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Abstract

This paper investigates the nonlinear analysis, in the form of nonlinear modes, of mechanical systems undergoing unilateral frictionless contact. The nonlinear eigenproblem is introduced by a Rayleigh quotient minimization with inequality constraints formulated in the frequency domain. An augmented Lagrangian procedure is used for the calculation of the nonlinear contact forces. The efficiency of the proposed method for large-scale mechanical systems involving non-smooth nonlinear terms is shown. An industrial application consisting of a compressor blade in contact with a rigid casing is proposed. Sensitivity of the nonlinear modal parameters to contact is illustrated.

1 Introduction

The concept of nonlinear modes is now commonly accepted as a relevant approach for the analysis of nonlinear dynamical systems [1]. While the variety of analytical methods developed during the past decades have contributed to strengthen the theoretical background on nonlinear modes, new challenges appear when it comes to applying this tool to large scale systems [2]. An encouraging perspective is however brought by numerical methods such as invariant manifold approach [3], asymptotic-numerical methods [4], shooting and continuation [5] or Fourier methods [6].

Dealing with non-smooth nonlinearities yields other challenges both on numerical analysis and phenomenological aspects. Concerning contact nonlinearities, difficulties come from their essential non-smooth nature. As contact is defined by a multi-valued law which takes the form of inequality constraints, solution methods are often related to optimization [7,8]. Dual formulations then provide physical sense to Lagrange multipliers. When it comes to numerical formulations, a popular and robust approach to deal with contact constraints is to introduce augmented Lagrangian functionals [9–11].

In contact dynamics, most integrated approaches are based on time integration and alternatives are unfortunately seldom explored [12, 13]. Thus, considering analysis of contact dynamics in the framework of nonlinear modes seems an interesting yet challenging approach. In this paper, we investigate the dynamics of linear elastic and autonomous flexible structures with contact interfaces. The target application concerns rotating components in turbomachinery which can be subject to intermittent contacts (or impacts) with surrounding stationary structures. Complex phenomena often occurs in such applications [14–16] and a modal approach appears interesting in order to describe their essential nature in particular.

The proposed approach is first described in a continuum mechanics formalism in which the problem of finding eigensolutions can be regarded as a mixed frequency-time boundary value problem. The solution is sought in the form of a Fourier series and the eigenvalue problem reduces to finding critical values of a generalized Rayleigh quotient. An inequality constrained optimization problem then arises which is solved using an augmented Lagrangian approach. Then, approximations for numerical treatments and algorithm are described.

A numerical application is proposed in which the first nonlinear mode of a turbomachinery compressor blade with contact occurring at its tip edge is explored. Effects of contact on modal parameters are investigated.

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2   Theoretical formulation

A flexible structure in intermittent contact with a rigid foundation is investigated. Since attention is given to eigensolutions of the nonlinear problem [1], no external forces are applied such as the final system of equations is autonomous. Usual linear damping is neglected as well as friction.

   Two main difficulties arise. First, the non-differentiability of the non-smooth contact conditions with respect to the displacement field, makes any linearization impossible. Second, as in every modal formulation, displacements and frequency of the motion are to be determined.

2.1   Unilateral frictionless contact formulation

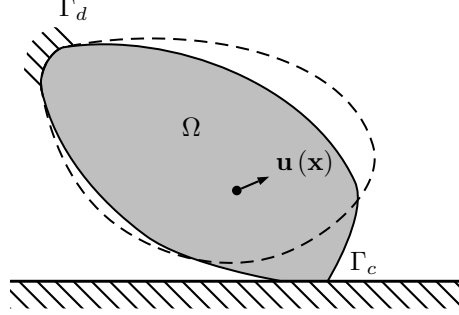A flexible component in contact with a rigid foundation is considered as illustrated in Fig. 1. Contact

Figure 1: Notations for contact formulation

forces acting on interface Γ_c are defined according to a continuous gap function separating the possibly contacting bodies:

$$\mathbf{g}(\mathbf{u}) = \mathbf{u}(\mathbf{x}) \cdot \mathbf{n} - \mathbf{g}_0(\mathbf{x}) \tag{1}$$

where $g_0(\mathbf{x})$ represents the initial positive gap, $\mathbf{n}$, the outward normal to Γ_c, and $\mathbf{u} = (u_1, u_2, u_3)$ the displacement field of a material point $\mathbf{x} = (x_1, x_2, x_3)$. Contact conditions [17], referred to as Kuhn-Tucker optimality conditions are such that for all $\mathbf{x} \in \Gamma_c$:

$$\tau_N \leq 0, \ \ \mathbf{g}(\mathbf{u}) \leq 0, \ \ \mathbf{g}(\mathbf{u}) \cdot \tau_N = 0 \tag{2}$$

where $\tau_N = \sigma \cdot \mathbf{n}$ stands for the contact pressure acting on Γ_c.

2.2   Boundary value problem

The linearized strain tensor mapping displacements to deformations is defined as $\epsilon_{kl} = \frac{1}{2}\left(\partial u_k/\partial x_l + \partial u_l/\partial x_k\right)$ and the constitutive linear Hooke law, as $\sigma_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij}$. The strong formulation of the problem with Dirichlet and Signorini boundary conditions takes the following form:

$$\rho\ddot{\mathbf{u}} - \operatorname{div}\boldsymbol{\sigma}(\mathbf{u}) = 0 \quad \text{on } \Omega \times \mathbb{R}_*^+ \tag{3a}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_d \times \mathbb{R}_*^+ \tag{3b}$$

$$\mathbf{g}(\mathbf{u}) \leq 0, \quad \tau_N \leq 0 \ \ \text{and} \ \ \mathbf{g}(\mathbf{u}) \cdot \tau_N = 0 \quad \text{on } \Gamma_c \times \mathbb{R}_*^+ \tag{3c}$$

   Since the subsequent developments involve variational forms, we introduce the space of admissible displacement fields:

$$V = \left\{ \mathbf{u} = (u_1, u_2, u_3), \ u_i \in H^1(\Omega), \ \mathbf{u} = 0 \text{ on } \Gamma_d \right\} \tag{4}$$

equipped with a classical inner product (denoted $\langle .,. \rangle$ in the following), and the usual linear forms on $V \times V$:

$$m(\mathbf{u}, \mathbf{v}) = \int_\Omega \rho \mathbf{u} \mathbf{v} \, d\mathbf{x} \tag{5a}$$

$$k(\mathbf{u}, \mathbf{v}) = \int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v}) \, d\mathbf{x} \tag{5b}$$

2

2.3   Mixed frequency-time eigenvalue problem

We are interested in finding periodic and non-trivial solutions of problem (3). As periodic but not necessarily , it seems natural to expand the displacement field in the frequency-domain using Fourier series:

$$\mathbf{u}(t) = \sum_{n \in \mathbb{Z}} \hat{\mathbf{u}}_n e^{jn\omega t} \text{ with } \hat{\mathbf{u}}_n = \frac{1}{T} \int_T \mathbf{u}(t) e^{-jn\omega t} \, dt \tag{6}$$

in which $\omega = 2\pi/T$ is the fundamental eigenfrequency of the motion to be determined.

The eigenvalue problem consists in finding $\{\omega, \hat{\mathbf{u}}\}$, with $\hat{\mathbf{u}} = \{\hat{\mathbf{u}}_n, \ n \in \mathbb{Z}\}$ such as,

$$-\operatorname{div} \boldsymbol{\sigma}(\hat{\mathbf{u}}_n) = (n\omega)^2 \rho \hat{\mathbf{u}}_n \quad \text{on } \Omega \times \mathbb{Z} \tag{7a}$$

$$\hat{\mathbf{u}}_n = 0 \quad \text{on } \Gamma_d \times \mathbb{Z} \tag{7b}$$

$$\mathbf{g}(\hat{\mathbf{u}}) \leq 0, \quad \tau_N \leq 0 \quad \text{and} \quad \mathbf{g}(\hat{\mathbf{u}}) \cdot \tau_N = 0 \quad \text{on } \Gamma_c \times [0, T] \tag{7c}$$

in which the gap function is now expressed using Eq. (6) as:

$$\mathbf{g}(\hat{\mathbf{u}}) = \sum_{n \in \mathbb{Z}} \hat{\mathbf{u}}_n \cdot \mathbf{n} \ e^{jn\omega t} - \mathbf{g}_0 \tag{8}$$

Note that while Eqs. (7a) and (7b) are formulated using frequency-domain fields, Eq. (7c) involves time-domain fields only. Thus Problem (7) can be regarded as a mixed time-frequency domains boundary value problem.

2.4   Rayleigh quotient and optimization

Based on the previous eigenvalue problem and using the linear forms defined in Eqs. (5), we introduce the following generalized Rayleigh quotient:

$$\mathrm{r}(\hat{\mathbf{u}}) = \sum_{n \in \mathbb{Z}} \frac{\mathrm{k}(\hat{\mathbf{u}}_n, \hat{\mathbf{u}}_n)}{\mathrm{m}(\hat{\mathbf{u}}_n, \hat{\mathbf{u}}_n)} \tag{9}$$

An eigenvector $\hat{\mathbf{u}}$ is a critical point of this Rayleigh quotient and the associated eigenvalue is such that $\omega^2 = \mathrm{r}(\hat{\mathbf{u}})$. Here, due to the Signorini conditions, critical points of the Rayleigh quotient are solutions of a constrained minimization problem:

$$\min_{\hat{\mathbf{u}}_n \in V} \mathrm{r}(\hat{\mathbf{u}}) \quad \text{s.t.} \quad \mathbf{g}(\hat{\mathbf{u}}) \leq 0 \ \text{ on } \Gamma_c \quad \forall t \in [0, T] \tag{10}$$

Optimality conditions for this constrained minimization problem imply that there exists a Lagrange multiplier time-function $\lambda(t) > 0$ such that:

$$\nabla_{\hat{\mathbf{u}}_n} \mathrm{r}(\hat{\mathbf{u}}) + \int_T \langle \lambda(t), \nabla_{\hat{\mathbf{u}}_n} \mathbf{g}(\hat{\mathbf{u}}) \rangle \, dt = 0 \quad \forall n \in \mathbb{Z} \tag{11a}$$

$$\lambda(t) > 0 \ \text{ and } \ \langle \lambda(t), \mathbf{g}(\hat{\mathbf{u}}) \rangle = 0 \quad \forall t \in [0, T] \tag{11b}$$

The constraint $\mathbf{g}(\mathbf{u}) \leq 0$ has been transferred to the Lagrange multiplier $\lambda(t)$ and the integral in Eq. (11a) can be regarded as a continuous equivalent to traditional weighted sum of individual constraints by the Lagrange multipliers.

Expanding Eqs. (11) leads to a variational formulation:

Find $\{\hat{\mathbf{u}}_n, \ n \in \mathbb{Z}\}$ *such as* $\forall \mathbf{v} \in V$

$$2 \, \mathrm{m}(\hat{\mathbf{u}}_n, \hat{\mathbf{u}}_n)^{-1} \left(\mathrm{k}(\hat{\mathbf{u}}_n, \mathbf{v}) - n^2 \omega^2(\hat{\mathbf{u}}) \, \mathrm{m}(\hat{\mathbf{u}}_n, \mathbf{v}) \right) + \int_T \langle \boldsymbol{\lambda}, \mathbf{v} \rangle e^{jn\omega t} \, dt = 0 \quad \forall n \in \mathbb{Z} \tag{12a}$$

$$\boldsymbol{\lambda} > 0 \ \text{ and } \ \langle \boldsymbol{\lambda}, \mathbf{g}(\hat{\mathbf{u}}) \rangle = 0 \quad \forall t \in [0, T] \tag{12b}$$

2.5 Augmented Lagrangian

The technique of augmented Lagrange multipliers converts an initially constrained minimization problem into a saddle point problem for which the addition of penalty terms improves convergence and benefits computationally due to subsequent convex properties. The respective Uzawa-like solution algorithm performs descent on the primal displacement variable and ascent on the dual Lagrange multipliers (coinciding with contact forces when convergence is reached).

For problem (10), an augmented Lagrangian can be defined as [18]:

$$L_\kappa\left(\hat{\mathbf{u}}, \theta\right) = \mathrm{r}\left(\hat{\mathbf{u}}\right) + \frac{1}{2}\int_T \|\sqrt{\kappa}\left(\mathbf{g}\left(\hat{\mathbf{u}}\right) + \theta\right)\|_+^2 \, \mathrm{d}t \tag{13}$$

in which $\kappa\theta$ represents a Lagrange multiplier and κ, a diagonal matrix of positive penalty parameters.

During an iteration along the primal variable, the solution algorithm holds estimated values of dual parameters θ and κ and searches for $\hat{\mathbf{u}}\left(\theta, \kappa\right)$ that minimizes the augmented Lagrange functional (13) such as:

$$\nabla_{\hat{\mathbf{u}}_n} L_\kappa\left(\hat{\mathbf{u}}, \theta\right) = 0 \quad \forall n \in \mathbb{Z} \tag{14}$$

or, equivalently:

Find $\{\hat{\mathbf{u}}_n, \ n \in \mathbb{Z}\}$ such as $\forall \mathbf{v} \in V$

$$2\,\mathrm{m}\left(\hat{\mathbf{u}}_n, \hat{\mathbf{u}}_n\right)^{-1}\left(\mathrm{k}\left(\hat{\mathbf{u}}_n, \mathbf{v}\right) - n^2\omega^2\left(\hat{\mathbf{u}}\right)\mathrm{m}\left(\hat{\mathbf{u}}_n, \mathbf{v}\right)\right) + \int_T \langle \kappa\left(\mathbf{g}\left(\hat{\mathbf{u}}\right) + \boldsymbol{\theta}\right)_+, \mathbf{v}\rangle e^{\mathrm{j}n\omega t}\,\mathrm{d}t = 0 \tag{15}$$

Parameter θ is then adjusted as follows:

$$\theta \leftarrow \theta + \max_{\mathbf{x},t}\left(\mathbf{g}\left(\hat{\mathbf{u}}\right), -\theta\right) \tag{16}$$

while the penalty coefficient κ can be updated if condition (16) did not improve the constraints.

2.6 Mode normalization

Since the frequency of the periodic motion is an unknown of the eigenvalue problem, Eq. (15) has more unknowns than equations. This under-determination can be overcome by conducting a mode normalization through one (since the system is conservative) master coordinate which amplitude defines the modal amplitude and which governs all remaining dependent variables. This amplitude is denoted by q and modal parameters $\hat{\mathbf{u}}_n(q)$ and $\omega(q)$ are now fully defined by its value.

3 Numerical approximations and algorithm

In the framework of finite-element method, we consider a discretized displacement field $\mathbf{U}$ together with consistent mass $\mathbf{M}$ and stiffness $\mathbf{K}$ structural matrices based on (5). Contact constraints (1) can be re-written as:

$$\mathbf{g}\left(\mathbf{U}\right) = \mathbf{A}\mathbf{U} - \mathbf{g}_0 \tag{17}$$

where rectangular matrix $\mathbf{A}$ restricts the displacement vector $\mathbf{U}$ to contact degrees-of-freedom.

Furthermore, a uniform time discretization is introduced:

$$t = \{t_k = kT/m, \ k = 1,\ldots,m\} \tag{18}$$

such as $\bar{\mathbf{u}}_k = \mathbf{u}\left(t_k\right)$ are discrete time values of $\mathbf{u}$ and the Fourier series of Eq. (6) is truncated up to N-th order. Hence, frequency-domain and time-domain variables are now related to each other in a discrete form:

$$\hat{\mathbf{u}}_n = \frac{1}{T}\sum_{k=1}^m \bar{\mathbf{u}}_k e^{-\mathrm{j}\frac{2\pi kn}{m}} \quad \text{and} \quad \bar{\mathbf{u}}_k = \sum_{n=-N}^N \hat{\mathbf{u}}_n e^{\mathrm{j}\frac{2\pi kn}{m}} \tag{19}$$

In practical applications, it is usually required to have N significantly smaller than m.

The final eigenvalue problem is obtained from Eq. (15) in which space, time and frequency approximations are introduced and then reduces to $2N + 1$ coupled sub-problems:

$$2\mathbf{M}^{-1}\left(\mathbf{K} - (n\omega)^2\mathbf{M}\right)\hat{\mathbf{U}}_n + \mathbf{A}^T\sum_{k=1}^{m}\kappa\left(\mathbf{g}\left(\hat{\mathbf{U}}\right) + \theta_k\right)_+ e^{\mathrm{j}\frac{2\pi kn}{m}} = 0 \tag{20}$$

The proposed algorithm 1 based on the one detailed in [19] also features Powell's measure of convergence:

$$G = \max_{\mathrm{dof}}\left|\max_k\left(\mathbf{g}\left(\bar{\mathbf{u}}_k\right), -\theta_k\right)\right| \tag{21}$$

and an adaptive procedure for updating penalty parameters and Lagrange multipliers.

Algorithm 1 Frequency-time domain augmented Lagrangian

Set $i = 0$, $i_{\max}$, $G^{(0)} = \infty$, $\theta = \theta^{(1)}$, $\kappa = \kappa^{(1)}$
Choose $\alpha > 1$, $\beta > 1$, $\epsilon > 0$
while $i < i_{\max}$ **do**
    Find $\{\hat{\mathbf{u}}_n\}_{n=-N,\dots,N}$ solution of Eq. (20)
    Evaluate $\{\mathbf{g}(\bar{\mathbf{u}}_k)\}_{k=1,\dots,m}$
    Set $\mathbf{L} = \{l : |\max(\mathbf{g}(\bar{\mathbf{u}}_k), -\theta_k)| \geq G^{(i)}/\alpha\}$
    Set $\bar{G} = \max_{\mathrm{dof}}|\max_k(\mathbf{g}(\bar{\mathbf{u}}_k), -\theta_k)|$
    if $\bar{G} \geq G^{(i)}$ **then**
        $\forall l \in \mathbf{L}$, $\kappa_{k,l} \leftarrow \beta\kappa_{k,l}$ and $\theta_{k,l} \leftarrow \theta_{k,l}/\beta$
    else
        $i \leftarrow i + 1$
        $\theta_k^{(i)} \leftarrow \theta_k$, $\kappa_k^{(i)} \leftarrow \kappa_k$ and $G^{(i)} \leftarrow \bar{G}$
        if $G^{(i)} \leq \epsilon$ **then**
            break
        end if
        $\theta_k \leftarrow \theta_k^{(i)} + \max(\mathbf{g}(\bar{\mathbf{u}}_k), -\theta_k)$
        if $G^{(i)} \geq G^{(i-1)}/\alpha$ **then**
            $\forall l \in \mathbf{L}$, $\kappa_{k,l} \leftarrow \beta\kappa_{k,l}$
        end if
    end if
end while

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4   Analysis of contact in compressor blades

As an illustration of the proposed technique, we consider a turbomachinery compressor blade with a rigid contact interface at its tip edge. The finite element model of this blade is shown in Fig. 2; nodes in red define the contact interface. This model is reduced by means of a Craig-Bampton procedure composed of 72 constraints modes associated with displacements of 24 retained nodes belonging to contact interface and 40 component normal modes. Contact occurs in the radial direction only and a uniform initial gap is assumed. In the sequel, all displacements and velocities are normalized with respect to this initial gap and nonlinear frequencies are normalized with respect to the first linear frequency of the blade.

Figs. 3 display modal parameters of the first nonlinear mode (associated with first bending mode of the blade). In Fig. 3a, which represents the evolution of eigenfrequency as function of modal amplitude, two particular regions with distinct dynamical behaviors can be distinguished. The first refers to modal amplitude values such as the gap is always open and is characterized by a linear dynamics and constant frequency. The second, when the modal amplitude is such that the contact can occur at the interface, is highly nonlinear. The frequency-dependent amplitude phenomenon with stiffening effects is retrieved. Fig. 3b shows a manifold for this mode, that is the displacement of one nonlinear coordinate plotted versus the displacement and velocity of a linear one (the master coordinate here). The effect of contact, which limits the amplitude of the nonlinear coordinate, is noticeable.
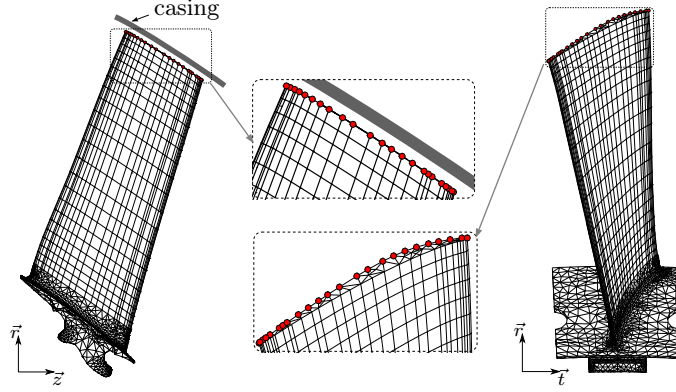
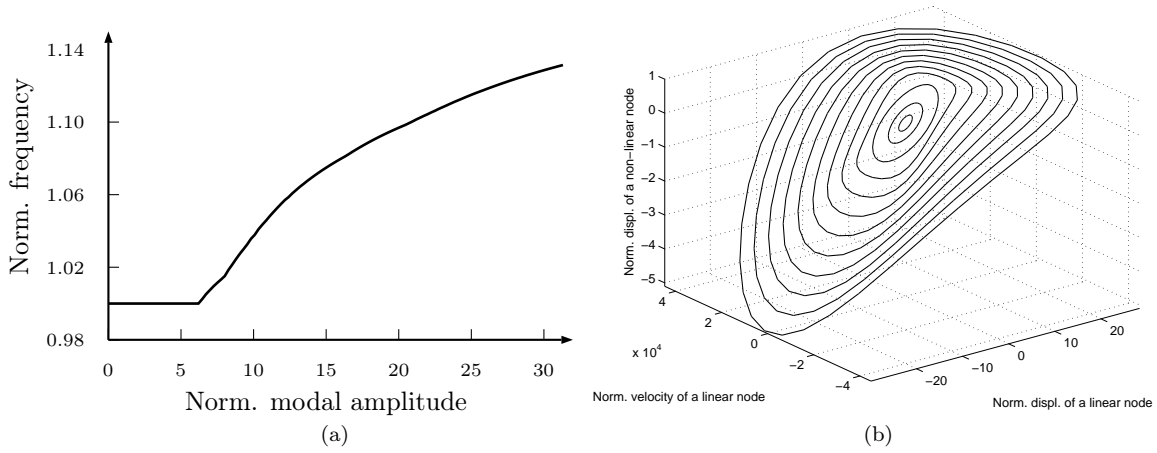Figure 2: Blade model and details of the contact interface

(a)

(b)

Figure 3: Modal parameters of the first nonlinear mode: (a) eigen-frequency / modal amplitude curve and (b) manifold.

In order to investigate in more details the effects of contact on modal parameters, Figs. 4 display the evolution of radial displacements of all edge nodes as function of time (one period) and for three selected values of the modal amplitude (with respect to Fig. 3a). Note that contact occurs when the radial displacements reach a unit value due to the chosen normalization. In Fig. 4a, the modal amplitude has not reached the limit contact amplitude yet and no contact occurs. For the amplitude considered in Fig. 4b, contact occurs on the two last two nodes of the edge and during about 15% of time in one period. In the largest modal amplitude depicted Fig. 4c, contact occurs on the last 4 nodes and also on the first 5 nodes of the edge and these two contact region are out of phase. These observations bring further explanations to the stiffening effects of contact which was noticed on Fig. 3a. Indeed, stiffening is related to both the size of the actual contact interface (*i.e.* the number of nodes for which contact occurs) and the duration of contact.

Finally, Figs. 5 display the mode shapes on the finite element mesh for the three previous modal amplitude values. The changes in shape due to contact are substantial in particular during initial times of the period and can result in stress concentrations.

5   Conclusions

A methodology for modal analysis of elastic structures with contact interfaces was proposed. Based on the definition of nonlinear modes as periodic solutions of the autonomous dynamical system, the
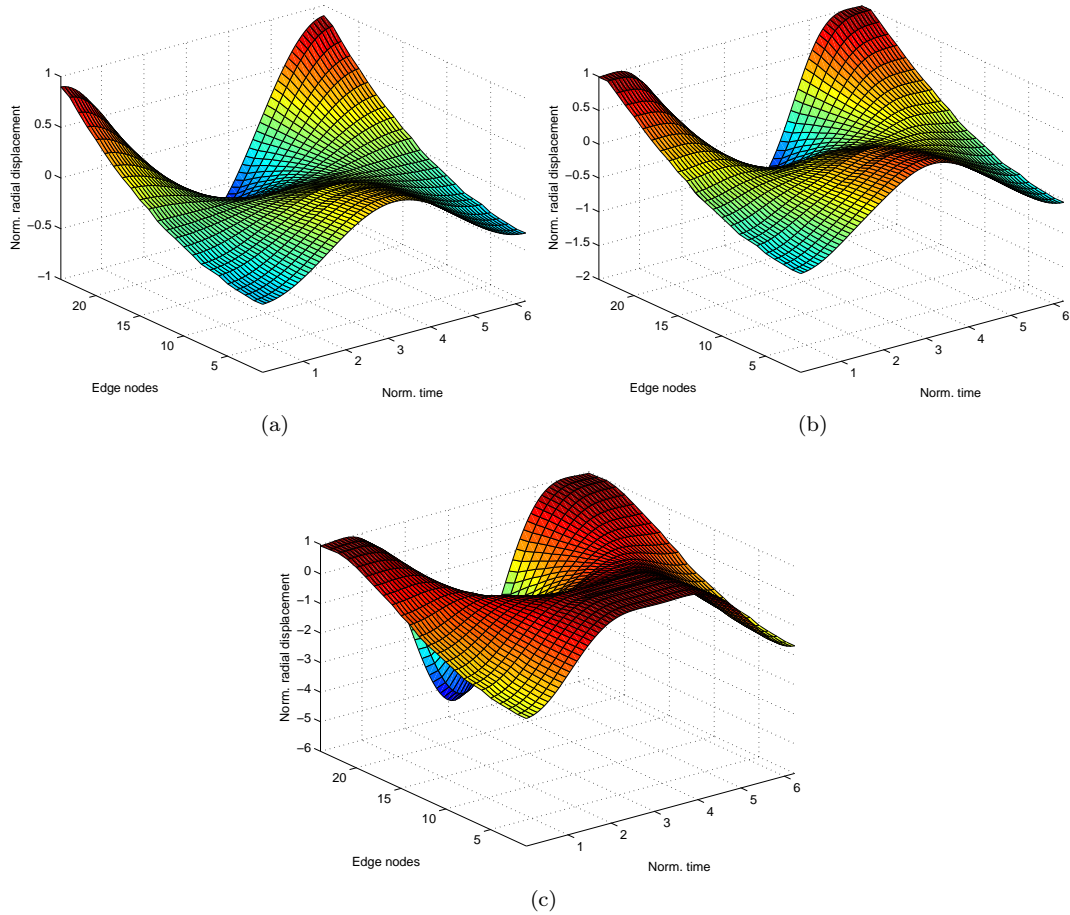
Figure 4: Radial displacements of edge nodes for three modal amplitudes: (a) 5.7, (b) 9.3 and (c) 31.3

eigenvalue problem is defined in a continuum mechanics framework with Signorini boundary conditions. A generalized Rayleigh quotient functional is introduced, which minimization under constraints is formulated using Lagrange multipliers and solved using an augmented Lagrangian approach. Numerical approximations in space, time and frequency are then introduced for the sake of numerical treatments.

As an application, a turbomachinery compressor blade in contact with a rigid casing is considered. The first nonlinear mode is studied using the proposed approach and the influence of contact (which depends on the amplitude of the motion) is investigated on modal parameters such as frequency and shapes.

Based on these encouraging developments, future works will focus on further investigations on nonlinear dynamics to include stability and bifurcations analysis and also more complex systems which include several deformable bodies in respective contacts. This tool can be used to predict dangerous interactions zones between blades and surrounding casings. In the future, friction effects and as well as wear of contact interfaces can also be studied.
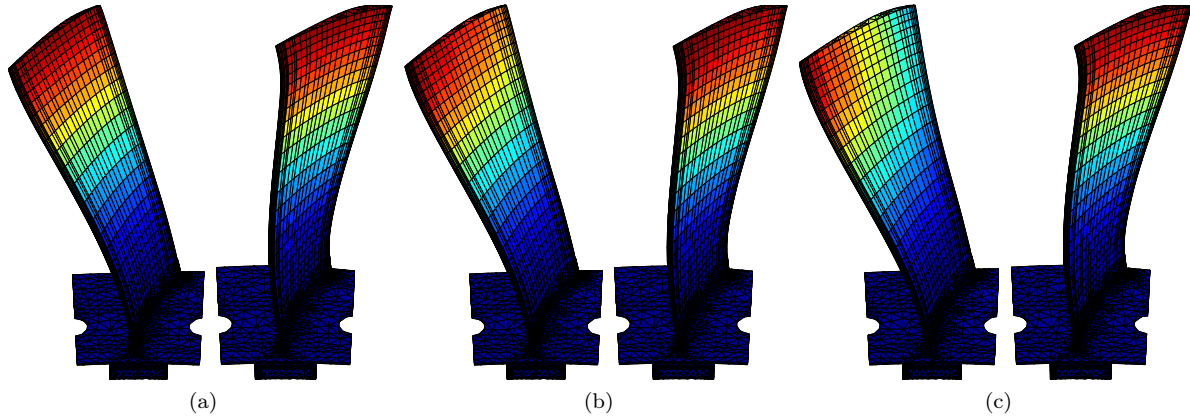
Acknowledgment

Figure 5: Mode shapes for three modal amplitudes: (a) 5.7, (b) 9.3 and (c) 31.3 and at $t = 0$ and $t = T/2$.

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