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A CONSTRUCTION OF THE ROUGH PATH ABOVE FRACTIONAL BROWNIAN MOTION USING VOLTERRA'S REPRESENTATION

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Abstract. This note is devoted to construct a rough path above a multidimensional fractional Brownian motion B with any Hurst parameter $H \in (0,1)$, by means of its representation as a Volterra Gaussian process. This approach yields some algebraic and computational simplifications with respect to [18], where the construction of a rough path over B was first introduced.

1. Introduction

Rough paths analysis is a theory introduced by Terry Lyons in the pioneering paper [11] which aims to solve differential equations driven by functions with finite p-variation with $p > 1$, or by Hölder continuous functions of order $\gamma \in (0,1)$. One possible shortcut to the rough path theory is the following summary (see [7, 8, 9, 12] for a complete construction). Given a γ-Hölder d-dimensional process $X = (X(1), \dots, X(d))$ defined on an arbitrary interval $[0,T]$, assume that one can define some iterated integrals of the form

$$\mathbf{X}^{\mathbf{n}}_{st}(i_1, \dots, i_n) = \int_{s \le u_1 < \cdots < u_n \le t} dX_{u_1}(i_1)\, dX_{u_2}(i_2) \cdots dX_{u_n}(i_n),$$

for $0 \le s < t \le T$, $n \le \lfloor 1/\gamma \rfloor$ and $i_1, \dots, i_n \in \{1, \dots, d\}$. As long as X is a nonsmooth function, the integral above cannot be defined rigorously in the Riemann sense (and not even in the Riemann-Stieltjes sense if $\gamma \le 1/2$). However, it is reasonable to assume that some elements $\mathbf{X}^{\mathbf{n}}$ can be constructed, sharing the following three properties with usual iterated integrals (here and in the sequel, we denote by $\mathcal{S}_{k,T} = \{(u_1, \dots, u_n) : 0 \le u_1 < \cdots < u_n \le T\}$ the k^{th} order simplex on $[0,T]$):

(1) *Regularity:* Each component of $\mathbf{X}^{\mathbf{n}}$ is $n\gamma$-Hölder continuous (in the sense of the Hölder norm introduced in (9)) for all $n \le \lfloor 1/\gamma \rfloor$, and $\mathbf{X}^{\mathbf{1}}_{st} = X_t - X_s$.

(2) *Multiplicativity*: Letting $(\delta \mathbf{X}^{\mathbf{n}})_{sut} := \mathbf{X}^{\mathbf{n}}_{st} - \mathbf{X}^{\mathbf{n}}_{su} - \mathbf{X}^{\mathbf{n}}_{ut}$ for $(s,u,t) \in \mathcal{S}_{3,T}$, one requires

$$(\delta \mathbf{X}^{\mathbf{n}})_{sut}(i_1, \dots, i_n) = \sum_{n_1=1}^{n-1} \mathbf{X}^{\mathbf{n_1}}_{su}(i_1, \dots, i_{n_1}) \mathbf{X}^{\mathbf{n-n_1}}_{ut}(i_{n_1+1}, \dots, i_n). \tag{1}$$

(3) *Geometricity:* For any n, m such that $n + m \le \lfloor 1/\gamma \rfloor$and $(s,t) \in \mathcal{S}_{2,T}$, we have:

$$\mathbf{X}^{\mathbf{n}}_{st}(i_1, \dots, i_n)\, \mathbf{X}^{\mathbf{m}}_{st}(j_1, \dots, j_m) = \sum_{\bar{k} \in \mathrm{Sh}(\bar{\imath}, \bar{\jmath})} \mathbf{X}^{\mathbf{n+m}}_{st}(k_1, \dots, k_{n+m}), \tag{2}$$

Date: September 7, 2009.

2000 *Mathematics Subject Classification.* 60H05, 60H07, 60G15.

Key words and phrases. Rough paths theory; Fractional Brownian motion; Multiple stochastic integrals.

where, for two tuples $\bar{\imath}, \bar{\jmath}$, $\Sigma_{(\bar{\imath},\bar{\jmath})}$ stands for the set of permutations of the indices contained in $(\bar{\imath}, \bar{\jmath})$, and $\mathrm{Sh}(\bar{\imath}, \bar{\jmath})$ is a subset of $\Sigma_{(\bar{\imath},\bar{\jmath})}$ defined by:

$$\mathrm{Sh}(\bar{\imath}, \bar{\jmath}) = \left\{\sigma \in \Sigma_{(\bar{\imath},\bar{\jmath})};\ \sigma \text{ does not change the orderings of } \bar{\imath} \text{ and } \bar{\jmath}\right\}.$$

We shall call the family $\{\mathbf{X}^{\mathbf{n}};\ n \leq \lfloor 1/\gamma \rfloor\}$ a rough path over X (it is also referred to as the signature of X in [7]).

Once a rough path over X is defined, the theory described in [7, 8, 12] can be seen as a procedure which allows us to construct, starting from the family $\{\mathbf{X}^{\mathbf{n}};\ n \leq \lfloor 1/\gamma \rfloor\}$, the complete stack $\{\mathbf{X}^{\mathbf{n}};\ n \geq 1\}$. Furthermore, with the rough path over X in hand, one can also define rigorously and solve differential equations driven by X.

The above general framework leads thus naturally to the question of a rough path construction for standard stochastic processes. The first example one may have in mind concerning this issue is arguably the case of a d-dimensional fractional Brownian motion (fBm) $B = (B(1), \dots, B(d))$ with Hurst parameter $H \in (0, 1)$. This is a Gaussian process with zero mean whose components are independent and with covariance function given by

$$\mathbf{E}\left(B_t(i)B_s(i)\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \qquad s, t \in \mathbb{R}_+.$$

For $H = \frac{1}{2}$ this is just the usual Brownian motion. For any $H \in (0, 1)$, the variance of the increments of B is then given by

$$\mathbf{E}\left[\left(B_t(i) - B_s(i)\right)^2\right] = (t-s)^{2H}, \qquad (s, t) \in \mathcal{S}_{2,T}, \quad i = 1, \dots, d,$$

and this implies that almost surely the trajectories of the fBm are γ-Hölder continuous for any $\gamma < H$, which justifies the fact that the fBm is the canonical example for a rough path construction.

The first successful rough path analysis for B has been implemented in [5] by means of a linearization of the fBm path, and it leads to the construction of a family $\{\mathbf{B}^{\mathbf{1}}, \mathbf{B}^{\mathbf{2}}, \mathbf{B}^{\mathbf{3}}\}$ satisfying (1) and (2), for any $H > 1/4$ (see also [6] for a generalized framework). Some other constructions can be found in [13, 15] by means of stochastic analysis methods, and in [17] thanks to complex analysis tools. In all those cases, the barrier $H > 1/4$ remains, and it has long been believed that this was a natural boundary, in terms of regularity, for an accurate rough path construction.

Let us describe now several recent attempts to go beyond the threshold $H = 1/4$. First, the complex analysis methods used in [16] allowed the authors to build a rough path above a process Γ called analytic fBm, which is a complex-valued process whose real and imaginary parts are fBm, for any value of $H \in (0, 1)$. It should be mentioned however that $\Re\Gamma$ and $\Im\Gamma$ are not independent, and thus the arguments in [16] cannot be extrapolated to the real-valued fBm. Then, a series of brilliant ideas developed in [18, 19] lead to the rough path construction in the real-valued case. We will try now to summarize briefly, in very vague terms, this series of ideas (see Section 3 for a more detailed didactic explanation):

(i) Consider a smooth approximation B^ε of the fBm B, and the corresponding approximation $\mathbf{B}^{\mathbf{n},\varepsilon}$ of $\mathbf{B}^{\mathbf{n}}$. Clearly $\mathbf{B}^{\mathbf{n},\varepsilon}$ satisfies relation (1), but may diverge as $\varepsilon \to 0$ whenever $H < 1/4$. Then, one can decompose $\mathbf{B}^{\mathbf{n},\varepsilon}_{st}$ as $\mathbf{B}^{\mathbf{n},\varepsilon}_{st} = \mathbf{A}^{\mathbf{n},\varepsilon}_{st} + \mathbf{C}^{\mathbf{n},\varepsilon}_{st}$, where $\mathbf{C}^{\mathbf{n},\varepsilon}$ is the increment of a function f, namely $\mathbf{C}^{\mathbf{n},\varepsilon}_{st} = f_t - f_s$, and $\mathbf{A}^{\mathbf{n},\varepsilon}$ is obtained as a boundary term in the integrals defining $\mathbf{B}^{\mathbf{n},\varepsilon}$. As explained in Section 3, a typical example of such a decomposition is given (for $n = 2$) by $\mathbf{A}^{\mathbf{2},\varepsilon}_{st} = -B^\varepsilon_s \otimes \delta B^\varepsilon_{st}$ and $\mathbf{C}^{\mathbf{2},\varepsilon}_{st} = \int_s^t B^\varepsilon_u \otimes dB^\varepsilon_u$,

and in this case $f_t = \int_0^t B_u^\varepsilon \otimes dB_u^\varepsilon$. Then it can be easily checked, thanks to the relation $\mathbf{C}_{st}^{\mathbf{n},\varepsilon} = f_t - f_s$, that $\mathbf{C}_{st}^{\mathbf{n},\varepsilon} - \mathbf{C}_{su}^{\mathbf{n},\varepsilon} - \mathbf{C}_{ut}^{\mathbf{n},\varepsilon} = 0$ for any $(s,u,t) \in \mathcal{S}_{3,T}$. This means that $\mathbf{C}^{\mathbf{n},\varepsilon}$ does not affect the multiplicative property (1) of $\mathbf{B}^{\mathbf{n},\varepsilon}$. On the other hand, the boundary term $\mathbf{A}_{st}^{\mathbf{n},\varepsilon}$ is usually easily seen to be convergent as $\varepsilon \to 0$ to some limit $\mathbf{A}_{st}^{\mathbf{n}}$. Then, the limit $\mathbf{A}_{st}^{\mathbf{n}}$ should fulfill the desired multiplicative property, but it does not exhibit the desired Hölder regularity $k\gamma$ for any $\gamma < H$. It should also be noticed that $\mathbf{A}_{st}^{\mathbf{n},\varepsilon}$ is not the only natural increment sharing the multiplicative property with $\mathbf{B}^{\mathbf{n},\varepsilon}$. We refer to Section 3 for further details, but let us mention that another possibility for $n = 2$ is the boundary term $\delta X_{st}^\varepsilon \otimes X_t^\varepsilon$, which is easily seen to satisfy relation (1).

(ii) The essential point in Unterberger's method is then the following. Consider a series representation of the fBm $\mathbf{B} = \sum_k \mathbf{B}_k$. We can carry out the above program for each component $\mathbf{B}_k$ by choosing a particular boundary term. Then, it turns out that there is a choice of the boundary term for each component $\mathbf{B}_k$ such that their sum satisfies the desired Hölder and multiplicative properties. This idea has been successfully implemented in [18, 19] using an entire series representation, providing a construction of a rough path associated to B. However, this construction is rather long and intricate, first because the entire series representation is obtained by a deterministic change of variable involving the Cayley's transform and second because the changes in the order of integration in the multiple integrals are coded by admissible cuts in some trees associated to multiple integrals. This language, well-known by theoretical Physicits (see e.g. [4], or [9] in the rough paths context), may sound however difficult to the noninitiated reader. Another construction involving the harmonizable representation of fBm is outlined in [20], but this method does not avoid the use of tree-based expansions.

The purpose of the current paper is to take up the program initiated in [18], and construct a rough path over B in a rather simple way, using the stochastic integral representation of the fBm as a Volterra Gaussian process. We know that (see [14, Proposition 5.1.3] for a justification) for $H < 1/2$, each component $B(i)$ of B can be written as

$$B_t(i) = \int_{\mathbb{R}} K(t,u)\, dW_t(i), \qquad t \geq 0, \tag{3}$$

where $W = (W(1), \ldots, W(d))$ is a d-dimensional Wiener process, and where the Volterra-type kernel K is defined on $\mathbb{R}_+ \times \mathbb{R}_+$ by

$$K(t,u) = c_H \Big[\left(\frac{u}{t}\right)^{\frac{1}{2}-H} (t-u)^{H-\frac{1}{2}} + \left(\frac{1}{2} - H\right) u^{\frac{1}{2}-H} \int_s^u v^{H-\frac{3}{2}} (v-s)^{H-\frac{1}{2}}\, dv \Big] \mathbf{1}_{\{0<u<t\}}, \tag{4}$$

with a strictly positive constant c_H, whose exact value is irrelevant for our purposes. Then, we show that the simple trick described at point (ii) above can be applied in a straightforward way using the Volterra representation, leading to a simple general formula for the multiple integrals $\mathbf{B}^{\mathbf{n}}$. To be more specific, let us describe the main result of this paper.

Theorem 1.1. *Let B be a d-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$, admitting representation (3). For $2 \leq n \leq \lfloor 1/H \rfloor$, any tuple $(i_1, \ldots, i_n)$*

of elements of $\{1,\dots,d\}$, $1\le j\le n$ *and* $(s,t)\in\mathcal{S}_{2,T}$, *set*

$$\hat{\mathbf{B}}^{\mathbf{n},j}_{st}(i_1,\dots,i_n)$$
$$=(-1)^{j-1}\int_{A^n_j}\prod_{l=1}^{j-1}K(s,u_l)\,[K(t,u_j)-K(s,u_j)]\prod_{l=j+1}^{n}K(t,u_l)\,dW_{u_1}(i_1)\cdots dW_{u_n}(i_n),\quad(5)$$

where the kernel K *is given by (4) and* A^n_j *is the subset of* $[s,t]^n$ *defined by*

$$A^n_j=\{(u_1,\dots,u_n)\in[0,t]^n;\,u_j=\min(u_1,\dots,u_n),\,u_1>\cdots>u_{j-1},\ \text{and}\ u_{j+1}<\cdots<u_n\}.$$

Notice that the multiple stochastic integral in (5) is understood in the Stratonovich sense, and is well-defined as a $L^2(\Omega)$ *random variable as long as* $n\le\lfloor 1/H\rfloor$. *Set also* $\mathbf{B}^{\mathbf{1}}_{st}(i)=B_t(i)-B_s(i)$, *and for* $2\le n\le\lfloor 1/H\rfloor$,

$$\mathbf{B}^{\mathbf{n}}_{st}(i_1,\dots,i_n)=\sum_{j=1}^{n}\hat{\mathbf{B}}^{\mathbf{n},j}_{st}(i_1,\dots,i_n).$$

Then the family $\{\mathbf{B}^{\mathbf{n}};\,1\le n\le\lfloor 1/H\rfloor\}$ *defines a rough path over* B, *in the sense that* $\mathbf{B}^{\mathbf{n}}$ *is almost surely* $n\gamma$*-Hölder continuous for any* $\gamma<H$, *and that it satisfies relations (1) and (2).*

As announced above, formula (5) defines in a compact and simple way the (substitute to) iterated integrals of B with respect to itself. Furthermore, this formula also yields a reasonably short way to estimate the moments of $\mathbf{B}^{\mathbf{n}}_{st}$, and thus its Hölder regularity. It should be mentioned however that our construction is not as general as the one proposed in [19], though it can be extended to a general class of Gaussian Volterra processes (see Remark 2.6).

In the case $1/4<H\le 1/2$, let us also say a word about the relationship between the processes $\mathbf{B}^{\mathbf{n}}$ we have produced and the ones constructed in the aforementioned references [5, 13, 17], which shall be denoted by $\mathbf{B}^{\mathbf{n},\mathrm{p}}$ (where p stands for pathwise). First of all, let us recall that only $n=2$ has to be considered for $1/3<H\le 1/2$, while $n=2,3$ corresponds to the rougher case $1/4<H\le 1/3$. Then an easy comparison can be established for $H=1/2$. Indeed, a slight extension of our construction also allows to define $\mathbf{B}^{\mathbf{2}}$ for the usual Brownian motion, and it is readily checked in this case that $\mathbf{B}^{\mathbf{2}}$ coincides with the usual Stratonovich Levy area. Whenever $1/4<H<1/2$, the situation is less clear: one the one hand, we know that $\delta\mathbf{B}^{\mathbf{n}}=\delta\mathbf{B}^{\mathbf{n},\mathrm{p}}$, and it can be seen from this relation that $\mathbf{B}^{\mathbf{n}}$ and $\mathbf{B}^{\mathbf{n},\mathrm{p}}$ only differ by the increment of a function $f^{\mathbf{n}}$. On the other hand, it is still an open question for us to clearly identify the function $f^{\mathbf{n}}$. We are thus left with two candidates $\mathbf{B}^{\mathbf{n}}$ and $\mathbf{B}^{\mathbf{n},\mathrm{p}}$ for the rough path construction, $\mathbf{B}^{\mathbf{n},\mathrm{p}}$ having the advantage of being produced as a limit taken on some smooth approximations of the fBm path.

Here is how our article is divided: some preliminary results, including algebraic integration vocabulary, some estimates on the kernel K and Itô-Stratonovich corrections, are given in Section 2. Then the basic ideas of the construction are implemented in Section 3 on second order iterated integrals. This section is thus intended as a didactic introduction to the construction, and could be enough for a first quick glimpse at the topic. Then we give all the details concerning the general iterated integral definition and prove Theorem 1.1 in Section 4.

2. Preliminaries

This section is first devoted to recall some notational conventions for a special subset (called set of increments) of functions of several variables. These conventions are taken from the algebraic integration theory as explained in [8, 10]. We will then recall some basic estimates on iterated Stratonovich integrals with respect to the Wiener process, which turn out to be useful for the remainder of the article.

2.1. **Some algebraic integration vocabulary.** The current section is not intended as an introduction to algebraic integration, which would be useless for our purposes. However, we shall use in the sequel some notation taken from this method of rough paths analysis, and we shall proceed to recall them now.

The algebraic integration setting is based on the notion of increment, together with an elementary operator δ acting on them. The notion of increment can be introduced in the following way: for an arbitrary real number $T > 0$, a vector space V, and an integer $k \geq 1$, we denote by $\mathcal{S}_{k,T}$ the k^{th} order simplex on $[0,T]$, and by $\mathcal{C}_k(V)$ the set of continuous functions $g : \mathcal{S}_{k,T} \to V$ such that $g_{t_1 \cdots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i \leq k-1$. Such a function will be called a $(k-1)$-*increment*, and we will set $\mathcal{C}_*(V) = \cup_{k \geq 1} \mathcal{C}_k(V)$. The operator δ alluded to above can be seen as an operator acting on k-increments, and is defined as follows on $\mathcal{C}_k(V)$:

$$\delta : \mathcal{C}_k(V) \to \mathcal{C}_{k+1}(V), \qquad (\delta g)_{t_1 \cdots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^i g_{t_1 \cdots \hat{t}_i \cdots t_{k+1}}, \tag{6}$$

where $\hat{t}_i$ means that this particular argument is omitted. Then a fundamental property of δ, which is easily verified, is that $\delta\delta = 0$, where $\delta\delta$ is considered as an operator from $\mathcal{C}_k(V)$ to $\mathcal{C}_{k+2}(V)$. We will denote $\mathcal{ZC}_k(V) = \mathcal{C}_k(V) \cap \mathrm{Ker}\delta$.

Some simple examples of actions of δ, which will be the ones we will really use throughout the paper, are obtained by letting $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$. Then, for any $(s,u,t) \in \mathcal{S}_{3,T}$, we have

$$(\delta g)_{st} = g_t - g_s, \quad \text{and} \quad (\delta h)_{sut} = h_{st} - h_{su} - h_{ut}, \tag{7}$$

and in this particular case, it can be trivially checked that for any $g \in \mathcal{C}_1$, one has $\delta\delta g = 0$. Conversely, any $h \in \mathcal{ZC}_2$ can be written as $h = \delta g$ for an element $g \in \mathcal{C}_1$. In the sequel of the paper, we shall write for two elements $h^1, h^2 \in \mathcal{C}_2$

$$h^1 \overset{\mathcal{ZC}_2}{=} h^2, \quad \text{iff} \quad h^1 = h^2 + z \quad \text{with} \quad z \in \mathcal{ZC}_2. \tag{8}$$

Otherwise stated, $h^1 \overset{\mathcal{ZC}_2}{=} h^2$ iff $\delta h^1 = \delta h^2$.

Notice that our future discussions will rely on some analytical assumptions made on elements of $\mathcal{C}_k(V)$. Suppose V is equipped with a norm $|\cdot|$. We measure the size of the increments by Hölder norms defined in the following way: for $g \in \mathcal{C}_2(V)$ let

$$\|g\|_\mu \equiv \sup_{(s,t) \in \mathcal{S}_{2,T}} \frac{|g_{st}|}{|t-s|^\mu}, \quad \text{and} \quad \mathcal{C}_2^\mu(V) = \left\{ g \in \mathcal{C}_2(V);\, \|g\|_\mu < \infty \right\}. \tag{9}$$

With this notation, we also set $\mathcal{C}_1^\mu(V) = \{ f \in \mathcal{C}_1(V);\, \|\delta f\|_\mu < \infty \}$ (notice that the sup norm of f is not taken into account in this definition). In the same way, for $h \in \mathcal{C}_3(V)$,

set

$$\|h\|_{\gamma,\rho} = \sup_{(s,u,t)\in\mathcal{S}_{3,T}} \frac{|h_{sut}|}{|u-s|^{\gamma}|t-u|^{\rho}}, \tag{10}$$

$$\|h\|_{\mu} \equiv \inf\left\{\sum_i \|h_i\|_{\rho_i,\mu-\rho_i};\, h=\sum_i h_i,\, 0<\rho_i<\mu\right\}, \tag{11}$$

where the last infimum is taken over all sequences $\{h_i \in \mathcal{C}_3(V)\}$ such that $h=\sum_i h_i$ and for all choices of the numbers $\rho_i \in (0,\mu)$. Then $\|\cdot\|_\mu$ is easily seen to be a norm on $\mathcal{C}_3(V)$, and we set

$$\mathcal{C}_3^{\mu}(V) := \{h\in\mathcal{C}_3(V);\, \|h\|_\mu<\infty\}.$$

In order to avoid ambiguities, we shall denote by $\mathcal{N}[f;\mathcal{C}_j^{\mu}(V)]$ the μ-Hölder norm (or semi-norm) on the space $\mathcal{C}_j(V)$, for $j=1,2,3$.

In order to define our increments $\mathbf{B}_{st}^{\mathbf{n}}$ for all s,t almost surely, we will also need the following definition, which has already been introduced in [16, Section 1.3]: for $j\geq 1$ and $\beta>0$, let $\mathcal{C}_j^{\mathrm{m},\beta}(V)$, where m stands for *multiparametric*, be the subspace of $\mathcal{C}_j(V)$ induced by the semi-norm:

$$\mathcal{N}[h;\mathcal{C}_j^{\mathrm{m},\beta}(V)] = \sup\left\{\frac{|h_{s_1+\varepsilon,\dots,s_j+\varepsilon}-h_{s_1,\dots,s_j}|}{\varepsilon^{\beta}};\, \varepsilon\in[0,1],\ s_1,\dots,s_j\in[0,T]\right\}. \tag{12}$$

Let us also mention that when $V=\mathbb{R}$, we will simply denote the spaces $\mathcal{C}_j(V), \mathcal{C}_j^{\mu}(V)$ and $\mathcal{C}_j^{\mathrm{m},\beta}(V)$ by $\mathcal{C}_j, \mathcal{C}_j^{\mu}, \mathcal{C}_j^{\mathrm{m},\beta}$ respectively.

Finally the lemma below, borrowed from [8, Lemma 4], will be an essential tool for the analysis of Hölder type regularity of our increments:

Lemma 2.1. *Let $\kappa>0$ and $p\geq 1$. Let $R\in\mathcal{C}_2(\mathbb{R}^l)$, with $\delta R\in\mathcal{C}_3^{\kappa}(\mathbb{R}^l)$ in the sense given by (11). If*

$$\int_{\mathcal{S}_{2,T}} \frac{|R_{uv}|^{2p}}{|u-v|^{2\kappa p+4}}\,du\,dv<\infty,$$

then $R\in\mathcal{C}_2^{\kappa}(\mathbb{R}^l)$. In particular, there exists a constant $C_{\kappa,p,l}>0$, such that

$$\mathcal{N}\big[R;\mathcal{C}_2^{\kappa}(\mathbb{R}^l)\big] \leq C_{\kappa,p,l}\left(\int_{\mathcal{S}_{2,T}} \frac{|R_{uv}|^{2p}}{|u-v|^{2\kappa p+4}}\,du\,dv\right)^{\frac{1}{2p}} + C_{\kappa,p,l}\,\mathcal{N}\big[\delta R;\mathcal{C}_3^{\kappa}(\mathbb{R}^l)\big].$$

2.2. **Analytic bounds on the fractional Brownian kernel.** We gather in this section some technical bounds on the kernel K involved in the Volterra representation of B, for which we use the following convention (valid until the end of the article): for two positive quantities a and b, we write $a\lesssim b$ whenever there exists a universal constant C such that $a\leq C\,b$.

First, a classical bound on K is the following:

Lemma 2.2. *Let K be the fBm kernel defined by (4). Then for any $0<u<t$, one has*

$$K(t,u)\lesssim (t-u)^{H-\frac{1}{2}}+u^{H-\frac{1}{2}}, \quad \text{and} \quad \partial_t K(t,u)\lesssim \left(\frac{u}{t}\right)^{\frac{1}{2}-H}(t-u)^{H-\frac{3}{2}}. \tag{13}$$

The following simple integral estimate on K also turns out to be useful:

Lemma 2.3. *Let $0<v<t\leq T$. Then $\int_v^t K^2(t,w)\,dw\lesssim (t-v)^{2H}$.*

Proof. Invoking the bound (13) on K, we have

$$\int_v^t K^2(t,w)\,dw \lesssim \int_v^t \left[(t-w)^{H-\frac{1}{2}} + w^{H-\frac{1}{2}}\right]^2 dw \\ \lesssim \int_v^t (t-w)^{2H-1}\,dw + \int_v^t w^{2H-1}\,dw \lesssim (t-v)^{2H} - \left[t^{2H} - v^{2H}\right].$$

Furthermore, since $a^\alpha - b^\alpha \leq (a-b)^\alpha$ for any $0 \leq b < a$ and $\alpha \in (0,1)$, we end up with $\int_v^t K^2(t,w)\,dw \lesssim (t-v)^{2H}$, which is our claim.

□

We shall also use a slightly more elaborated result on K:

Lemma 2.4. *Let $0 < s < t \leq T$, assume $H < 1/2$, and consider the quantity*

$$I_{st} = \int_0^t \left[K(t,u_1) - K(s,u_1)\right]^2 \left(\int_{u_1}^t K^2(t,u_2)\,du_2\right) du_1,$$

where we recall that we have used the convention $K(t,u) = K(t,u)\mathbf{1}_{[0,t)}(u)$. Then $|I_{st}| \lesssim |t-s|^{4H}$.

Proof. According to the fact that $K(t,u) = 0$ whenever $u \geq t$, we obtain the expression

$$I_{st} = \int_0^s \left[K(t,u_1) - K(s,u_1)\right]^2 \left(\int_{u_1}^t K^2(t,u_2)\,du_2\right) du_1 \\ + \int_s^t K^2(t,u_1) \left(\int_{u_1}^t K^2(t,u_2)\,du_2\right) du_1 := I_{st}^1 + I_{st}^2.$$

Let us bound now the first of those terms: thanks to Lemma 2.3, one can write $\int_{u_1}^t K^2(t,u_2)\,du_2 \lesssim (t-u_1)^{2H}$. Moreover, for $0 \leq u < s$ the bound (13) on $\partial_t K(t,u)$ yields

$$|K(t,u) - K(s,u)| = \left|\int_s^t \partial_v K(v,u) dv\right| \lesssim (s-u)^{H-\frac{1}{2}} - (t-u)^{H-\frac{1}{2}}, \tag{14}$$

and thus, putting these two estimates together, we obtain:

$$I_{st}^1 \lesssim \int_0^s \left[(s-u)^{H-\frac{1}{2}} - (t-u)^{H-\frac{1}{2}}\right]^2 (t-u)^{2H} du.$$

Performing the changes of variable $v = s-u$ and $y = v/(t-s)$, we end up with

$$I_{st}^1 \lesssim (t-s)^{4H} \int_0^{s/(t-s)} \left[(1+y)^{H-\frac{1}{2}} - y^{H-\frac{1}{2}}\right]^2 (1+y)^{2H}\,dy.$$

Furthermore, it is easily checked that $\int_0^\infty \left[(1+y)^{H-\frac{1}{2}} - y^{H-\frac{1}{2}}\right]^2 (1+y)^{2H}\,dy$ is a convergent integral whenever $H < 1/2$, which gives the desired bound for I_{st}^1. The term I_{st}^2 is in fact easier to handle, and we leave those details to the reader for the sake of conciseness. Then, the estimates on I_{st}^1 and I_{st}^2 yield our claim.

□

Finally, the following related integral bound also turns out to be an important estimate for the analysis of n^{th} order iterated integrals:

Lemma 2.5. *Suppose that $2kH < 1$. For $A > 0$, set*

$$\beta_A = \int_0^A \left[y^{H-\frac{1}{2}} - (1+y)^{H-\frac{1}{2}} \right] \left[y^{H-\frac{1}{2}} + (A-y)^{H-\frac{1}{2}} \right] y^{2(k-1)H} dy.$$

Then $\sup_{A>0} \beta_A < \infty$.

Proof. We can write $\beta_A \leq \alpha_A + \gamma_A$, with

$$\begin{aligned} \alpha_A &= \int_0^\infty \left[y^{H-\frac{1}{2}} - (1+y)^{H-\frac{1}{2}} \right] y^{2(k-1)H+H-\frac{1}{2}} \, dy \\ \gamma_A &= \int_0^A \left[y^{H-\frac{1}{2}} - (1+y)^{H-\frac{1}{2}} \right] (A-y)^{H-\frac{1}{2}} y^{2(k-1)H} \, dy. \end{aligned}$$

One can check easily, as in the proof of Lemma 2.4, that α_A is finite as long as $2kH < 1$. On the other hand, an obvious change of variables yields

$$\gamma_A = A^{2kH} \int_0^1 h_A(y) \, (1-y)^{H-\frac{1}{2}} y^{2(k-1)H} dy, \tag{15}$$

where the (positive) function h_A is defined on $\mathbb{R}_+$ by $h_A(y) = y^{H-\frac{1}{2}} - (\frac{1}{A} + y)^{H-\frac{1}{2}}$. We now use two elementary estimates:

$$h_A(y) \leq \left(\frac{1}{2} - H \right) \frac{y^{H-\frac{3}{2}}}{A}, \quad \text{and} \quad h_A(y) \leq y^{H-\frac{1}{2}},$$

and we obtain

$$\begin{aligned} h_A(y) &= h_A(y)^{2kH} \, h_A(y)^{1-2kH} \\ &\leq \left(\left(\frac{1}{2} - H \right) \frac{1}{A} y^{H-\frac{3}{2}} \right)^{2kH} y^{(H-\frac{1}{2})(1-2kH)} = \frac{c_{H,k} \, y^{(1-2k)H-\frac{1}{2}}}{A^{2kH}}, \end{aligned}$$

where $c_{H,k} = (\frac{1}{2} - H)^{2kH}$. Plugging this bound into (15), we get

$$\gamma_A \leq c_{H,k} \int_0^1 (1-y)^{H-\frac{1}{2}} y^{-H-\frac{1}{2}} dy.$$

This last integral being finite, our claim is now proved. □

Remark 2.6. The reader can check that the only properties of the kernel K used in the sequel are

$$K(t,u) \lesssim (t-u)^{H-\frac{1}{2}} + u^{H-\frac{1}{2}}, \quad \text{and} \quad \partial_t K(t,u) \lesssim (t-u)^{H-\frac{3}{2}}$$

for some $H \in (0, 1/2)$. Our rough path construction is thus valid for any Gaussian Volterra process defined by a kernel satisfying the above estimates.

2.3. **Contraction of Stratonovich iterated integrals.** An important tool in our analysis of iterated integrals will be a general formula of Itô-Stratonovich corrections for iterated integrals. This kind of result has already been obtained in the literature, and for our purposes, it will be enough to use a particular case of [2, Proposition 1], recalled here for further use. Note that we need an additional notation for this intermediate result: we set dY for the Stratonovich type differential with respect to a process Y, while the Itô type differential is denoted by ∂Y.

Proposition 2.7. *Let $Y = (Y(1), \dots, Y(n))$ be a n-dimensional martingale of Gaussian type, defined on an interval $[s,t]$, of the form $Y_u(j) = \int_s^u \psi_v(j)\, dW_v(i_j)$ for a family of $L^2([s,t])$ functions $(\psi(1), \dots, \psi(n))$, a set of indices $(i_1, \dots, i_n)$ belonging to $\{1, \dots, d\}^n$, and where we recall that $(W(1), \dots, W(d))$ is a d-dimensional Wiener process. Then the following decomposition holds true:*

$$\int_{s \le u_1 < \cdots < u_n \le t} dY_{u_1}(i_1) \cdots dY_{u_n}(i_n) = \sum_{k=\lfloor n/2 \rfloor}^{n} \frac{1}{2^{n-k}} \sum_{\nu \in D_n^k} J_{st}(\nu).$$

In the above formula, the sets D_n^k are subsets of $\{1,2\}^k$ given by

$$D_n^k = \left\{ \nu = (n_1, \dots, n_k); \sum_{j=1}^{k} n_j = n \right\},$$

and the Itô-type multiple integrals $J_{st}(\nu)$ are defined as follows:

$$J_{st}(\nu) = \int_{s \le u_1 < \cdots < u_k \le t} \partial Z_{u_1}(1) \cdots \partial Z_{u_k}(k),$$

where, setting $\sum_{l=1}^{j} n_l = m(j)$, we have

$$Z(j) = Y(i_{m(j)}) \quad \text{if} \quad n_j = 1,$$

and

$$Z_u(j) = \left(\int_s^u \psi_v(m(j)-1) \psi_v(m(j))\, dv \right) \mathbf{1}_{(i_{m(j)-1} = i_{m(j)})} \quad \text{if} \quad n_j = 2.$$

The previous Itô-Stratonovich decomposition allows us to bound the second order moment of iterated Stratonovich integrals in the following way:

Lemma 2.8. *Let $\varphi \in L^2([s,t])$. Consider the Stratonovich iterated integral*

$$I_{st}^n(\varphi) = \int_{s < u_1 < \cdots < u_n < t} \prod_{i=1}^{n} \varphi(u_i)\, dW_{u_1}(i_1) \cdots dW_{u_n}(i_n).$$

Then,

$$\mathbf{E}\left[I_{st}^n(\varphi)^2 \right] \le C \left(\int_s^t \varphi(u)^2 du \right)^n, \tag{16}$$

where the constant C depends on n and the multiindex $(i_1, \dots, i_n)$.

Proof. By Proposition 2.7, we can decompose the Stratonovich integral $I_{st}^n(\varphi)$ into a sum of Itô integrals:

$$I_{st}^n(\varphi) = \sum_{k=\lfloor n/2 \rfloor}^{n} \frac{1}{2^{n-k}} \sum_{\nu \in D_n^k} J_{st}(\nu),$$

and it suffices to consider each Itô integral $J_{st}(\nu)$. Then we proceed by recurrence with respect to k, with the notation of Proposition 2.7. Suppose first that $n_k = 1$. Then,

$$J_{st}(\nu) = \int_s^t J_{su}(\nu') \varphi(u) \partial_u W(i_n),$$

where $\nu' = (n_1, \dots, n_{k-1})$. As a consequence,

$$\mathbf{E}[J_{st}(\nu)^2] = \int_s^t \mathbf{E}\left[J_{su}(\nu')^2 \right] \varphi(u)^2 du \le \sup_{s \le u \le t} \mathbf{E}\left[J_{su}(\nu')^2 \right] \int_s^t \varphi(u)^2 du.$$

On the other hand, if $n_k = 2$, then $J_{st}(\nu) = \int_s^t J_{su}(\nu'')\varphi(u)^2 du$, with $\nu'' = (n_1, \dots, n_{k-2})$, and again

$$\mathbf{E}[J_{st}(\nu)^2] \leq \sup_{s\leq u\leq t} \mathbf{E}\left[J_{su}(\nu'')^2\right] \left(\int_s^t \varphi(u)^2 du\right)^2.$$

By recurrence we obtain (16), where $C = \left(\sum_{k=\lfloor n/2\rfloor}^n \frac{\left|D_n^k\right|}{2^{n-k}}\right)^2$.

□

3. Iterated integrals of order 2

In this section, we will define the element $\mathbf{B^2}$ announced in Theorem 1.1. The study of this particular case will (hopefully) allow us to introduce many of the technical ingredients needed for the general case in a didactic way.

3.1. **Heuristic considerations.** Let us first specify what is meant by an iterated integral of order 2: according to the definitions contained in the Introduction, we are searching for a process $\{\mathbf{B}^{\mathbf{2}}_{st}(i_1, i_2);\ (s,t) \in \mathcal{S}_{2,T},\ 1 \leq i_1, i_2 \leq d\}$ satisfying:
(i) The regularity condition $\mathbf{B^2} \in \mathcal{C}_2^{2\gamma}(\mathbb{R}^{d,d})$.
(ii) The multiplicative property

$$\delta\mathbf{B}^{\mathbf{2}}_{sut}(i_1, i_2) = \mathbf{B}^{\mathbf{1}}_{su}(i_1)\, \mathbf{B}^{\mathbf{1}}_{ut}(i_2) = [B_u(i_1) - B_s(i_1)]\ [B_t(i_2) - B_u(i_2)]\,, \tag{17}$$

which should be satisfied almost surely for all $(s,u,t) \in \mathcal{S}_{3,T}$ and $1 \leq i_1, i_2 \leq d$.
(iii) The geometric relation, which can be read here as:

$$\mathbf{B}^{\mathbf{2}}_{st}(i_1, i_2) + \mathbf{B}^{\mathbf{2}}_{st}(i_2, i_1) = \mathbf{B}^{\mathbf{1}}_{st}(i_1)\, \mathbf{B}^{\mathbf{1}}_{st}(i_2), \quad (s,t) \in \mathcal{S}_{2,T}, \quad 1 \leq i_1, i_2 \leq d. \tag{18}$$

In order to construct this kind of element, let us start with some heuristic considerations, similar to the starting point of [18]: assume for the moment that X is a smooth d-dimensional function defined on $[0,T]$. Then the natural notion of iterated integral of order 2 for X is obviously an element $\hat{\mathbf{X}}^{\mathbf{2}}$, defined in the Riemann sense by

$$\hat{\mathbf{X}}^{\mathbf{2}}_{st}(i_1, i_2) = \int_{s\leq u_1\leq u_2\leq t} dX_{u_1}(i_1)\, dX_{u_2}(i_2) = \int_s^t [X_u(i_1) - X_s(i_1)]\ dX_u(i_2). \tag{19}$$

We shall now decompose $\hat{\mathbf{X}}^{\mathbf{2}}$ into terms of the form $\mathbf{A^2}$ and $\mathbf{C^2}$ as explained in the Introduction. In our case, this can be done in two ways: first, equation (19) immediately yields

$$\hat{\mathbf{X}}^{\mathbf{2}}_{st}(i_1, i_2) = \hat{\mathbf{A}}^{\mathbf{2},2}_{st} + \hat{\mathbf{C}}^{\mathbf{2},2}_{st}, \quad \text{with} \quad \hat{\mathbf{A}}^{\mathbf{2},2}_{st} = -X_s(i_1)\, \delta X_{st}(i_2),\ \hat{\mathbf{C}}^{\mathbf{2},2}_{st} = \int_s^t X_u(i_1)\, dX_u(i_2),$$

where we have called those quantities $\hat{\mathbf{A}}^{\mathbf{2},2}$ and $\hat{\mathbf{C}}^{\mathbf{2},2}$ because they involve increments of the second component $X(i_2)$ of X. Notice now that $\hat{\mathbf{C}}^{\mathbf{2},2}$ is the increment of a function f defined as $f_t = \int_0^t X_u(i_1)\, dX_u(i_2)$. Hence, according to convention (8), one can write $\hat{\mathbf{X}}^{\mathbf{2}}(i_1, i_2) \overset{\mathcal{ZC}_2}{=} \hat{\mathbf{A}}^{\mathbf{2},2}$. By inverting the order of integration in u_1, u_2 thanks to Fubini's theorem, we also obtain

$$\hat{\mathbf{X}}^{\mathbf{2}}_{st}(i_1, i_2) = \hat{\mathbf{A}}^{\mathbf{2},1}_{st} + \hat{\mathbf{C}}^{\mathbf{2},1}_{st}, \quad \text{with} \quad \hat{\mathbf{A}}^{\mathbf{2},1}_{st} = \delta X_{st}(i_1)\, X_t(i_2),\ \hat{\mathbf{C}}^{\mathbf{2},1}_{st} = -\int_s^t X_u(i_2)\, dX_u(i_1),$$

and thus $\hat{\mathbf{X}}^{\mathbf{2}}(i_1, i_2) \overset{\mathcal{ZC}_2}{=} \hat{\mathbf{A}}^{\mathbf{2},1}$.

Let us go back now to the case of the d-dimensional fBm B. If we wish the iterated integral $\mathbf{B^2}$ we are constructing to behave in a similar manner as a Riemann type integral, then one should also have the relation:

$$\mathbf{B^2}(i_1,i_2) \overset{\mathcal{ZC}_2}{=} \mathbf{A^{2,2}}, \quad \text{and} \quad \mathbf{B^2}(i_1,i_2) \overset{\mathcal{ZC}_2}{=} \mathbf{A^{2,1}},$$

with $\mathbf{A}^{\mathbf{2},2}_{st} = -B_s(i_1)\,\delta B_{st}(i_2)$ and $\mathbf{A}^{\mathbf{2},1}_{st} = \delta B_{st}(i_1)\,B_t(i_2)$. This means in particular, according to the fact that $\delta|_{\mathcal{ZC}_2} = 0$, that both $\mathbf{A^{2,1}}$ and $\mathbf{A^{2,2}}$ satisfy the multiplicative relation (17), as it can be easily checked by direct computations. However, this naive decomposition has an important drawback: the increments $\mathbf{A^{2,1}}$ and $\mathbf{A^{2,2}}$ only belong to $\mathcal{C}_2^\gamma$, instead of $\mathcal{C}_2^{2\gamma}$, for any $\gamma < H$ (this point was also stressed in [18]).

Our construction diverges from [18] in the way we cope with the regularity problem mentioned above. Indeed, we start from the following observation: invoking the representation (3) of B, one can write

$$\begin{aligned}\mathbf{A}^{\mathbf{2},2}_{st} &= -B_s(i_1)\,\delta B_{st}(i_2) = -\int_{\mathbb{R}} K(s,u_1) dW_{u_1}(i_1) \int_{\mathbb{R}} \left[K(t,u_2) - K(s,u_2)\right] dW_{u_2}(i_2) \\ &= -\int_{\mathbb{R}^2} K(s,u_1)\left[K(t,u_2) - K(s,u_2)\right] dW_{u_1}(i_1) dW_{u_2}(i_2),\end{aligned}$$

where we recall that the stochastic differentials dW are defined in the Stratonovich sense. In the same way, we get

$$\mathbf{A}^{\mathbf{2},1}_{st} = \int_{\mathbb{R}^2} \left[K(t,u_1) - K(s,u_1)\right] K(t,u_2)\, dW_{u_1}(i_1) dW_{u_2}(i_2).$$

The idea in order to transform $\mathbf{A^{2,1}}, \mathbf{A^{2,2}}$ into $\mathcal{C}_2^{2\gamma}$ increments is then to replace the integrals over $\mathbb{R}^2$ above by integrals on the simplex, as mentioned in the Introduction. Namely, we set now

$$\hat{\mathbf{B}}^{\mathbf{2},1}_{st}(i_1,i_2) = \int_{u_1<u_2} \left[K(t,u_1) - K(s,u_1)\right] K(t,u_2)\, dW_{u_1}(i_1) dW_{u_2}(i_2) \tag{20}$$

$$\hat{\mathbf{B}}^{\mathbf{2},2}_{st}(i_1,i_2) = -\int_{u_2<u_1} K(s,u_1)\left[K(t,u_2) - K(s,u_2)\right] dW_{u_1}(i_1) dW_{u_2}(i_2), \tag{21}$$

and notice that these formulas are a particular case of (5) for $n = 2$. We shall see that $\hat{\mathbf{B}}^{\mathbf{2},1}(i_1,i_2)$ and $\hat{\mathbf{B}}^{\mathbf{2},2}(i_1,i_2)$ are elements of $\mathcal{C}_2^{2\gamma}$, but they do not satisfy the multiplicative and geometric property anymore. However, it is now easily conceived, by some symmetry arguments, that the sum of these last two terms do satisfy the desired algebraic properties again. Indeed, we set now

$$\mathbf{B}^{\mathbf{2}}_{st}(i_1,i_2) = \hat{\mathbf{B}}^{\mathbf{2},1}_{st}(i_1,i_2) + \hat{\mathbf{B}}^{\mathbf{2},2}_{st}(i_1,i_2), \tag{22}$$

and we claim that $\mathbf{B^2}$ is a $\mathcal{C}_2^{2\gamma}(\mathbb{R}^{d,d})$ increment which fulfills relations (17) and (18). The remainder of this section is devoted to prove these claims.

3.2. **Properties of the second order increment.** It is obviously essential for the following developments to check that $\mathbf{B^2}$ is a well defined object in $L^2(\Omega)$. The next proposition asserts the existence of $\mathbf{B}^{\mathbf{2}}_{st}$ as a L^2 random variable for all s,t in the interval $[0,T]$.

Proposition 3.1. *Let $H < 1/2$, $(s,t) \in \mathcal{S}_{2,T}$ and $\mathbf{B}^{\mathbf{2}}_{st}$ be the matrix valued random variable defined by (22). Then $\mathbf{B}^{\mathbf{2}}_{st}(i_1,i_2) \in L^2(\Omega;\mathbb{R}^{d,d})$ and $\mathbf{E}[|\mathbf{B}^{\mathbf{2}}_{st}|^2] \lesssim (t-s)^{4H}$.*

Proof. Assume first $i_1 \neq i_2$. We shall focus on the relation $\mathbf{E}[(\mathbf{B}^{\mathbf{2},1}_{st}(i_1,i_2))^2] \lesssim (t-s)^{4H}$, the bound on $\mathbf{B}^{\mathbf{2},2}_{st}$ being obtained in a similar way. Now Stratonovich and Itô type integrals coincide when $i_1 \neq i_2$, and according to expression (20) we have

$$\begin{aligned}&\mathbf{E}\left[\left(\mathbf{B}^{\mathbf{2},1}_{st}(i_1,i_2)\right)^2\right]\\ &= \int_{u_1<u_2} \left[K(t,u_1)\mathbf{1}_{[0,t]}(u_1) - K(s,u_1)\mathbf{1}_{[0,s]}(u_1)\right]^2 K^2(t,u_2)\mathbf{1}_{[0,t]}(u_2)\, du_1 du_2,\end{aligned}$$

which is exactly the quantity I_{st} studied at Lemma 2.4. The desired bound follows from Lemma 2.4.

Let us now treat the case $i_1 = i_2 = i$, still concentrating our efforts on the inequality $\mathbf{E}[(\mathbf{B}^{\mathbf{2},1}_{st}(i,i))^2] \lesssim (t-s)^{4H}$. In this context, Proposition 2.7 yields the decomposition $\mathbf{B}^{\mathbf{2},1}_{st}(i,i) = M_{st} + V_{st}$, with

$$\begin{aligned}M_{st} &= \int_{u_1<u_2} [K(t,u_1) - K(s,u_1)]\, K(t,u_2)\, \partial W_{u_1}(i)\, \partial W_{u_2}(i)\\ V_{st} &= \frac{1}{2}\int_{u_1<u_2} [K(t,u_1) - K(s,u_1)]\, K(t,u_2)\, du_1\, du_2,\end{aligned}$$

where we stress the fact that V_{st} is a deterministic correction term. It is thus obviously enough to obtain the bounds $\mathbf{E}[M^2_{st}] \lesssim (t-s)^{4H}$ and $V^2_{st} \lesssim (t-s)^{4H}$ separately, the first of these bounds being obtained by evaluating I_{st} in Lemma 2.4 again. As far as V_{st} is concerned, Jensen's inequality allows us to assert

$$V^2_{st} \lesssim \int_{u_1<u_2} [K(t,u_1) - K(s,u_1)]^2\, K(t,u_2)^2\, du_1\, du_2 = I_{st},$$

which trivially finishes the proof.

□

A second technical step in the study of $\mathbf{B}^{\mathbf{2}}$ is to prove that this increment belongs to a space of the form $\mathcal{C}_2^{\mathrm{m},\beta}(\mathbb{R}^{d,d})$.

Proposition 3.2. *Let $\mathbf{B}^{\mathbf{2}}$ be the iterated integral increment defined by Equation (22). Then there exists a version of $\mathbf{B}^{\mathbf{2}}$ such that $\mathbf{B}^{\mathbf{2}} \in \mathcal{C}_2^{\mathrm{m},\beta}(\mathbb{R}^{d,d})$ for any $\beta < H$.*

Proof. Thanks to the multiparametric version of Kolmogorov's criterion [3], it is enough to check that

$$\mathbf{E}\left[|\mathbf{B}^{\mathbf{2}}_{s+h,t+k} - \mathbf{B}^{\mathbf{2}}_{st}|^p\right] \lesssim |h|^p + |k|^p, \tag{23}$$

for $(s,t) \in \mathcal{S}_{2,T}$, h,k such that $s+h, t+k \in [0,T]$, $s+h < t+k$, and p large enough. Furthermore, invoking the fact that $\mathbf{B}^{\mathbf{2}}_{st}$ is an element of the second chaos of W, on which all L^p norms are equivalent, it is enough to check relation (23) for $p = 2$. This last computation is however very similar to the one contained in the proof of Proposition 3.1, and we omit it here for the sake of conciseness.

□

We are now equipped with the continuous version of $\mathbf{B}^{\mathbf{2}}$ exhibited in the last proposition, on which we will work without further mention, and we are now ready to prove the algebraic relations satisfied by our second order increment.

Proposition 3.3. *The increment $\mathbf{B}^{\mathbf{2}}$ defined by (22) satisfies relations (17) and (18).*

Proof. Recall that we are now dealing with a continuous version of $\mathbf{B^2}$. In fact, one can easily modify the arguments of Proposition 3.2 in order to get a continuous version of the pair $(\mathbf{B^1}, \mathbf{B^2})$. This means that it is enough to check relations (17) and (18) for some fixed $0 \leq s < u < t \leq T$.

Let us then verify (17) for $s, u, t \in [0, T]$ such that $s < u < t$. It is readily seen, by writing the definitions of $\mathbf{B}^{\mathbf{2},1}_{st}(i_1, i_2), \mathbf{B}^{\mathbf{2},1}_{su}(i_1, i_2)$ and $\mathbf{B}^{\mathbf{2}}_{ut}(i_1, i_2)$, that

$$\delta\mathbf{B}^{\mathbf{2},1}_{sut}(i_1, i_2) = \int_{u_1<u_2} [K(u, u_1) - K(s, u_1)] [K(t, u_2) - K(u, u_2)] \, dW_{u_1}(i_1) dW_{u_2}(i_2),$$

the right hand side of this equality being well defined as a L^2 random variable (a fact which can be shown similarly to Proposition 3.1). Along the same lines, we also get

$$\delta\mathbf{B}^{\mathbf{2},2}_{sut}(i_1, i_2) = \int_{u_1>u_2} [K(u, u_1) - K(s, u_1)] [K(t, u_2) - K(u, u_2)] \, dW_{u_1}(i_1) dW_{u_2}(i_2),$$

and thus

$$\begin{aligned} \delta\mathbf{B}^{\mathbf{2}}_{sut}(i_1, i_2) &= \delta\mathbf{B}^{\mathbf{2},1}_{sut}(i_1, i_2) + \delta\mathbf{B}^{\mathbf{2},2}_{sut}(i_1, i_2) \\ &= \int_{\mathbb{R}^2} [K(u, u_1) - K(s, u_1)] [K(t, u_2) - K(u, u_2)] \, dW_{u_1}(i_1) dW_{u_2}(i_2) \\ &= \mathbf{B}^{\mathbf{1}}_{su}(i_1) \, \mathbf{B}^{\mathbf{1}}_{ut}(i_2), \end{aligned}$$

which is relation (17). Relation (18) is shown thanks to the same kind of elementary considerations, and its proof is left to the reader.

□

Finally, let us close this section by giving the proof of the announced regularity result on $\mathbf{B^2}$.

Proposition 3.4. *The increment $\mathbf{B^2}$ is almost surely an element of $\mathcal{C}_2^{2\gamma}(\mathbb{R}^{d,d})$, for any $\gamma < H$.*

Proof. Consider a fixed Hölder exponent $\gamma < H$. The proof of this result is based on Lemma 2.1, which can be read here as $\mathcal{N}[\mathbf{B^2}; \mathcal{C}_2^{2\gamma}(\mathbb{R}^{d,d})] \lesssim A + D$, with

$$A = \left(\int_{\mathcal{S}_{2,T}} \frac{|\mathbf{B}^{\mathbf{2}}_{uv}|^{2p}}{|u - v|^{4\gamma p + 4}} \, du \, dv \right)^{\frac{1}{2p}}, \quad \text{and} \quad D = \mathcal{N}\big[\delta\mathbf{B^2}; \mathcal{C}_3^{2\gamma}(\mathbb{R}^{d,d})\big].$$

Let us first deal with the term D above: we have seen that $\mathbf{B^2}$ satisfies the multiplicative property (17), which can be summarized as $\delta\mathbf{B^2} = \delta B \otimes \delta B$. Furthermore, $B \in \mathcal{C}_1^{\gamma}(\mathbb{R}^d)$ for any $\gamma < H$, and thus, for any $1 \leq i_1, i_2 \leq d$ and $0 \leq s < u < t \leq T$

$$|\delta\mathbf{B}^{\mathbf{2}}_{sut}(i_1, i_2)| = |\delta B_{su}(i_1)| \, |\delta B_{ut}(i_2)| \leq \mathcal{N}^2[B; \mathcal{C}_1^{\gamma}(\mathbb{R}^d)] \, |u - s|^{\gamma} |t - u|^{\gamma}.$$

In other words, the quantity $\|\delta\mathbf{B^2}\|_{\gamma,\gamma}$ defined by (10) is almost surely finite, and according to definition (11), we obtain that D is also almost surely finite.

We will now show that A is finite almost surely when p is large enough, by proving that $\mathbf{E}[A] < \infty$. Indeed, invoking Jensen's inequality we obtain:

$$\mathbf{E}\,[A] \leq \left(\int_{\mathcal{S}_{2,T}} \frac{\mathbf{E}\,[|\mathbf{B}^{\mathbf{2}}_{uv}|^{2p}]}{|u - v|^{4\gamma p + 4}} \, du dv \right)^{\frac{1}{2p}} \lesssim \left(\int_{\mathcal{S}_{2,T}} \frac{\mathbf{E}^p\,[|\mathbf{B}^{\mathbf{2}}_{uv}|^{2}]}{|u - v|^{4\gamma p + 4}} \, du dv \right)^{\frac{1}{2p}}, \tag{24}$$

where we have used the fact that $\mathbf{B^2}$ belongs to the second chaos of W, on which all the L^p norms are equivalent. On the other hand, Proposition 3.1 gives $\mathbf{E}^p\left[|\mathbf{B}^{\mathbf{2}}_{uv}|^2\right] \lesssim |u-v|^{4pH}$, and plugging this inequality into (24), we obtain that $\mathbf{E}[A]$ is finite as long as $p > 1/(H-\gamma)$.

□

In conclusion, putting together the last two propositions, we have constructed an element $\mathbf{B^2}$ which satisfies the properties (i)–(iii) given at the beginning of the section, for any $H < 1/2$.

4. General case

The aim of this section is to prove Theorem 1.1 in its full generality. Recall that we define our substitute $\mathbf{B^n}$ to n^{th} order integrals in the following way: for $2 \le n \le \lfloor 1/H \rfloor$, any tuple $(i_1, \dots, i_n)$ of elements of $\{1, \dots, d\}$, $1 \le j \le n$ and $(s,t) \in \mathcal{S}_{2,T}$, set

$$\begin{aligned}&\hat{\mathbf{B}}^{\mathbf{n},j}_{st}(i_1,\dots,i_n)\\ &\quad = (-1)^{j-1}\int_{A^n_j}\prod_{l=1}^{j-1}K(s,u_l)\,[K(t,u_j)-K(s,u_j)]\prod_{l=j+1}^{n}K(t,u_l)\,dW_{u_1}(i_1)\cdots dW_{u_n}(i_n),\end{aligned} \tag{25}$$

where the kernel K is given by (4) and A^n_j is the subset of $[s,t]^n$ defined by

$$A^n_j = \left\{(u_1,\dots,u_n)\in[0,t]^n;\, u_j = \min(u_1,\dots,u_n),\, u_1 > \cdots > u_{j-1},\text{ and } u_{j+1} < \cdots < u_n\right\}.$$

The increment $\mathbf{B^n}$ is then given by

$$\mathbf{B}^{\mathbf{n}}_{st}(i_1,\dots,i_n) = \sum_{j=1}^{n-1}\hat{\mathbf{B}}^{\mathbf{n},j}_{st}(i_1,\dots,i_n). \tag{26}$$

It is obviously harder to reproduce the heuristic considerations leading to this expression than in Section 3.1. Let us just mention that the same kind of changes in the order of integration allows us to produce some increments similar to $\mathbf{A}^{\mathbf{2},1}, \mathbf{A}^{\mathbf{2},2}$. Then the reordering trick yields some terms of the form $\hat{\mathbf{B}}^{\mathbf{n},j}_{st}$. After observing the form of several of these terms, the general expression (25) is then intuited in a natural way.

Notation: in order to write shorter formulas in the computations below, we use the following conventions in the sequel, whenever possible:

(i) A product of kernels of the form $\prod_{j=1}^n K(\tau_j, u_j)$ will simply be denoted by $\prod_{j=1}^n K_{\tau_j}$, meaning that the variable u_j has to be understood according to the position of the kernel K in the product.

(ii) In the same context, we will also set δK_{st} for a quantity of the form $K(t,u_j)-K(s,u_j)$.

(iii) Furthermore, when all the τ_j are equal to the same instant t, we write $\prod_{j=1}^n K(t,u_j) = K_t^{\otimes n}$.

(iv) Finally, we will also shorten the notations for the increments of the Wiener process W, and simply write dW for $\prod_{j=1}^n dW_{u_j}(i_j)$.

All these conventions allow us, for instance, to summarize formula (25) into

$$\hat{\mathbf{B}}^{\mathbf{n},j}_{st}(i_1,\dots,i_n) = (-1)^{j-1}\int_{A^n_j} K_s^{\otimes(j-1)}\,\delta K_{st}\,K_t^{\otimes(n-j)}\,dW. \tag{27}$$

4.1. **Moments of the $\mathbf{n^{th}}$ order integrals.** As in Section 3.2, an important step of our analysis is a control of the second moment of $\mathbf{B^n}$. This is given in the following proposition.

Proposition 4.1. *For $n \leq \lfloor \frac{1}{H} \rfloor$, let $\mathbf{B}^n_{st}$ be defined by (26). Then for $(s,t) \in \mathcal{S}_{2,T}$, we have*

$$\mathbf{E}\left[|\mathbf{B}^n_{st}|^2\right] \leq C(t-s)^{2nH},$$

for a strictly positive constant C.

Proof. Thanks to decomposition (26), it suffices to show that for any fixed family of indexes $i_1, \dots, i_n \in \{1, \dots, d\}$ and for any $1 \leq j \leq n-1$, we have

$$\mathbf{E}\left[\left|\hat{\mathbf{B}}^{\mathbf{n},j}_{st}(i_1, \dots, i_n)\right|^2\right] \leq C(t-s)^{2nH}.$$

Invoking now expression (25) for $\hat{\mathbf{B}}^{\mathbf{n},j}$ and decomposing the integral over the region A_j appearing in the definition of $\hat{\mathbf{B}}^{\mathbf{n},j}_{st}(i_1, \dots, i_n)$ into integrals over the simplex, it suffices to show an inequality of the type

$$\mathbf{E}\left[(Q_{st})^2\right] \leq C(t-s)^{2nH}, \quad \text{with} \quad Q_{st} = \int_{0<u_1<\cdots<u_n<t} \delta K_{st} \prod_{i=2}^{n} K_{\tau_i}\, dW. \tag{28}$$

Notice that in the expression above, we made use of the notation introduced at the beginning of the current section, and for $i = 1, \dots, n$, we assume $\tau_i = s$ or t. We concentrate our efforts now in proving (28).

Let us further decompose Q into $Q = Q^1 + Q^2$, where

$$Q^1_{st} = \int_{s<u_1<\cdots<u_n<t} K_t^{\otimes n}\, dW, \quad \text{and} \quad Q^2_{st} = \int_{0<u_1<\cdots<u_n<t, u_1<s} \delta K_{st} \prod_{i=2}^{n} K_{\tau_i}\, dW. \tag{29}$$

Notice that in Q^1_{st} we have assumed the $\tau_i = t$ for all i, since otherwise this term vanishes. Moreover, the term Q^1_{st} can be handled using the properties of the multiple Stratonovich integrals established in Lemma 2.8, and applying the estimate obtained in Lemma 2.3. This yields easily the relation $\mathbf{E}[(Q^1_{st})^2] \lesssim (t-s)^{2nH}$.

Concerning Q^2_{st}, one can write $Q^2_{st} = \sum_{j=1}^{n} B^j_{st}$ where

$$B^j_{st} = \int_{0<u_1<\cdots<u_j<s<u_{j+1}<\cdots<u_n<t} \delta K_{st} \prod_{i=2}^{n} K_{\tau_i}\, dW.$$

Notice that in the above equation $\tau_i = t$ if $i = j+1, \dots, n$, since we have again $B^j_{st} = 0$ otherwise. Each term B^j_{st} can thus be written as the product of two factors: $B^j_{st} = C^j_{st} D^j_{st}$, where for $j \geq 2$

$$C^j_{st} = \int_{0<u_1<\cdots<u_j<s} \delta K_{st} \prod_{i=2}^{j} K_{\tau_i}\, dW, \quad \text{and} \quad D^j_{st} = \int_{s<u_{j+1}<\cdots<u_n<t} K_t^{\otimes(n-j)}\, dW,$$

and for $j = 1$, $C^1_{st} = \int_0^s \delta K_{st} dW$ and D^1_{st} is given by the above formula.

The random variables C^j_{st} and D^j_{st} are independent, and $\mathbf{E}[(D^j_{st})^2]$ can be bounded easily like $\mathbf{E}[(Q^1_{st})^2]$. Hence we obtain

$$\mathbf{E}\left[(B^j_{st})^2\right] = \mathbf{E}\left[(C^j_{st})^2\right] \mathbf{E}\left[(D^j_{st})^2\right] \leq C\mathbf{E}\left[(C^j_{st})^2\right]\ (t-s)^{2(n-j)H}. \tag{30}$$

In order to bound the second moment of C_{st}^j, we express this factor as a sum of Itô integrals by means of Proposition 2.7. To do this, we give up for a moment our convention on products of increments, and we define, for $u \in [0,s]$ and $l = 2, \dots, j$, the processes

$$Y_u(1) = \int_0^u [K(t,v) - K(s,v)]\, dW_v(i_1) \quad \text{and} \quad Y_u(l) = \int_0^u K(\tau_l, v) dW_v(i_l).$$

Then, the processes $\{Y_u(l);\, 0 \le u \le s\}$ are Gaussian martingales and

$$C_{st}^j = \int_{0<u_1<\cdots<u_j<s} dY_{u_1}(1) dY_{u_1}(2) \cdots dY_{u_l}(l).$$

Thus, a direct application of Proposition 2.7 yields

$$C_{st}^j = \sum_{k=\lfloor j/2 \rfloor}^{j} \frac{1}{2^{j-k}} \sum_{\nu \in D_j^k} J_{0s}(\nu), \quad \text{where} \quad J_{0s}(\nu) = \int_{0<u_1<\cdots<u_k<s} \partial Z_{u_1}(1) \cdots \partial Z_{u_k}(k),$$

for $\nu = (j_1, \dots, j_k)$. Thus, setting $\sum_{l=1}^h j_l = m(h)$, we have $Z(h) = Y(i_{m(h)})$ if $j_h = 1$, and $Z_u(h) = \langle Y(m(h)-1), Y(m(h)) \rangle_u$ if $j_h = 2$ and $i_{m(h)-1} = i_{m(h)}$, where $\langle \cdot, \cdot \rangle$ designates the bracket of two continuous martingales. We are going to estimate $\mathbf{E}[J_{0s}(\nu)^2]$ using a recursive argument. This will be done in several steps:

Step 1: Suppose $j_k, j_{k-1}, \dots, j_1 = 2$. Then $j = 2k$, and we can assume that $i_m = i_{m-1}$ for $m = 2, 4, \dots, 2k$, otherwise $J_{0s}(\nu) = 0$. The term $J_{0s}(\nu)$ is deterministic and it can be expressed as follows:

$$J_{0s}(\nu) = \int_{0<u_1<\cdots<u_k<s} [K(t,u_1) - K(s,u_1)]\, K(\tau_2, u_1) \times \prod_{h=2}^{k} K(\tau_{2h-1}, u_h) K(\tau_{2h}, u_h) du_1 \cdots du_k.$$

As a consequence, owing to (13) and (14), we have

$$|J_{0s}(\nu)| \le C \int_{0<u_1<\cdots<u_k<s} \varphi_{u_1}^{(1)} \prod_{h=2}^{k} \varphi_{u_h}^{(2)}\, du_1 \cdots du_k, \tag{31}$$

where

$$\varphi_{u_1}^{(1)} = \left[(s-u_1)^{H-\frac{1}{2}} - (t-u_1)^{H-\frac{1}{2}} \right] \left[(s-u_1)^{H-\frac{1}{2}} + u_1^{H-\frac{1}{2}} \right],$$

and

$$\varphi_{u_h}^{(2)} = \left[(s-u_h)^{H-\frac{1}{2}} + u_h^{H-\frac{1}{2}} \right]^2.$$

Moreover, the integral of $\prod_{h=2}^{k}\varphi_{u_h}^{(2)}$ is easily bounded: indeed, we have

$$\begin{aligned}
&\int_{u_1<u_2<\cdots<u_k<s}\prod_{h=2}^{k}\varphi_{u_h}^{(2)}\,du_2\cdots du_k\\
\leq&\int_{u_1<u_2<\cdots<u_k<s}\prod_{h=2}^{k}\left[(s-u_h)^{H-\frac{1}{2}}+(u_h-u_{h-1})^{H-\frac{1}{2}}\right]^2 du_2\cdots du_k\\
\leq&C\int_{u_1<u_2<\cdots<u_k<s}\prod_{h=2}^{k}\left[(u_{h+1}-u_h)^{2H-1}+(u_h-u_{h-1})^{2H-1}\right] du_2\cdots du_k\\
\leq&C(s-u_1)^{2(k-1)H},
\end{aligned}$$

with the convention $u_{k+1}=s$. Therefore, plugging this inequality into (31) and making the change of variables $s-u_1=v$ and $y=\frac{v}{t-s}$, we get

$$\begin{aligned}
|J_{0s}(\nu)|&\leq C\int_0^s\left[(s-u_1)^{H-\frac{1}{2}}-(t-u_1)^{H-\frac{1}{2}}\right]\left[(s-u_1)^{H-\frac{1}{2}}+u_1^{H-\frac{1}{2}}\right](s-u_1)^{2(k-1)H}du_1\\
&=C\int_0^s\left[v^{H-\frac{1}{2}}-(t-s+v)^{H-\frac{1}{2}}\right]\left[v^{H-\frac{1}{2}}+(s-v)^{H-\frac{1}{2}}\right]v^{2(k-1)H}dv\\
&=C(t-s)^{2kH}\int_0^{s/(t-s)}\left[y^{H-\frac{1}{2}}-(1+y)^{H-\frac{1}{2}}\right]\left[y^{H-\frac{1}{2}}+\left(\frac{s}{t-s}-y\right)^{H-\frac{1}{2}}\right]y^{2(k-1)H}dy.
\end{aligned}$$

We are now in a position to use Lemma 2.5 with $A=s/(t-s)$, and we obtain

$$|J_{0s}(\nu)|\leq C(t-s)^{2kH}, \tag{32}$$

which implies that $J_{0s}(\nu)^2\leq C(t-s)^{2jH}$, owing to the fact that $2k=j$.

Step 2: Suppose that $j_k=1$. Then Proposition 2.7 gives

$$J_{0s}(\nu)=\int_{0<u_1<\cdots<u_k<s}\partial Z_{u_1}(1)\cdots\partial Z_{u_{k-1}}(k-1)K(\tau_j,u_k)\,\partial W_u(i_j),$$

and

$$\begin{aligned}
\mathbf{E}\left[J_{0s}(\nu)^2\right]&=\int_0^s\mathbf{E}\left(J_{0u}(\nu')^2\right)K(\tau_j,u)^2du\\
&\leq\int_0^s\mathbf{E}\left(J_{0u}(\nu')^2\right)\left((s-u)^{2H-1}+u^{2H-1}\right)du,
\end{aligned}$$

with $\nu'=(j_1,\ldots,j_{k-1})$. This relation allows to set an induction procedure, as we shall see later.

Step 3: Suppose that $j_k,j_{k-1},\ldots,j_{b+1}=2$ *and* $j_b=1$, *where* $b\geq 2$. We assume that $i_{m(h)}=i_{m(h)-1}$ for $h=b+1,\ldots,k$. Here again, Proposition 2.7 implies

$$J_{0s}(\nu)=\int_{0<u_1<\cdots<u_k<s}\partial Z_{u_1}(1)\cdots\partial Z_{u_b}(b)\prod_{h=b+1}^{k}K(\tau_{m(h)-1},u_h)K(\tau_{m(h)},u_h)du_1\cdots du_k,$$

and Fubini's theorem yields

$$J_{0s}(\nu)=\int_0^s J_{0u_b}(\nu')K(\tau_{m(h)},u_b)G(u_b)dW_{u_b}(i_{m(h)}),$$

with $\nu' = (j_1, \dots, j_{b-1})$, and where

$$G(u_b) = \int_{0<u_b<u_{b+1}<\cdots<u_k<s} \prod_{h=b+1}^{k} K(\tau_{m(h)-1}, u_h) K(\tau^{m(h)}, u_h) du_{b+1} \cdots du_k.$$

As for the previous bound (32) we obtain

$$|G(u_b)| \leq C(s-u_b)^{2(k-b)H}.$$

Therefore

$$\begin{aligned} \mathbf{E}\left[J_{0s}(\nu)^2\right] &= \int_0^s \mathbf{E}\left[J_{0u_b}(\nu')^2\right] K(\tau_{m(h)}, u_b)^2 G(u_b)^2 du_b \\ &\leq C \int_0^s \mathbf{E}\left[J_{0u_b}(\nu'')^2\right] \left[(s-u_b)^{2H-1} + u_b^{2H-1}\right] (s-u_b)^{4(k-b)H} du_b. \end{aligned}$$

Notice that the above inequality includes the inequality obtained in Step 2, which corresponds to the case $b = k$.

Step 4: Suppose that $j_k, j_{k-1}, \dots, j_{b+1} = 2$, $j_b = 1$, $j_{b-1}, j_{b-2}, \dots, j_{c+1} = 2$ *and* $j_c = 1$, *where* $2 \leq c \leq b$. We assume also that $i_{m(h)} = i_{m(h)-1}$ for $h = c+1, \dots, b-1, b+1, \dots, k$. By the same arguments as in Step 2 we obtain

$$\begin{aligned} \mathbf{E}\left[J_{0s}(\nu)^2\right] &\leq C \int_{0<u_c<u_b<s} \mathbf{E}\left[J_{0u_c}(\nu')^2\right] \left[(u_b-u_c)^{2H-1} + u_c^{2H-1}\right] (u_b-u_c)^{4(b-c)H} \\ &\quad \times \left[(s-u_b)^{2H-1} + u_b^{2H-1}\right] (s-u_b)^{4(k-b)H} du_c du_b, \end{aligned}$$

with $\nu' = (j_1, \dots, j_{c-1})$. Replacing u_b^{2H-1} by $(u_b-u_c)^{2H-1}$ and integrating with respect to u_b yields

$$\mathbf{E}\left[J_{0s}(\nu)^2\right] \leq C \int_0^s \mathbf{E}\left[J_{0u_c}(\nu')^2\right] \left[(s-u_c)^{2H-1} + u_c^{2H-1}\right] (s-u_c)^{4(k-c)H+2H} du_c.$$

Step 5: Iteration scheme. Iterating the argument in Step 4, we reduce the size of ν' until we obtain a multiindex of length r such that $\nu' = (1, 2, \dots, 2)$ or $\nu' = (2, 2, \dots, 2)$, with $j_{r+1} = 1$, and we obtain an estimate of the form

$$\mathbf{E}\left[J_{0s}(\nu)^2\right] \leq C \int_0^s \mathbf{E}\left[J_{0u}(\nu')^2\right] \left[(s-u)^{2H-1} + u^{2H-1}\right] (s-u)^{2H \sum_{l=r+2}^{k} j_l} du, \tag{33}$$

Suppose first that $\nu' = (1, 2, \dots, 2)$. Then,

$$\begin{aligned} J_{0s}(\nu') &= \int_{0<u_1<\cdots<u_r<u} [K(t, u_1) - K(s, u_1)] \\ &\quad \times \prod_{h=2}^{r} K(\tau_{m(h)-1}, u_h) K(\tau_{m(h)}, u_h) dW_{u_1}(i_1) du_2 \cdots du_r. \end{aligned}$$

and by Fubini's theorem

$$J_{0s}(\nu') = \int_0^u [K(t, u_1) - K(s, u_1)] F(u_1) dW_{u_1}(i_1),$$

where

$$F(u_1) = \int_{u_1<u_2<\cdots<u_r<u} \prod_{h=2}^{r} K(\tau_{m(h)-1}, u_h) K(\tau_{m(h)}, u_h) du_2 \cdots du_r.$$

As in the proof of (32) we get

$$|F(u_1)| \leq C(u-u_1)^{2(r-1)H}.$$

Therefore,

$$\mathbf{E}\left[J_{0s}(\nu')^2\right] \leq C\int_0^u [(t-u_1)^{H-\frac{1}{2}} - (s-u_1)^{H-\frac{1}{2}}]^2 (u-u_1)^{4(r-1)H} du_1. \tag{34}$$

Substituting (34) into (33) yields, after integrating in the variable u,

$$\mathbf{E}\left[J_{0s}(\nu)^2\right] \leq C\int_0^s [(t-u)^{H-\frac{1}{2}} - (s-u)^{H-\frac{1}{2}}]^2 (s-u)^{2(j-1)H} du.$$

Performing the changes of variables $v = s-u$ and $y = v/(t-s)$, we end up with

$$\mathbf{E}\left[J_{0s}(\nu)^2\right] \leq C(t-s)^{2jH}\int_0^{s/(t-s)} [(1+t)^{H-\frac{1}{2}} - y^{H-\frac{1}{2}}]^2 y^{2(j-1)H} dy \leq C(t-s)^{2jH}, \tag{35}$$

where the last step is obtained thanks to a slight variation of Lemma 2.5.

If $\nu' = (2,2,\dots,2)$, then we proceed as in Step 1 and we obtain

$$|J_{0u}(\nu')| \leq C\int_0^u [(s-u_1)^{H-\frac{1}{2}} - (t-u_1)^{H-\frac{1}{2}}][(u-u_1)^{H-\frac{1}{2}} + u_1^{H-\frac{1}{2}}](u-u_1)^{2(r-1)H} du_1. \tag{36}$$

Substituting (36) into (33), integrating first in the variable u and using the same arguments as in Step 1 we obtain also the estimate

$$\mathbf{E}\left[J_{0s}(\nu)^2\right] \leq C(t-s)^{2jH}. \tag{37}$$

Step 6: Conclusion. Our bounds (35) and (37) on $J_{0s}(\nu)$ yields the same kind of estimate for the term C^j_{st}. Thus relation (30) gives $B^j_{st} \lesssim (t-s)^{2nH}$. This estimate can now be plugged into the definition (29) of Q^2, then in the definition of Q, which leads to our claim (28). The proof is now finished.

□

4.2. **Proof of Theorem 1.1.** Before we prove our main theorem, we need a last elementary technical ingredient, which relies on the notational convention given at the beginning of the current section.

Lemma 4.2. *For $n \geq 3$, $j = 2,\dots,n-1$ and $0 \leq s < t \leq T$, set*

$$M^{n,j}_{st} = K_s^{\otimes(j-1)} \,\delta K_{st}\, K_t^{\otimes(n-j)}.$$

Recall that for an element $M \in \mathcal{C}_2$, δM is defined by (7). Then

$$\delta M^{n,j}_{sut} = -\sum_{m=1}^{j-1} K_s^{\otimes(m-1)} \,\delta K_{su}\, K_u^{\otimes(j-1-m)} \,\delta K_{ut}\, K_t^{\otimes(n-j)} + K_s^{\otimes(j-1)} \,\delta K_{su} \sum_{m=1}^{n-j} K_u^{\otimes(m-1)} \,\delta K_{ut}\, K_t^{\otimes(n-j-m)}.$$

The relation still holds true for $j \in \{1,n\}$ and $n = 2$, with the convention $K^{\otimes 0} = \mathbf{1}$ and $\delta K^{\otimes 0} = 0$.

Proof. This proof is completely elementary, and included here for the sake of completeness, since it uses heavily the notation of Section 2.1.

First, if a, b, c are 3 increments in $\mathcal{C}_1$, and if we define $N \in \mathcal{C}_2$ by $N_{st} = a_s\, \delta b_{st}\, c_t$, then a simple application of Definition (7) gives

$$\delta N_{sut} = -\delta a_{su}\, \delta b_{ut}\, c_t + a_s\, \delta b_{su}\, \delta c_{ut}.$$

Our claim is thus proved by applying this relation to $a = K^{\otimes(j-1)}$, $b = K$, $c = K^{\otimes(n-j)}$, and observing that $[\delta K^{\otimes l}]_{st} = \sum_{p=1}^{l} K_s^{\otimes(p-1)}\, \delta K_{st}\, K_t^{\otimes(l-p)}$.

□

Proof of Theorem 1.1. The structure of the proof is the same as in the second order case of Section 3.2: we first prove that (a modification of) $\mathbf{B}^{\mathbf{n}}$ is almost surely an element of $\mathcal{C}_2^{\mathrm{m},\beta}((\mathbb{R}^d)^{\otimes n})$ for any $\beta < H$, applying Kolmogorov's criterion and with the same kind of computations as for Proposition 4.1. This allows us to reduce the algebraic relations (1) and (2) to the case of some fixed s, u, t. We will first focus on (1).

Step 1: Proof of the mutliplicative property (1). Fix $(s, u, t) \in \mathcal{S}_{3,T}$. Recall that $\hat{\mathbf{B}}_{st}^{\mathbf{n},j}$ is defined by (27). Therefore, invoking Lemma 4.2, $\delta\hat{\mathbf{B}}^{\mathbf{n},j}$ is given by

$$\begin{aligned}\delta\hat{\mathbf{B}}_{sut}^{\mathbf{n},j}(i_1,\dots,i_n) = (-1)^j \int_{A_j^n} \sum_{m=1}^{j-1} K_s^{\otimes(m-1)}\, \delta K_{su}\, K_u^{\otimes(j-1-m)}\, \delta K_{ut}\, K_t^{\otimes(n-j)}\, dW \\ + (-1)^{j-1} \int_{A_j^n} K_s^{\otimes(j-1)}\, \delta K_{su} \sum_{m=1}^{n-j} K_u^{\otimes(m-1)}\, \delta K_{ut}\, K_t^{\otimes(n-j-m)}\, dW. \quad (38)\end{aligned}$$

On the other hand, set $Z_{sut} = \sum_{n_1=1}^{n-1} \mathbf{B}_{su}^{\mathbf{n_1}} \mathbf{B}_{ut}^{\mathbf{n-n_1}}$. One can easily check that

$$\begin{aligned} Z_{sut} &= \sum_{n_1=1}^{n-1} \sum_{k=1}^{n_1} \sum_{h=1}^{n-n_1} \hat{\mathbf{B}}_{su}^{\mathbf{n_1},k} \hat{\mathbf{B}}_{ut}^{\mathbf{n-n_1},h} \quad (39)\\ &= \sum_{n_1=1}^{n-1} \sum_{k=1}^{n_1} \sum_{h=1}^{n-n_1} (-1)^{k+h} \int_{A_{k,h}(n_1)} K_s^{\otimes(k-1)}\, \delta K_{su}\, K_u^{\otimes(n_1-k+h-1)}\, \delta K_{ut}\, K_t^{\otimes(n-n_1-h)}\, dW, \end{aligned}$$

where $A_{k,h}(n_1)$ is the set defined by

$$\begin{aligned} A_{k,h}(n_1) = A_k^{n_1} \times A_h^{n-n_1} = \Big\{ (u_1,\dots,u_n);\, u_k < u_{k+1} < \cdots < u_{n_1},\, u_k < u_{k-1} < \cdots < u_1, \\ u_{n_1+h} < u_{n_1+h+1} < \cdots < u_n,\, u_{n_1+h} < u_{n_1+h-1} < \cdots < u_{n_1+1} \Big\}. \end{aligned}$$

We want to show that (39) and (38) coincide.

In order to follow the computations below, it might be useful to keep in mind an illustration of the coordinates ordering on a set of the form $A_{k,h}(m)$, for which an example is provided at Figure 1 (note that the ordering between u_m and u_{m+1} is not specified).

Notice that on the set $A_{k,h}(n_1) \cap \{u_k < u_{n_1+h}\}$ the minimum of the coordinates is u_k, and on the set $A_{k,h}(n_1) \cap \{u_{n_1+h} < u_k\}$ the minimum is u_{n_1+h}. Define

$$A_{h,k}^1(n_1) = A_{k,h}(n_1) \cap \{u_k < u_{n_1+h}\}, \quad \text{and} \quad A_{h,k}^2(n_1) = A_{k,h}(n_1) \cap \{u_{n_1+h} < u_k\}.$$

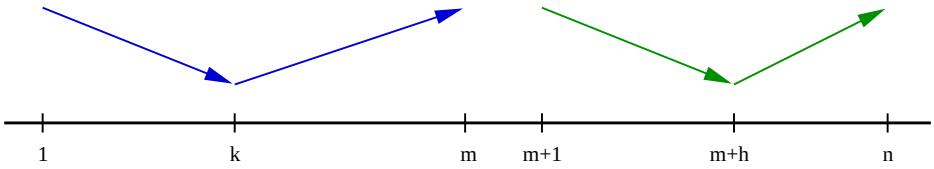

FIGURE 1. Coordinates ordering on $A_{k,h}(m)$.

Consider now the decomposition $Z = Z^1 + Z^2$, where

$$Z^i_{sut} = \sum_{n_1=1}^{n-1} \sum_{k=1}^{n_1} \sum_{h=1}^{n-n_1} (-1)^{k+h} \int_{A^i_{k,h}(n_1)} K_s^{\otimes(k-1)} \, \delta K_{su} \, K_u^{\otimes(n_1-k+h-1)} \, \delta K_{ut} \, K_t^{\otimes(n-n_1-h)} \, dW.$$

We fix j and we try to compute the contribution of Z^i_{sut} on the set A^n_j for $i = 1, 2$. This contribution will be the sum of the integrals on the set $A^n_j \cap A^i_{k,h}(n_1)$, for each $k = 1, \dots, n_1$, $h = 1, \dots, n - n_1$ and for each $n_1 = 1, \dots, n-1$.

Notice first that the intersection $A^n_j \cap A^1_{k,h}(n_1)$ is non empty only if $k = j$, $h = 1$, and $u_m < u_{m+1}$ which also implies $j \leq n_1$. Moreover, in this case we have $A^1_{j,1}(n_1) \cap \{u_m < u_{m+1}\} = A^n_j$. In this way we obtain that the contribution of Z^1_{sut} on A^n_j is

$$\begin{aligned} &(-1)^{j-1} \sum_{n_1=j}^{n-1} \int_{A^n_j} K_s^{\otimes(j-1)} \, \delta K_{su} \, K_u^{\otimes(n_1-j)} \, \delta K_{ut} \, K_t^{\otimes(n-n_1-1)} \, dW \\ =&(-1)^{j-1} \sum_{m=1}^{n-j} \int_{A^n_j} K_s^{\otimes(j-1)} \, \delta K_{su} \, K_u^{\otimes(m-1)} \, \delta K_{ut} \, K_t^{\otimes(n-m-j)} \, dW, \end{aligned} \tag{40}$$

where we have used the simple change of variables $m = n_1 - j + 1$. In the same manner, on the set $A^n_j \cap A^2_{k,h}(n_1)$ we have $k = n_1$, $h = j$, which also implies $n_1 \leq j$. Therefore, the contribution of Z^2_{sut} on A^n_j is

$$(-1)^{j} \sum_{n_1=1}^{j-1} \int_{A^n_j} K_s^{\otimes(n_1-1)} \, \delta K_{su} \, K_u^{\otimes(j-1-n_1)} \, K_t^{\otimes(n-j)} \, dW. \tag{41}$$

One can now easily verify that the sum of (40) and (41) is equal to the term (38).

It remains to prove that the contribution of Z_{sut} to the set $(\cup_j A^n_j)^c$ is zero. For this, observe that $(\cup_j A^n_j)^c$ can be split into slices $D_{k,p,h}$ of the following form: for $1 \leq k \leq p \leq n-1$, we assume that $u_k < u_{k-1} < \cdots < u_1$ and $u_k < u_{k+1} < \cdots < u_p$ but $u_p > u_{p+1}$. Suppose also that $1 \leq h \leq n - p$ and that u_{p+h} is the minimum of the coordinates $u_{p+1}, \dots, u_n$. Then, for $D_{k,p,h}$ to be a subset of $\cup_{n_1=1}^n \cup_{k,h} A_{k,h}(n_1)$, we need the further condition $u_{p+h} < u_{p+h+1} < \cdots < u_n$ and $u_{p+h} < u_{p+h-1} < \cdots < u_{p+1}$. With all these constraints in mind, it is easily seen that $D_{k,p,h}$ corresponds to two possible choices of set $A_{k,h}(n_1)$. Indeed, we have

$$D_{k,p,h} = A_{k,h}(p) = A_{k,h+1}(p-1).$$

Going back now to the expression (39) of Z_{sut}, it is readily checked that the two contributions, respectively on $A_{k,h}(p)$ and $A_{k,h+1}(p-1)$, yield two terms with opposite sign, which cancel out in the sum.

Step 2: Proof of the geometric property (2). Fix n, m such that $n+m \leq \lfloor 1/\gamma \rfloor$ and let $(s,t) \in \mathcal{S}_{2,T}$. Consider the product

$$\begin{aligned}&\mathbf{B}^{\mathbf{n}}_{st}(i_1,\dots,i_n)\,\mathbf{B}^{\mathbf{m}}_{st}(j_1,\dots,j_m)\\&=\sum_{j=1}^{n}\sum_{h=1}^{m}(-1)^{j+h}\left(\int_{A_j^n} K_s^{\otimes(j-1)}\delta K_{st}K_t^{\otimes(n-j)}dW\right)\left(\int_{A_h^m} K_s^{\otimes(h-1)}\delta K_{st}K_t^{\otimes(m-h)}dW\right),\end{aligned}$$

where we have used notation (27) and where we recall that the sets A_j^n and A_h^m are defined by

$$A_j^n = \{u \in [0,t]^n : u_j < u_{j-1} < \cdots < u_1, u_j < u_{j+1} < \cdots < u_n\},$$
$$A_h^m = \{v \in [0,t]^m : v_h < v_{h-1} < \cdots < v_1, v_h < v_{h+1} < \cdots < v_m\}.$$

The product of the two Stratonovich integrals can be expressed as a Stratonovich integral on the region $A_j^n \times A_h^m$ with respect to the differential

$$dW_{u_1}(i_1)\cdots dW_{u_n}(i_n)dW_{v_1}(j_1)\cdots dW_{v_m}(j_m).$$

We will make use of the notation $z=(u,v)$, where $z_\alpha = u_\alpha$, for $\alpha = 1,\dots,n$ and $z_\alpha = v_{\alpha-n}$ for $\alpha = n+1,\dots,n+m$. Like in Step 1, the region $A_j^n \times A_h^m$ can be first decomposed into the union of the disjoint regions $D_{j,h}$ and $E_{j,h}$, corresponding respectively to the additional constraints $\{u_j < v_h\}$ and $\{u_j > v_h\}$ (notice that this decomposition is valid up to the set $\{u_j = v_h\}$, whose contribution to the stochastic integral is null).

Consider first the case $\{u_j < v_h\}$. On $D_{j,h}$ the minimum of all the coordinates z_α is z_j. Then $D_{j,h}$ can be further decomposed into the disjoint union of the sets

$$\begin{aligned}D^{\pi}_{j,h,1} &= \{z \in [0,t]^{n+m} : z_j < z_{\alpha_{j+h-2}} < \cdots < z_{\alpha_1},\, z_j < z_{\beta_1} < \cdots < z_{\beta_{n-j+1+m-h}}\}\\&\quad \cap\{z_{n+h} < z_{n+h-1}\},\end{aligned}$$

where

$$\pi(1,\dots,n+m) = (\alpha_1,\dots,\alpha_{j+h-2}, j, \beta_1,\dots,\beta_{n-j+1+m-h})$$

runs over all permutations of the coordinates $1,\dots,n+m$ such that $\pi(j+h-1)=j$ and:
(i) $\alpha_1,\dots,\alpha_{j+h-2}$ is a permutation of the coordinates $1,\dots,j-1$ and $n+1,\dots,n+h-1$ that keeps the orderings $z_1 > \cdots > z_{j-1}$ and $z_{n+1} > \cdots > z_{n+h-1}$.
(ii) $\beta_1,\dots,\beta_{n-j+1+m-h}$ is a permutation of the coordinates $j+1,\dots,n$ and $n+h,\dots,n+m$ that keeps the orderings $z_{j+1} < \cdots < z_n$ and $z_{n+h} < \cdots < z_{n+m}$.
Moreover, $D_{j,h}$ can be also be decomposed into the disjoint union of the sets

$$\begin{aligned}D^{\tilde{\pi}}_{j,h,2} &= \{z \in [0,t]^{n+m} : z_j < z_{\alpha_{j+h-1}} < \cdots < z_{\alpha_1}, z_j < z_{\beta_1} < \cdots < z_{\beta_{n-j+m-h}}\}\\&\quad \cap\{z_{n+h} < z_{n+h+1}\},\end{aligned}$$

where

$$\tilde{\pi}(1,\dots,n+m) = (\alpha_1,\dots,\alpha_{j+h-1}, j, \beta_1,\dots,\beta_{n-j+m-h})$$

runs over all permutations of the coordinates $1,\dots,n+m$ such that $\tilde{\pi}(j+h)=j$ and:
(i) $\alpha_1,\dots,\alpha_{j+h-1}$ is a permutation of the coordinates $1,\dots,j-1$ and $n+1,\dots,n+h$ that keeps the orderings $z_1 > \cdots > z_{j-1}$ and $z_{n+1} > \cdots > z_{n+h}$.
(ii) $\beta_1,\dots,\beta_{n-j+m-h}$ is a permutation of the coordinates $j+1,\dots,n$ and $n+h+1,\dots,n+m$ that keeps the orderings $z_{j+1} < \cdots < z_n$ and $z_{n+h+1} < \cdots < z_{n+m}$.

Then, on the set $D_{j,h}$ we write

$$\begin{aligned}&K_s^{\otimes(j-1)}\delta K_{st}K_t^{\otimes(n-j)}K_s^{\otimes(h-1)}\delta K_{st}K_t^{\otimes(m-h)}\\&= K_s^{\otimes(j-1)}\delta K_{st}K_t^{\otimes(n-j)}K_s^{\otimes(h-1)}K_t^{\otimes(m-h+1)} - K_s^{\otimes(j-1)}\delta K_{st}K_t^{\otimes(n-j)}K_s^{\otimes h}K_t^{\otimes(m-h)},\end{aligned}$$

and the integral

$$I_{j,h} := \int_{D_{j,h}} K_s^{\otimes(j-1)}\delta K_{st}K_t^{\otimes(n-j)}K_s^{\otimes(h-1)}\delta K_{st}K_t^{\otimes(m-h)}dW$$

can be expressed as the sum $I_{j,h} = I_{j,h}^+ + I_{j,h}^-$, with

$$\begin{aligned}I_{j,h}^+ = \sum_\pi \int_{D_{j,h,1}^\pi} (-1)^{j+h-2} \prod_{l=1}^{j+h-2} K(s,z_{\alpha_l})\delta K_{st}(z_j)\\ \times \prod_{l=1}^{n-j+1+m-h} K(t,z_{\beta_l})\, dW_{z_1}(i_1)\cdots dW_{z_{n+m}}(i_{n+m}),\end{aligned}$$

and

$$\begin{aligned}I_{j,h}^- = \sum_{\tilde\pi} \int_{D_{j,h,2}^{\tilde\pi}} (-1)^{j+h-1} \prod_{l=1}^{j+h-1} K(s,z_{\alpha_l})\delta K_{st}(z_j)\\ \times \prod_{l=1}^{n-j+m-h} K(t,z_{\beta_l})\, dW_{z_1}(i_1)\cdots dW_{z_{n+m}}(i_{n+m}).\end{aligned}$$

Let us handle first the term $I_{j,h}^+$: consider a permutation σ of $1,\dots,n+m$ which maps $\alpha_1,\dots,\alpha_{j+h-2}$ into $1,\dots,j+h-2$ and $\beta_1,\dots,\beta_{n-j+1+m-h}$ into $j+h,\dots,n+m$, with the additional condition $\sigma(j) = j+h-1$. If we make this permutation in the coordinates of $I_{j,h}^+$ we obtain

$$\begin{aligned}I_{j,h}^+ = \int_{A_{j+h-1}^{n+m}\cap\{z_\nu<z_\eta\}} (-1)^{j+h-2} \prod_{l=1}^{j+h-2} K(s,z_l)\delta K_{st}(z_{j+h-1})\\ \times \prod_{l=j+h}^{n+m} K(t,z_l)\, dW_{z_1}(k_1)\cdots dW_{z_{n+m}}(k_{n+m}),\end{aligned}$$

where $k_1,\dots,k_{n+m}$ is a permutation of the indexes $i_1,\dots,i_n,j_1,\dots,j_m$ preserving the orderings of the indexes $i_1,\dots,i_n$ and $j_1,\dots,j_m$, and which puts i_j in the $j+h-1$ place, and where ν,η are defined by

$$\nu = \min\{i \geq j+h : k_i \in \{j_1,\dots,j_m\}\},\quad \eta = \max\{i \leq j+h-2 : k_i \in \{j_1,\dots,j_m\}\}.$$

In the same way, we can consider a permutation σ in the coordinates z_i which maps $\alpha_1,\dots,\alpha_{j+h-1}$ into $1,\dots,j+h-1$ and $\beta_1,\dots,\beta_{n-j+m-h}$ into $j+h+1,\dots n+m$, and

$\sigma(j) = j + h$. If we make this permutation in the coordinates of $I^-_{j,h}$ we obtain

$$I^-_{j,h} = \int_{A^{n+m}_{j+h} \cap \{z_\nu > z_\eta\}} (-1)^{j+h-1} \prod_{l=1}^{j+h-1} K(s, z_l) \delta K_{st}(z_{j+h-1}) \times \prod_{l=j+h+1}^{n+m} K(t, z_l) dW_{z_1}(k_1) \cdots dW_{z_{n+m}}(k_{n+m}),$$

where $k_1, \ldots, k_{n+m}$ is a permutation of the indexes $i_1, \ldots, i_n, j_1, \ldots, j_m$ which does not change the orderings of the indexes $i_1, \ldots, i_n$ and and $j_1, \ldots, j_m$, and where ν, η are now defined by

$$\nu = \min\{i \geq j + h + 1 : k_i \in \{j_1, \ldots, j_m\}\}, \quad \eta = \max\{i \leq j + h - 1 : k_i \in \{j_1, \ldots, j_m\}\}.$$

When we sum these integrals over all permutations σ of the above type, and also over j and h, we obtain $\sum_{\bar{k} \in \mathrm{Sh}(\bar{\imath}, \bar{\jmath})} \mathbf{B}^{\mathbf{n+m},1}_{st}(k_1, \ldots, k_{n+m})$, where

$$\mathbf{B}^{\mathbf{n+m},1}_{st}(k_1, \ldots, k_{n+m}) = \sum_{p=1, k_p \in \{i_1, \ldots, i_n\}}^{n+m} \int_{A^{n+m}_p} (-1)^{p-1} \prod_{l=1}^{p-1} K(s, z_l) \delta K_{st}(z_p) \times \prod_{l=p+1}^{n+m} K(t, z_l) dW_{z_1}(k_1) \cdots dW_{z_{n+m}}(k_{n+m}),$$

In a similar manner we could show that the sum of the integrals over $E_{j,h}$ give rise to $\sum_{\bar{k} \in \mathrm{Sh}(\bar{\imath}, \bar{\jmath})} \mathbf{B}^{\mathbf{n+m},2}_{st}(k_1, \ldots, k_{n+m})$, for $h = 1, \ldots, m$, where

$$\mathbf{B}^{\mathbf{n+m},2}_{st}(k_1, \ldots, k_{n+m}) = \sum_{p=1, k_p \in \{j_1, \ldots, j_m\}}^{n+m} \int_{A^{n+m}_p} (-1)^{p-1} \prod_{l=1}^{p-1} K(s, z_l) \delta K_{st}(z_p) \times \prod_{l=p+1}^{n+m} K(t, z_l) dW_{z_1}(k_1) \cdots dW_{z_{n+m}}(k_{n+m}).$$

Taking into account the two contributions $\mathbf{B}^{\mathbf{n+m},1}_{st}$ and $\mathbf{B}^{\mathbf{n+m},2}_{st}$, the proof of the geometric property is now easily finished.

Step 3: Proof of the regularity property. Like in Proposition 3.4, the fact that $\mathbf{B}^{\mathbf{n}}$ belongs to $\mathcal{C}_2^{n\gamma}$ for any $\gamma < H$ is an easy consequence of the moment estimate of Proposition 4.1, plus a simple induction procedure.

Indeed, assume that $\mathbf{B}^{\mathbf{k}} \in \mathcal{C}_2^{k\gamma}((\mathbb{R}^d)^{\otimes k})$ for any $k \leq n - 1$. Then Lemma 2.1 gives here that $\mathcal{N}[\mathbf{B}^{\mathbf{n}}; \mathcal{C}_2^{n\gamma}(\mathbb{R}^{d,d})] \lesssim A + D$, with

$$A = \left(\int_{\mathcal{S}_{2,T}} \frac{|\mathbf{B}^{\mathbf{n}}_{uv}|^{2p}}{|u - v|^{2n\gamma p + 4}} \, du \, dv \right)^{\frac{1}{2p}}, \quad \text{and} \quad D = \mathcal{N}\big[\delta \mathbf{B}^{\mathbf{n}}; \mathcal{C}_3^{n\gamma}(\mathbb{R}^{d,d})\big].$$

Furthermore, since we have seen that $\mathbf{B}^{\mathbf{n}}$ satisfies the multiplicative property (1), then D is easily shown to be almost surely finite thanks to our induction hypothesis. Finally, the quantity $\mathbf{E}[A]$ can be bounded along the same lines as in Proposition 3.4, except that Proposition 4.1 is used instead of Proposition 3.1.

□

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