However, the line bundles on \tilde{Z} are antipreserved by the involution (which acts, for example because we can choose an equivariant resolution \tilde{Z} [4]), because $\text{Pic}^0(X) = 0$. Thus, the image of the map (5.3) lies in the Prym variety $\text{Prym}(Z_Q/X)$. This proves the following bound.

Lemma 5.3. *The dimension of $\text{Pic}^0(\tilde{Z})$ is less than or equal to the dimension of $\text{Prym}(Z_Q/X)$ which is 5.*

Proof. The curve $X \cap Q$ is a plane quintic so it has genus $g = 6$, and its Pryms have dimension $g - 1$. □

Corollary 5.4. *The dimension of the locus of obstructed bundles of type (ii) is ≤ 13 .* □

Question 5.5. *What is A ?*

If $A = 0$ this would give a better bound on the dimension of the locus of obstructed bundles of type (ii). On the other hand, if $A > 0$ this would indicate the presence of some interesting examples of irregular surfaces. So, in any case it would be interesting to determine A . However, the answer would require first understanding the possible singularities of $X \cap H$ for all possible quadrics $H \subset \mathbb{P}^3$, when X is the fixed general quintic surface. This goes beyond the scope of the present paper.

5.3. Case (i)—generalities. Suppose $P \subset X$ is a zero-dimensional subscheme. Recall that P is *totally superfluous for quadrics in X* if it is nonempty, and for any subscheme $P' \subset P$ with $\text{length}(P/P') = 1$, the map

$$H^0(X, \mathcal{J}_{P'}(2)) \rightarrow H^0(X, \mathcal{J}_P(2))$$

is an isomorphism. In other words P and P' impose the same conditions on quadrics on X . Given that

$$H^0(X, \mathcal{O}_X(2)) \cong H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)),$$

it is equivalent to say that P is totally superfluous for quadrics in \mathbb{P}^3 .

Corollary 5.6. *Suppose P is a zero-dimensional subscheme. Then there exists an exact sequence*

$$(5.4) \quad 0 \rightarrow \mathcal{O}_X(-1) \rightarrow E \rightarrow \mathcal{J}_P \rightarrow 0$$

with E locally free, if and only if P is a l.c.i. totally superfluous for quadrics in X . In particular, in case (i) of Theorem 5.1, we can add the condition that P is totally superfluous for quadrics.

For any locally free extension fitting into (5.4) with P nonempty, the bundle E is stable and obstructed with $c_1(E) = \mathcal{O}_X(-1)$ and $c_2(E) = d$.

Proof. The first part is just Lemma 3.1. For the last paragraph, the co-obstruction ϕ is given by the obvious factorization. The fact that E is always stable comes from $H^0(E) = H^0(\mathcal{J}_P) = 0$. \square

Apply the discussion at and above Lemma 3.2, with $L = \mathcal{O}_X(-1)$ and $L' = \mathcal{O}_X$:

$$(5.5) \quad E^* \cong E(1),$$

giving a symmetry of cohomology dimensions

$$h^0(E(1)) = h^0(E^*) = h^2(E(1)) = h^2(E^*);$$

$End^0(E) = Sym^2(E)(1)$; and there are exact sequences

$$(5.6) \quad 0 \rightarrow E \rightarrow End^0(E) \rightarrow \mathcal{J}_P^2(1) \rightarrow 0$$

and

$$(5.7) \quad 0 \rightarrow \mathcal{O}(-1) \rightarrow End^0(E) \rightarrow \mathcal{G} \rightarrow 0.$$

Here \mathcal{G} is as defined in Lemma 3.2 and we have the further exact sequence

$$(5.8) \quad 0 \rightarrow \mathcal{G} \rightarrow E(1) \rightarrow \underline{Ext}^2(\mathcal{O}_{2P}(1), \mathcal{O}_X) \rightarrow 0.$$

Corollary 5.7. *Then the map $Ad(\phi)$ fits into the diagram*

$$\begin{array}{ccccc} & & H^0(\mathcal{O}_{2P}(2))^* & & \\ & & \downarrow & & \\ \dots & H^1(End^0(E)) & \longrightarrow & H^1(\mathcal{G}) & \longrightarrow & H^0(\mathcal{O}(2))^* & \dots \\ & & & \downarrow & & & \\ & & & H^1(E(1)) & \longrightarrow & H^1(End^0(E) \otimes K_X) & \\ & & & \downarrow & & & \\ & & & 0 & & & \end{array}$$

where the horizontal and vertical sequences are exact.

Proof. Apply Proposition 3.3. \square

Lemma 5.8. *Let $2P$ denote the fat subscheme defined by \mathcal{J}_P^2 . Suppose that there are no quadrics of \mathbb{P}^3 passing through $2P$. Then the obstruction is unique up to scalar, i.e. $h^2(End^0(E)) = 1$. Furthermore, the map*

$$Ad(\phi) : H^1(End^0(E)) \rightarrow H^1(E(1))$$

is surjective and the map

$$H^1(E(1)) \rightarrow H^1(End^0(E) \otimes K_X)$$

is injective.

Proof. Calculate with the exact sequence from (5.6)

$$0 \rightarrow H^0(E(1)) \rightarrow H^0(\text{End}^0(E)(1)) \rightarrow H^0(\mathcal{J}_P^2(2)).$$

The assumption that there are no quadrics passing through $2P$ says that $H^0(\mathcal{J}_P^2(2)) = 0$. We have seen above that $H^0(E(1)) = \mathbb{C}$ so

$$H^2(\text{End}^0(E)) \cong H^0(\text{End}^0(E)(1))^* \cong \mathbb{C},$$

i.e. the obstruction is unique.

Consider the diagonal map in the diagram of Corollary 5.7,

$$H^0(\mathcal{O}_{2P}(2))^* \rightarrow H^0(\mathcal{O}(2))^*.$$

By Remark 3.4 this is dual to the restriction of quadrics to $2P$, so the hypothesis implies that the dual map is injective. Hence the diagonal map is surjective. A diagram chase then shows that the other diagonal map $H^1(\text{End}^0(E)) \rightarrow H^1(E(1))$ is surjective.

For the last map, use the long exact sequence for (5.6) which may be written

$$H^0(\mathcal{J}_P^2(2)) \rightarrow H^1(E(1)) \rightarrow H^1(\text{End}^0(E)(1)) \rightarrow H^1(\mathcal{J}_P^2(2)).$$

Again the hypothesis says that $H^0(\mathcal{J}_P^2(2)) = 0$ so we get the required injectivity. \square

Lemma 5.9. *Suppose P is totally superfluous for quadrics, and is not contained in a plane. Suppose given a vector bundle E in an extension of the form (5.4). Then $h^0(E(1)) = 1$ so the exact sequence (5.4) and in particular P , are determined by E . The extension class is determined up to a scalar multiple.*

Let d denote the length of P , then $h^1(E(1)) = d - 8$ and in particular $d \geq 8$.

Proof. The exact sequence (5.4) together with the exact sequence for $\mathcal{J}_P \subset \mathcal{O}_X$ give

$$0 \rightarrow H^0(\mathcal{O}_X) \rightarrow H^0(E(1)) \rightarrow H^0(\mathcal{J}_P(1)),$$

but the assumption that P is not contained in a plane implies $H^0(\mathcal{J}_P(1)) = 0$. Thus $\mathbb{C} = H^0(\mathcal{O}_X) \cong H^0(E(1))$ and the map $\mathcal{O}_X(-1) \rightarrow E$ is unique up to scalar multiple. Given E , fix the unique map and let \mathcal{J} denote the quotient $E/\mathcal{O}_X(-1)$. Then \mathcal{J}^{**} is a line bundle, but since $\det(E) \cong \mathcal{O}_X(-1)$ by hypothesis we get $\mathcal{J}^{**} = \mathcal{O}_X$ so \mathcal{J} is the ideal of a zero-dimensional subscheme P . The extension class is determined up to the scalar automorphisms of $\mathcal{O}_X(-1)$ and \mathcal{J} .

To show that $h^1(E(1)) = d - 8$ consider long exact sequence

$$0 \rightarrow H^1(E(1)) \rightarrow H^1(\mathcal{J}_P(1)) \rightarrow H^2(\mathcal{O}_X) \rightarrow H^2(E(1)) \rightarrow H^2(\mathcal{J}_P(1)) \rightarrow 0.$$

Note that $h^2(\mathcal{J}_P(1)) = h^2(\mathcal{O}_X(1)) = 1$ and similarly by duality and Proposition 5.9 $h^2(E(1)) = h^0(E^*) = h^0(E(1)) = 1$. It follows that the last map of the above long exact sequence is an isomorphism, in particular its kernel is zero so the sequence becomes

$$0 \rightarrow H^1(E(1)) \rightarrow H^1(\mathcal{J}_P(1)) \rightarrow H^2(\mathcal{O}_X) \rightarrow 0.$$

On the other hand, by assumption $H^0(\mathcal{J}_P(1)) = 0$ and recall that $H^1(\mathcal{O}_X(1)) = 0$, so we have the sequence

$$0 \rightarrow H^0(\mathcal{O}_X(1)) \rightarrow H^0(P, \mathcal{O}_P(1)) \rightarrow H^1(\mathcal{J}_P(1)) \rightarrow 0.$$

Using $h^2(\mathcal{O}_X) = h^0(\mathcal{O}_X(1)) = 4$ and P has d points, this gives

$$h^1(\mathcal{J}_P(1)) = d - 4, \quad h^1(E(1)) = d - 8.$$

This implies in particular that $d \geq 8$. \square

Corollary 5.10. *In the situations of both Lemmas 5.8 and 5.9, either*

$$\dim_E(M) = 4d - 20$$

and the moduli space M has a hypersurface singularity at E , or else

$$\dim_E(M) = 4d - 19$$

and the moduli space is smooth at E . The quadratic term of the Kuranishi map has rank $h^1(E(1)) = d - 8$; in particular if $d > 8$ then the moduli space has the expected dimension.

Proof. If the Kuranishi map vanishes identically then M is smooth of dimension one more than the expected dimension at E ; if not then M has the expected dimension at E with a hypersurface singularity.

The quadratic term of the Kuranishi map has the same rank as the rank of the linear map $Ad(\phi)$. In the situation of Lemma 5.8, this rank is $h^1(E(1))$. If $h^1(E(1)) > 0$ then the Kuranishi map has to be nontrivial so $\dim_E(M)$ is the expected dimension. \square

Lemma 5.11. *Suppose given a family of extensions of the form (5.4), where the family of totally superfluous subschemes P are parametrized by a variety F of dimension $f \leq 2d$, and the general member P of this family imposes c conditions on quadrics. Then the total dimension of the family is $\leq f + d - c - 1$. If the P are not in a plane then the dimension is equal to $f + d - c - 1$.*

Proof. For a given P in the family F , the dimension of the space of extensions is $d - c$ by Lemma 3.1, and taking into account the scalar multiple the full dimension of the family of bundles E is bounded by $f + d - c - 1$.

If we are in the situations of Lemma 5.9, then for each obstructed bundle near E the subscheme P is uniquely determined; and the extension class is determined up to scalar multiple, so the dimension estimate is an equality. \square

Lemma 5.12. *In the situation of Corollary 5.6 including the hypothesis that P is totally superfluous for quadrics, suppose P is contained in, and spans a plane. Then the space of obstructions has dimension 3, and a general co-obstruction corresponds to a reducible spectral curve whose determinant β is the square of the linear form defining the plane containing P .*

Proof. Suppose $I \subset \mathbb{P}^3$ is a plane with $P \subset X \cap I$. The long exact sequence of (5.4) gives

$$0 \rightarrow \mathbb{C} \rightarrow H^0(E(1)) \rightarrow H^0(\mathcal{J}_P(1)) \rightarrow 0$$

and $H^0(\mathcal{J}_P(1)) = \mathbb{C}$. Thus $H^0(E(1)) = \mathbb{C}^2$ and we get an injection

$$\mathcal{O}_X(-1)^{\oplus 2} \hookrightarrow E.$$

Using $E(1) \cong E^*$ we get dually $h : E \hookrightarrow \mathcal{O}_X^{\oplus 2}$. From this we can easily construct a whole 3-dimensional $\mathfrak{sl}(2)$ of maps $E \rightarrow E(1)$. The cokernel of h is a sheaf of pure dimension 1 whose cycle class has to be the same as C , on the other hand it goes through the points P and by hypothesis P spans I . These imply that the cokernel of h is a rank one torsion-free sheaf on C , so E is obtained as an elementary transformation as in §4. The elements of the constructed $\mathfrak{sl}(2)$ of co-obstructions, are matrices vanishing along C so they all have $\det(\psi) = \alpha^2$ where α is the equation of C .

Some further exact sequences show that the space of co-obstructions has dimension ≤ 3 : the exact sequence corresponding to (5.6) tensored with $K_X = \mathcal{O}_X(1)$ is

$$0 \rightarrow H^0(E(1)) \rightarrow H^0(\text{End}^0(E)(1)) \rightarrow H^0(\mathcal{J}_P^2(2)) \rightarrow H^1(E(1));$$

the hypothesis that P spans the plane I implies that $2P$ is contained in a unique quadric, namely $2I$, so $h^0(\mathcal{J}_P^2(2)) = 1$; and the long exact sequence of (5.4) gives

$$0 \rightarrow \mathbb{C} \rightarrow H^0(E(1)) \rightarrow H^0(\mathcal{J}_P(1)) \rightarrow 0$$

with $H^0(\mathcal{J}_P(1)) = \mathbb{C}$. Putting these together gives a bound of 3 for the dimension of the space of co-obstructions. Since we have already constructed a three dimensional space, the dimension must be 3 and a general element is one of our constructed endomorphisms. This completes the proof. \square

6. EXAMPLES OF OBSTRUCTED BUNDLES: CASE (I)

In this section we discuss the construction and further classification of examples of obstructed bundles on our very general quintic surface X , falling into case (i) of Theorem 5.1.

To start off, note that by taking a line $D \subset \mathbb{P}^3$ and looking at 4 points in $X \cap D$, we get a totally superfluous subscheme P : vanishing along D imposes only 3 conditions on quadrics, and any three of the points will do the job. This gives a construction of a stable bundle with $c_2 = 4$. However this case falls outside most of our other hypotheses, so in the current paper we don't investigate too much further the moduli space in the range of very small c_2 . For the examples, we also avoid the more delicate case of subschemes P contained in a plane.

6.1. Points on the normal cubic curve. One way to construct subschemes P which are totally superfluous for quadrics, is by taking subcollections of points in $X \cap N$ where $N \subset \mathbb{P}^3$ is a curve. If N is a rational curve and $d > h^0(\mathcal{O}_N(2))$ then any collection of d points on N is totally superfluous for quadrics, indeed any $d - 1$ of the points already force a quadric to vanish on N , because any collection of points satisfies the maximal rank property for line bundles on $N \cong \mathbb{P}^1$.

Some possible choices for N would be a line as above, or two skew lines, or three lines meeting in a rational stick-figure. These basic examples suggest looking more generally at the rational normal cubic curve

$$N = \{[1 : t : t^2 : t^3]\} \subset \mathbb{P}^3$$

which is embedded by the full linear system $|\mathcal{O}_{\mathbb{P}^1}(3)|$. The restriction of a quadric to N is a section of $\mathcal{O}_N(6)$, so points on N can impose a maximum of

$$h^0(N, \mathcal{O}_N(6)) = 7$$

conditions on quadrics. As soon as there are $d \geq 8$ points, the resulting subscheme is totally superfluous.

The choice of $N \subset \mathbb{P}^3$ has $f = 12$ parameters, as can be seen in the following way. The group of automorphisms of \mathbb{P}^3 fixing a given N is $PSL(2) \subset PSL(4)$ with the embedding given by identifying \mathbb{C}^4 with the symmetric cube of the standard representation of $SL(2)$:

$$\mathbb{C}^4 = \text{Sym}^3(\mathbb{C}^2), \quad SL(\mathbb{C}^2) \rightarrow SL(\mathbb{C}^4).$$

Any other rational normal cubic is a translate of N , since they have the invariant characterization as being given by complete linear systems of degree 3 on \mathbb{P}^1 . The space of translates of N is $PSL(4)/PSL(2)$ which has dimension $15 - 3 = 12$.

The number of conditions imposed by a subscheme P is equal to 7, so by Lemma 5.11, the dimension of the space of obstructed bundles is $d + 4$.

In the first case $d = 8$, we get a 12-dimensional family, indeed there is a unique extension for each P . In this case by (5.1) the expected dimension is also 12.

For the bundles in this family, the dimension of the space of obstructions is 1, indeed whenever P forces a quadric form to vanish on a curve in \mathbb{P}^3 then quadrics vanishing on the double $2P$ of P in X , must in fact vanish on the double of P in \mathbb{P}^3 because they also vanish in transverse directions to X . It now suffices to have 4 points spanning \mathbb{P}^3 in order to insure that there are no quadrics passing through $2P$. Then apply Lemma 5.8.

Hence, the dimension of $M(2, -1, 8)$ at a general E in our family, is either 12 or 13; but in any case the Zariski tangent space has dimension 13. If the dimension is 12 this would imply that the component is generically non-reduced.

Conjecture 6.1. *The dimension of $M(2, -1, 8)$ at a general E in our family is 12, so the family obtained from points on rational normal cubics is a generically non-reduced component of $M(2, -1, 8)$.*

We will give a heuristic discussion assuming that we know a better bound of ≤ 12 in Corollary 5.4 (a finer analysis should be able to do that). Before doing that, we construct a 13-dimensional family.

6.2. Points on normal elliptic curves. Suppose Y is a smooth elliptic curve, with a divisor of degree 4 denoted $\mathcal{O}_Y(1)$. Notice that no basepoint is chosen on Y , and all divisors of a fixed nonzero degree are related by translation automorphisms of Y , so the choice of $(Y, \mathcal{O}_Y(1))$ corresponds to a single parameter (the j -invariant of the elliptic curve). Now $H^0(\mathcal{O}_Y(1)) \cong \mathbb{C}^4$, and the choice of such an identification up to scalars is a 15-dimensional space. Any such choice gives an embedding of Y as a normal elliptic curve of degree 4 in \mathbb{P}^3 . There is no connected subgroup of $PSL(4)$ fixing Y , so the dimension of the family of all smooth normal elliptic curves of degree 4, is 16. Now, choose a collection P' of 7 points among the 20 in $X \cap Y$. This is a discrete choice out of C_{20}^7 possibilities. There is a unique nonzero section of $\mathcal{O}_Y(2)$ vanishing on these points, and this section has an additional zero denoted $q = q(Y, P') \in Y$. Let F be the family of all choices (Y, P') such that $q(Y, P') \in X$ also. This is a codimension 1 condition (and nontrivial, since X is general), so F has dimension 15. We think of F as the family of (Y, P) such that $P \subset X \cap Y$ has 8 points and such that $\mathcal{O}_Y(P) \cong \mathcal{O}_Y(2)$.

The projection from F to the Hilbert scheme of subschemes $P \subset X$, has as fiber the collection of all degree four elliptic curves Y' passing through P . Any such P imposes 7 conditions on quadrics of \mathbb{P}^3 , and any of the 8 points is superfluous. There is a 3-dimensional space of quadrics passing through P .

We claim that for any elliptic curve Y' of degree 4 passing through P' , two of the quadrics will vanish on Y' . Indeed, a ninth point on Y' imposes an additional condition, so there is a two dimensional space of quadrics passing also through the ninth point, but a degree consideration shows that these must contain Y' . We conclude that any point (Y', P) in F corresponds to a choice of two dimensional

subspace of the \mathbb{C}^3 of quadrics passing through P . In particular, the fiber of (Y', P) lying over P has dimension 2, so the family of subschemes P has dimension 13.

General subschemes P in this family satisfy the condition of Lemma 5.8, so the space of obstructions is one-dimensional and the moduli space has dimension ≤ 13 ; thus our construction gives an irreducible component of $M(2, -1, 8)$ of dimension 13. At a general point (or in fact, any point at which the obstruction space is unidimensional) the Kuranishi map vanishes entirely, that is to say all higher order obstructions vanish and the moduli space is smooth but not of the expected dimension.

6.3. Disjointness of the two components. We now give an heuristic argument as to why the 12-dimensional locus of obstructed bundles obtained using the rational normal cubics, shouldn't be in the closure of the 13-dimensional family constructed using elliptic curves. This would prove Conjecture 6.1 modulo a better bound of ≤ 12 for the dimension of the type (ii) singular locus. In this case we would have at least two irreducible components of $M(2, -1, 8)$, one nonreduced of the expected dimension 12 and the other smooth of dimension 13.

Suppose N is a rational normal cubic, and P_0 is a subscheme of 8 points, and E_0 is a bundle obtained as an extension as usual.

If the dimension of the moduli space at E_0 is 13, then since this is bigger than the expected dimension all bundles near E_0 in $M(2, -1, 8)$ must be obstructed. By semicontinuity of the dimension of the obstruction space, they all have obstruction space of dimension 1. At a general point near P_0 , we are either in case (i) or case (ii) of Theorem 5.1. Assuming that the dimension of the space of obstructed bundles in case (ii) is ≤ 12 , it implies that the general nearby point must be in case (i). In particular, the nearby bundle corresponds to an extension of the form (5.4) for a subscheme P_t of length 8. Since P_t must still be totally superfluous, it has to impose 7 conditions on quadrics, and the 3-dimensional space of quadrics vanishing on P_t is a deformation of the 3-dimensional space of those vanishing on P_0 . We can choose a 2 dimensional subspace whose intersection is a curve of the form $N \cup D$ where D is a line going through two points of N . By modifying the choice of 2-dimensional subspace we may assume that D doesn't meet P_0 .

Now $Y_0 := N \cup D$ can be viewed as a degenerate elliptic curve, and when the two quartics are generalized, it smooths out to a family of elliptic curves Y_t as in §6.2. Given 8 points in $X \cap N$, is it possible that we get a smoothing Y_t of $Y_0 = N \cup D$ such that the collection of 8 points $P_0 \subset N$ deforms out to a subscheme P_t of 8 of the required kind on Y ?

Following one of the points as an origin, we can consider $N - N \cap D$ as a Neron-type group scheme limit of the family of elliptic curves Y . The divisors P_t are supposed to be linearly equivalent to 2 times a linear section; but a linear section has a point which goes to the component $D \subset Y_0$ in the limit, so it shouldn't be possible for a deformation P_t of a divisor O_0 in the finite part of our group scheme, to remain linearly equivalent to twice a linear section.

Assuming the above argument, we obtain a nonreduced component. However, the quadratic term of the obstruction map vanishes, because $Ad(\phi) : H^1(End^0(E)) \rightarrow H^1(End^0(E) \otimes K_X)$ factors through $H^1(E(1))$ which has dimension $d - 8 = 0$. So, the first nonzero term of the Kuranishi map should have degree at least 3.

6.4. Nine points on an elliptic curve. As we saw above, the family of elliptic curves of degree 4 in \mathbb{P}^3 is 16 dimensional. Denote by F the family of (Y, P) where Y is such an elliptic curve and $P \subset X \cap Y$ is a choice of 9 out of the 20 intersection points. The choice of P is discrete so F has 16 dimensions. The space of quadrics on Y has dimension 8, so P imposes 8 conditions, and if Y is in general position (indeed, if it is not in the hypersurface considered previously) then any 8 points will also impose 8 conditions so P is totally superfluous. A general choice will satisfy this general position property because X was assumed general to start with. The number of extra points is 1 so the extension class is unique up to scalars. We obtain in this way a 16 dimensional family of obstructed bundles; at a general point the dimension of the space of obstructions is 1.

By Corollary 5.4, the dimension of the space of obstructed bundles of type (ii) is ≤ 13 . We can again argue that the dimension of $M(2, -1, 9)$ at a general member (E_0, Y_0, P_0) of the family, is 16. All nearby bundles E_t would be obstructed, with one-dimensional obstruction space, so coming from subschemes P_t ; however, since P_t must also be totally superfluous, it can still impose only 8 conditions on quadrics. We get a 2-dimensional family V_t of quadrics passing through P_t , but this must be a smooth deformation of the 2-dimensional family of quadrics V_0 going through P_0 , i.e. the equations of Y_0 . Note that Y_0 is a complete intersection of any two elements forming a basis of V_0 , so the intersection of two basis elements of V_t is again an elliptic curve of degree 4, Y_t and we have $P_t \subset Y_t$. Thus, any deformation must remain inside our given family, so we have constructed a 16-dimensional component of $M(2, -1, 9)$. The points of this component are generically obstructed, so we get a non-reduced component.

The quadratic term of the obstruction map is nonzero: by Corollary 5.10, $Ad(\phi)$ has rank $d - 8 = 1$. Therefore the quadratic term of the Kuranishi map is nonzero at a general point. Hence, at a general point of this component the moduli space is given by a single equation $x^2 = 0$.

The special cases of 9 points on a rational normal cubic, or 9 points arrayed on 2 lines (in groups of 4 and 5) or 3 meeting lines (in groups of 2 or 3) are limits of the general family we constructed here. Indeed, the elliptic curves can degenerate, and since our 9 points are general on Y , the points can go into the various different components of the degeneration in various different ways. The argument given in the previous subsection used a linear equivalence condition on the divisors to argue that the points couldn't all specialize to N , but for a general collection of points that argument no longer applies.

It would be interesting to be able to find another component which is generically smooth of the expected dimension.

6.5. Classification for case (i). Motivated by the above examples, we state a finer classification result for the obstructed bundles of type (i).

Theorem 6.2. *Suppose $X \subset \mathbb{P}^3$ is a very general quintic surface. Suppose E is an obstructed vector bundle on X of type (i) from Theorem 5.1 fitting into an extension (5.2) with a subscheme P of length d , totally superfluous for quadrics. Then either:*

- (a)— P is supported on $X \cap I$ for $I \subset \mathbb{P}^3$ a plane, or else (a') on $X \cup 2I$ where $2I$ denotes the plane doubled, or else (a'') on $X \cap (I \cup D)$ where D is a line not contained in I ;*
- (b)— $d \leq 20$ and P imposes ≤ 8 conditions on quadrics, it is supported on a zero-dimensional intersection $X \cap Y$ for $Y \subset \mathbb{P}^3$ a possibly degenerate genus one degree*

four curve which is the complete intersection of two set-theoretically transverse quadrics;

(c)— $d \geq 10$ and P is supported on $Z = X \cap H$ where H is a reduced quadric surface, and H is unique i.e. P imposes 9 conditions on quadrics; or

(d)— $d \geq 11$ and P is general i.e. it imposes all 10 conditions on quadrics.

Proof. Let c denote the number of conditions imposed on quadrics, and $V \cong \mathbb{C}^{10-c}$ the space of quadrics vanishing on P . If $c = 10$ then $d \geq 11$ by the totally superfluous condition and we are in case (d), which was considered for example by O’Grady [30, (3.29)].

If $c = 9$ then $d \geq 10$ and $\dim(V) = 1$ and the unique element of V up to scalars, determines a unique quadric H such that $P \subset X \cap H$. This gives (c) if the quadric is reduced, or (a’) if the quadric is of the form $2I$ for a plane I .

Suppose $c \leq 8$, then $\dim(V) \geq 2$ and we can choose two linearly independent elements. These correspond to quadrics H_1 and H_2 . Suppose first of all that $Y = H_1 \cap H_2$ has dimension 1. Then it has degree 4 and is a possibly degenerate version of an elliptic curve; and $P \subset Y \cap X$.

The fact that X is general implies that Y is not contained in X and indeed meets X in a finite subscheme. This subscheme has length 20, so the length of P is ≤ 20 . This gives (b).

Suppose in the previous situation, on the other hand, that H_1 and H_2 contain a common component of dimension 2 in \mathbb{P}^3 , which must be a plane I . We can write $H_i = I \cup U_i$ with U_i also being distinct planes; then $P \subset H_1 \cap H_2 = I \cup D$ where $D = U_1 \cap U_2$ is a line. This gives (a’). \square

Proposition 6.3. *Suppose $d \geq 6$. Letting e denote the dimension of the space of obstructed bundles of given type at E , we have the following estimates in the different cases of the previous theorem:*

- (a) $e \leq 2d - 1$;
- (b) $e \leq d + 12$;
- (c) $e = 2d - 1$;
- (d) $e = 3d - 11$.

Proof. Let f be the dimension of the space of subschemes P (i.e. of some irreducible component of the space of totally superfluous subschemes); let c be the number of conditions imposed by P on quadrics. Recall from Lemma 5.11 that $e \leq f + d - c - 1$.

In all cases (a) the number of conditions is at least $c \geq 3$, indeed a single point cannot be totally superfluous so $d \geq 2$, but any two points even infinitesimally close impose at least 2 conditions so in turn $d \geq 3$, but again any three points impose at least 3 conditions. In cases (a) and (a’) we have $f \leq 3 + d$ which gives the estimate $e \leq 2d - 1$. If we are in case (a’’) but not in any other, then all quadrics passing through P must also pass through $I \cup D$, in particular there are at least 2 additional points of P on $D - (D \cap I)$. This also implies that $c = 8$. The dimension of the family of possible $I \cup D$ is 7, and for each one the points on $X \cap D$ are determined up to a discrete choice; whereas the dimension of the family of at most $d - 2$ points on $I \cap P$ is $\leq d - 2$. This gives $f \leq d + 5$, and $e = f + d - 9 \leq 2d - 4$.

In case (b), the family of choices of elliptic curve has dimension ≤ 16 . The choice of P is discrete once the elliptic curve is fixed (except in some very degenerate configurations but which also don’t contribute more than 16 to the dimension). So

$f \leq 16$. The number of conditions imposed is at least 3 (we think probably ≥ 6 in fact), so we get $e \leq d + 12$ and hopefully even $e \leq d + 9$.

In case (c) note that the family of such P has dimension $d + 9$ consisting of all pairs (H, P) with H arbitrary and P a general divisor of degree d on Z . For each P imposing $c = 9$ conditions, the dimension of the space of extensions up to scalar is $d - 10$, so the dimension of this component of the obstructed locus is $2d - 1$.

In case (d) the family of choices of P has dimension $2d$ and for each choice, the dimension of the space of extensions up to scalars is $d - 11$. The dimension of this component of the space of obstructed bundles is $3d - 11$. \square

Theorem 6.4. *If $d \geq 11$ then all irreducible components of $M_X(2, -1, d)$ are good, i.e. generically smooth of the expected dimension.*

If $d \geq 12$ then the $3d - 11$ -dimensional component Σ of the locus of obstructed bundles consisting of bundles of type (d) is nonempty, and is the biggest irreducible component of the singular locus of the moduli space. The singularities of M along a general point of Σ are ordinary quadratic double points in the transverse direction.

Proof. Use the dimension estimates of Proposition 6.3, and the estimate of Corollary 5.4 for the locus of obstructed bundles of type (ii). Together these say that the dimension of the locus of obstructed bundles is bounded by

$$\max(3d - 11, 2d - 1, d + 12, 13).$$

If $d \geq 11$ then

$$\max(3d - 11, 2d - 1, d + 12, 13) < 4d - 20.$$

By the standard argument this shows that the moduli space is good.

If $d \geq 12$ then

$$\max(2d - 1, d + 12, 13) < 3d - 11$$

so the dimension of the other components is strictly smaller than the dimension of Σ . This shows that Σ is the unique irreducible component of largest dimension. Notice that at points where the dimension of the space of extensions might jump, we are actually in one of the other cases (a)–(c) so the dimension is smaller.

Recall now the description of the quadratic term in the obstruction map at a general point of Σ where there is a single obstruction ϕ . The quadratic term is a symmetric bilinear form on $H^1(\text{End}^0(E))$ whose rank is equal to the rank of the linear map $\text{Ad}(\phi)$ which factors as

$$H^1(\text{End}^0(E)) \rightarrow H^1(E(1)) \rightarrow H^1(\text{End}^0(E) \otimes K_X).$$

Since all quadrics passing through $2P$ must vanish, Corollary 5.10, $\text{Ad}(\phi)$ and the quadratic term of the Kuranishi map have rank $d - 8$.

The Zariski tangent space at E has dimension equal to the expected dimension plus the number of obstructions, i.e. $4d - 19$. The component of Σ has dimension $3d - 11$, so the transverse direction in the Zariski tangent space has dimension $d - 8$; it follows that the singularity is an ordinary double point in the transverse direction, noting that the quadratic form has to vanish in the directions along Σ . In other words, the equations for M are locally of the form

$$x_1^2 + \dots + x_{d-8}^2 = 0$$

in terms of a local coordinate system x_1, \dots, x_{4d-19} such that x_1, \dots, x_{d-8} are the coordinate functions defining Σ . \square

Remark 6.5. *If we can get the better condition $e \leq d + 9$ in case (b) above, then we could improve the conditions of the theorem to $d \geq 10$ for goodness and $d \geq 11$ for Σ being the biggest piece of the singular locus. The example for $d = 9$ shows that $d = 10$ would be sharp.*

The result of Theorem 6.4 improves upon the sharp bound asked for by O’Grady in his Question, [31, p. 112]:

—“is $M(\xi)$ good if $\Delta_\xi > \text{rk}(\xi)(p_g + 1)$?”

For $\xi = (2, -1, d)$ we have $\text{rk}(\xi) = 2$, $\Delta_\xi = d - \frac{5}{4}$ and $p_g = 4$ for our quintic surfaces so the sharp bound asked for by O’Grady would say $d > 11\frac{1}{4}$ i.e. $d \geq 12$. The result of Theorem 6.4 improves this by 1. Remark 6.5 would improve it by 2. It is clear from our discussion that the phase transition occurs at $c_2(E) = \text{rk}(\xi)(p_g + 1) = 10$.

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