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On certain time- and space-fractional evolution systems

Ahmad Z. FINO^{†*} and Mokhtar KIRANE[†]

Abstract

In this paper, we investigate the local existence and the finite-time blow-up of solutions of semilinear parabolic system with nonlocal in time nonlinearity. In addition, we also give the blow-up rate and necessary conditions for local and global existence.

Keywords: Parabolic system, local existence, mild and weak solution, blow-up, blow-up rate, maximal regularity, interior regularity, schauder's estimates, Riemann-Liouville fractional integrals and derivatives.

1 Introduction

This article is concerned to study the following semilinear parabolic system with nonlocal in time nonlinearity

$$\begin{cases} u_t - \Delta u = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} |v|^{p-1} v(s) ds & x \in \mathbb{R}^N, t > 0, \\ v_t - \Delta v = \frac{1}{\Gamma(1-\delta)} \int_0^t (t-s)^{-\delta} |u|^{q-1} u(s) ds & x \in \mathbb{R}^N, t > 0, \end{cases} \quad (1.1)$$

supplemented with the initial conditions

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \mathbb{R}^N, \quad (1.2)$$

where $u_0, v_0 \in C_0(\mathbb{R}^N)$, $\gamma, \delta \in (0, 1)$ and $p, q > 1$.

Here $-\Delta$ stands the Laplacian operator with $D(-\Delta) = H^2(\mathbb{R}^N)$, where $H^2(\mathbb{R}^N)$ is the standard Sobolev space, Γ is the Euler gamma function. The space $C_0(\mathbb{R}^N)$ denotes the set of all continuous functions decay to zero at infinity.

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Our analysis is based on the observation that the nonlinear differential system (1.1) can be written in the form:

$$\begin{cases} u_t - \Delta u = J_{0|t}^\alpha (|v|^{p-1}v) & x \in \mathbb{R}^N, t > 0, \\ v_t - \Delta v = J_{0|t}^\beta (|u|^{q-1}u) & x \in \mathbb{R}^N, t > 0, \end{cases} \quad (1.3)$$

where $\alpha := 1 - \gamma \in (0, 1)$, $\beta := 1 - \delta \in (0, 1)$, and $J_{0|t}^\theta$ is the Riemann-Liouville fractional integral defined in (2.8) for all $\theta \in (0, 1)$.

In the case of a single equation, Cazenave, Deicktsein and Weissler [3] addressed the local, global existence and blow-up questions while in the recent work of Fino and Kirane [5] we can find the blow-up rate and a necessary condition for the local and the global existence. The equation is

$$\begin{cases} u_t - \Delta u = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} |u|^{p-1} u(s) ds & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x) & x \in \mathbb{R}^N, \end{cases} \quad (1.4)$$

where $u_0 \in C_0(\mathbb{R}^N)$, $\gamma \in (0, 1)$, $p > 1$ and $u \in C([0, T], C_0(\mathbb{R}^N))$ for all $0 < T < \infty$. The principal results are as follows:

In the paper [3], it was shown that

- (i) If $p \leq p_* := \max\{1/\gamma; 1 + 2(2-\gamma)/(N-2+2\gamma)_+\}$ and $u_0 \geq 0$, $u_0 \not\equiv 0$, then u blows up in finite time.
- (ii) If $p > p_*$ and $u_0 \in L^{q_{sc}}(\mathbb{R}^N)$ (where $q_{sc} = N(p-1)/(4-2\gamma)$) with $\|u_0\|_{L^{q_{sc}}}$ sufficiently small, then u exists globally.

It was shown later in [5] that in the case $p \leq 1 + 2(2-\gamma)/(N-2+2\gamma)_+$ or $p < (1/\gamma)$ and $u_0 \geq 0$, $u_0 \not\equiv 0$ that there exist two positive constants $c, C > 0$ such that

$$c(T^* - t)^{-\frac{2-\gamma}{p-1}} \leq \sup_{x \in \mathbb{R}^N} u(x, t) \leq C(T^* - t)^{-\frac{2-\gamma}{p-1}} \quad \text{for all } t \in (0, T^*),$$

where T^* denotes the maximal time of local existence.

In this paper, we generalize the work of [3] and [5] to 2×2 systems. The main results of this article are:

If $u_0, v_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ are such that $u_0, v_0 \geq 0$ and $u_0, v_0 \not\equiv 0$, and if

$$\frac{N}{2} \leq \max \left\{ \frac{(2-\delta)p + (1-\gamma)pq + 1}{pq-1}; \frac{(2-\gamma)q + (1-\delta)pq + 1}{pq-1} \right\}$$

or

$$p < \frac{1}{\delta} \quad \text{and} \quad q < \frac{1}{\gamma}$$

then any solution (u, v) to (1.1) – (1.2) blows-up in a finite time.

To understand the behavior of (u, v) near blowing-up time, the first step usually consists in deriving a bound for the blow-up rate. So in the case of the blowing-up, in a finite time $T_{\max} := T^*$, of solutions we have

$$\begin{cases} c_1(T^* - t)^{-\alpha_1} \leq \sup_{\mathbb{R}^N} u(., t) \leq C_1(T^* - t)^{-\alpha_1}, & t \in (0, T^*) \\ c_2(T^* - t)^{-\alpha_2} \leq \sup_{\mathbb{R}^N} v(., t) \leq C_2(T^* - t)^{-\alpha_2}, & t \in (0, T^*), \end{cases}$$

where

$$\alpha_1 := \frac{(2 - \gamma) + (2 - \delta)p}{pq - 1} \quad \text{and} \quad \alpha_2 := \frac{(2 - \delta) + (2 - \gamma)q}{pq - 1}.$$

The organization of this paper is as follows. In section 2, some preliminaries are set. In Section 3, a local existence theorem of solutions for the parabolic system (1.1) – (1.2) is proved. Section 4 contains a blow-up result of solutions for (1.1) – (1.2). Section 5 is dedicated to the blow-up rate of solutions. Finally, we give a necessary conditions for local and global existence in section 6.

2 Preliminaries

In this section, we present some definitions and results concerning the laplacian, fractional integrals and fractional derivatives needed to prove the main results.

First, if we take the heat equation

$$u_t - \Delta u = 0, \quad x \in \mathbb{R}^N, \quad t > 0, \quad (2.1)$$

then, the fundamental solution G_t of Eq. (2.1) is given by

$$G_t(x) := G(t, x) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}. \quad (2.2)$$

It is well-known that this function satisfies

$$G_t \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), \quad G_t(x) \geq 0, \quad \int_{\mathbb{R}^N} G_t(x) dx = 1, \quad (2.3)$$

for all $x \in \mathbb{R}^N$ and $t > 0$. Hence, using the Young inequality for convolution, we have

$$\|G_t * v\|_r \leq \|v\|_r, \quad (2.4)$$

for any $v \in L^r(\mathbb{R}^N)$ and any $1 \leq r \leq \infty$, $t > 0$.

The semigroup on $L^2(\mathbb{R}^N)$ generated by the laplacian Δ is $e^{t\Delta}v := G_t * v$ for all $v \in L^2(\mathbb{R}^N)$, $t > 0$ where $u * v$ stands for the convolution of u and v . Moreover, as $(-\Delta)$ is a self-adjoint operator with $D(-\Delta) = H^2(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} u(x)(-\Delta)v(x) dx = \int_{\mathbb{R}^N} v(x)(-\Delta)u(x) dx, \quad (2.5)$$

for all $u, v \in H^2(\mathbb{R}^N)$.

Next, if $AC[0, T]$ is the space of all functions which are absolutely continuous on $[0, T]$ with $0 < T < \infty$, then, for $f \in AC[0, T]$, the left-handed and right-handed Riemann-Liouville fractional derivatives $D_{0|t}^\alpha f(t)$ and $D_{t|T}^\alpha f(t)$ of order $\alpha \in (0, 1)$ are defined by (see [7])

$$D_{0|t}^\alpha f(t) := DJ_{0|t}^{1-\alpha} f(t), \quad (2.6)$$

$$D_{t|T}^\alpha f(t) := -\frac{1}{\Gamma(1-\alpha)} D \int_t^T (s-t)^{-\alpha} f(s) ds, \quad (2.7)$$

for all $t \in [0, T]$, where $D := d/(dt)$ is the usual derivative, and

$$J_{0|t}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \quad (2.8)$$

is the Riemann-Liouville fractional integral defined in [7].

Now, for every $f, g \in C([0, T])$, such that $D_{0|t}^\alpha f(t), D_{t|T}^\alpha g(t)$ exist and are continuous, for all $t \in [0, T]$, $0 < \alpha < 1$, we have the formula of integration by parts (see (2.64) p. 46 in [10])

$$\int_0^T (D_{0|t}^\alpha f)(s)g(s) ds = \int_0^T f(s) (D_{t|T}^\alpha g)(s) ds. \quad (2.9)$$

Note also that, for all $f \in AC^2[0, T]$, we have (see 2.2.30 in [7])

$$-D.D_{t|T}^\alpha f = D_{t|T}^{1+\alpha} f, \quad (2.10)$$

where

$$AC^2[0, T] := \{f : [0, T] \rightarrow \mathbb{R} \text{ and } Df \in AC[0, T]\}.$$

Moreover, for all $1 \leq q \leq \infty$, the following equalities (see [Lemma 2.4 p.74][7])

$$D_{0|t}^\alpha J_{0|t}^\alpha = Id_{L^q(0, T)} \quad (2.11)$$

hold almost everywhere on $[0, T]$. Later on, we will use the following results.

• If $w_1(t) = (1 - t/T)_+^\sigma$, $t \geq 0$, $T > 0$, $\sigma \gg 1$, where $(\cdot)_+$ is the positive part, then

$$D_{t|T}^\alpha w_1(t) = \frac{(1-\alpha+\sigma)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)} T^{-\alpha} \left(1 - \frac{t}{T}\right)_+^{\sigma-\alpha}, \quad (2.12)$$

$$D_{t|T}^{\alpha+1} w_1(t) = \frac{(1-\alpha+\sigma)(\sigma-\alpha)\Gamma(\sigma+1)}{\Gamma(2-\alpha+\sigma)} T^{-\alpha-1} \left(1 - \frac{t}{T}\right)_+^{\sigma-\alpha-1}, \quad (2.13)$$

for all $\alpha \in (0, 1)$; so

$$\left(D_{t|T}^\alpha w_1\right)(T) = 0 \quad ; \quad \left(D_{t|T}^\alpha w_1\right)(0) = C T^{-\alpha}, \quad (2.14)$$

where $C = (1 - \alpha + \sigma)\Gamma(\sigma + 1)/\Gamma(2 - \alpha + \sigma)$; indeed, using the Euler change of variable $y = (s - t)/(T - t)$, we get

$$\begin{aligned} D_{t|T}^\alpha w_1(t) &:= -\frac{1}{\Gamma(1 - \alpha)} D \left[\int_t^T (s - t)^{-\alpha} \left(1 - \frac{s}{T}\right)^\sigma ds \right] \\ &= -\frac{T^{-\sigma}}{\Gamma(1 - \alpha)} D \left[(T - t)^{1 - \alpha + \sigma} \int_0^1 (y)^{-\alpha} (1 - y)^\sigma ds \right] \\ &= +\frac{(1 - \alpha + \sigma)B(1 - \alpha; \sigma + 1)}{\Gamma(1 - \alpha)} T^{-\sigma} (T - t)^{\sigma - \alpha}, \end{aligned}$$

where $B(\cdot; \cdot)$ stands for the beta function. Then, (2.12) follows using the relation

$$B(1 - \alpha; \sigma + 1) = \frac{\Gamma(1 - \alpha)\Gamma(\sigma + 1)}{\Gamma(2 - \alpha + \sigma)}.$$

Moreover, (2.13) follows from (2.10) applied to (2.12).

• If $w_2(t) = (1 - t^2/T^2)_+^\ell$, $T > 0$, $\ell \gg 1$, then, with an easily computation and using the change of variable $y = (s - t)/(T - t)$, we have

$$D_{t|T}^\alpha w_2(t) = \frac{T^{-\alpha}}{\Gamma(1 - \alpha)} \sum_{k=0}^{\ell} C_1(\ell, k, \alpha) \left(1 - \frac{t}{T}\right)^{\ell + k - \alpha}, \quad (2.15)$$

$$D_{t|T}^{1+\alpha} w_2(t) = \frac{T^{-\alpha-1}}{\Gamma(1 - \alpha)} \sum_{k=0}^{\ell} C_2(\ell, k, \alpha) \left(1 - \frac{t}{T}\right)^{\ell + k - \alpha - 1}, \quad (2.16)$$

for all $-T \leq t \leq T$, $\alpha \in (0, 1)$, where

$$\begin{cases} C_1(\ell, k, \alpha) := c_\ell^k (1 - \alpha + \ell + k) 2^{\ell - k} (-1)^k \frac{\Gamma(k + \ell + 1)\Gamma(1 - \alpha)}{\Gamma(k + \ell + 2 - \alpha)}, \\ C_2(\ell, k, \alpha) := (\ell + k - \alpha) C_1(\ell, k, \alpha), \\ c_\ell^k := \frac{\ell!}{(\ell - k)!k!}; \end{cases}$$

so

$$\left(D_{t|T}^\alpha w_2\right)(T) = 0 \quad ; \quad \left(D_{t|T}^\alpha w_2\right)(-T) = C_3(\ell, k, \alpha) T^{-\alpha}, \quad (2.17)$$

where

$$C_3(\ell, k, \alpha) := \frac{2^{2\ell - \alpha}}{\Gamma(1 - \alpha)} \sum_{k=0}^{\ell} c_\ell^k (1 - \alpha + \ell + k) 2^{\ell - k} (-1)^k \frac{\Gamma(k + \ell + 1)\Gamma(1 - \alpha)}{\Gamma(k + \ell + 2 - \alpha)}.$$

3 Local existence

In this section, we derive the existence of a local mild solution for the system (1.1) – (1.2). First, we give the definition of the mild solution.

Definition 1 (*Mild solution*)

Let $u_0, v_0 \in L^\infty(\mathbb{R}^N)$ and $T > 0$. We say that $(u, v) \in L^\infty([0, T], L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N))$ is a mild solution of (1.1) – (1.2) if

$$\begin{cases} u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} J_{0|s}^\alpha(|v|^{p-1}v) ds, & t \in [0, T], \\ v(t) = e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta} J_{0|s}^\beta(|u|^{q-1}u) ds, & t \in [0, T]. \end{cases} \quad (3.1)$$

Theorem 1 (*Local existence of a mild solution*)

Given $u_0, v_0 \in C_0(\mathbb{R}^N)$ and $p, q > 1$, there exist a maximal time $T_{\max} > 0$ and a unique mild solution $(u, v) \in C([0, T_{\max}], C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N))$ to the system (1.1) – (1.2). In addition, if $u_0, v_0 \geq 0$, $u_0, v_0 \not\equiv 0$, then $u(t), v(t) > 0$ for all $0 < t < T_{\max}$.

Moreover, if $u_0, v_0 \in L^r(\mathbb{R}^N)$ for $1 \leq r < \infty$, then $u, v \in C([0, T_{\max}], L^r(\mathbb{R}^N))$.

Proof For arbitrary $T > 0$, we define the Banach space

$$E_T = \{(u, v) \in L^\infty((0, T), C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N)); \ |||(u, v)||| \leq 2(\|u_0\|_\infty + \|v_0\|_\infty)\}, \quad (3.2)$$

where $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathbb{R}^N)}$ and $|||\cdot|||$ is the norm of E_T defined by:

$$|||(u, v)||| := \|u\|_1 + \|v\|_1 := \|u\|_{L^\infty((0, T) \times \mathbb{R}^N)} + \|v\|_{L^\infty((0, T) \times \mathbb{R}^N)}.$$

Next, for every $(u, v) \in E_T$, we define $\Psi(u, v) := (\Psi_1(u, v), \Psi_2(u, v))$, where

$$\Psi_1(u, v) := e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} J_{0|s}^\alpha(|v|^{p-1}v) ds, \quad t \in (0, T)$$

and

$$\Psi_2(u, v) := e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta} J_{0|s}^\beta(|u|^{q-1}u) ds, \quad t \in (0, T).$$

We will prove the local existence by the Banach fixed point theorem.

• **$\Psi : \mathbf{E}_T \rightarrow \mathbf{E}_T$** : Let $(u, v) \in E_T$, using (2.4), we obtain

$$\begin{aligned} |||\Psi(u, v)||| &\leq \|u_0\|_\infty + C_1 \left\| \int_0^t \int_0^s (s-\sigma)^{-\gamma} \|v(\sigma)\|_\infty^p d\sigma ds \right\|_{L^\infty(0, T)} \\ &+ \|v_0\|_\infty + C_2 \left\| \int_0^t \int_0^s (s-\sigma)^{-\delta} \|u(\sigma)\|_\infty^q d\sigma ds \right\|_{L^\infty(0, T)} \\ &= \|u_0\|_\infty + C_1 \left\| \int_0^t \int_\sigma^t (s-\sigma)^{-\gamma} \|v(\sigma)\|_\infty^p ds d\sigma \right\|_{L^\infty(0, T)} \\ &+ \|v_0\|_\infty + C_2 \left\| \int_0^t \int_\sigma^t (s-\sigma)^{-\delta} \|u(\sigma)\|_\infty^q ds d\sigma \right\|_{L^\infty(0, T)}, \end{aligned}$$

where

$$C_1 := \frac{1}{\Gamma(1-\gamma)}, \quad C_2 := \frac{1}{\Gamma(1-\delta)}.$$

So, using the fact that $(u, v) \in E_T$, we get

$$\begin{aligned}
|||\Psi(u, v)||| &\leq \|u_0\|_\infty + C_3 T^{2-\gamma} \|v\|_1^p \\
&\quad + \|v_0\|_\infty + C_4 T^{2-\delta} \|u\|_1^q \\
&\leq (\|u_0\|_\infty + \|v_0\|_\infty) + \max\left\{C_3 T^{2-\gamma} \|v\|_1^{p-1}; C_4 T^{2-\delta} \|u\|_1^{q-1}\right\} (\|v\|_1 + \|u\|_1) \\
&\leq (\|u_0\|_\infty + \|v_0\|_\infty) + 2T(u_0, v_0)(\|u_0\|_\infty + \|v_0\|_\infty),
\end{aligned}$$

where

$$C_3 := \frac{C_1}{(1-\gamma)(2-\gamma)}, \quad C_4 := \frac{C_2}{(1-\delta)(2-\delta)}$$

and

$$T(u_0, v_0) := \max\left\{C_3 T^{2-\gamma} 2^{p-1} (\|u_0\|_\infty + \|v_0\|_\infty)^{p-1}; C_4 T^{2-\delta} 2^{q-1} (\|u_0\|_\infty + \|v_0\|_\infty)^{q-1}\right\}.$$

Now, if we choose T such that

$$2T(u_0, v_0) \leq 1, \tag{3.3}$$

we conclude that $\|\Psi(u, v)\|_1 \leq 2(\|u_0\|_\infty + \|v_0\|_\infty)$, and then $\Psi(u, v) \in E_T$.

• **Ψ is a Contraction map:** For $(u, v), (\tilde{u}, \tilde{v}) \in E_T$, we have

$$\begin{aligned}
|||\Psi(u, v) - \Psi(\tilde{u}, \tilde{v})||| &\leq C_1 \left\| \int_0^t \int_0^s (s-\sigma)^{-\gamma} \| |v|^{p-1} v(\sigma) - |\tilde{v}|^{p-1} \tilde{v}(\sigma) \|_\infty d\sigma ds \right\|_{L^\infty(0, T)} \\
&\quad + C_2 \left\| \int_0^t \int_0^s (s-\sigma)^{-\delta} \| |u|^{q-1} u(\sigma) - |\tilde{u}|^{q-1} \tilde{u}(\sigma) \|_\infty d\sigma ds \right\|_{L^\infty(0, T)} \\
&= C_1 \left\| \int_0^t \int_\sigma^t (s-\sigma)^{-\gamma} \| |v|^{p-1} v(\sigma) - |\tilde{v}|^{p-1} \tilde{v}(\sigma) \|_\infty ds d\sigma \right\|_{L^\infty(0, T)} \\
&\quad + C_2 \left\| \int_0^t \int_\sigma^t (s-\sigma)^{-\delta} \| |u|^{q-1} u(\sigma) - |\tilde{u}|^{q-1} \tilde{u}(\sigma) \|_\infty ds d\sigma \right\|_{L^\infty(0, T)}.
\end{aligned}$$

Now, by the same computation as above, we conclude

$$\begin{aligned}
|||\Psi(u, v) - \Psi(\tilde{u}, \tilde{v})||| &\leq C_3 T^{2-\gamma} \| |v|^{p-1} v - |\tilde{v}|^{p-1} \tilde{v} \|_1 + C_4 T^{2-\delta} \| |u|^{q-1} u - |\tilde{u}|^{q-1} \tilde{u} \|_1 \\
&\leq C(p) C_3 T^{2-\gamma} \max\{\|v\|_1^{p-1}, \|\tilde{v}\|_1^{p-1}\} \|v - \tilde{v}\|_1 \\
&\quad + C(q) C_4 T^{2-\delta} \max\{\|u\|_1^{q-1}, \|\tilde{u}\|_1^{q-1}\} \|u - \tilde{u}\|_1 \\
&\leq C(p, q) T(u_0, v_0) |||(u, v) - (\tilde{u}, \tilde{v})||| \\
&\leq \frac{1}{2} |||(u, v) - (\tilde{u}, \tilde{v})|||,
\end{aligned}$$

thanks to the standard estimate:

$$\| |u|^{p-1} u - |v|^{p-1} v \| \leq C(p) |u - v| (|u|^{p-1} + |v|^{p-1}) \quad \text{for every } u, v \text{ and all } p > 1,$$

and the choice of T :

$$\max\{C(p, q), 1\}T(u_0, v_0) \leq \frac{1}{2}. \quad (3.4)$$

Then, by the Banach fixed point theorem, the system (1.1) – (1.2) admits a unique mild solution $(u, v) \in E_T$.

Note that, for later use, it is sufficient to take the space

$$\{(u, v) \in L^\infty((0, T), C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N)); \|u\|_1 \leq 2\|u_0\|_\infty \text{ and } \|v\|_1 \leq 2\|v_0\|_\infty\}$$

instead of E_T and to choose $T > 0$ small enough so that

$$\begin{cases} C(p)2^p C_3 T^{2-\gamma} \|v_0\|_\infty^p \leq \frac{1}{2^\eta} \|u_0\|_\infty, \\ C(q)2^q C_4 T^{2-\delta} \|u_0\|_\infty^q \leq \frac{1}{2^\eta} \|v_0\|_\infty, \end{cases} \quad (3.5)$$

where $\eta > 0$ is a positive constant large enough such that

$$\max \left\{ \frac{\Gamma(1-\delta)}{C(q)2^{pq+\eta(q+1)+q-p}}, \frac{\Gamma(1-\gamma)}{C(p)2^{pq+\eta(p+1)+p-q}} \right\} \leq \frac{1}{2}.$$

Thus, the conditions (3.5) imply that

$$\begin{cases} 2^{pq+\eta q+\eta+q} (C(p)C_3)^q C(q)C_4 T^{(2-\delta)+(2-\gamma)q} \|v_0\|_\infty^{pq-1} \leq 1, \\ 2^{pq+\eta p+\eta+p} (C(q)C_4)^p C(p)C_3 T^{(2-\gamma)+(2-\delta)p} \|u_0\|_\infty^{pq-1} \leq 1. \end{cases} \quad (3.6)$$

Now, using the uniqueness, we conclude the existence of a solution on a maximal interval $[0, T_{\max})$ where

$$T_{\max} := \sup \{T > 0 : (u, v) \text{ is a solution to (1.1) – (1.2) in } E_T\} \leq \infty.$$

Moreover, a simple calculation allows us to prove that $(u, v) \in C([0, T_{\max}), C_0(\mathbb{R}^N) \times C_0(\mathbb{R}^N))$.

• **Positivity of solutions:** If $u_0, v_0 \geq 0$, then we can construct a nonnegative solution on some interval $[0, T]$ by applying the fixed-point argument in the set $E_T^+ = \{(u, v) \in E_T; u, v \geq 0\}$. In particular, it follows from (3.1) that $u(t) \geq e^{t\Delta} u_0 > 0$ and $v(t) \geq e^{t\Delta} v_0 > 0$ on $(0, T]$. It is not difficult, by uniqueness and contradiction's principle, to deduce that (u, v) stays positive on $(0, T_{\max})$.

• **Regularity of solutions:** If $u_0, v_0 \in L^r(\mathbb{R}^N)$, for $1 \leq r < \infty$, then by repeating the fixed point argument in the space

$$\begin{aligned} E_{T,r} &:= \{(u, v) \in L^\infty((0, T), (C_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)) \times (C_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N))) : \\ &\quad |||(u, v)||| \leq 2(\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty}), |||(u, v)|||_r \leq 2(\|u_0\|_{L^r} + \|v_0\|_{L^r})\}, \end{aligned}$$

instead of E_T , where

$$\| (u, v) \|_r := \|u\|_{L^\infty((0, T), L^r(\mathbb{R}^N))} + \|v\|_{L^\infty((0, T), L^r(\mathbb{R}^N))},$$

and by estimating $\|u^p\|_{L^r(\mathbb{R}^N)}$ by $\|u\|_{L^\infty(\mathbb{R}^N)}^{p-1} \|u\|_{L^r(\mathbb{R}^N)}$, the same for v , in the contraction mapping argument, using (2.4), we obtain a unique solution in $E_{T,r}$. Thus, as above, we conclude that

$$(u, v) \in C([0, T_{\max}), (C_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)) \times (C_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N))).$$

□

In the case of $u_0, v_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ (the case which will be used below), we say that (u, v) is a global solution if $T_{\max} = \infty$, while in the case of $T_{\max} < \infty$ we say that (u, v) blows up in a finite time or that (u, v) is non-global and in this case we have

$$\lim_{t \rightarrow T_{\max}} \left(\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} + \|v(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \right) + \left(\|u(\cdot, t)\|_{L^2(\mathbb{R}^N)} + \|v(\cdot, t)\|_{L^2(\mathbb{R}^N)} \right) = \infty.$$

4 Blow-up of solutions

In this section, we prove a blow-up result for system (1.1)–(1.2). First we give the definition of the weak solution and a lemma asserting that the mild solution is the weak solution. Hereafter

$$\int_{Q_T} = \int_0^T \int_{\mathbb{R}^N} dx dt \quad \text{for all } T > 0; \quad \int_{\mathbb{R}^N} = \int_{\mathbb{R}^N} dx.$$

Definition 2 (Weak solution)

Let $T > 0$ and $u, v \in L^\infty([0, T], L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N))$. We say that $U := (u, v)$ is a weak solution of the problem (1.1)–(1.2) if

$$\begin{aligned} \int_{\mathbb{R}^N} u_0(x) \varphi(x, 0) + \int_{Q_T} J_{0|t}^\alpha (|v|^{p-1} v)(x, t) \varphi(x, t) &= - \int_{Q_T} u(x, t) \Delta \varphi(x, t) \\ &- \int_{Q_T} u(x, t) \varphi_t(x, t), \end{aligned} \quad (4.1)$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} v_0(x) \psi(x, 0) + \int_{Q_T} J_{0|t}^\beta (|u|^{q-1} u)(x, t) \psi(x, t) &= - \int_{Q_T} v(x, t) \Delta \psi(x, t) \\ &- \int_{Q_T} v(x, t) \psi_t(x, t), \end{aligned} \quad (4.2)$$

for any $\varphi, \psi \in C^1([0, T], H^2(\mathbb{R}^N))$ such that $\varphi(x, T) = \psi(x, T) = 0$, $x \in \mathbb{R}^N$, where $Q_T := [0, T] \times \mathbb{R}^N$, $\alpha = 1 - \gamma$ and $\beta = 1 - \delta$.

Lemma 1 (Mild \rightarrow Weak)

Let $T > 0$ and $U := (u, v)$, where $u, v \in L^\infty([0, T], L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N))$. If U is a mild solution of (1.1)–(1.2), then U be a weak solution of (1.1)–(1.2), for all $T > 0$.

Proof Let $T > 0$, $u, v \in L^\infty([0, T], L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N))$ and $U := (u, v)$ be a solution of (3.1). For $\varphi, \psi \in C^1([0, T], H^2(\mathbb{R}^N))$ such that $\varphi(x, T) = \psi(x, T) = 0$ for all $x \in \mathbb{R}^N$, we have, after multiplying the first (resp. second) equation in (3.1) by φ (resp. ψ) and integrate over \mathbb{R}^N ,

$$\int_{\mathbb{R}^N} u(x, t) \varphi(x, t) = \int_{\mathbb{R}^N} T(t) u_0(x) \varphi(x, t) + \int_{\mathbb{R}^N} \int_0^t T(t-s) J_{0|s}^\alpha (|v|^{p-1} v)(x, t) ds \varphi(x, t),$$

and

$$\int_{\mathbb{R}^N} v(x, t) \psi(x, t) = \int_{\mathbb{R}^N} T(t) v_0(x) \psi(x, t) + \int_{\mathbb{R}^N} \int_0^t T(t-s) J_{0|s}^\beta (|u|^{q-1} u)(x, t) ds \psi(x, t),$$

where $\alpha = 1 - \gamma$ and $\beta = 1 - \delta$. So after differentiation in time, we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} u(x, t) \varphi(x, t) &= \int_{\mathbb{R}^N} \frac{d}{dt} (T(t) u_0(x) \varphi(x, t)) \\ &+ \int_{\mathbb{R}^N} \frac{d}{dt} \int_0^t T(t-s) J_{0|s}^\alpha (|v|^{p-1} v)(x, s) ds \varphi(x, t), \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} v(x, t) \psi(x, t) &= \int_{\mathbb{R}^N} \frac{d}{dt} (T(t) v_0(x) \psi(x, t)) \\ &+ \int_{\mathbb{R}^N} \frac{d}{dt} \int_0^t T(t-s) J_{0|s}^\beta (|u|^{q-1} u)(x, s) ds \psi(x, t). \end{aligned} \quad (4.4)$$

Now, using (2.5) and the property of the semigroup $T(t)$ founded in [2, Chapter 3] with the negative self-adjoint operator Δ , we get:

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{d}{dt} (T(t) u_0(x) \varphi(x, t)) &= \int_{\mathbb{R}^N} \Delta (T(t) u_0(x)) \varphi(x, t) dx + \int_{\mathbb{R}^N} T(t) u_0(x) \varphi_t(x, t) \\ &= \int_{\mathbb{R}^N} T(t) u_0(x) \Delta \varphi(x, t) + \int_{\mathbb{R}^N} T(t) u_0(x) \varphi_t(x, t), \end{aligned} \quad (4.5)$$

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{d}{dt} (T(t) v_0(x) \psi(x, t)) &= \int_{\mathbb{R}^N} \Delta (T(t) v_0(x)) \psi(x, t) dx + \int_{\mathbb{R}^N} T(t) v_0(x) \psi_t(x, t) \\ &= \int_{\mathbb{R}^N} T(t) v_0(x) \Delta \psi(x, t) + \int_{\mathbb{R}^N} T(t) v_0(x) \psi_t(x, t), \end{aligned} \quad (4.6)$$

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{d}{dt} \int_0^t T(t-s) f_1(x, s) ds \varphi(x, t) &= \int_{\mathbb{R}^N} f_1(x, t) \varphi(x, t) + \int_{\mathbb{R}^N} \int_0^t \Delta (T(t-s) f_1(x, s)) ds \varphi(x, t) \\ &+ \int_{\mathbb{R}^N} \int_0^t T(t-s) f_1(x, s) ds \varphi_t(x, t) \\ &= \int_{\mathbb{R}^N} f_1(x, t) \varphi(x, t) + \int_{\mathbb{R}^N} \int_0^t T(t-s) f_1(x, s) ds \Delta \varphi(x, t) \\ &+ \int_{\mathbb{R}^N} \int_0^t T(t-s) f_1(x, s) ds \varphi_t(x, t), \end{aligned} \quad (4.7)$$

and, similarly,

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{d}{dt} \int_0^t T(t-s) f_2(x, s) ds \psi(x, t) &= \int_{\mathbb{R}^N} f_2(x, t) \psi(x, t) + \int_{\mathbb{R}^N} \int_0^t T(t-s) f_2(x, s) ds \Delta \psi(x, t) \\ &+ \int_{\mathbb{R}^N} \int_0^t T(t-s) f_2(x, s) ds \psi_t(x, t), \end{aligned} \quad (4.8)$$

where

$$f_1 := J_{0|t}^\alpha (|u|^{p-1}u) \in L^\infty(0, T; L^2(\mathbb{R}^N)) \quad \text{and} \quad f_2 := J_{0|t}^\beta (|u|^{q-1}u) \in L^\infty(0, T; L^2(\mathbb{R}^N)).$$

Thus, using (3.1) and (4.5) – (4.8), we conclude that (4.3) and (4.4) imply

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} u(x, t) \varphi(x, t) &= \int_{\mathbb{R}^N} u(x, t) \Delta \varphi(x, t) + \int_{\mathbb{R}^N} u(x, t) \varphi_t(x, t) \\ &+ \int_{\mathbb{R}^N} f_1(x, t) \varphi(x, t), \end{aligned} \quad (4.9)$$

and

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} v(x, t) \psi(x, t) &= \int_{\mathbb{R}^N} v(x, t) \Delta \psi(x, t) + \int_{\mathbb{R}^N} v(x, t) \psi_t(x, t) \\ &+ \int_{\mathbb{R}^N} f_2(x, t) \psi(x, t). \end{aligned} \quad (4.10)$$

Finally, after integrating in time over $[0, T]$ and using the fact that $\varphi(x, T) = \psi(x, T) = 0$ for all $x \in \mathbb{R}^N$, we conclude the result. \square

Theorem 2 *Let $u_0, v_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ be such that $u_0, v_0 \geq 0$ and $u_0, v_0 \not\equiv 0$. If*

$$\frac{N}{2} \leq \max \left\{ \frac{(2-\delta)p + (1-\gamma)pq + 1}{pq-1}; \frac{(2-\gamma)q + (1-\delta)pq + 1}{pq-1} \right\} \quad (4.11)$$

or

$$p < \frac{1}{\delta} \quad \text{and} \quad q < \frac{1}{\gamma} \quad (4.12)$$

then any mild solution (u, v) to (1.1) – (1.2), blows-up in a finite time.

Proof The proof is by contradiction. Suppose that (u, v) is a global mild solution to (1.1)–(1.2), then (u, v) is a solution of (1.1)–(1.2) where $u, v \in C([0, T], C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N))$, for all $T > 0$, such that $u(t), v(t) > 0$ for all $t \in [0, T]$.

Thus, using Lemma 1, we conclude that u and v verify (4.1) and (4.2), respectively, for all $\varphi, \psi \in C^1([0, T], H^2(\mathbb{R}^N))$ such that $\varphi(x, T) = \psi(x, T) = 0$, $x \in \mathbb{R}^N$.

Now, we take $\varphi(x, t) = D_{t|T}^\alpha (\tilde{\varphi}(x, t)) := D_{t|T}^\alpha \left((\varphi_1(x))^\ell \varphi_2(t) \right)$ and $\psi(x, t) = D_{t|T}^\beta (\tilde{\varphi}(x, t))$, for $\alpha := 1 - \gamma$ and $\beta := 1 - \delta$, with $\varphi_1(x) := \Phi(|x|/(T^{1/2}))$, $\varphi_2(t) := (1 - t/T)_+^\eta$, where $\ell \geq pq/((p-1)(q-1))$, $\eta \gg 1$, $1 \leq B < T$ large enough such as in the case of $T \rightarrow \infty$

we don't have $B \rightarrow \infty$ in the same time, and Φ is a smooth nonnegative non-increasing function such that

$$\Phi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2, \end{cases}$$

$0 \leq \Phi \leq 1$, $|\Phi'(r)| \leq C_1/r$, for all $r > 0$. So, using (2.14), we obtain

$$\begin{aligned} & \int_{\Omega} u_0(x) D_{t|T}^{\alpha} \tilde{\varphi}(x, 0) + \int_{\Omega_T} J_{0|t}^{\alpha}(v^p)(x, t) D_{t|T}^{\alpha} \tilde{\varphi}(x, t) \\ &= - \int_{\Omega_T} u(x, t) \Delta D_{t|T}^{\alpha} \tilde{\varphi}(x, t) - \int_{\Omega_T} u(x, t) D D_{t|T}^{\alpha} \tilde{\varphi}(x, t), \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} & \int_{\Omega} v_0(x) D_{t|T}^{\beta} \tilde{\varphi}(x, 0) + \int_{\Omega_T} J_{0|t}^{\beta}(u^q)(x, t) D_{t|T}^{\alpha} \tilde{\varphi}(x, t) \\ &= - \int_{\Omega_T} v(x, t) \Delta D_{t|T}^{\beta} \tilde{\varphi}(x, t) - \int_{\Omega_T} v(x, t) D D_{t|T}^{\beta} \tilde{\varphi}(x, t), \end{aligned} \quad (4.14)$$

where

$$\Omega_T := [0, T] \times \Omega, \quad \text{for } \Omega = \{x \in \mathbb{R}^N ; |x| \leq 2T^{1/2}\}, \quad \int_{\Omega_T} = \int_{\Omega_T} dx dt \quad \text{and} \quad \int_{\Omega} = \int_{\Omega} dx.$$

Furthermore, using (2.9) and (2.14) in the left hand sides of (4.13) and (4.14) while in the right hand sides we use (2.10), we conclude that

$$\begin{aligned} & C T^{-\alpha} \int_{\Omega} u_0(x) \varphi_1^{\ell}(x) + \int_{\Omega_T} D_{0|t}^{\alpha} J_{0|t}^{\alpha}(v^p)(x, t) \tilde{\varphi}(x, t) \\ &= - \int_{\Omega_T} u(x, t) \Delta D_{t|T}^{\alpha} \tilde{\varphi}(x, t) + \int_{\Omega_T} u(x, t) D_{t|T}^{1+\alpha} \tilde{\varphi}(x, t), \end{aligned} \quad (4.15)$$

and

$$\begin{aligned} & C T^{-\beta} \int_{\Omega} v_0(x) \varphi_1^{\ell}(x) + \int_{\Omega_T} D_{0|t}^{\beta} J_{0|t}^{\alpha}(u^q)(x, t) \tilde{\varphi}(x, t) \\ &= - \int_{\Omega_T} v(x, t) \Delta D_{t|T}^{\beta} \tilde{\varphi}(x, t) + \int_{\Omega_T} v(x, t) D_{t|T}^{1+\beta} \tilde{\varphi}(x, t). \end{aligned} \quad (4.16)$$

Moreover, from (2.11), we may write

$$\begin{aligned} & \int_{\Omega_T} v^p(x, t) \tilde{\varphi}(x, t) + C T^{-\alpha} \int_{\Omega} u_0(x) \varphi_1^{\ell}(x) \\ &= - \int_{\Omega_T} u(x, t) \Delta \varphi_1^{\ell}(x) D_{t|T}^{\alpha} \varphi_2(t) + \int_{\Omega_T} u(x, t) D_{t|T}^{1+\alpha} \tilde{\varphi}(x, t), \end{aligned} \quad (4.17)$$

and

$$\begin{aligned} & \int_{\Omega_T} u^q(x, t) \tilde{\varphi}(x, t) + C T^{-\beta} \int_{\Omega} v_0(x) \varphi_1^{\ell}(x) \\ &= - \int_{\Omega_T} v(x, t) \Delta \varphi_1^{\ell}(x) D_{t|T}^{\beta} \varphi_2(t) + \int_{\Omega_T} v(x, t) D_{t|T}^{1+\beta} \tilde{\varphi}(x, t). \end{aligned} \quad (4.18)$$

Then, the inequality $(-\Delta)(\varphi_1^\ell) \leq \ell\varphi_1^{\ell-1}(-\Delta)\varphi_1$ allows us to write:

$$\begin{aligned}
& \int_{\Omega_T} v^p(x, t) \tilde{\varphi}(x, t) + C T^{-\alpha} \int_{\Omega} u_0(x) \varphi_1^\ell(x) \\
& \leq C \int_{\Omega_T} u(x, t) \varphi_1^{\ell-1}(x) \left| (-\Delta)\varphi_1(x) D_{t|T}^\alpha \varphi_2(t) \right| \\
& + \int_{\Omega_T} u(x, t) \varphi_1^\ell(x) \left| D_{t|T}^{1+\alpha} \varphi_2(t) \right| \\
& = C \int_{\Omega_T} u(x, t) \tilde{\varphi}^{1/q} \tilde{\varphi}^{-1/q} \varphi_1^{\ell-1}(x) \left| (-\Delta)\varphi_1(x) D_{t|T}^\alpha \varphi_2(t) \right| \\
& + \int_{\Omega_T} u(x, t) \tilde{\varphi}^{1/q} \tilde{\varphi}^{-1/q} \varphi_1^\ell(x) \left| D_{t|T}^{1+\alpha} \varphi_2(t) \right| \tag{4.19}
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\Omega_T} u^q(x, t) \tilde{\varphi}(x, t) dx dt + C T^{-\beta} \int_{\Omega} v_0(x) \varphi_1^\ell(x) \\
& \leq C \int_{\Omega_T} v(x, t) \varphi_1^{\ell-1}(x) \left| (-\Delta)\varphi_1(x) D_{t|T}^\beta \varphi_2(t) \right| \\
& + \int_{\Omega_T} v(x, t) \varphi_1^\ell(x) \left| D_{t|T}^{1+\beta} \varphi_2(t) \right| \\
& = C \int_{\Omega_T} v(x, t) \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} \varphi_1^{\ell-1}(x) \left| (-\Delta)\varphi_1(x) D_{t|T}^\beta \varphi_2(t) \right| \\
& + \int_{\Omega_T} v(x, t) \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} \varphi_1^\ell(x) \left| D_{t|T}^{1+\beta} \varphi_2(t) \right| \tag{4.20}
\end{aligned}$$

Therefore, as $u_0, v_0 \geq 0$, using Hölder's inequality, we conclude that

$$\int_{\Omega_T} v^p(x, t) \tilde{\varphi}(x, t) \leq \left(\int_{\Omega_T} u^q(x, t) \tilde{\varphi}(x, t) \right)^{1/q} \mathcal{A}, \tag{4.21}$$

$$\int_{\Omega_T} u^q(x, t) \tilde{\varphi}(x, t) \leq \left(\int_{\Omega_T} v^p(x, t) \tilde{\varphi}(x, t) \right)^{1/p} \mathcal{B}, \tag{4.22}$$

where

$$\mathcal{A} := C \left(\int_{\Omega_T} \varphi_1^\ell \varphi_2^{-\frac{1}{q-1}} \left| D_{t|T}^{1+\alpha} \varphi_2 \right|^{\tilde{q}} \right)^{1/\tilde{q}} + C \left(\int_{\Omega_T} \varphi_1^{\ell-\tilde{q}} \varphi_2^{-\frac{1}{q-1}} \left| \Delta_x \varphi_1 D_{t|T}^\alpha \varphi_2 \right|^{\tilde{q}} \right)^{1/\tilde{q}},$$

and

$$\mathcal{B} := C \left(\int_{\Omega_T} \varphi_1^\ell \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{1+\beta} \varphi_2 \right|^{\tilde{p}} \right)^{1/\tilde{p}} + C \left(\int_{\Omega_T} \varphi_1^{\ell-\tilde{p}} \varphi_2^{-\frac{1}{p-1}} \left| \Delta_x \varphi_1 D_{t|T}^\beta \varphi_2 \right|^{\tilde{p}} \right)^{1/\tilde{p}},$$

with $\tilde{p} := p/(p-1)$ and $\tilde{q} := q/(q-1)$. Now, combining (4.21) and (4.22), we get

$$\begin{cases} \left(\int_{\Omega_T} v^p(x, t) \tilde{\varphi}(x, t) \right)^{1-1/pq} \leq \mathcal{B}^{1/q} \mathcal{A}, \\ \left(\int_{\Omega_T} u^q(x, t) \tilde{\varphi}(x, t) \right)^{1-1/pq} \leq \mathcal{A}^{1/p} \mathcal{B}. \end{cases} \quad (4.23)$$

At this stage, we introduce the scaled variables: $\tau = T^{-1}t$, $\xi = T^{-1/2}x$; using formula (2.12) and (2.13) in the right hand-side of (4.23), we obtain:

$$\begin{cases} \left(\int_{\Omega_T} v^p(x, t) \tilde{\varphi}(x, t) \right)^{1-1/pq} \leq CT^{\theta_1}, \\ \left(\int_{\Omega_T} u^q(x, t) \tilde{\varphi}(x, t) \right)^{1-1/pq} \leq CT^{\theta_2}, \end{cases} \quad (4.24)$$

where

$$\theta_1 := \left(-(1+\alpha)\tilde{q} + \left(1 + \frac{N}{2}\right) \right) \frac{1}{\tilde{q}} + \left(-(1+\beta)\tilde{p} + \left(1 + \frac{N}{2}\right) \right) \frac{1}{q\tilde{p}}, \quad (4.25)$$

and

$$\theta_2 := \left(-(1+\beta)\tilde{p} + \left(1 + \frac{N}{2}\right) \right) \frac{1}{\tilde{p}} + \left(-(1+\alpha)\tilde{q} + \left(1 + \frac{N}{2}\right) \right) \frac{1}{p\tilde{q}}. \quad (4.26)$$

Note that inequality (4.11) is equivalent to $\theta_1 \leq 0$ or $\theta_2 \leq 0$. So, we have to distinguish three cases:

- The case $\theta_1 < 0$ (resp. $\theta_2 < 0$): We pass to the limit in the first equation (resp. second equation) in (4.24), as T goes to ∞ ; we get

$$\lim_{T \rightarrow \infty} \int_0^T \int_{|x| \leq 2T^{1/2}} v^p(x, t) \tilde{\varphi}(x, t) dx dt = 0.$$

(resp.

$$\lim_{T \rightarrow \infty} \int_0^T \int_{|x| \leq 2T^{1/2}} u^q(x, t) \tilde{\varphi}(x, t) dx dt = 0.)$$

Using the dominated convergence theorem and the continuity in time and space of v (resp. u), we infer that

$$\int_{Q_\infty} v^p(x, t) dx dt = 0 \quad \implies \quad v \equiv 0.$$

(resp.

$$\int_{Q_\infty} u^q(x, t) dx dt = 0 \quad \implies \quad u \equiv 0.)$$

Moreover, from (4.22) (resp. (4.21)), we infer that

$$\int_0^T \int_{|x| \leq 2T^{1/2}} u^q(x, t) \tilde{\varphi}(x, t) dx dt = 0,$$

(resp.

$$\int_0^T \int_{|x| \leq 2T^{1/2}} v^p(x, t) \tilde{\varphi}(x, t) dx dt = 0),$$

and by the same argument as above, we conclude that $u \equiv v \equiv 0$; contradiction.

• The case $\theta_1 = 0$ (resp. $\theta_2 = 0$) : In this case, using (4.24) as $T \rightarrow \infty$, we conclude that

$$v \in L^p(0, \infty; L^p(\mathbb{R}^N)), \quad (4.27)$$

(resp.

$$u \in L^q(0, \infty; L^q(\mathbb{R}^N)). \quad (4.28)$$

Now, we take $\varphi_1(x) := \Phi(|x|/(B^{-1/2}T^{1/2}))$ instead of the one chosen above, where $1 \leq B < T$ large enough such that when $T \rightarrow \infty$ we don't have $B \rightarrow \infty$ in the same time, then if we repeat the same calculation as above and taking account of the support of Δ , we obtain (as in (4.19) – (4.20))

$$\begin{aligned} \int_{\Sigma_B} v^p \tilde{\varphi} &\leq C \int_{\Sigma_B} u \tilde{\varphi}^{1/q} \tilde{\varphi}^{-1/q} (\varphi_1(x))^\ell \left| D_{t|T}^{1+\alpha} \varphi_2(t) \right| \\ &+ C \int_{\Delta_B} u \tilde{\varphi}^{1/q} \tilde{\varphi}^{-1/q} (\varphi_1(x))^{\ell-1} \left| (-\Delta_x) \varphi_1(x) D_{t|T}^\alpha \varphi_2(t) \right|, \end{aligned} \quad (4.29)$$

and

$$\begin{aligned} \int_{\Sigma_B} u^q \tilde{\varphi} &\leq C \int_{\Sigma_B} v \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} (\varphi_1(x))^\ell \left| D_{t|T}^{1+\beta} \varphi_2(t) \right| \\ &+ C \int_{\Delta_B} v \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} (\varphi_1(x))^{\ell-1} \left| (-\Delta_x) \varphi_1(x) D_{t|T}^\beta \varphi_2(t) \right|, \end{aligned} \quad (4.30)$$

where

$$\Sigma_B := [0, T] \times \left\{ x \in \mathbb{R}^N ; |x| \leq 2B^{-1/2}T^{1/2} \right\}, \quad \int_{\Sigma_B} = \int_{\Sigma_B} dx dt,$$

and

$$\Delta_B := [0, T] \times \left\{ x \in \mathbb{R}^N ; B^{-1/2}T^{1/2} \leq |x| \leq 2B^{-1/2}T^{1/2} \right\}, \quad \int_{\Delta_B} = \int_{\Delta_B} dx dt.$$

On the other hand, as $U = (u, v)$ is a global solution then, u (resp. v) verifies (4.1) (resp. (4.2)) locally and in particular on Δ_B . Thus, we obtain

$$\begin{aligned} \int_{\Delta_B} v^p \tilde{\varphi} &\leq C \int_{\Delta_B} u \tilde{\varphi}^{1/q} \tilde{\varphi}^{-1/q} (\varphi_1(x))^\ell \left| D_{t|T}^{1+\alpha} \varphi_2(t) \right| \\ &+ C \int_{\Delta_B} u \tilde{\varphi}^{1/q} \tilde{\varphi}^{-1/q} (\varphi_1(x))^{\ell-1} \left| (-\Delta_x) \varphi_1(x) D_{t|T}^\alpha \varphi_2(t) \right|, \end{aligned} \quad (4.31)$$

and

$$\begin{aligned} \int_{\Delta_B} u^q \tilde{\varphi} &\leq C \int_{\Delta_B} v \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} (\varphi_1(x))^\ell \left| D_{t|T}^{1+\beta} \varphi_2(t) \right| \\ &+ C \int_{\Delta_B} v \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} (\varphi_1(x))^{\ell-1} \left| (-\Delta_x) \varphi_1(x) D_{t|T}^\beta \varphi_2(t) \right|. \end{aligned} \quad (4.32)$$

At this stage, we set

$$\mathcal{U}_1 := \int_{\Sigma_B} u^q \tilde{\varphi} dx dt, \quad \mathcal{U}_2 := \int_{\Delta_B} u^q \tilde{\varphi} dx dt,$$

and

$$\mathcal{V}_1 := \int_{\Sigma_B} v^p \tilde{\varphi} dx dt, \quad \mathcal{V}_2 := \int_{\Delta_B} v^p \tilde{\varphi} dx dt.$$

Then by taking the Hölder inequality in (4.29), (4.30), (4.31) and (4.32), we infer

$$\begin{cases} \mathcal{V}_1 \leq \mathcal{U}_1^{1/q} \mathcal{A}_1 + \mathcal{U}_2^{1/q} \mathcal{C}_1, \\ \mathcal{U}_1 \leq \mathcal{V}_1^{1/p} \mathcal{B}_1 + \mathcal{V}_2^{1/p} \mathcal{C}_2, \end{cases} \quad (4.33)$$

and

$$\begin{cases} \mathcal{V}_2 \leq \mathcal{U}_2^{1/q} \mathcal{A}_2 + \mathcal{U}_1^{1/q} \mathcal{C}_1, \\ \mathcal{U}_2 \leq \mathcal{V}_2^{1/p} \mathcal{B}_2 + \mathcal{V}_1^{1/p} \mathcal{C}_2, \end{cases} \quad (4.34)$$

where

$$\begin{aligned} \mathcal{A}_1 &:= C \left(\int_{\Sigma_B} \varphi_1^\ell \varphi_2^{-\frac{1}{q-1}} \left| D_{t|T}^{1+\alpha} \varphi_2 \right|^{\tilde{q}} \right)^{1/\tilde{q}}, \\ \mathcal{A}_2 &:= C \left(\int_{\Delta_B} \varphi_1^\ell \varphi_2^{-\frac{1}{q-1}} \left| D_{t|T}^{1+\alpha} \varphi_2 \right|^{\tilde{q}} \right)^{1/\tilde{q}} \leq \mathcal{A}_1, \\ \mathcal{B}_1 &:= C \left(\int_{\Sigma_B} \varphi_1^\ell \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{1+\beta} \varphi_2 \right|^{\tilde{p}} \right)^{1/\tilde{p}}, \\ \mathcal{B}_2 &:= C \left(\int_{\Delta_B} \varphi_1^\ell \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{1+\beta} \varphi_2 \right|^{\tilde{p}} \right)^{1/\tilde{p}} \leq \mathcal{B}_1, \\ \mathcal{C}_1 &:= C \left(\int_{\Delta_B} \varphi_1^{\ell-\tilde{q}} \varphi_2^{-\frac{1}{q-1}} \left| \Delta_x \varphi_1 D_{t|T}^\alpha \varphi_2 \right|^{\tilde{q}} \right)^{1/\tilde{q}}, \end{aligned}$$

and

$$\mathcal{C}_2 := C \left(\int_{\Delta_B} \varphi_1^{\ell-\tilde{p}} \varphi_2^{-\frac{1}{p-1}} \left| \Delta_x \varphi_1 D_{t|T}^\beta \varphi_2 \right|^{\tilde{p}} \right)^{1/\tilde{p}}.$$

Combining (4.33) and (4.34), we obtain

$$\begin{aligned} \mathcal{V}_1 &\leq \mathcal{V}_1^{1/pq} \mathcal{B}_1^{1/q} \mathcal{A}_1 + \mathcal{V}_2^{1/pq} \mathcal{C}_2^{1/q} \mathcal{A}_1 \\ &+ \mathcal{V}_2^{1/pq} \mathcal{B}_2^{1/q} \mathcal{C}_1 + \mathcal{V}_1^{1/pq} \mathcal{C}_2^{1/q} \mathcal{C}_1 \end{aligned} \quad (4.35)$$

and

$$\begin{aligned} \mathcal{U}_1 &\leq \mathcal{U}_1^{1/pq} \mathcal{A}_1^{1/p} \mathcal{B}_1 + \mathcal{U}_2^{1/pq} \mathcal{C}_1^{1/p} \mathcal{B}_1 \\ &\quad + \mathcal{U}_2^{1/pq} \mathcal{A}_2^{1/p} \mathcal{C}_2 + \mathcal{U}_2^{1/pq} \mathcal{C}_1^{1/p} \mathcal{C}_2. \end{aligned} \quad (4.36)$$

To estimate the first term in the right-hand sides of (4.35) and (4.36), we apply Young's inequality

$$ab \leq \frac{1}{pq} a^{pq} + \frac{pq-1}{pq} b^{\frac{pq}{pq-1}} \quad p > 1, q > 1, \quad a > 0, b > 0.$$

This yields

$$\left(1 - \frac{1}{pq}\right) \mathcal{V}_1 \leq \mathcal{B}_1^{\frac{p}{pq-1}} \mathcal{A}_1^{\frac{pq}{pq-1}} + \mathcal{V}_2^{1/pq} \left[\mathcal{C}_2^{1/q} \mathcal{A}_1 + \mathcal{B}_2^{1/q} \mathcal{C}_1 + \mathcal{C}_2^{1/q} \mathcal{C}_1 \right], \quad (4.37)$$

and

$$\left(1 - \frac{1}{pq}\right) \mathcal{U}_1 \leq \mathcal{A}_1^{\frac{q}{pq-1}} \mathcal{B}_1^{\frac{pq}{pq-1}} + \mathcal{U}_2^{1/pq} \left[\mathcal{C}_1^{1/p} \mathcal{B}_1 + \mathcal{A}_2^{1/p} \mathcal{C}_2 + \mathcal{C}_1^{1/p} \mathcal{C}_2 \right]. \quad (4.38)$$

Using the definition of φ and applying the following change of variables

$$\tau = T^{-1}t, \quad \xi = \left(\frac{T}{B}\right)^{-1/2} x,$$

in the integrals in $\mathcal{A}_i, \mathcal{B}_i$ and \mathcal{C}_i for $i = 1, 2$, we get

$$\mathcal{V}_1 \leq CT^{\theta_1 \frac{pq}{pq-1}} B^{\delta_1 \frac{pq}{pq-1}} + \mathcal{V}_2^{1/pq} \left[CT^{\theta_1} B^{\delta_2} + CT^{\theta_1} B^{\delta_3} + CT^{\theta_1} B^{\delta_4} \right], \quad (4.39)$$

and

$$\mathcal{U}_1 \leq CT^{\theta_2 \frac{pq}{pq-1}} B^{\eta_1 \frac{pq}{pq-1}} + \mathcal{U}_2^{1/pq} \left[CT^{\theta_2} B^{\eta_2} + CT^{\theta_2} B^{\eta_3} + CT^{\theta_2} B^{\eta_4} \right], \quad (4.40)$$

where

$$\begin{cases} \delta_1 := -\frac{N}{2} \left(\frac{1}{q\tilde{p}} + \frac{1}{\tilde{q}} \right), & \delta_2 := \frac{1}{q} - \frac{N}{2} \left(\frac{1}{q\tilde{p}} + \frac{1}{\tilde{q}} \right), \\ \delta_3 := 1 - \frac{N}{2} \left(\frac{1}{q\tilde{p}} + \frac{1}{\tilde{q}} \right), & \delta_4 := 1 + \frac{1}{q} - \frac{N}{2} \left(\frac{1}{q\tilde{p}} + \frac{1}{\tilde{q}} \right), \end{cases}$$

and

$$\begin{cases} \eta_1 := -\frac{N}{2} \left(\frac{1}{p\tilde{q}} + \frac{1}{\tilde{p}} \right), & \eta_2 := \frac{1}{p} - \frac{N}{2} \left(\frac{1}{p\tilde{q}} + \frac{1}{\tilde{p}} \right), \\ \eta_3 := 1 - \frac{N}{2} \left(\frac{1}{p\tilde{q}} + \frac{1}{\tilde{p}} \right), & \eta_4 := 1 + \frac{1}{p} - \frac{N}{2} \left(\frac{1}{p\tilde{q}} + \frac{1}{\tilde{p}} \right). \end{cases}$$

Let us recall that $\theta_1 = 0$ (resp. $\theta_2 = 0$) imply that

$$\mathcal{V}_1 \leq CB^{\delta_1 \frac{pq}{pq-1}} + \mathcal{V}_2^{1/pq} \left[CB^{\delta_2} + CB^{\delta_3} + CB^{\delta_4} \right], \quad (4.41)$$

(resp.

$$\mathcal{U}_1 \leq CB^{\eta_1 \frac{pq}{pq-1}} + \mathcal{U}_2^{1/pq} \left[CB^{\eta_2} + CB^{\eta_3} + CB^{\eta_4} \right], \quad (4.42)$$

Now, as $v \in L^p(0, \infty; L^p(\mathbb{R}^N))$, (resp. $u \in L^q(0, \infty; L^q(\mathbb{R}^N))$), we have

$$\lim_{T \rightarrow \infty} \mathcal{V}_2 = 0,$$

(resp.

$$\lim_{T \rightarrow \infty} \mathcal{U}_2 = 0,)$$

then taking the limit as $T \rightarrow \infty$ in (4.41), respectively in (4.42), taking into account the dominated convergence theorem, we conclude that

$$\int_0^\infty \int_{\mathbb{R}^N} v^p(x, t) dx dt \leq CB^{\delta_1 \frac{pq}{pq-1}}. \quad (4.43)$$

(resp.

$$\int_0^\infty \int_{\mathbb{R}^N} u^q(x, t) dx dt \leq CB^{\eta_1 \frac{pq}{pq-1}}.) \quad (4.44)$$

Finally, as $\delta_1 < 0$ and $\eta_1 < 0$, taking the limit for $B \rightarrow \infty$ in (4.43), respectively in (4.44) and using the continuity of u and v , we conclude that

$$v \equiv 0 \quad \text{or} \quad u \equiv 0,$$

and (4.33) implies that $u \equiv v \equiv 0$, which is a contradiction.

• The case $p < (1/\delta)$ and $q < (1/\gamma)$: We repeat the same procedure as in the case ($\theta_1 < 0$ and $\theta_2 < 0$) by choosing the following test function $\tilde{\varphi}(x, t) = (\varphi_1(x))^\ell \varphi_2(t)$ where $\varphi_1(x) = \Phi(|x|/R)$, $\varphi_2(t) = (1 - t/T)_+^r$, $r \gg 1$ and $R \in (0, T)$ large enough such that in the case of $T \rightarrow \infty$ we don't have $R \rightarrow \infty$ in the same time, with the same functions Φ as above. Then, as in (4.23), we obtain

$$\begin{cases} \left(\int_{C_T} v^p \tilde{\varphi}(x, t) \right)^{1-1/pq} \leq \mathcal{D}_2^{1/q} \mathcal{D}_1, \\ \left(\int_{C_T} u^q \tilde{\varphi}(x, t) \right)^{1-1/pq} \leq \mathcal{D}_1^{1/p} \mathcal{D}_2, \end{cases} \quad (4.45)$$

where

$$C_T := [0, T] \times \{x \in \mathbb{R}^N ; |x| \leq 2R\} \quad \int_{C_T} = \int_{C_T} dx dt,$$

$$\mathcal{D}_1 := C \left(\int_{C_T} \varphi_1^\ell \varphi_2^{-\frac{1}{q-1}} \left| D_{t|T}^{1+\alpha} \varphi_2 \right|^{\tilde{q}} \right)^{1/\tilde{q}} + C \left(\int_{C_T} \varphi_1^{\ell-\tilde{q}} \varphi_2^{-\frac{1}{q-1}} \left| \Delta_x \varphi_1 D_{t|T}^\alpha \varphi_2 \right|^{\tilde{q}} \right)^{1/\tilde{q}},$$

and

$$\mathcal{D}_2 := C \left(\int_{C_T} \varphi_1^\ell \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{1+\beta} \varphi_2 \right|^{\tilde{p}} \right)^{1/\tilde{p}} + C \left(\int_{C_T} \varphi_1^{\ell-\tilde{p}} \varphi_2^{-\frac{1}{p-1}} \left| \Delta_x \varphi_1 D_{t|T}^\beta \varphi_2 \right|^{\tilde{p}} \right)^{1/\tilde{p}}.$$

Then, the new variables $\xi = R^{-1}x$, $\tau = T^{-1}t$ and (2.12) – (2.13) allow us to write:

$$\begin{cases} \left(\int_{C_T} v^p \tilde{\varphi}(x, t) \right)^{1-1/pq} \leq C_1(T, R), \\ \left(\int_{C_T} u^q \tilde{\varphi}(x, t) \right)^{1-1/pq} \leq C_2(T, R), \end{cases} \quad (4.46)$$

where

$$\begin{aligned} C_1(T, R) &:= CT^{\alpha_1} R^{\beta_1} + CT^{\alpha_2} R^{\beta_2} + CT^{\alpha_3} R^{\beta_3} + CT^{\alpha_4} R^{\beta_4}, \\ C_2(T, R) &:= CT^{\gamma_1} R^{\sigma_1} + CT^{\gamma_2} R^{\sigma_2} + CT^{\gamma_3} R^{\sigma_3} + CT^{\gamma_4} R^{\sigma_4}, \end{aligned}$$

with

$$\begin{cases} \alpha_1 := \frac{1}{q} \left[\frac{1}{\tilde{p}} - (1 + \beta) \right] + \left[\frac{1}{\tilde{q}} - (1 + \alpha) \right], & \alpha_2 := \frac{1}{q} \left[\frac{1}{\tilde{p}} - (1 + \beta) \right] + \left[\frac{1}{\tilde{q}} - \alpha \right], \\ \alpha_3 := \frac{1}{q} \left[\frac{1}{\tilde{p}} - \beta \right] + \left[\frac{1}{\tilde{q}} - (1 + \alpha) \right], & \alpha_4 := \frac{1}{q} \left[\frac{1}{\tilde{p}} - \beta \right] + \left[\frac{1}{\tilde{q}} - \alpha \right], \\ \gamma_1 := \frac{1}{p} \left[\frac{1}{\tilde{q}} - (1 + \alpha) \right] + \left[\frac{1}{\tilde{p}} - (1 + \beta) \right], & \gamma_2 := \frac{1}{p} \left[\frac{1}{\tilde{q}} - (1 + \alpha) \right] + \left[\frac{1}{\tilde{p}} - \beta \right], \\ \gamma_3 := \frac{1}{p} \left[\frac{1}{\tilde{q}} - \alpha \right] + \left[\frac{1}{\tilde{p}} - (1 + \beta) \right], & \gamma_4 := \frac{1}{p} \left[\frac{1}{\tilde{q}} - \alpha \right] + \left[\frac{1}{\tilde{p}} - \beta \right], \\ \beta_1 := \frac{N}{\tilde{p}q} + \frac{N}{\tilde{q}}, & \beta_2 := \frac{N}{\tilde{p}q} + \frac{N}{\tilde{q}} - 2, \\ \beta_3 := \frac{1}{q} \left[\frac{N}{\tilde{p}} - 2 \right] + \frac{N}{\tilde{q}}, & \beta_4 := \frac{1}{q} \left[\frac{N}{\tilde{p}} - 2 \right] + \frac{N}{\tilde{q}} - 2, \end{cases}$$

and

$$\begin{cases} \beta_1 := \frac{N}{\tilde{q}p} + \frac{N}{\tilde{p}}, & \beta_2 := \frac{N}{\tilde{q}p} + \frac{N}{\tilde{p}} - 2, \\ \beta_3 := \frac{1}{p} \left[\frac{N}{\tilde{q}} - 2 \right] + \frac{N}{\tilde{p}}, & \beta_4 := \frac{1}{p} \left[\frac{N}{\tilde{q}} - 2 \right] + \frac{N}{\tilde{p}} - 2. \end{cases}$$

Taking the limit as $T \rightarrow \infty$, we infer, as

$$p < \frac{1}{\delta} \iff \frac{1}{\tilde{p}} - \beta < 0 \quad \text{and} \quad q < \frac{1}{\gamma} \iff \frac{1}{\tilde{q}} - \alpha < 0,$$

that

$$\begin{cases} \left(\int_0^\infty \int_{|x| \leq 2R} v^p \tilde{\varphi}(x, t) dx dt \right)^{1-1/pq} = 0, \\ \left(\int_0^\infty \int_{|x| \leq 2R} u^q \tilde{\varphi}(x, t) dx dt \right)^{1-1/pq} = 0. \end{cases}$$

Finally, by taking $R \rightarrow \infty$, we get a contradiction with the fact that $u(x, t) > 0$ and $v(x, t) > 0$ for all $x \in \mathbb{R}^N$, $t > 0$. \square

Remarks 1 *We can extend our analysis to the following system*

$$\begin{cases} u_t - \Delta u = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\psi_1(x, s) |v|^{p-1} v(s)}{(t-s)^\gamma} ds & x \in \mathbb{R}^N, t > 0, \\ v_t - \Delta v = \frac{1}{\Gamma(1-\delta)} \int_0^t \frac{\psi_2(x, s) |u|^{q-1} u(s)}{(t-s)^\delta} ds & x \in \mathbb{R}^N, t > 0, \end{cases}$$

where $\psi_1, \psi_2 \in L^1_{Loc}(\mathbb{R}^N \times (0, \infty))$, $\psi_1(\cdot, t), \psi_2(\cdot, t) \geq 0$ for all $t \geq 0$, $\psi \not\equiv 0$, and for all $0 < R, B < T$, $\tau \in [0, 1]$, $\xi \in [0, 2]$, $1 < p$, $0 < \beta \leq 2$ and $0 < \gamma, \delta < 1$, we have

$$\begin{cases} \psi_1(B^{-1/2}T^{1/2}\xi, T\tau) \geq C > 0, \\ \psi_2(B^{-1/2}T^{1/2}\xi, T\tau) \geq C > 0, \end{cases}$$

if (p, q) verifies (4.11), while

$$\begin{cases} \psi_1(R\xi, T\tau) \geq C > 0, \\ \psi_2(R\xi, T\tau) \geq C > 0, \end{cases}$$

if (p, q) verifies (4.12).

5 Blow-up Rate

In this section, we study the blow-up rate for the parabolic system (1.1) – (1.2) requiring the following regularity of the initial data:

$$u_0, v_0 \geq 0, \quad u_0, v_0 \not\equiv 0 \quad \text{and} \quad u_0, v_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N). \quad (5.1)$$

First, we give a lemma which will play a crucial role in the sequel.

Lemma 2 *Let u, v be a nonnegative classical global solution of*

$$\begin{cases} u_t = \Delta u + J_{-\infty|t}^\alpha(v^p) & \text{in } \mathbb{R}^N \times \mathbb{R}, \\ v_t = \Delta v + J_{-\infty|t}^\beta(u^q) & \text{in } \mathbb{R}^N \times \mathbb{R}. \end{cases} \quad (5.2)$$

Then, for

$$\frac{N}{2} \leq \max \left\{ \frac{(2-\delta)p + (1-\gamma)pq + 1}{pq-1}; \frac{(2-\gamma)q + (1-\delta)pq + 1}{pq-1} \right\}, \quad (5.3)$$

or

$$p < \frac{1}{\delta} \quad \text{and} \quad q < \frac{1}{\gamma}, \quad (5.4)$$

we have $u \equiv v \equiv 0$.

Proof Its sufficient to observe that

$$J_{-\infty|t}^\alpha(v^p) \geq J_{-T|t}^\alpha(v^p) \quad \text{and} \quad J_{-\infty|t}^\beta(u^q) \geq J_{-T|t}^\beta(u^q).$$

Then, by repeating the same computations as in Theorem 1 with $\varphi_2(t) := (1 - t^2/T^2)_+^\eta$, $\eta \gg 1$, taking into account (2.15) – (2.17), and by taking

$$(\varphi_3(t)\varphi_1^\ell(x))^{1/q}(\varphi_3(t)\varphi_1^\ell(x))^{-1/q} \quad \text{instead of} \quad \varphi^{1/q}\varphi^{-1/q}$$

in (4.19) – (4.29) – (4.31) and

$$(\varphi_3(t)\varphi_1^\ell(x))^{1/p}(\varphi_3(t)\varphi_1^\ell(x))^{-1/p} \quad \text{instead of} \quad \varphi^{1/p}\varphi^{-1/p}$$

in (4.20) – (4.30) – (4.32), where $\ell \gg 1$ and $\varphi_3(t) := (1 - \frac{t}{T})_+^\eta$, to use the Hölder's inequality, we conclude the result.

Finally, we note that here, we need also to use the fact that $\varphi_3(t)\varphi_1^\ell(x) \leq \varphi(x, t)$ before the combination as in (4.21) – (4.22) and (4.33) – (4.34). \square

Theorem 3 *Let*

$$\alpha_1 := \frac{(2 - \gamma) + (2 - \delta)p}{pq - 1} \quad \text{and} \quad \alpha_2 := \frac{(2 - \delta) + (2 - \gamma)q}{pq - 1}.$$

If

$$\frac{N}{2} \leq \max \left\{ \frac{(2 - \delta)p + (1 - \gamma)pq + 1}{pq - 1}; \frac{(2 - \gamma)q + (1 - \delta)pq + 1}{pq - 1} \right\},$$

or

$$p < \frac{1}{\delta} \quad \text{and} \quad q < \frac{1}{\gamma},$$

and (u, v) *be the blowing-up solution of (1.1) – (1.2) – (5.1) in a finite time* $T_{\max} := T^*$, *then there are constants* $c_i, C_i > 0$, *for* $i = 1, 2$, *such that*

$$\begin{cases} c_1(T^* - t)^{-\alpha_1} \leq \sup_{\mathbb{R}^N} u(., t) \leq C_1(T^* - t)^{-\alpha_1}, & t \in (0, T^*), \\ c_2(T^* - t)^{-\alpha_2} \leq \sup_{\mathbb{R}^N} v(., t) \leq C_2(T^* - t)^{-\alpha_2}, & t \in (0, T^*). \end{cases} \quad (5.5)$$

Proof We decompose the proof into two parts:

- The upper blow-up rate estimate: Let

$$M_1(t) := \sup_{\mathbb{R}^N \times (0, t]} u \quad \text{and} \quad M_2(t) := \sup_{\mathbb{R}^N \times (0, t]} v, \quad t \in (0, T^*).$$

First, we show that there is $\eta \in (0, 1)$ such that

$$\eta \leq M_1(t)^{-1/\alpha_1} M_2(t)^{1/\alpha_2} \leq \eta^{-1}, \quad t \in \left(\frac{T^*}{2}, T^* \right). \quad (5.6)$$

Indeed, we proceed by contradiction, as in [4][Proof of (2.3)]. If (5.6) were false then there would exist a sequence $t_n \rightarrow T^*$ such that

$$M_1(t_n)^{-1/\alpha_1} M_2(t_n)^{1/\alpha_2} \rightarrow 0 \quad \text{or} \quad M_2(t_n)^{-1/\alpha_2} M_1(t_n)^{1/\alpha_1} \rightarrow 0. \quad (5.7)$$

We will study the first case in (5.7) while the second one can be treated by proceeding in the same way and by changing the role of u and v . Then, using the fact that the condition (5.1) and Theorem 1 imply $M_2 > 0$, we infer that M_1 diverges as $t_n \rightarrow T^*$. So, for each t_n , we choose

$$(\widehat{x}_n, \widehat{t}_n) \in \mathbb{R}^N \times (0, t_n] \quad \text{such that} \quad u(\widehat{x}_n, \widehat{t}_n) \geq \frac{1}{2} M_1(t_n). \quad (5.8)$$

Obviously, $M_1(t_n) \rightarrow \infty$ implies $\widehat{t}_n \rightarrow T^*$.

We rescale the solution (u, v) about the corresponding point $(\widehat{x}_n, \widehat{t}_n)$ with the scaling factor $\lambda_n := \lambda(t_n) := (1/(2A)M_1(t_n))^{-1/(2\alpha_1)}$, for $A \geq 1$, as follows:

$$\varphi^{\lambda_n}(y, s) := \lambda_n^{2\alpha_1} u(\lambda_n y + \widehat{x}_n, \lambda_n^2 s + \widehat{t}_n), \quad (y, s) \in \mathbb{R}^N \times I_n(T^*), \quad (5.9)$$

$$\psi^{\lambda_n}(y, s) := \lambda_n^{2\alpha_2} v(\lambda_n y + \widehat{x}_n, \lambda_n^2 s + \widehat{t}_n), \quad (y, s) \in \mathbb{R}^N \times I_n(T^*), \quad (5.10)$$

where $I_n(t) := (-\lambda_n^{-2}\widehat{t}_n, \lambda_n^{-2}(t - \widehat{t}_n))$ for all $t > 0$. Then $(\varphi^{\lambda_n}, \psi^{\lambda_n})$ is a mild solution of the system

$$\varphi_s = \Delta \varphi + J_{-\lambda_n^{-2}\widehat{t}_n|s}^\alpha(\psi^p) \quad \text{in } \mathbb{R}^N \times I_n(T^*), \quad (5.11)$$

$$\psi_s = \Delta \psi + J_{-\lambda_n^{-2}\widehat{t}_n|s}^\beta(\varphi^q) \quad \text{in } \mathbb{R}^N \times I_n(T^*), \quad (5.12)$$

i.e., for $G(t) := G(x, t) := (4\pi t)^{-N/2} e^{-|x|^2/4t}$, we have

$$\varphi^{\lambda_n}(s) = G(s + \lambda_n^{-2}\widehat{t}_n) * \varphi^{\lambda_n}(-\lambda_n^{-2}\widehat{t}_n) + \int_{-\lambda_n^{-2}\widehat{t}_n}^s G(s - \sigma) * J_{-\lambda_n^{-2}\widehat{t}_n|\sigma}^\alpha((\psi^{\lambda_n})^p) d\sigma, \quad (5.13)$$

$$\psi^{\lambda_n}(s) = G(s + \lambda_n^{-2}\widehat{t}_n) * \psi^{\lambda_n}(-\lambda_n^{-2}\widehat{t}_n) + \int_{-\lambda_n^{-2}\widehat{t}_n}^s G(s - \sigma) * J_{-\lambda_n^{-2}\widehat{t}_n|\sigma}^\beta((\varphi^{\lambda_n})^q) d\sigma, \quad (5.14)$$

in $\mathbb{R}^N \times I_n(T^*)$ such that $\varphi^{\lambda_n}(0, 0) \geq A$, and

$$\begin{cases} 0 \leq \varphi^{\lambda_n} \leq \lambda_n^{2\alpha_1} M_1(t_n) = 2A, \\ 0 \leq \psi^{\lambda_n} \leq \lambda_n^{2\alpha_2} M_2(t_n) = (2A)^{\alpha_2/\alpha_1} M_2(t_n) M_1(t_n)^{-\alpha_2/\alpha_1}, \end{cases}$$

in $\mathbb{R}^N \times (-\lambda_n^{-2}\widehat{t}_n, 0]$, where $*$ is the space convolution.

So, as in Lemma 1, $(\varphi^{\lambda_n}, \psi^{\lambda_n})$ is a weak solution of (5.11) – (5.12).

Now, involving the interior regularity (cf. [8][Theorem 10.1 p. 204]), there exists $\mu \in (0, 1)$ such that for any $K > 0$, the sequences $\varphi^{\lambda_n}, \psi^{\lambda_n}$ are bounded in the $C^{\mu, \mu/2}(\overline{S}_{2K})$ -norm by a constant independent on n , where

$$S_K := \{(x, t) \in \mathbb{R}^N \times \mathbb{R}; |x| < K, -K < t \leq 0\}.$$

We notice here that we have to use the Maximal regularity theory to obtain such regularity of $(\varphi^{\lambda_n}, \psi^{\lambda_n})$ to apply the interior regularity (for more precise information, see [5][Proof of

Th. 3].

Moreover, Schauder's estimates imply that the $C^{2+\mu, 1+\mu/2}(\overline{S}_K)$ -norm of $\varphi^{\lambda_n}, \psi^{\lambda_n}$ is uniformly bounded. Thus, there is a subsequence converging to a solution (φ, ψ) of the following system

$$\varphi_s = \Delta\varphi + J_{-\infty|t}^\alpha(\psi^p), \quad \psi_s = \Delta\psi + J_{-\infty|t}^\beta(\varphi^q), \quad \text{in } \mathbb{R}^N \times (-\infty, 0], \quad (5.15)$$

such that $\varphi(0, 0) \geq A$, and

$$0 \leq \varphi \leq 2A, \quad 0 \leq \psi \leq \lim_{n \rightarrow \infty} (2A)^{\alpha_2/\alpha_1} M_2(t_n) M_1(t_n)^{-\alpha_2/\alpha_1} = 0.$$

Combining the last inequality with the system (5.15), we obtain $\varphi \equiv 0$ on $\mathbb{R}^N \times (-\infty, 0]$. This contradicts $\varphi(0, 0) \geq A$. Consequently, (5.6) holds.

On the other hand, obviously, M_1, M_2 are positive, continuous and nondecreasing on $(0, T^*)$. Moreover, as $\lim_{t \rightarrow T^*} M_1(t) = \infty$, this allow us, for all $t_0 \in (0, T^*)$, to define

$$t_0^+ := t^+(t_0) := \max\{t \in (t_0, T^*) : M_1(t) = 2M_1(t_0)\}. \quad (5.16)$$

Let

$$\lambda(t_0) := \left(\frac{1}{2A} M_1(t_0) \right)^{-1/(2\alpha_1)}. \quad (5.17)$$

We claim that

$$\lambda^{-2}(t_0)(t_0^+ - t_0) \leq D, \quad t_0 \in \left(\frac{T^*}{2}, T^* \right), \quad (5.18)$$

where $0 < D < \infty$ is a positive constant which does not depend on t_0 .

Indeed, if (5.18) were false, then there would exist a sequence $t_n \rightarrow T^*$ such that

$$\lambda_n^{-2}(t_n^+ - t_n) \rightarrow \infty,$$

where $\lambda_n = \lambda(t_n)$ and $t_n^+ = t^+(t_n)$. For each t_n choose $(\widehat{x}_n, \widehat{t}_n)$ as in (5.8) and rescale (u, v) around $(\widehat{x}_n, \widehat{t}_n)$ as in (5.9)-(5.10). Then $(\varphi^{\lambda_n}, \psi^{\lambda_n})$ is a mild solution (so weak solution) of the systems (5.11)-(5.12), in $\mathbb{R}^N \times I_n(T^*)$ such that $\varphi(0, 0) \geq A$, and in $\mathbb{R}^N \times I_n(t_n^+)$ we have from (5.6) and the definition of t_n^+ the following inequality

$$\begin{aligned} 0 &\leq \varphi^{\lambda_n} \leq \lambda_n^{2\alpha_1} M_1(t_n^+) = \lambda_n^{2\alpha_1} 2M_1(t_n) := 4A, \\ 0 &\leq \psi^{\lambda_n} \leq \lambda_n^{2\alpha_2} M_2(t_n^+) \\ &\leq \lambda_n^{2\alpha_2} \eta^{-\alpha_2} (M_1(t_n^+))^{\alpha_2/\alpha_1} = (4A)^{\alpha_2/\alpha_1} \eta^{-\alpha_2}. \end{aligned}$$

Interior regularity and uniform Schauder estimates for $(\varphi^{\lambda_n}, \psi^{\lambda_n})$ yield a subsequence converging in $C_{loc}^{2+\mu, 1+\mu/2}(\mathbb{R}^N \times \mathbb{R}) \times C_{loc}^{2+\mu, 1+\mu/2}(\mathbb{R}^N \times \mathbb{R})$ to a solution (φ, ψ) of

$$(\varphi)_s = \Delta\varphi + J_{-\infty|t}^\alpha(\psi^p), \quad (\psi)_s = \Delta\psi + J_{-\infty|t}^\beta(\varphi^q), \quad \text{in } \mathbb{R}^N \times \mathbb{R}, \quad (5.19)$$

such that $\varphi(0, 0) \geq A$ and

$$0 \leq \varphi \leq 4A, \quad 0 \leq \psi \leq (4A)^{\alpha_2/\alpha_1} \eta^{-\alpha_2}.$$

It follows from Lemma 2 that $\varphi \equiv \psi \equiv 0$. This is a contradiction with $\varphi(0,0) \geq A \geq 1$. Now (5.18) is true.

Next we use an idea from Hu [6]. From (5.17) and (5.18) it follows that

$$(t_0^+ - t_0) \leq D(2A)^{1/\alpha_1} M_1(t_0)^{-1/\alpha_1} \quad \text{for any } t_0 \in \left(\frac{T^*}{2}, T^*\right).$$

Fix $t_0 \in (T^*/2, T^*)$ and denote $t_1 = t_0^+, t_2 = t_1^+, t_3 = t_2^+, \dots$. Then

$$\begin{aligned} t_{j+1} - t_j &\leq D(2A)^{1/\alpha_1} M_1(t_j)^{-1/\alpha_1}, \\ M_1(t_{j+1}) &= 2M_1(t_j), \end{aligned}$$

$j = 0, 1, 2, \dots$. Consequently,

$$\begin{aligned} T^* - t_0 &= \sum_{j=0}^{\infty} (t_{j+1} - t_j) \leq D(2A)^{1/\alpha_1} \sum_{j=0}^{\infty} M_1(t_j)^{-1/\alpha_1} \\ &= D(2A)^{1/\alpha_1} M_1(t_0)^{-1/\alpha_1} \sum_{j=0}^{\infty} 2^{-j/\alpha_1}. \end{aligned}$$

We conclude that

$$u(x, t_0) \leq M_1(t_0) \leq C(T^* - t_0)^{-\alpha_1}, \quad \forall t_0 \in (0, T^*)$$

where

$$C_1 = 2A \left(D \sum_{j=0}^{\infty} 2^{-j/\alpha_1} \right)^{\alpha_1},$$

and consequently

$$\sup_{\mathbb{R}^N} u(\cdot, t) \leq C_1 (T^* - t)^{-\alpha_1}, \quad \forall t \in (0, T^*).$$

Finally, (5.6) implies

$$v(x, t_0) \leq M_2(t_0) \leq C_2 (T^* - t_0)^{-\alpha_2}, \quad \forall t_0 \in (0, T^*)$$

where

$$C_2 = \eta^{-\alpha_2} C_1^{\alpha_2/\alpha_1},$$

and consequently

$$\sup_{\mathbb{R}^N} v(\cdot, t) \leq C_2 (T^* - t)^{-\alpha_2}, \quad \forall t \in (0, T^*).$$

• The lower blow-up rate estimate: If we repeat the same proof of the local existence in Theorem 1, by taking $\|u\|_1 \leq \theta_1$ and $\|v\| \leq \theta_2$ instead of $\|(u, v)\| \leq 2(\|u_0\|_\infty + \|v_0\|_\infty)$ in the space E_T in (3.2) for all positive constants $\theta_1, \theta_2 > 0$ and all $0 < t < T$, then the condition (3.3) of T will be:

$$\|u_0\|_\infty + C_3 T^{2-\gamma} \theta_2^p \leq \theta_1 \quad \text{and} \quad \|v_0\|_\infty + C_4 T^{2-\delta} \theta_1^q \leq \theta_2. \quad (5.20)$$

Then, by the same reasoning, we infer that $\|u(t)\|_\infty \leq \theta_1$ and $\|v(t)\|_\infty \leq \theta_2$ for all $0 < t < T$. Consequently, if

$$\|u_0\|_\infty + C_3 t^{2-\gamma} \theta_2^p \leq \theta_1 \quad \text{and} \quad \|v_0\|_\infty + C_4 t^{2-\delta} \theta_1^q \leq \theta_2,$$

then $\|u(t)\|_\infty \leq \theta_1$ and $\|v(t)\|_\infty \leq \theta_2$. Applying this to any point in the trajectories, we see that if $0 \leq s < t$ and

$$(t-s)^{2-\gamma} \leq \frac{\theta_1 - \|u(s)\|_\infty}{C_3 \theta_2^p} \quad \text{and} \quad (t-s)^{2-\delta} \leq \frac{\theta_2 - \|v(s)\|_\infty}{C_4 \theta_1^q}, \quad (5.21)$$

then $\|u(t)\|_\infty \leq \theta_1$ and $\|v(t)\|_\infty \leq \theta_2$, for all $0 < t < T$.

Moreover, if $0 \leq s < T^*$, $\|u(s)\|_\infty < \theta_1$ and $\|v(s)\|_\infty < \theta_2$, then:

$$(T^* - s)^{2-\gamma} > \frac{\theta_1 - \|u(s)\|_\infty}{C_3 \theta_2^p} \quad \text{and} \quad (T^* - s)^{2-\delta} > \frac{\theta_2 - \|v(s)\|_\infty}{C_4 \theta_1^q}. \quad (5.22)$$

Indeed, by contradiction. Suppose that for some $\theta_1 > \|u(s)\|_\infty, \theta_2 > \|v(s)\|_\infty$ and all $t \in (s, T^*)$ we have

$$(t-s)^{2-\gamma} \leq \frac{\theta_1 - \|u(s)\|_\infty}{C_3 \theta_2^p} \quad \text{or} \quad (t-s)^{2-\delta} \leq \frac{\theta_2 - \|v(s)\|_\infty}{C_4 \theta_1^q},$$

and then, using (5.21), we infer that $\|u(t)\|_\infty \leq \theta_1$ or $\|v(t)\|_\infty \leq \theta_2$ for all $t \in (s, T^*)$.

Contradiction with the fact that $\|u(t)\|_\infty \rightarrow \infty$ and $\|v(t)\|_\infty \rightarrow \infty$ as $t \rightarrow T^*$.

Next, for example, letting $\theta_1 = 2\|u(s)\|_\infty$ and $\theta_2 = 2\|v(s)\|_\infty$ in (5.22), we see that for $0 < s < T^*$ we have:

$$(T^* - s)^{2-\gamma} > \frac{\|u(s)\|_\infty}{2^p C_3 \|v(s)\|_\infty^p} \quad \text{and} \quad (T^* - s)^{2-\delta} > \frac{\|v(s)\|_\infty}{2^q C_4 \|u(s)\|_\infty^q},$$

which, after combination, imply

$$c_1 (T^* - t)^{-\alpha_1} < \|u(s)\|_\infty \quad \text{and} \quad c_2 (T^* - t)^{-\alpha_2} < \|v(s)\|_\infty,$$

where

$$c_1 := \left(2^{p(1+q)} C_3 C_4^p \right)^{-\frac{1}{pq-1}}, \quad c_2 := \left(2^{q(1+p)} C_4 C_3^q \right)^{-\frac{1}{pq-1}},$$

and by the positivity of u and v we get

$$c_1 (T^* - s)^{-\alpha_1} < \sup_{x \in \mathbb{R}^N} u(x, s) \quad \text{and} \quad c_2 (T^* - s)^{-\alpha_2} < \sup_{x \in \mathbb{R}^N} v(x, s), \quad \forall s \in (0, T^*) \quad (5.23)$$

□

6 Necessary conditions for local and global existence

A necessary condition for the existence of local and global solutions to the problem (1.1) – (1.2) are presented in this section. We obtain that these conditions depend on the behavior of the initial conditions when x goes to infinity.

Theorem 4 (*Necessary conditions for global existence*)

Let $u_0, v_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, $u_0, v_0 \geq 0$, and $p, q > 1$. If (u, v) is a mild global solution to problem (1.1) – (1.2), then there is a positive constant $C > 0$ such that

$$\lim_{|x| \rightarrow \infty} (u_0(x)|x|^{2\alpha_1}) \leq C \quad \text{and} \quad \lim_{|x| \rightarrow \infty} (v_0(x)|x|^{2\alpha_2}) \leq C, \quad (6.1)$$

where $C > 0$ is a real positive number which may change from line to line.

Proof Let (u, v) be a global mild solution to (1.1) – (1.2), then $u \in C([0, R]; C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N))$ for all $R > 0$ sufficiently large. So we repeat the same calculation as in the proof of Theorem 2 by taking $\tilde{\varphi}(x, t) := \varphi_1(x/R) \varphi_2(t)$ where $\varphi_2(t) := (1 - t/(R^2)) \ell_+$ instead of the one chosen in Theorem 2, where, for $x \in \mathbb{R}^N$, $0 \leq \varphi_1 \in H^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is the first eigenfunction, for $-\Delta_x$, relative to the first eigenvector $\lambda_1 := \inf\{\|u\|_{H^1}; \|u\|_{L^2} = 1 \text{ and } u = 0 \text{ in } B_2^c\}$ where B_2 stands for the ball of center 0 and radius 2 and B_2^c for its complementary.

Then, as in (4.19) – (4.20), we have

$$\begin{aligned} & \int_{\Sigma_1} v^p \tilde{\varphi} dx dt + C R^{-2\alpha} \int_{|x| \leq 2R} u_0(x) \varphi_1(x/R) dx \\ & \leq C \int_{\Sigma_1} u \tilde{\varphi}^{1/q} \tilde{\varphi}^{-1/q} \varphi_1(x/R) \left| D_{t|R^2}^{1+\alpha} \varphi_2(t) \right| dx dt \\ & + C \int_{\Sigma_1} u \tilde{\varphi}^{1/q} \tilde{\varphi}^{-1/q} \left| (-\Delta_x) \varphi_1(x/R) D_{t|R^2}^\alpha \varphi_2(t) \right| dx dt, \end{aligned} \quad (6.2)$$

and

$$\begin{aligned} & \int_{\Sigma_1} u^q \tilde{\varphi} dx dt + C R^{-2\beta} \int_{|x| \leq 2R} v_0(x) \varphi_1(x/R) dx \\ & \leq C \int_{\Sigma_1} v \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} \varphi_1(x/R) \left| D_{t|R^2}^{1+\beta} \varphi_2(t) \right| dx dt \\ & + C \int_{\Sigma_1} v \tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} \left| (-\Delta_x) \varphi_1(x/R) D_{t|R^2}^\beta \varphi_2(t) \right| dx dt. \end{aligned} \quad (6.3)$$

where $\alpha := 1 - \gamma$, $\beta := 1 - \delta$ and

$$\Sigma_1 := \{(x, t) \in \mathbb{R}^N \times [0, \infty); |x| \leq 2R, t \leq R^2\}.$$

Hence, using Hölder's inequality, we get

$$\mathcal{V}_3 + C R^{-2\alpha} \int_{|x| \leq 2R} u_0(x) \varphi_1(x/R) dx \leq \mathcal{U}_3^{1/q} \left[\mathcal{E}_1^{1/\tilde{q}} + \mathcal{F}_1^{1/\tilde{q}} \right], \quad (6.4)$$

and

$$\mathcal{U}_3 + CR^{-2\beta} \int_{|x| \leq 2R} v_0(x) \varphi_1(x/R) dx \leq \mathcal{V}_3^{1/p} \left[\mathcal{E}_2^{1/\tilde{p}} + \mathcal{F}_2^{1/\tilde{p}} \right], \quad (6.5)$$

where

$$\begin{aligned} \mathcal{U}_3 &:= \int_{\Sigma_1} u^q \tilde{\varphi} dx dt, & \mathcal{V}_3 &:= \int_{\Sigma_1} v^p \tilde{\varphi} dx dt \\ \mathcal{E}_1 &:= C \left(\int_{\Sigma_1} \varphi_1 \varphi_2^{-\frac{1}{q-1}} \left| D_{t|R^2}^{1+\alpha} \varphi_2 \right|^{\tilde{q}} dx dt \right)^{1/\tilde{q}}, \\ \mathcal{E}_2 &:= C \left(\int_{\Sigma_1} \varphi_1 \varphi_2^{-\frac{1}{p-1}} \left| D_{t|R^2}^{1+\beta} \varphi_2 \right|^{\tilde{p}} dx dt \right)^{1/\tilde{p}}, \\ \mathcal{F}_1 &:= C \left(\int_{\Sigma_1} (\varphi_1 \varphi_2)^{-\frac{1}{q-1}} \left| (-\Delta_x) \varphi_1 D_{t|R^2}^\alpha \varphi_2 \right|^{\tilde{q}} dx dt \right)^{1/\tilde{q}}, \end{aligned}$$

and

$$\mathcal{F}_2 := C \left(\int_{\Sigma_1} (\varphi_1 \varphi_2)^{-\frac{1}{p-1}} \left| (-\Delta_x) \varphi_1 D_{t|R^2}^\beta \varphi_2 \right|^{\tilde{p}} dx dt \right)^{1/\tilde{p}}.$$

Combining (6.4) and (6.5) leads to

$$\mathcal{V}_3 + CR^{-2\alpha} \int_{|x| \leq 2R} u_0(x) \varphi_1(x/R) dx \leq \mathcal{V}_3^{1/pq} \left[\left(\mathcal{E}_2^{1/q\tilde{p}} + \mathcal{F}_2^{1/q\tilde{p}} \right) \left(\mathcal{E}_1^{1/\tilde{q}} + \mathcal{F}_1^{1/\tilde{q}} \right) \right], \quad (6.6)$$

and

$$\mathcal{U}_3 + CR^{-2\beta} \int_{|x| \leq 2R} v_0(x) \varphi_1(x/R) dx \leq \mathcal{U}_3^{1/pq} \left[\left(\mathcal{E}_1^{1/p\tilde{q}} + \mathcal{F}_1^{1/p\tilde{q}} \right) \left(\mathcal{E}_2^{1/\tilde{p}} + \mathcal{F}_2^{1/\tilde{p}} \right) \right]. \quad (6.7)$$

Moreover, using Young's inequality in the right-hand side of (6.6) – (6.7), allows us to obtain

$$\mathcal{V}_3 + CR^{-2\alpha} \int_{|x| \leq 2R} u_0(x) \varphi_1(x/R) dx \leq \mathcal{V}_3 + \left[\left(\mathcal{E}_2^{1/q\tilde{p}} + \mathcal{F}_2^{1/q\tilde{p}} \right) \left(\mathcal{E}_1^{1/\tilde{q}} + \mathcal{F}_1^{1/\tilde{q}} \right) \right]^{\frac{pq}{pq-1}},$$

and

$$\mathcal{U}_3 + CR^{-2\beta} \int_{|x| \leq 2R} v_0(x) \varphi_1(x/R) dx \leq \mathcal{U}_3 + \left[\left(\mathcal{E}_1^{1/p\tilde{q}} + \mathcal{F}_1^{1/p\tilde{q}} \right) \left(\mathcal{E}_2^{1/\tilde{p}} + \mathcal{F}_2^{1/\tilde{p}} \right) \right]^{\frac{pq}{pq-1}}.$$

These imply

$$CR^{-2\alpha} \int_{|x| \leq 2R} u_0(x) \varphi_1(x/R) dx \leq \left[\left(\mathcal{E}_2^{\frac{p-1}{pq-1}} + \mathcal{F}_2^{\frac{p-1}{pq-1}} \right) \left(\mathcal{E}_1^{\frac{p(q-1)}{pq-1}} + \mathcal{F}_1^{\frac{p(q-1)}{pq-1}} \right) \right], \quad (6.8)$$

and

$$CR^{-2\beta} \int_{|x| \leq 2R} v_0(x) \varphi_1(x/R) dx \leq \left[\left(\mathcal{E}_1^{\frac{q-1}{pq-1}} + \mathcal{F}_1^{\frac{q-1}{pq-1}} \right) \left(\mathcal{E}_2^{\frac{q(p-1)}{pq-1}} + \mathcal{F}_2^{\frac{q(p-1)}{pq-1}} \right) \right]. \quad (6.9)$$

Now, if we take the scaled variables $\tau = R^{-2}t$, $\xi = R^{-1}x$ and use the fact that $(-\Delta_x)\varphi_1(x/R) = R^{-2}\lambda_1\varphi_1(x/R)$, we arrive at

$$CR^{-2\alpha} \int_{|\xi| \leq 2} u_0(R\xi)\varphi_1(\xi) d\xi \leq \bar{C}_1(R) \int_{|\xi| \leq 2} \varphi_1(\xi) d\xi, \quad (6.10)$$

and

$$CR^{-2\beta} \int_{|\xi| \leq 2} v_0(R\xi)\varphi_1(\xi) d\xi \leq \bar{C}_2(R) \int_{|\xi| \leq 2} \varphi_1(\xi) d\xi, \quad (6.11)$$

where

$$\bar{C}_1(R) := CR^{-2\frac{(1+\beta p)+p(1+\alpha q)}{pq-1}} \quad \text{and} \quad \bar{C}_2(R) := CR^{-2\frac{(1+\alpha q)+q(1+\beta p)}{pq-1}}.$$

If we arrange the variable R in the right-hand side of (6.10)-(6.11), we get

$$\begin{aligned} \int_{|\xi| \leq 2} u_0(R\xi)\varphi_1(\xi) &\leq \bar{C}_3(R) \int_{|\xi| \leq 2} \varphi_1(\xi) = \bar{C}_3(R) \int_{|\xi| \leq 2} |R\xi|^{2\alpha_1} |R\xi|^{-2\alpha_1} \varphi_1(\xi) \\ &\leq C \int_{|\xi| \leq 2} |R\xi|^{-2\alpha_1} \varphi_1(\xi), \end{aligned} \quad (6.12)$$

and

$$\begin{aligned} \int_{|\xi| \leq 2} v_0(R\xi)\varphi_1(\xi) &\leq \bar{C}_4(R) \int_{|\xi| \leq 2} \varphi_1(\xi) = \bar{C}_4(R) \int_{|\xi| \leq 2} |R\xi|^{2\alpha_2} |R\xi|^{-2\alpha_2} \varphi_1(\xi) \\ &\leq C \int_{|\xi| \leq 2} |R\xi|^{-2\alpha_2} \varphi_1(\xi), \end{aligned} \quad (6.13)$$

where $\bar{C}_3(R) := CR^{2\alpha}\bar{C}_1(R) = CR^{-2\alpha_1}$ and $\bar{C}_4(R) := CR^{2\beta}\bar{C}_2(R) = CR^{-2\alpha_2}$. Using the estimate

$$\begin{aligned} \inf_{|\xi| > 1} (u_0(R\xi)|R\xi|^{2\alpha_1}) \int_{|\xi| \leq 2} |R\xi|^{-2\alpha_1} \varphi_1(\xi) &\leq \int_{1 < |\xi| \leq 2} u_0(R\xi)\varphi_1(\xi) \\ &\leq \int_{|\xi| \leq 2} u_0(R\xi)\varphi_1(\xi) \end{aligned}$$

and

$$\begin{aligned} \inf_{|\xi| > 1} (v_0(R\xi)|R\xi|^{2\alpha_2}) \int_{|\xi| \leq 2} |R\xi|^{-2\alpha_2} \varphi_1(\xi) &\leq \int_{1 < |\xi| \leq 2} v_0(R\xi)\varphi_1(\xi) \\ &\leq \int_{|\xi| \leq 2} v_0(R\xi)\varphi_1(\xi) \end{aligned}$$

in the right-hand side of (6.12) and (6.13), we conclude, after dividing by $\int_{|\xi| \leq 2} |R\xi|^{-2\alpha_1} \varphi_1(\xi)$ and $\int_{|\xi| \leq 2} |R\xi|^{-2\alpha_2} \varphi_1(\xi)$, that

$$\inf_{|\xi| > 1} (u_0(R\xi)|R\xi|^{2\alpha_1}) \leq C, \quad (6.14)$$

and

$$\inf_{|\xi|>1} (v_0(R\xi)|R\xi|^{2\alpha_2}) \leq C. \quad (6.15)$$

Passing to the limit in (6.14) – (6.15), as $R \rightarrow \infty$, and taking account of the continuity of u_0 and v_0 , we obtain

$$\lim_{|x| \rightarrow \infty} (u_0(x)|x|^{2\alpha_1}) = \liminf_{|x| \rightarrow \infty} (u_0(x)|x|^{2\alpha_1}) \leq C,$$

and

$$\lim_{|x| \rightarrow \infty} (v_0(x)|x|^{2\alpha_2}) = \liminf_{|x| \rightarrow \infty} (v_0(x)|x|^{2\alpha_2}) \leq C.$$

□

Corollary 1 (*sufficient conditions for the nonexistence of global solution*)

Let $u_0, v_0 \in C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, $u_0, v_0 \geq 0$ and $p, q > 1$. If

$$\lim_{|x| \rightarrow \infty} (u_0(x)|x|^{2\alpha_1}) = \lim_{|x| \rightarrow \infty} (v_0(x)|x|^{2\alpha_2}) = +\infty,$$

then the system (1.1) – (1.2) cannot have a global solution. □

Finally, a necessary conditions are given for local existence. We obtain a similar estimate of T founded in the proof of Theorem 1, as $|x|$ goes to infinity.

Theorem 5 (*Necessary conditions for local existence*)

Let $u_0, v_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, $u_0, v_0 \geq 0$, and $p, q > 1$. If (u, v) is a local solution to system (1.1) – (1.2) on $[0, T]$ where $0 < T < \infty$, then we have the estimates

$$\lim_{|x| \rightarrow \infty} u_0(x) \leq C T^{-\alpha_1} \quad \text{and} \quad \lim_{|x| \rightarrow \infty} v_0(x) \leq C T^{-\alpha_2}, \quad (6.16)$$

for some positive constant $C > 0$. Note that, if $A := \lim_{|x| \rightarrow \infty} u_0(x)$ and $B := \lim_{|x| \rightarrow \infty} v_0(x)$ then we obtain, a similar estimate founded in (3.6),

$$\begin{cases} C^{-(pq-1)} T^{(2-\gamma)+(2-\delta)p} A^{pq-1} \leq 1, \\ C^{-(pq-1)} T^{(2-\delta)+(2-\gamma)q} B^{pq-1} \leq 1. \end{cases}$$

Proof Take here, for $R > 0$ sufficiently large, $\varphi(x, t) := \varphi_1(x/R)\varphi_2(t)$ where $\varphi_2(t) := (1 - t/T)_+^\ell$, instead of the one chosen in Theorem 4. Then, as (6.8) – (6.9), we obtain, with $\Sigma_2 := \{(x, t) \in \mathbb{R}^N \times [0, \infty); |x| \leq 2R, t \leq T\}$,

$$CT^{-\alpha} \int_{|x| \leq 2R} u_0(x)\varphi_1(x/R) dx \leq \left[\left(\mathcal{G}_2^{\frac{p-1}{pq-1}} + \mathcal{H}_2^{\frac{p-1}{pq-1}} \right) \left(\mathcal{G}_1^{\frac{p(q-1)}{pq-1}} + \mathcal{H}_1^{\frac{p(q-1)}{pq-1}} \right) \right], \quad (6.17)$$

and

$$CT^{-\beta} \int_{|x| \leq 2R} v_0(x)\varphi_1(x/R) dx \leq \left[\left(\mathcal{G}_1^{\frac{q-1}{pq-1}} + \mathcal{H}_1^{\frac{q-1}{pq-1}} \right) \left(\mathcal{G}_2^{\frac{q(p-1)}{pq-1}} + \mathcal{H}_2^{\frac{q(p-1)}{pq-1}} \right) \right], \quad (6.18)$$

where

$$\begin{aligned}\mathcal{G}_1 &:= C \left(\int_{\Sigma_2} \varphi_1 \varphi_2^{-\frac{1}{q-1}} \left| D_{t|T}^{1+\alpha} \varphi_2 \right|^{\tilde{q}} dx dt \right)^{1/\tilde{q}}, \\ \mathcal{G}_2 &:= C \left(\int_{\Sigma_2} \varphi_1 \varphi_2^{-\frac{1}{p-1}} \left| D_{t|T}^{1+\beta} \varphi_2 \right|^{\tilde{p}} dx dt \right)^{1/\tilde{p}}, \\ \mathcal{H}_1 &:= C \left(\int_{\Sigma_2} (\varphi_1 \varphi_2)^{-\frac{1}{q-1}} \left| (-\Delta_x) \varphi_1 D_{t|T}^\alpha \varphi_2 \right|^{\tilde{q}} dx dt \right)^{1/\tilde{q}},\end{aligned}$$

and

$$\mathcal{H}_2 := C \left(\int_{\Sigma_2} (\varphi_1 \varphi_2)^{-\frac{1}{p-1}} \left| (-\Delta_x) \varphi_1 D_{t|T}^\beta \varphi_2 \right|^{\tilde{p}} dx dt \right)^{1/\tilde{p}},$$

with $\alpha := 1 - \gamma$, $\beta := 1 - \delta$, $\tilde{p} := p/(p-1)$ and $\tilde{q} := q/(q-1)$. If we take the scaled variables $\tau = T^{-1}t$, $\xi = R^{-1}x$, taking into account the fact that $(-\Delta_x)\varphi_1(x/R) = R^{-2}\lambda_1\varphi_1(x/R)$, then (6.17) – (6.18) implies

$$CT^{-\alpha} \int_{|\xi| \leq 2} u_0(R\xi) \varphi_1(\xi) d\xi \leq \bar{C}_3(R, T) \int_{|\xi| \leq 2} \varphi_1(\xi) d\xi, \quad (6.19)$$

and

$$CT^{-\beta} \int_{|\xi| \leq 2} v_0(R\xi) \varphi_1(\xi) d\xi \leq \bar{C}_4(R, T) \int_{|\xi| \leq 2} \varphi_1(\xi) d\xi, \quad (6.20)$$

where

$$\bar{C}_3(R, T) := C \left(T^{-\frac{1+\beta p}{pq-1}} + T^{-\frac{1-p(1-\beta)}{pq-1}} R^{-\frac{2p}{pq-1}} \right) \left(T^{-\frac{(1+\alpha q)p}{pq-1}} + T^{-\frac{(1-q(1-\alpha))p}{pq-1}} R^{-\frac{2pq}{pq-1}} \right),$$

and

$$\bar{C}_4(R, T) := C \left(T^{-\frac{1+\alpha q}{pq-1}} + T^{-\frac{1-q(1-\alpha)}{pq-1}} R^{-\frac{2q}{pq-1}} \right) \left(T^{-\frac{(1+\beta p)q}{pq-1}} + T^{-\frac{(1-p(1-\beta))q}{pq-1}} R^{-\frac{2pq}{pq-1}} \right).$$

Thus

$$\int_{|\xi| \leq 2} u_0(R\xi) \varphi_1(\xi) d\xi \leq \bar{C}_5(R, T) \int_{|\xi| \leq 2} \varphi_1(\xi) d\xi, \quad (6.21)$$

and

$$\int_{|\xi| \leq 2} v_0(R\xi) \varphi_1(\xi) d\xi \leq \bar{C}_6(R, T) \int_{|\xi| \leq 2} \varphi_1(\xi) d\xi, \quad (6.22)$$

where

$$\bar{C}_5(R, T) := CT^\alpha \bar{C}_3(R, T) \quad \text{and} \quad \bar{C}_6(R, T) := CT^\beta \bar{C}_4(R, T).$$

Using the estimate

$$\begin{aligned}\inf_{|\xi| > 1} (u_0(R\xi)) \int_{|\xi| \leq 2} \varphi_1(\xi) d\xi &\leq \int_{1 < |\xi| \leq 2} u_0(R\xi) \varphi_1(\xi) d\xi \\ &\leq \int_{|\xi| \leq 2} u_0(R\xi) \varphi_1(\xi) d\xi\end{aligned}$$

(resp.

$$\begin{aligned} \inf_{|\xi|>1} (v_0(R\xi)) \int_{|\xi|\leq 2} \varphi_1(\xi) d\xi &\leq \int_{1<|\xi|\leq 2} v_0(R\xi)\varphi_1(\xi) d\xi \\ &\leq \int_{|\xi|\leq 2} v_0(R\xi)\varphi_1(\xi) d\xi \end{aligned}$$

in the left-hand side of (6.21) (resp. (6.22)), we conclude, after dividing by the term $\int_{|\xi|\leq 2} u_0(R\xi)\varphi_1(\xi) d\xi$ (resp. $\int_{|\xi|\leq 2} v_0(R\xi)\varphi_1(\xi) d\xi$), that

$$\inf_{|\xi|>1} u_0(R\xi) \leq \overline{C}_5(R, T), \quad (6.23)$$

and

$$\inf_{|\xi|>1} v_0(R\xi) \leq \overline{C}_6(R, T). \quad (6.24)$$

Passing to the limit in (6.23) – (6.24), as $R \rightarrow \infty$, into account the continuity of u_0 and v_0 , we arrive at

$$\lim_{|x|\rightarrow\infty} u_0(x) = \liminf_{|x|\rightarrow\infty} u_0(x) \leq C T^{-\alpha_1},$$

and

$$\lim_{|x|\rightarrow\infty} v_0(x) = \liminf_{|x|\rightarrow\infty} v_0(x) \leq C T^{-\alpha_2}.$$

□

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