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► **To cite this version:**

Mohamed Darouach. Complements to full order observer design for linear systems with unknown inputs. Applied Mathematics Letters, 2009, 22 (7), pp.1107-1111. 10.1016/j.aml.2008.11.004 . hal-00413700

HAL Id: hal-00413700

<https://hal.science/hal-00413700>

Submitted on 4 Sep 2009

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Complements to Full Order Observers Design for Linear Systems with Unknown Inputs

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Abstract

This note completes the obtained results in [1]. It presents a full-order observers for linear systems with unknown inputs in the state and in the measurement equations. It gives a more general approach for the observers design than that presented in [1] and it shows that all the obtained results are independent of the choice of the generalized inverses considered in the observers design. Continuous and discrete time systems are considered.

Keywords : Full order observer, Linear systems, Unknown input, Stability, Existence conditions, Strong* detectability.

1 Introduction and preliminaries

The problem of observing the state vector of a deterministic linear time-invariant multivariable system has been the object of numerous studies ever since the original work of Luenberger [4], [5] first appeared. The Observers problem is to reconstruct (or to estimate) the state or a linear combination of the states of the system using the input and output measurements. During the past decades Luenberger observer has been extended to systems with unknown inputs. Many complete results on these observers can be founded in the literature (see [1], [2], [7], [8], [9], [10], [11], [12], [13] and references therein). The problem is of great importance in practice since there are many situations where the disturbances or partial inputs are inaccessible. In these cases a conventional Luenberger observer cannot be applied. Others applications of the unknown inputs observers are in the fault detection and isolation (FDI) problems [9], [11]. This note completes and extends the approach developed in [1] to the observers design for general linear systems for which the unknown input affect also the measurements. The results are related to the strong* detectability developed in [2]. It is also shown that the observers design is invariant to the choice of the generalized inverses. A straightforward algebraic method for the design of these observers is given. Continuous-time as well as discrete-time systems are considered.

Consider the following dynamical system

$$\sigma x = Ax + Fd \tag{1a}$$

$$y = Cx + Gd \tag{1b}$$

where σ denotes the time derivative d/dt for the continuous-time systems and a forward unit time shift δ for the discrete-time systems, $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the output vector and $d \in \mathbb{R}^m$ is the unknown input. Matrices A , F , C , and G are known constant and of appropriate dimensions.

Now define the following sets: $D_c = \{\lambda \in \mathbb{C}, \text{Re}(\lambda) \geq 0\}$, $\bar{D}_c = \{\lambda \in \mathbb{C}, \text{Re}(\lambda) < 0\}$, $D_d = \{\lambda \in \mathbb{C}, |\lambda| \geq 1\}$ and $\bar{D}_d = \{\lambda \in \mathbb{C}, |\lambda| < 1\}$.

The following definitions and theorems are useful for the sequel (see reference [2]).

Definition 1. *The system (1) or the quadruple (A, F, C, G) is strongly detectable if $y(t) = 0$ for $t > 0$ implies $\lim_{t \rightarrow \infty} x(t) = 0$ for all $u(t)$ and $x(0)$.*

Definition 2. The system (1) or the quadruple (A, F, C, G) is strong* detectable if $\lim_{t \rightarrow \infty} y(t) = 0$ implies $\lim_{t \rightarrow \infty} x(t) = 0$ for all $u(t)$ and $x(0)$.

Theorem 1. The system (1) or the quadruple (A, F, C, G) is strongly detectable if and only if all its zeros s satisfy $s \in \bar{D}_c$ in the continuous case and $s \in \bar{D}_d$ in the discrete time case or equivalently

$$\text{rank} \begin{bmatrix} sI - A & -F \\ C & G \end{bmatrix} = n + \text{rank} \begin{bmatrix} F \\ G \end{bmatrix}, \forall s \in D_c \text{ for the continuous case } (\forall s \in D_d \text{ for the discrete case}).$$

Theorem 2. The system (1) or the quadruple (A, F, C, G) is strong* detectable if and only if it is strongly detectable and in addition

$$\text{rank} \begin{bmatrix} CF & G \\ G & 0 \end{bmatrix} = \text{rank } G + \text{rank} \begin{bmatrix} F \\ G \end{bmatrix} \quad (2)$$

Now define a generalized inverse of a matrix M as a matrix M^+ such that $MM^+M = M$. Then, the following theorem gives the general solution X to equation $MXM = M$ [3].

Theorem 3. Let M^+ be a particular generalized inverse of the matrix M , then a general solution to equation $MXM = M$ is given by

$$X = M^+ + \Lambda - M^+M\Lambda MM^+$$

where Λ is an arbitrary matrix of appropriate dimension.

2 Full order unknown input observers design

Consider the linear time-invariant multivariable system described by

$$\sigma x = Ax + Bu + Fd \quad (3a)$$

$$y = Cx + Gd \quad (3b)$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$ are the state and the output vectors of the system, $u \in \mathbb{R}^m$ is the known input and $d \in \mathbb{R}^q$ is the unknown input. Matrices A, F, C , and G are known constant and of appropriate dimensions.

Our aim is to design an observer of the form

$$\sigma \zeta = N\zeta + Jy + Hu \quad (4a)$$

$$\hat{x} = \zeta - Ey \quad (4b)$$

where $\hat{x} \in \mathbb{R}^n$ is the estimate of x . N, J , and E are constant matrices of appropriate dimensions to be determined.

Remark 1 No assumption is made on the rank of the matrices F and G □

Define the following matrix

$$P = I + EC$$

The following proposition gives the conditions for system (4) to be a full order observer for system (3).

Proposition 1. The full-order observer (4) will estimate (asymptotically) $x(t)$ if the following conditions hold

- 1) N is a stability matrix
- 2) $PA - NP - JC = 0$
- 3) $PF - NEG - JG = 0$
- 4) $EG = 0$
- 5) $H = PB$

Proof. Assume that condition 4) of is satisfied, then the observer reconstruction error is

$$e = x - \hat{x} = Px - \zeta + EGd = Px - \zeta \quad (5)$$

then, the dynamic of this estimation error is given by the following equation

$$\sigma e = Ne + (PA - NP - JC)x + (PB - H)u + (PF - NEG - JG)d$$

If conditions 1), 2), 3), and 5) are satisfied, then $\lim_{t \rightarrow \infty} e(t) = 0$ for any $x(0)$, $\hat{x}(0)$, $d(t)$, and $u(t)$. Hence $\hat{x}(t)$ in (4) is an estimate of $x(t)$. This completes the proof. \square

From proposition 1, the design of the observer (4) is reduced to find the matrices N , J , E , and H so that conditions 1) – 5) are satisfied.

Now using the definition of P , equations 2)-4) can be written as

$$N = A + [E \quad K] \begin{bmatrix} CA \\ C \end{bmatrix} \quad (6)$$

$$[E \quad K] \Sigma = \Theta \quad (7)$$

where $K = -J - NE$, $\Sigma = \begin{bmatrix} CF & G \\ G & 0 \end{bmatrix}$ and $\Theta = [-F \quad 0]$.

One can see that upon matrix E is determined, we can deduce the value of matrix H from equation 5) of proposition 1.

The necessary and sufficient condition for the existence of the solution of equation (7) can then be given by the following lemma.

Lemma 1. *The necessary and sufficient condition for the existence of the solution to (7) is given by (2).*

Proof. From the general solution of linear matrix equations [3], there exists a solution to (7) if and only if:

$$\Theta \Sigma^+ \Sigma = \Theta \quad (8)$$

where Σ^+ is any generalized inverse matrix of Σ satisfying $\Sigma \Sigma^+ \Sigma = \Sigma$. Equation (8) is also equivalent to

$$\text{rank} \begin{bmatrix} \Sigma \\ \Theta \end{bmatrix} = \text{rank} [\Sigma] \quad (9)$$

or equivalently

$$\text{rank} \begin{bmatrix} I & 0 & C \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \Sigma \\ \Theta \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & G \\ G & 0 \\ -F & 0 \end{bmatrix} = \text{rank} [\Sigma]$$

which proves the lemma. \square

From [3], under condition (2) the general solution of equation (7) is

$$[E \quad K] = \Theta \Sigma^+ - Z(I - \Sigma \Sigma^+) \quad (10)$$

where Z is an arbitrary matrix of appropriate dimension.

Inserting (10) into (6) yields

$$N = \mathbf{A}_1 - Z\mathbf{B}_1 \quad (11)$$

where

$$\mathbf{A}_1 = A + \Theta \Sigma^+ \begin{bmatrix} CA \\ C \end{bmatrix} \quad (12)$$

and

$$\mathbf{B}_1 = (I - \Sigma \Sigma^+) \begin{bmatrix} CA \\ C \end{bmatrix} \quad (13)$$

The necessary and sufficient condition for the existence of the matrix parameter Z such that N is Hurwitz, which guarantees that condition 1) of proposition 1 is satisfied, is given by the following lemma.

Lemma 2. *Under assumption (2), there exists a matrix Z such that the matrix N in (11) is Hurwitz if and only if the quadruple (A, F, C, G) is strongly detectable.*

Proof. From (10), the matrix N is Hurwitz if and only if the pair $(\mathbf{B}_1, \mathbf{A}_1)$ is detectable or equivalently $\text{rank} \begin{bmatrix} sI - \mathbf{A}_1 \\ \mathbf{B}_1 \end{bmatrix} = n, \forall s \in D_c$ ($s \in D_d$ for the discrete case).

Now, define the following regular matrices $S_1 = \begin{bmatrix} I & 0 & 0 \\ C & I & -sI \\ 0 & 0 & I \end{bmatrix}$ and $S_2 = \begin{bmatrix} I & 0 \\ -\Sigma^+ \begin{bmatrix} C \\ CA \end{bmatrix} & I \end{bmatrix}$ and the full column rank matrix $S_3 = \begin{bmatrix} I & -\Theta\Sigma^+ \\ 0 & I - \Sigma\Sigma^+ \\ 0 & \Sigma\Sigma^+ \end{bmatrix}$, then we have

$$\text{rank} \begin{bmatrix} sI - A & \Theta \\ \begin{bmatrix} CA \\ C \end{bmatrix} & \Sigma \end{bmatrix} = \text{rank } S_1 \begin{bmatrix} sI - A & \Theta \\ \begin{bmatrix} CA \\ C \end{bmatrix} & \Sigma \end{bmatrix} = \text{rank} \begin{bmatrix} sI - A & -F \\ C & G \end{bmatrix} + \text{rank } G \quad (14)$$

On the other hand

$$\text{rank} \begin{bmatrix} sI - A & \Theta \\ \begin{bmatrix} CA \\ C \end{bmatrix} & \Sigma \end{bmatrix} = \text{rank } S_3 \begin{bmatrix} sI - A & \Theta \\ \begin{bmatrix} CA \\ C \end{bmatrix} & \Sigma \end{bmatrix} S_2 = \text{rank} \begin{bmatrix} sI - \mathbf{A}_1 \\ \mathbf{B}_1 \end{bmatrix} + \text{rank } \Sigma \quad (15)$$

by using (2) the lemma is proved. \square

The following lemma proves that the obtained results are independent of the choice of the generalized inverse.

Lemma 3. *Under assumption (2), the observer design is independent of the choice of the generalized inverse.*

Proof. From theorem 3, equation (10) and the fact that $\Theta\Sigma\Sigma^+ = \Theta$, the new matrix \bar{N} obtained for N , by using the general solution to equation $\Sigma X \Sigma = \Sigma$, is given by

$$\bar{N} = \bar{\mathbf{A}} - Z\bar{\mathbf{B}}$$

where $\bar{\mathbf{A}} = \mathbf{A}_1 + \Theta\Lambda\mathbf{B}_1$ and $\bar{\mathbf{B}} = (I - \Sigma\Lambda)\mathbf{B}_1$. In this case one can see that the new matrix \bar{N} has the same form as N . Then, there exists a matrix parameter Z such that \bar{N} is Hurwitz if and only if the pair $(\bar{\mathbf{B}}, \bar{\mathbf{A}})$ is detectable.

Now, define the following full column rank matrices $\bar{S}_2 = \begin{bmatrix} I & 0 \\ -X \begin{bmatrix} C \\ CA \end{bmatrix} & I \end{bmatrix}$, $\bar{S}_3 = \begin{bmatrix} I & -\Theta X \\ 0 & I - \Sigma X \\ 0 & \Sigma X \end{bmatrix}$, where

X is the general solution to $\Sigma X \Sigma = \Sigma$ given by Theorem 3. In this case by replacing S_2 by \bar{S}_2 and S_3 by \bar{S}_3 in (15) we obtain

$$\text{rank} \begin{bmatrix} sI - A & \Theta \\ \begin{bmatrix} CA \\ C \end{bmatrix} & \Sigma \end{bmatrix} = \text{rank } \bar{S}_3 \begin{bmatrix} sI - A & \Theta \\ \begin{bmatrix} CA \\ C \end{bmatrix} & \Sigma \end{bmatrix} \bar{S}_2 = \text{rank} \begin{bmatrix} sI - \bar{\mathbf{A}} \\ \bar{\mathbf{B}} \end{bmatrix} + \text{rank } \Sigma$$

From (14) we can see that the pair $(\bar{\mathbf{B}}, \bar{\mathbf{A}})$ is detectable if and only if the quadruple (A, F, C, G) is strong* detectable. The lemma is proved. \square

From the above results we have the following theorem.

Theorem 4. *The full order observer (4) will estimate (asymptotically) $x(t)$ if the quadruple (A, F, C, G) is strong* detectable.*

The design of the observer (4) can be obtained as follows:

- 1) Verify the strong* detectability of the quadruple (A, F, C, D) .
- 2) Compute \mathbf{A}_1 and \mathbf{B}_1 given by (12) and (13).
- 3) Find matrix Z by pole placement to determine N (11).
- 4) Compute K and E given by (10), then calculate $J = K + NE$ and $H = (I + EC)B$.

Remark 2 If $G = 0$, conditions of theorem 1 or equivalently the strong* detectability of (A, F, C) reduce to: $\text{rank } CF = \text{rank } F$ and $\text{rank} \begin{bmatrix} sI - A & -F \\ C & 0 \end{bmatrix} = \text{rank } F + n$, which correspond to those obtained in [1] when F is of full column rank. \square

The following remark gives an intuitive interpretation of the above observer design.

Remark 3 First we can see that the observer (4) uses only the knowledge of the known inputs and outputs.

Now let us consider system (1), from condition 4) of proposition 1, we note that $EG = 0$, then $Ey = ECx$ and $E\sigma y = EC\sigma x = ECAx + ECBu + ECFd$.

Define the following variables $\bar{y} = \begin{bmatrix} \sigma y \\ y \end{bmatrix}$ and $\bar{d} = \begin{bmatrix} d \\ \alpha \end{bmatrix}$, where $\alpha(t)$ is a fictive variable with the same dimension as the variable d . Then by pre-multiplying \bar{y} by $\begin{bmatrix} E & K \end{bmatrix}$ we obtain the following system

$$\sigma x = Ax + Bu + Fd = Ax + Bu - \Theta \bar{d} \quad (16a)$$

$$\begin{bmatrix} E & K \end{bmatrix} \bar{y} = \begin{bmatrix} E & K \end{bmatrix} \begin{bmatrix} ECA \\ C \end{bmatrix} x + \begin{bmatrix} E & K \end{bmatrix} \begin{bmatrix} CB \\ 0 \end{bmatrix} u + \begin{bmatrix} E & K \end{bmatrix} \Sigma \bar{d} \quad (16b)$$

From equation (7) we have $\begin{bmatrix} E & K \end{bmatrix} \Sigma = \Theta$, then from (16b) we obtain

$$\Theta \bar{d} = \begin{bmatrix} E & K \end{bmatrix} \bar{y} - \begin{bmatrix} E & K \end{bmatrix} \begin{bmatrix} CA \\ C \end{bmatrix} x - \begin{bmatrix} E & K \end{bmatrix} \begin{bmatrix} CB \\ 0 \end{bmatrix} u$$

Inserting this value into equation (16a) gives

$$\sigma x = Nx + PBu - \begin{bmatrix} E & K \end{bmatrix} \bar{y}$$

where $N = A + \begin{bmatrix} E & K \end{bmatrix} \begin{bmatrix} CA \\ C \end{bmatrix}$ and $P = I + \begin{bmatrix} E & K \end{bmatrix} \begin{bmatrix} C \\ 0 \end{bmatrix}$. Let $x = \zeta - Ey$ and $J = -K - NE$, we obtain the following system

$$\sigma \zeta = N\zeta + PBu + Jy \quad (17a)$$

$$x = \zeta - Ey \quad (17b)$$

which is the observer (4). \square

3 Conclusion

In this note, we have presented a complement to the results obtained in [1] for the full-order observers design for linear multivariable systems. The existence and stability conditions are given, and generalize those adopted in [1].

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