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► **To cite this version:**

Jean Marc Owo. BACKWARD DOUBLY STOCHASTIC INTEGRAL EQUATIONS OF THE VOLTERRA TYPE. 2009. hal-00413703v2

**HAL Id: hal-00413703**

**<https://hal.science/hal-00413703v2>**

Preprint submitted on 20 Sep 2009 (v2), last revised 13 Aug 2011 (v3)

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## BACKWARD DOUBLY STOCHASTIC INTEGRAL EQUATIONS OF THE VOLTERRA TYPE

JEAN-MARC OWO

ABSTRACT. In this paper, we study backward doubly stochastic integral equations of the Volterra type ( BDSIEVs in short). Under uniform Lipschitz assumptions, we establish an existence and uniqueness result.

### 1. INTRODUCTION

Backward doubly stochastic differential equations (BDSDEs for short) are equations with two different directions of stochastic integrals, i.e., the equations involve both a standard (forward) stochastic integral  $dW_t$  and a backward stochastic integral  $dB_t$ :

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))ds + \int_t^T g(s, Y(s), Z(s))dB_s - \int_t^T Z(s)dW_s.$$

This kind of equation was introduced by Pardoux and Peng [2] in 1994. They proved the existence and uniqueness of solutions for BDSDEs under uniform Lipschitz conditions. Many others investigations concerned BDSDEs were made with weaker conditions namely by Zhou and al. [9] in 2004 with non-Lipschitz assumptions which in turn were weakened by Han Baoyan and al. [3] in 2005 and recently by N'zi and Owo [4] (2008). In [10] (2005) Shi and al. weaken the uniform Lipschitz assumptions to linear growth and continuous conditions by virtue of the comparison theorem that is introduced by themselves. They obtain the existence of solutions to BDSDE but without uniqueness. Pursue their investigations on BDSDEs, N'zi and Owo [5] (2009) obtained recently an existence result with discontinuous conditions. Meanwhile, An other line of researches concerned with backward stochastic integral equations of Volterra type (BSIEVs for short) i.e., equations in form:

$$(1.1) \quad Y(t) = \xi + \int_t^T f(t, s, Y(s), Z(t, s))ds - \int_t^T Z(t, s)dW_s,$$

are lead by Lin [6] in (2002) under global Lipschitz condition on the drift which was recently weakened by A. Aman and M. N'zi [1] in (2005) to local Lipschitz condition.

Recently, general case of BSIEVs (1.1), has been studied by J. Yong [8], 2006 and [8] (2008).

The purpose of this paper is to generalize the theory of Volterra equations to backward doubly stochastic integral equations.

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*Key words and phrases.* Volterra integrals, backward stochastic integral, backward doubly stochastic Volterra integral equations.

Thus, we consider the following equation:

$$(1.2) \quad Y(t) = \xi + \int_t^T f(t, s, Y(s), Z(t, s))ds + \int_t^T g(t, s, Y(s), Z(t, s))dB_s \\ - \int_t^T Z(t, s)dW_s,$$

that we call backward doubly stochastic Volterra integral equations (in short BDSVIEs).

The present paper is organized as follows : in section 2, we deal with notations and recall some preliminaries result concern to BDSDE, the section 3 deals with the main result.

## 2. PRELIMINARIES

**2.1. Notations.** The Euclidean norm of a vector  $x \in \mathbb{R}^k$  will be denote by  $|x|$ , and for an element  $z \in \mathbb{R}^{d \times k}$  considered as a  $d \times k$  matrix, we define its Euclidean norm by  $\|z\| = \sqrt{\text{Tr}(zz^*)}$  and  $\langle z, y \rangle = \text{Tr}(zy^*)$ , where  $y^*$  is the transpose of  $y$ . Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $T$  be a fixed final time. Throughout this paper  $\{W_t; 0 \leq t \leq T\}$  and  $\{B_t; 0 \leq t \leq T\}$  will denote two mutually independent standard Brownian motion processes, with values  $\mathbb{R}^d$  and  $\mathbb{R}^l$ , respectively, defined on  $(\Omega, \mathcal{F}, \mathbb{P})$ .

Let  $\mathcal{N}$  denote the class of  $P$ -null sets of  $\mathcal{F}$ . For each  $(t, s) \in [0, T]^2$ , we define

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B, \text{ and } \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$$

where for any process  $\{x_t\}$ ;  $\mathcal{F}_{u,t}^x = \sigma\{x_r - x_u; u \leq r \leq t\} \vee \mathcal{N}$ ,  $\mathcal{F}_t^x = \mathcal{F}_{0,t}^x$ .

Define  $\mathcal{D} = \{(t, s) \in \mathbb{R}_+^2; 0 \leq t \leq s \leq T\}$  and denote by  $\mathcal{P}$  the  $\sigma$ -algebra of  $\mathcal{F}_{t \vee s}$ -progressively measurable subsets of  $\Omega \times \mathcal{D}$ .

For any  $(k, d) \in \mathbb{N}^{*2}$ , let  $\mathcal{M}^2(t, T, \mathbb{R}^k)$  (resp.  $\mathcal{M}^2(\mathcal{D}, \mathbb{R}^{k \times d})$ ) be the set of  $\mathbb{R}^k$ -valued (resp.  $\mathbb{R}^{k \times d}$ -valued),  $\mathcal{F}_s$ -progressively measurable processes  $\varphi(s)$  which are square-integrable with respect to  $\mathbb{P} \otimes \lambda \otimes \lambda$  (where  $\lambda$  denotes Lebesgue measure over  $[0, T]$ ).

Denote by  $\mathcal{S}^2([t, T], \mathbb{R}^k)$  the set of  $\mathbb{R}^{k \times d}$ -valued,  $\mathcal{F}_s$ -progressively measurable and continuous processes  $\varphi(s)$  such that  $\mathbb{E}(\sup_{t \leq s \leq T} |\varphi(s)|^2) < +\infty$ .

For  $S \in [0, T]$ , set  $\mathcal{D}_{S,T} = \{(t, s) \in \mathbb{R}_+^2; S \leq t \leq s \leq T\}$  and

$$\mathcal{H}^2(\mathcal{D}_{S,T}) = \mathcal{M}^2(S, T, \mathbb{R}^k) \times \mathcal{M}^2(\mathcal{D}_{S,T}, \mathbb{R}^{k \times d}),$$

with the norm

$$\|(y(\cdot), z(\cdot, \cdot))\|_{a, \mathcal{H}^2(\mathcal{D}_{S,T})}^2 = \mathbb{E} \left\{ \int_S^T e^{at} |y(t)|^2 dt + \int_S^T \int_t^T e^{as} \|z(t, s)\|^2 ds dt \right\},$$

where  $a \in \mathbb{R}^+$  will be chosen suitable. For  $a = 0$ ,

$$\|(y(\cdot), z(\cdot, \cdot))\|_{0, \mathcal{H}^2(\mathcal{D}_{S,T})}^2 = \mathbb{E} \left\{ \int_S^T |y(t)|^2 dt + \int_S^T \int_t^T \|z(t, s)\|^2 ds dt \right\}.$$

Note that for  $a > 0$ ,  $\|(y(\cdot), z(\cdot, \cdot))\|_{a, \mathcal{H}^2(\mathcal{D}_{S,T})}^2$  and  $\|(y(\cdot), z(\cdot, \cdot))\|_{0, \mathcal{H}^2(\mathcal{D}_{S,T})}^2$  are equivalent.

Let  $\mathcal{B}_k$  be the Borel  $\sigma$ -algebra of  $\mathbb{R}^k$ .

**2.2. Existence and uniqueness solution of BDSDE.** In this subsection, we recall an existence and uniqueness result for adapted solutions to BDSDE. Let us consider the following BDSDE:  $0 \leq t \leq T$ ,

$$(2.1) \quad Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) dB_s - \int_t^T Z_s dW_s.$$

Let us stand the following assumptions

( $H_1$ ) the maps  $f, g : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k (\mathbb{R}^{k \times l})$  are  $\mathcal{B}([0, T]) \otimes \mathcal{B}_k \otimes \mathcal{B}_{k \times d} \times \mathcal{F}_T$ -measurable such that  $f(\cdot, 0, 0) \in \mathcal{M}^2(0, T, \mathbb{R}^k)$  and  $g(\cdot, 0, 0) \in \mathcal{M}^2(0, T, \mathbb{R}^{k \times l})$ .

( $H_2$ ) there exists two constants  $C > 0$  and  $0 < \alpha < 1$  such that  $\forall (\omega, t) \in \Omega \times [0, T]$  and  $\forall (y_1, z_1), (y_2, z_2) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$

$$|f(t, y_1, z_1) - f(t, y_2, z_2)|^2 \leq C(|y_1 - y_2|^2 + \|z_1 - z_2\|^2)$$

$$\|g(t, y_1, z_1) - g(t, y_2, z_2)\|^2 \leq C|y_1 - y_2|^2 + \alpha \|z_1 - z_2\|^2$$

( $H_3$ )  $\xi$  is a square-integrable  $k$ -dimensional  $\mathcal{F}_T$ -measurable random vector.

**Theorem 2.1** (Pardoux and Peng [2]). *Under hypotheses ( $H_1$ ) and ( $H_2$ ), there exists a unique pair of process  $(Y, Z) \in \mathcal{S}^2([0, T], \mathbb{R}^k) \times \mathcal{M}^2(0, T, \mathbb{R}^{k \times d})$  which satisfies equation (2.1).*

### 3. EXISTENCE AND UNIQUENESS OF THE ADAPTED SOLUTION TO BDSVIE

In this section, we are concerned with solving the following BDSVIE

$$(3.1) \quad Y(t) = \xi + \int_t^T f(t, s, Y(s), Z(t, s)) ds + \int_t^T g(t, s, Y(s), Z(t, s)) dB_s - \int_t^T Z(t, s) dW_s,$$

for  $t \in [0, T]$ .

**Definition 3.1.** A pair of processes  $(Y(\cdot), Z(\cdot, \cdot)) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^k) \times L^2(0, T; L^2_{\mathcal{F}}(0, T; \mathbb{R}^{k \times d}))$  is called adapted solution of (3.1) if it satisfies (3.1), with  $Y(\cdot)$  being  $\mathbb{F}$ -adapted, and  $Z(t, \cdot)$  being  $\mathbb{F}$ -adapted for almost all  $t \in [0, T]$ .

**Hypotheses.** Let us stand the following assumptions ( $H$ ):

( $H'_1$ )  $f : \Omega \times [0, T] \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k$  is  $(\mathcal{P} \otimes \mathcal{B}_k \otimes \mathcal{B}_{k \times d} / \mathcal{B}_k)$ -measurable function satisfying:

(i)  $f(\cdot, \cdot, \cdot, 0, 0) \in \mathcal{M}^2(\mathcal{D}, \mathbb{R}^k)$

(ii) there exists a constant  $C > 0$  such that for all  $(\omega, (t, s)) \in \Omega \times \mathcal{D}$  and for all  $(y_1, z_1), (y_2, z_2) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$ ,

$$|f(\omega, t, s, y_1, z_1) - f(\omega, t, s, y_2, z_2)|^2 \leq C(|y_1 - y_2|^2 + \|z_1 - z_2\|^2)$$

( $H'_2$ )  $g : \Omega \times \mathcal{D} \times \mathbb{R}^k \rightarrow \mathbb{R}^{k \times l}$  is  $(\mathcal{P} \otimes \mathcal{B}_k / \mathcal{B}_{k \times l})$ -measurable function such that

$$g(\cdot, \cdot, 0) \in \mathcal{M}^2(\mathcal{D}, \mathbb{R}^{k \times l})$$

$$\|g(\omega, t, s, y_1, z_1) - g(\omega, t, s, y_2, z_2)\|^2 \leq C|y_1 - y_2|^2 + \alpha \|z_1 - z_2\|^2$$

for all  $(\omega, (t, s)) \in \Omega \times \mathcal{D}$  and for all  $y_1, y_2 \in \mathbb{R}^k$  and  $0 < \alpha < 1$ .

( $H'_3$ )  $\xi$  is a square-integrable  $k$ -dimensional  $\mathcal{F}_T$ -measurable random vector.

Our main result in this paper is the following Theorem.

**Theorem 3.2.** *Let ( $H'_1$ ), ( $H'_2$ ) and ( $H'_3$ ) hold. Then the BDSVIE (3.1) admits a unique adapted solution  $(Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^2[0, T]$  on  $[0, T]$ .*

Before we start proving the theorem, let us establish the same result on  $[S, T]$  for any  $S \in [0, T]$ , in case  $f$  and  $g$  do not depend on  $y$  and  $z$ .

We consider the equation

$$(3.2) \quad Y(t) = \xi + \int_t^T f(t, s)ds + \int_t^T g(t, s)dB_s - \int_t^T Z(t, s)dW_s, \quad t \in [S, T]$$

**Lemma 3.3.** *Let  $(H'_1)$ ,  $(H'_2)$  and  $(H'_3)$  hold. Then, BDSVIE (3.2) admits a unique adapted solution  $(Y(\cdot), Z(\cdot, \cdot)) \in L^2_{\mathcal{F}}(S, T; \mathbb{R}^k) \times L^2(S, T; L^2_{\mathcal{F}}(S, T; \mathbb{R}^{k \times d}))$  for any  $S \in [0, T]$  and*

$$\begin{aligned} & \mathbb{E} \int_S^T |Y(t)|^2 dt + \mathbb{E} \int_S^T \int_t^T |Z(t, s)|^2 ds dt \\ & \leq 3(T-S)\mathbb{E}|\xi|^2 + 3[(T-S) \vee 1] \mathbb{E} \int_S^T \int_t^T (|f(t, s)|^2 + |g(t, s)|^2) ds dt \end{aligned}$$

*Proof.* Let  $S \in [0, T]$ .

First, we consider the following family of BDSDEs parameterized by  $t \in [S, T]$ .

$$(3.3) \quad \lambda^t(r) = \xi + \int_r^T f(t, s)ds + \int_r^T g(t, s)dB_s - \int_r^T \mu^t(s)dW_s, \quad r \in [t, T].$$

For each fixed  $t \in [S, T]$ , by the theorem 2.1, the BDSDE (3.3) admits a unique adapted solution  $(\lambda^t(\cdot), \mu^t(\cdot)) \in \mathcal{S}^2([t, T], \mathbb{R}^k) \times \mathcal{M}^2(t, T, \mathbb{R}^{k \times d})$  on  $[t, T]$ .

Moreover, if  $t \mapsto f(t, s)$  and  $t \mapsto g(t, s)$  are continuous on  $[0, T]$ , then  $(t, s) \mapsto \lambda^t(s)$  is continuous on  $\mathcal{D}_{S, T}$  and  $t \mapsto \mu^t(\cdot)$  is continuous on  $[0, T]$ .

Next, define

$$\begin{cases} Y(t) = \lambda^t(t), & t \in [S, T] \\ Z(t, s) = \mu^t(s), & (t, s) \in \mathcal{D}_{S, T} \end{cases} .$$

Then (3.3) reads :

$$(3.4) \quad Y(t) = \xi + \int_t^T f(t, s)ds + \int_t^T g(t, s)dB_s - \int_t^T Z(t, s)dW_s, \quad t \in [S, T].$$

Thus, we prove the existence of adapted solution to BDSVIE (3.2) on  $[S, T]$ .

For the uniqueness, let us suppose that  $\{(Y'(t), Z'(t, s))\} \in \mathcal{M}^2(S, T, \mathbb{R}^k) \times \mathcal{M}^2(\mathcal{D}_{S, T}, \mathbb{R}^{k \times d})$  is an other adapted solution. Then we have  $\forall t \in [S, T]$

$$(3.5) \quad Y(t) - Y'(t) + \int_t^T [Z(t, s) - Z'(t, s)]dW(s) = 0.$$

Taking  $\mathbb{E}[\cdot | \mathcal{F}_t]$  in (3.5), we get  $Y(t) - Y'(t) = 0, \quad \forall t \in [S, T]$ .

Hence,

$$\mathbb{E} \int_S^T \int_t^T |Z(t, s) - Z'(t, s)|^2 ds dt = \mathbb{E} \int_S^T |Y(t) - Y'(t)|^2 dt = 0.$$

□

**Proof of Theorem 3.2.** For any  $\{(y(t), z(t, s))\} \in \mathcal{M}^2(S, T, \mathbb{R}^k) \times \mathcal{M}^2(\mathcal{D}_{S,T}, \mathbb{R}^{k \times d})$ , we consider the following BDSVIE:

$$(3.6) \quad Y(t) = \xi + \int_t^T f(t, s, y(s), z(t, s)) ds + \int_t^T g(t, s, y(s), z(t, s)) dB_s \\ - \int_t^T Z(t, s) dW_s, \quad t \in [S, T].$$

Thus, by lemma 3.3, BDSVIE (3.6) admits a unique adapted solution  $\{(Y(t), Z(t, s)); (t, s) \in \mathcal{D}_{S,T}\} \in \mathcal{M}^2(S, T, \mathbb{R}^k) \times \mathcal{M}^2(\mathcal{D}_{S,T}, \mathbb{R}^{k \times d})$  and

$$\begin{aligned} & \mathbb{E} \int_S^T |Y(t)|^2 dt + \mathbb{E} \int_S^T \int_t^T |Z(t, s)|^2 ds dt \\ & \leq 3(T-S)\mathbb{E}|\xi|^2 + 3[(T-S) \vee 1] \mathbb{E} \int_S^T \int_t^T (|f(t, s, 0, 0)|^2 + |g(t, s, 0, 0)|^2) ds dt \\ & + 3C(T-S)[(T-S) + 1] \mathbb{E} \int_S^T |y(t)|^2 dt + 3[(T-S)C + \alpha] \mathbb{E} \int_S^T \int_t^T |z(t, s)|^2 ds dt \\ & \leq 3(T-S)\mathbb{E}|\xi|^2 + 3[(T-S) \vee 1] \mathbb{E} \int_S^T \int_t^T (|f(t, s, 0, 0)|^2 + |g(t, s, 0, 0)|^2) ds dt \\ & + M(S, T) \left[ \mathbb{E} \int_S^T |y(t)|^2 dt + \mathbb{E} \int_S^T \int_t^T |z(t, s)|^2 ds dt \right], \end{aligned}$$

where  $M(S, T) = \max\{3C(T-S)[(T-S) + 1]; 3[(T-S)C + \alpha]\}$ .

Hence,

$$\begin{aligned} & \|(Y(\cdot), Z(\cdot, \cdot))\|_{0, \mathcal{H}^2(\mathcal{D}_{S,T})}^2 \\ & \leq 3(T-S)\mathbb{E}|\xi|^2 + 3[(T-S) \vee 1] \mathbb{E} \int_S^T \int_t^T (|f(t, s, 0, 0)|^2 + |g(t, s, 0, 0)|^2) ds dt \\ & + M(S, T) \|(y(\cdot), z(\cdot, \cdot))\|_{0, \mathcal{H}^2(\mathcal{D}_{S,T})}^2. \end{aligned}$$

Let us define a map  $\Theta : \mathcal{H}^2(\mathcal{D}_{S,T}) \rightarrow \mathcal{H}^2(\mathcal{D}_{S,T})$  by

$$(3.7) \quad \Theta(y(\cdot), z(\cdot, \cdot)) = (Y(\cdot), Z(\cdot, \cdot)), \quad \forall (y(\cdot), z(\cdot, \cdot)) \in \mathcal{H}^2(\mathcal{D}_{S,T}),$$

where  $(Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^2(\mathcal{D}_{S,T})$  is the adapted solution to BDSVIE (3.6).

The map  $\Theta$ , as defined, is a contraction when  $T - S > 0$  is small.

Indeed, let  $(y'(\cdot), z'(\cdot, \cdot)) \in \mathcal{H}^2(\mathcal{D}_{S,T})$  and  $(Y'(\cdot), Z'(\cdot, \cdot)) = \Theta(y'(\cdot), z'(\cdot, \cdot))$ .

By lemma 3.3, we know that, for  $(y(\cdot), z(\cdot, \cdot)) \in \mathcal{H}^2(\mathcal{D}_{S,T})$ , the adapted solution to BDSVIE (3.6) on  $[S, T]$  is defined by the unique form:

$$(3.8) \quad \begin{cases} Y(t) = \lambda^t(t), & t \in [S, T] \\ Z(t, s) = \mu^t(s), & (t, s) \in \mathcal{D}_{S,T} \end{cases},$$

where, for each  $t$  fixed in  $[S, T]$ ,  $(\lambda^t(\cdot), \mu^t(\cdot))$  is the unique adapted solution on  $[t, T]$  to the BDSDE:

$$\lambda^t(r) = \xi + \int_r^T f(t, s, y(s), z(t, s)) ds + \int_r^T g(t, s, y(s), z(t, s)) dB_s - \int_r^T \mu^t(s) dW_s,$$

for  $r \in [t, T]$ . Similarly, for  $(y'(\cdot), z'(\cdot, \cdot)) \in \mathcal{H}^2(\mathcal{D}_{S,T})$ , we can represent the solution  $(Y'(\cdot), Z'(\cdot, \cdot))$  by the form:

$$(3.9) \quad \begin{cases} Y'(t) = \lambda^t(t), & t \in [S, T] \\ Z'(t, s) = \mu^t(s), & (t, s) \in \mathcal{D}_{S,T} \end{cases},$$

where, for each  $t$  fixed in  $[S, T]$ ,  $(\lambda^t(\cdot), \mu^t(\cdot))$  is the unique adapted solution on  $[t, T]$  to the BDSDE:

$$\lambda^t(r) = \xi + \int_r^T f(t, s, y'(s), z'(t, s)) ds + \int_r^T g(t, s, y'(s), z'(t, s)) dB_s - \int_r^T \mu^t(s) dW_s,$$

for  $r \in [t, T]$ .

Then for  $r \in [t, T]$ , we have

$$(3.10) \quad \begin{aligned} \lambda^t(r) - \lambda^t(r) &= \int_r^T [f(t, s, y(s), z(t, s)) - f(t, s, y'(s), z'(t, s))] ds \\ &+ \int_r^T [g(t, s, y(s), z(t, s)) - g(t, s, y'(s), z'(t, s))] dB_s \\ &- \int_r^T [\mu^t(s) - \mu^t(s)] dW_s. \end{aligned}$$

Let  $a > 0$  and  $\theta > 0$ . Applying It's formula to  $e^{ar} |\lambda^t(r) - \lambda^t(r)|^2$ , we obtain

$$\begin{aligned} &\mathbb{E} e^{ar} |\lambda^t(r) - \lambda^t(r)|^2 + \mathbb{E} \int_r^T e^{as} |\mu^t(s) - \mu^t(s)|^2 ds + a \mathbb{E} \int_r^T e^{as} |\lambda^t(s) - \lambda^t(s)|^2 ds \\ &= 2 \mathbb{E} \int_r^T e^{as} (\lambda^t(s) - \lambda^t(s)) [f(t, s, y(s), z(t, s)) - f(t, s, y'(s), z'(t, s))] ds \\ &+ \mathbb{E} \int_r^T e^{as} |g(t, s, y(s), z(t, s)) - g(t, s, y'(s), z'(t, s))|^2 ds \\ &\leq \theta \mathbb{E} \int_r^T e^{as} |\lambda^t(s) - \lambda^t(s)|^2 ds + \frac{1}{\theta} \mathbb{E} \int_r^T e^{as} |f(t, s, y(s), z(t, s)) - f(t, s, y'(s), z'(t, s))|^2 ds \\ &+ \mathbb{E} \int_r^T e^{as} |g(t, s, y(s), z(t, s)) - g(t, s, y'(s), z'(t, s))|^2 ds. \end{aligned}$$

Using assumptions  $(H'_1)$  and  $(H'_2)$ , we have

$$(3.11) \quad \begin{aligned} &\mathbb{E} e^{ar} |\lambda^t(r) - \lambda^t(r)|^2 \\ &+ \mathbb{E} \int_r^T e^{as} |\mu^t(s) - \mu^t(s)|^2 ds + a \mathbb{E} \int_r^T e^{as} |\lambda^t(s) - \lambda^t(s)|^2 ds \\ &\leq \theta \mathbb{E} \int_r^T e^{as} |\lambda^t(s) - \lambda^t(s)|^2 ds + C \left( \frac{1}{\theta} + 1 \right) \mathbb{E} \int_r^T e^{as} |y(s) - y'(s)|^2 ds \\ &+ \left( \frac{1}{\theta} C + \alpha \right) \mathbb{E} \int_r^T e^{as} |z(t, s) - z'(t, s)|^2 ds. \end{aligned}$$

For fixed  $t \in [S, T]$ , taking  $r = t$  in (3.11) and using the representation (3.8) and (3.9), we obtain

$$\begin{aligned} & \mathbb{E}e^{at}|Y(t) - Y'(t)|^2 + \mathbb{E} \int_t^T e^{as}|Z(t, s) - Z'(t, s)|^2 ds + a\mathbb{E} \int_t^T e^{as}|\lambda^t(s) - \lambda'^t(s)|^2 ds \\ \leq & \theta\mathbb{E} \int_t^T e^{as}|\lambda^t(s) - \lambda'^t(s)|^2 ds + C\left(\frac{1}{\theta} + 1\right)\mathbb{E} \int_t^T e^{as}|y(s) - y'(s)|^2 ds \\ & + \left(\frac{1}{\theta}C + \alpha\right)\mathbb{E} \int_t^T e^{as}|z(t, s) - z'(t, s)|^2 ds. \end{aligned}$$

Thus, taking  $a > \theta$ , we have for every  $t \in [S, T]$

$$\begin{aligned} & \mathbb{E}e^{at}|Y(t) - Y'(t)|^2 + \mathbb{E} \int_t^T e^{as}|Z(t, s) - Z'(t, s)|^2 ds \\ (3.12) \quad & \leq C\left(\frac{1}{\theta} + 1\right)\mathbb{E} \int_t^T e^{as}|y(s) - y'(s)|^2 ds \\ & + \left(\frac{1}{\theta}C + \alpha\right)\mathbb{E} \int_t^T e^{as}|z(t, s) - z'(t, s)|^2 ds. \end{aligned}$$

By integrating (3.12) from  $S$  to  $T$ , yields

$$\begin{aligned} & \mathbb{E} \int_S^T e^{at}|Y(t) - Y'(t)|^2 dt + \mathbb{E} \int_S^T \int_t^T e^{as}|Z(t, s) - Z'(t, s)|^2 ds dt \\ (3.13) \quad & \leq (T - S)C\left(\frac{1}{\theta} + 1\right)\mathbb{E} \int_S^T e^{as}|y(s) - y'(s)|^2 ds \\ & + \left(\frac{1}{\theta}C + \alpha\right)\mathbb{E} \int_S^T \int_t^T e^{as}|z(t, s) - z'(t, s)|^2 ds dt. \end{aligned}$$

Consequently,

$$\begin{aligned} & \|\Theta(y(\cdot), z(\cdot, \cdot)) - \Theta(y'(\cdot), z'(\cdot, \cdot))\|_{a, \mathcal{H}^2(\mathcal{D}_{S,T})}^2 \\ & \equiv \|(Y(\cdot), Z(\cdot, \cdot)) - (Y'(\cdot), Z'(\cdot, \cdot))\|_{a, \mathcal{H}^2(\mathcal{D}_{S,T})}^2 \\ & \equiv \mathbb{E} \left\{ \int_S^T e^{at}|Y(t) - Y'(t)|^2 dt + \int_S^T \int_t^T e^{as}|Z(t, s) - Z'(t, s)|^2 ds dt \right\} \\ (3.14) \quad & \leq \Lambda \mathbb{E} \left\{ \int_S^T e^{as}|y(s) - y'(s)|^2 ds + \int_S^T \int_t^T e^{as}|z(t, s) - z'(t, s)|^2 ds dt \right\} \\ & \leq \Lambda \|(y(\cdot), z(\cdot, \cdot)) - (y'(\cdot), z'(\cdot, \cdot))\|_{a, \mathcal{H}^2(\mathcal{D}_{S,T})}^2, \end{aligned}$$

where  $\Lambda = \text{Max}((T - S)C(\frac{1}{\theta} + 1); (\frac{1}{\theta}C + \alpha))$

It is easy to see that for  $\theta > 0$  and  $S \in [0, T]$  such that  $\frac{C}{1-\alpha} < \theta < a$  and  $0 < T - S < \frac{\theta}{C(1+\theta)}$ , we have  $\Lambda < 1$ . Thus the map  $\Theta : \mathcal{H}^2(\mathcal{D}_{S,T}) \rightarrow \mathcal{H}^2(\mathcal{D}_{S,T})$  is a contraction. Hence, it admits a unique fixed point  $(Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^2(\mathcal{D}_{S,T})$  which is the unique adapted solution of BDSVIE (3.1) for  $t \in [S, T]$ .

To end the proof, let  $S' \in [0, S]$ .

From the above proof, we know that when  $(t, s) \in \mathcal{D}_{S,T}$  i.e  $(t, s) \in [S, T] \times [t, T]$ , there exists a unique adapted solution, therefore there exists unique  $Y(S)$ .

Now, we consider

$$(3.15) \quad \begin{aligned} \bar{Y}(t) &= Y(S) + \int_t^S f(t, s, \bar{Y}(s), \bar{Z}(t, s))ds + \int_t^S g(t, s, \bar{Y}(s), \bar{Z}(t, s))dB_s \\ &\quad - \int_t^S \bar{Z}(t, s)dW_s, \quad t \in [S', S]. \end{aligned}$$

Using the above procedure, we conclude that the equation (3.15) admits a unique adapted solution  $(\bar{Y}(\cdot), \bar{Z}(\cdot, \cdot)) \in \mathcal{H}^2(\mathcal{D}_{S', S})$  for  $t \in [S', S]$ . Therefore, we can deduce by induction, the existence and uniqueness of an adapted solution in  $\mathcal{H}^2(\mathcal{D})$  to BDSVIE (3.1) on  $\mathcal{D} = \mathcal{D}_{0, T}$   $\square$

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