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AXISYMMETRIC FORMULATION FOR BOUNDARY INTEGRAL EQUATION METHODS IN SCALAR POTENTIAL PROBLEMS

by L. KRAHENBUHL and A. NICOLAS

Abstract : Axisymmetric geometries often appear in electromagnetic device studies. The authors present an original formulation for Boundary Integral Equation methods in scalar potential problems. This technique requires only 2D boundary in the r-z plane and evaluation of the equations only on those boundaries.

INTRODUCTION

Scalar potential problems are described by Laplace's equation :

$$\Delta\psi = 0 \tag{1}$$

in the general 3D space.

When the geometry is an axisymmetric there are two possibilities :

- use cylindrical coordinates, express equation (1) in this set of coordinates and note that each quantity is invariant in the azimuthal direction
- Laplace equation becomes :

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial\psi}{\partial r} = 0 \tag{2}$$

Boundary Integral Equations have to be developed with operator :

$$\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r}$$

This gives complicated expressions.

- the second method we propose to develop is to express the BIE in 3D space and to integrate all the invariant quantities analytically before solving the equations.

AXISYMMETRIC FORMULATION

In a 3D space Laplace equation is transformed into BIE equation :

$$c.\psi = \iint_{\partial R} \left(\psi \cdot \frac{\partial G}{\partial n} - \epsilon \cdot \frac{\psi}{\mu_{1i}} \cdot G \right) . ds - \iint_{\partial R} H_{0n} \cdot G . ds \tag{3}$$

as shown in previous publications [1].

When ψ , ψ and H_{0n} are constant along one part of the boundary ∂R , partial integration of functions G and $\partial G/\partial n$ along that direction can be made. As an example it can be shown that partial integration of 3D Green's function

$$G = \frac{1}{4\pi r} \tag{4}$$

over a straight line gives 2D Green's function :

$$G_{2D} = - \frac{1}{2\pi} \cdot \log r \tag{5}$$

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In axisymmetric geometry physical quantities keep constant values on circular lines : Green's function G and derivative function $\partial G/\partial n$ have to be integrated on circles.

By this way a 2D Boundary Integral Equation can be generated in the r-z plane, similar to those in x-y plane [4].

$$c.\psi = \int_{\partial R} \left(\psi \cdot G'_{ax} - \epsilon \cdot \frac{\psi}{\mu_{1i}} \cdot G_{ax} \right) . dl - \int_{\partial R} H_{0n} \cdot G_{ax} . dl \tag{6}$$

It can be noticed that G_{ax} gives the potential of a uniform charge distribution on a circle. Classical development [2] can be applied to obtain function G_{ax} :

$$G_{ax} = \oint G . dl = \int_{\text{circle}} \frac{R \cdot d\theta}{4\pi \cdot \sqrt{D^2 - 2 \cdot R \cdot \rho \cdot (1 + \cos\theta)}} = \frac{R}{\pi \cdot D} K(k^2) \tag{7}$$

$$k = 2 \cdot \sqrt{R \cdot \rho} / D \tag{8}$$

where :

- K : complete elliptic integral of the first kind [3]
- ρ , R , D : as shown on Fig. 1

Function G'_{ax} is obtained by a similar development :

$$G'_{ax} = \oint \frac{\partial G}{\partial n} . dl \tag{9}$$

$$= - \frac{\cos\alpha}{2\pi D} K(k^2) + \frac{1}{2\pi D} \cdot [D \cdot \cos\alpha - 2 \cdot R \cdot \cos(\alpha - \nu)] \cdot E(k^2)$$

with

- E : complete elliptic integral of the second kind.
- α , ν , D' : as shown on Fig. 1

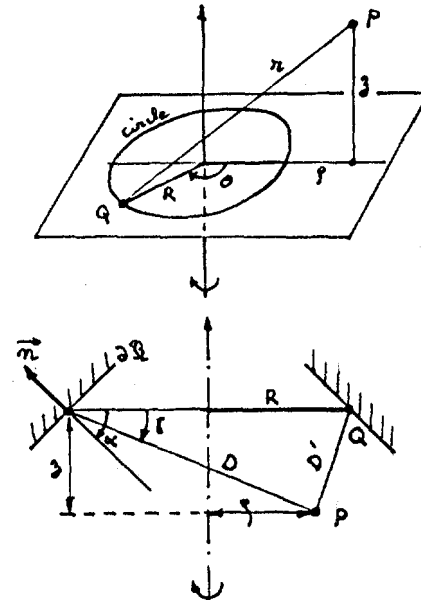


Figure 1

FLUX DENSITY CALCULATION

The induced magnetic field in a point P of the region A:

$$H_i = - \text{grad } \varphi \quad (10)$$

can be computed directly by integration of φ and ψ on the boundary δB :

$$\vec{H}_i = - \int_{\delta B} [\varphi \cdot \vec{\text{grad}} G'_{ax} - (\epsilon \frac{\phi}{\mu_r} + H_{on}) \cdot \vec{\text{grad}} G_{ax}] \cdot d\Omega \quad (11)$$

with

$$\frac{dG_{ax}}{d\rho} = \frac{R}{\pi D} \left(- \frac{R+\rho}{D^2} \cdot K + \frac{dK}{dk} \cdot \frac{dk}{d\rho} \right) \quad (12)$$

$$\frac{dG_{ax}}{dz} = \frac{R}{\pi D} \left(- \frac{z}{D^2} \cdot K + \frac{dK}{dk} \cdot \frac{dk}{dz} \right) \quad (13)$$

$$\frac{dG'_{ax}}{d\rho} = - \frac{\cos \alpha}{2 \cdot R} \cdot \frac{dG_{ax}}{d\rho} \quad (14)$$

$$+ \frac{D \cdot \cos \alpha - 2 \cdot R \cdot \cos(\alpha - \gamma)}{2 \cdot \pi \cdot D^{1/2}} \cdot \left(\frac{dE}{dk} \frac{dk}{d\rho} + \frac{2 \cdot (R-\rho)}{D^{1/2}} \cdot E \right)$$

$$+ \frac{1}{2 \cdot \pi \cdot D^{1/2}} \cdot \left[(\rho - R) \cdot \cos \alpha + \frac{2 \cdot R}{D} (R + \rho) \cdot \cos(\alpha - \gamma) \right] \cdot E$$

$$\frac{dG'_{ax}}{dz} = - \frac{\cos \alpha}{2 \cdot R} \cdot \frac{dG_{ax}}{dz} \quad (15)$$

$$+ \frac{D \cdot \cos \alpha - 2 \cdot R \cdot \cos(\alpha - \gamma)}{2 \cdot \pi \cdot D^{1/2}} \cdot \left(\frac{dE}{dk} \frac{dk}{dz} - \frac{2 \cdot z}{D^{1/2}} \cdot E \right)$$

$$+ \frac{1}{2 \cdot \pi \cdot D^{1/2}} \cdot \left[z \cdot \cos \alpha - 2 \cdot R \left(\sin \alpha - \frac{z}{D} \cdot \cos(\alpha - \gamma) \right) \right] \cdot E$$

$$\frac{dK}{dk} = \frac{1}{k} \cdot \left(\frac{E}{1-k^2} - K \right) \quad (16)$$

$$\frac{dE}{dk} = \frac{1}{k} \cdot (E - K) \quad (17)$$

$$\frac{dk}{dz} = - \frac{z \cdot k^3}{4 \cdot R \cdot \rho} \quad (18)$$

$$\frac{dk}{d\rho} = \frac{k}{2 \cdot \rho} - \frac{k^3}{4} \cdot \frac{R+\rho}{R \cdot \rho} \quad (19)$$

$$K = K(k^2) \quad E = E(k^2)$$

It must be noticed that these expressions depend on the same functions K and E as G_{ax} and G'_{ax} .

When the point P is on the boundary, a more simple expression can be used :

$$\vec{H}_i = - \left(\epsilon \frac{\phi}{\mu_r} + H_{on} \right) \cdot \vec{n} - \frac{\delta \phi}{\delta t} \cdot \vec{t} \quad (20)$$

an equivalent result is obtained in this way. However computing time is less important in this case.

NUMERICAL DEVELOPMENT

Numerical calculation of integrals K and E can be done in several ways :

- numerical integration ;
- asymptotic development [3] ;
- tabulation and interpolation.

As these integrals have to be computed a large number of times, the third solution appears to be the most convenient.

Function E is always regular, but K becomes singular when $r_{PQ} \rightarrow 0$. This singularity is similar to $\text{Log}(r)$ singularity, and Gaussian quadrature formulae with weight function $\text{Log}(t)$ are well matched for evaluation of the integral (6) of function K.

This development is now applied in PHIAx program, using classical techniques [1], [4].

NUMERICAL RESULT

The analytical solution is known for a ferromagnetic sphere in a constant field [5]. We have chosen this example to test the method.

The sphere is discretised into 4 finite elements of second order (fig. 2). The accuracy of the solution on the boundary is better than 0.4 % on the potential and 1. % on the normal flux density (fig. 3).

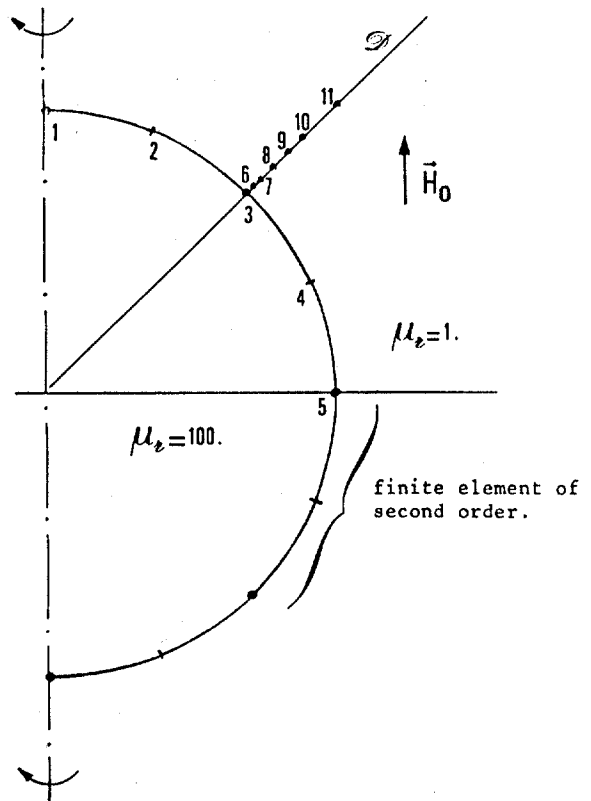


Fig. 2 : Finite element discretisation of the sphere.

Point	Potential			Flux density		
	Phiax progr.	Analit. value		Phiax progr.	Analit. value	
1	3.8808	3.8823	.4%	5.8881	5.8823	1.1%
2	3.5859	3.5868	.3%	5.4369	5.4346	.4%
3	2.7441	2.7452	.3%	4.1628	4.1595	.6%
4	1.4853	1.4857	.1%	2.2517	2.2511	.1%
5	.1E-04	0.	.04%	.4E-05	0.	

Fig.3 : Solution on the boundary.

The values of potential and flux density for 6 external points are also presented with comparison to analytical values (fig. 4). The points 6 and 7 are very near the boundary and the integration errors increase.

The same integrals give the angular factor "C" ($|4|$; eq. (6)). So this coefficient becomes a "goodness factor" of the method and allows the definition of a forbidden area around the boundary where potential and magnetic field cannot be computed.

Point	Potential	
	Phi _{ax} progr.	Analit. value
3	2.7441	2.7452
6	2.5948	2.6114
7	2.4736	2.4402
8	2.1434	2.1447
9	1.8999	1.8998
10	1.6938	1.6946
11	1.3715	1.3726

Point	H _ρ		H _z		Rel. error on 'c' see eq. (6)
	Phi _{ax} progr.	Analit. value	Phi _{ax} progr.	Analit. value	
3	2.9138	2.9118	0.9731	0.9706	
6	-2.0787	2.7014	-3.9932	0.9005	7.5% X
7	2.9622	2.4402	1.3445	0.8134	11.% X
8	2.0131	2.0107	0.6744	0.6702	1.%
9	1.6701	1.6763	0.5522	0.5588	} <1% }
10	1.4101	1.4122	0.4691	0.4707	
11	1.0284	1.0295	0.3427	0.3431	

Fig. 4 : Potential and flux density on the straight line \mathcal{D} .

X points in the forbidden area.

CONCLUSION

The method we developed and exposed is the complementary to the BIE programs already existing in 2D and 3D. It solves a large set of problems at a low computing cost and needs a very short geometry description time. With PHI2D, PHI3D and now this PHIAX packages it is possible to solve with the BIE method all magnetostatic or electrostatic problems.

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