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# Asymptotic near-efficiency of the “Gibbs-energy and empirical-variance” estimating functions for fitting Matérn models to a dense (noisy) series

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## Abstract

Let us call as “Gaussian Gibbs energy” the quadratic form appearing in the maximum likelihood (ML) criterion when fitting a zero-mean multidimensional Gaussian distribution to one realization. We consider a continuous-time Gaussian process  $Z$  which belongs to the Matérn family with known regularity index  $\nu \geq 1/2$ . For estimating the range and the variance of  $Z$  from observations on a dense regular grid, corrupted by a Gaussian white noise of known variance, we propose two simple estimating functions based on the conditional Gibbs energy mean (CGEM) and the empirical variance (EV). We show that the ratio of the large sample mean squared error of the resulting CGEM-EV estimate of the range-parameter to the one of its ML estimate, and the analog ratio for the variance-parameter, both converge (when the grid-step tends to 0) toward a constant, only function of  $\nu$ , surprisingly close to 1 provided  $\nu$  is not too large. This latter condition on  $\nu$  has not to be imposed to obtain the convergence to 1 of the analog ratio for the microergodic-parameter. Possible extensions of this approach, which may benefit from very easy numerical implementations, are briefly discussed.

## 1. Introduction

We consider time-series of length  $n$  obtained by observing on a dense regular grid, a continuous-time process  $Z$  which is Gaussian, has mean zero and an autocorrelation function which belongs to the Matérn family with regularity index  $\nu \geq 1/2$ . This family is commonly used, for instance in geostatistics; see e.g. Stein (1999), Guttorp and Gneiting (2006), Gaetan and Guyon (2010). Recall that  $\nu = 1/2$  correspond to the well known exponential autocorrelation (in other

words,  $Z$  is a stationary Ornstein-Uhlenbeck (O.U.) process). The definition of Matérn processes on  $\mathbb{R}$  can be easily formulated in terms of the Fourier transform of their autocorrelation function, namely the spectral density over  $(-\infty, +\infty)$ :

$$f_{\nu, b, \theta}^*(\omega) = b g_{\nu, \theta}^*(\omega), \quad \text{with } g_{\nu, \theta}^*(\omega) := \frac{C_\nu \theta^{2\nu}}{(\theta^2 + \omega^2)^{\frac{1}{2} + \nu}}. \quad (1.1)$$

In this paper the constant  $C_\nu = \frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}\Gamma(\nu)}$  is chosen so that  $\int_{-\infty}^{\infty} g_{\nu, \theta}^*(\omega) d\omega = 1$ . Thus  $b$  is the variance of  $Z(t)$  and  $\theta$  is the so-called “inverse-range parameter” (in fact, it is  $\nu^{1/2}/\theta$  which can be interpreted as an effective range or “correlation length” independently of  $\nu$ , cf Stein (1999, Section 2.10); we will often drop the term “inverse”).

We are concerned here with dense grid in the sense that the interval  $\delta$  between two successive observations is small relatively to  $1/\theta$ . The considered processes being mean square continuous, this means that two successive ideal (i.e. without measurement error) observations are “strongly” correlated. See Stein (1999, Chapter 3), for such a setting which shows that standard large- $n$  asymptotic analysis followed by a (less standard) small- $\delta$  analysis yields useful insights and good approximations for various real (finite-size) problems.

In this article, we assume that  $\nu$  is known and we mainly study settings where there are Gaussian i.i.d. measurement errors, or, equivalently for the parametric inference point of view we take here, there is a so-called nugget effect. Notice that introducing such a nugget effect is actually not too restrictive since, of course, data always have a finite number of digits, so one may easily accept to include, in the parametric model, a nugget with known variance equal to the squared finite precision; furthermore this is known to often remedy to numerical instabilities due to ill-conditioning of the linear systems we may encounter in the no-nugget case. However, in this first study, we restrict ourselves to the case where the suspected measurement errors have a known common variance. We thus assume that (possibly after appropriate rescaling) the observations is a vector  $\mathbf{y}$  of size  $n$  whose conditional law, given  $Z$ , is

$$\mathbf{y} | \mathbf{z} \sim N(\mathbf{z}, I_n) \text{ where } \mathbf{z} \sim N(0, b_0 R_{\theta_0}),$$

with  $I_n$  denoting the identity matrix and  $R_\theta$  the autocorrelation matrix of the column vector  $\mathbf{z} = (Z(\delta), Z(2\delta), \dots, Z(n\delta))^T$ , i.e. the Toeplitz matrix with coefficients given by

$$[R_\theta]_{j,k} = K_{\nu, \theta}(\delta|j - k|), \quad j, k = 1, \dots, n, \quad \text{where } K_{\nu, \theta}(t) = \int_{-\infty}^{\infty} g_{\nu, \theta}^*(\omega) \cos(\omega t) d\omega.$$

Recall that expressions of  $K_{\nu, \theta}$  in terms of Bessel functions are well known (see e.g. Stein (1999) or Rasmussen and Williams (2006) for explicit very simple expressions for  $\nu = 3/2$  and  $5/2$ ).

It is often of great interest to be able to “effectively reduce” the number of parameters, especially when computing the likelihood function is costly. This is classically done in the no-nugget case (i.e.  $bR_\theta + I_n$  is replaced by  $bR_\theta$ , and  $\mathbf{y}$  by  $\mathbf{z}$ , in the likelihood function): one can eliminate  $b$  from the numerical search by noticing that the maximizer  $b$  of the likelihood for any fixed  $\theta > 0$  is simply in this case:

$$\hat{b}(\theta) = (1/n)\mathbf{z}^T R_\theta^{-1} \mathbf{z}; \quad (1.2)$$

see Zhang (2004) also for numerical improvements produced by such a “concentration” of the likelihood (also called “profiling”).

Zhang and Zimmerman (2007) recently proposed to use the classical weighted least square method (not statistically fully efficient but much less costly than maximum likelihood (ML)) to estimate the range parameters, next, to plug-in these parameters (the  $\theta$  here) in the likelihood which is then maximized with respect to  $b$  (the solution, say  $\hat{b}_{\text{ML}}(\theta)$ , being either the explicit (1.2) in the no-nugget case, or obtained iteratively by Fisher scoring otherwise). The idea underlying this method is that, at least for the infill asymptotic context (i.e.  $\delta = 1/n$  and  $n$  large), even if  $\theta$  is fixed at a wrong value  $\theta_1$ , the product  $\hat{b}_{\text{ML}}(\theta_1)\theta_1^{2\nu}$  still remains an efficient estimator of  $c_0 := b_0\theta_0^{2\nu}$  (see Du et al. (2009), Wang and Loh (2011) for recent results of this type). As is now classical (Stein (1999)),  $c_0$  will be called the microergodic parameter of the Matern model (1.1). Zhang (2004) showed that a good estimation of  $c_0$  is more important than a joint estimation of  $b_0$  and  $\theta_0$  to obtain a good prediction of  $Z(\cdot)$  for dense sampling designs.

The method we propose here, firstly reverses the roles of variance and range, in that it is based on a very simple estimate for the variance, namely the empirical variance in the no-nugget case, and its corrected version for bias otherwise, which is simply defined by

$$\hat{b}_{\text{EV}} := n^{-1}\mathbf{y}^T \mathbf{y} - 1.$$

Secondly we propose to replace the maximization of the likelihood w.r.t.  $\theta$  by the simple following estimating equation in  $\theta$ , in the with-nugget case: solve, with  $b$  fixed at  $\hat{b}_{\text{EV}}$

$$\mathbf{y}^T A_{b,\theta} (I_n - A_{b,\theta}) \mathbf{y} = \text{tr} A_{b,\theta} \text{ where } A_{b,\theta} = bR_\theta (I_n + bR_\theta)^{-1}. \quad (1.3)$$

In the no-nugget case, this equation in  $\theta$  is simply replaced by  $\mathbf{z}^T R_\theta^{-1} \mathbf{z} = n\hat{b}_{\text{EV}}$ . One may call “Gaussian Gibbs energy” (GE in short) of the underlying discretely sampled process the quantity  $(1/n)\mathbf{z}^T R_\theta^{-1} \mathbf{z}$  and it is easily seen that  $(b/n) (\mathbf{y}^T A_{b,\theta} (I_n - A_{b,\theta}) \mathbf{y} + \text{tr}(I_n - A_{b,\theta}))$  is the conditional Gibbs energy mean (CGEM) obtained by taking the expectation of  $(1/n)\mathbf{z}^T R_\theta^{-1} \mathbf{z}$ , conditional on  $\mathbf{y}$ , for the candidate parameters  $b, \theta$ . So equation (1.3) in  $\theta$  will be called the CGEM-EV estimating equation (GEV in the no-nugget case) and we will denote by  $\hat{\theta}_{\text{GEV}}$  this new range parameter estimate.

Notice that it can be directly checked that equation (1.3) is equivalent to equate to 0 the derivative w.r.t.  $b$  (and not w.r.t.  $\theta$ !) of the likelihood. Our proposal is based on the following two ideas. First, since it is quite plausible that the above idea, underlying Zang and Zimmerman (2007)'s proposal, remains true for a random  $\theta$ , then, instead of "fixing"  $\theta$  at  $\theta_1$ , one may as well adjust  $\theta$  so that  $\hat{b}_{\text{ML}}(\theta)$  coincides with a given value  $b_1$  for the variance; and, denoting  $\hat{\theta}_1$  the so-obtained  $\theta$  (thus  $(b_1, \hat{\theta}_1)$  must cancel the  $b$ -derivative (1.3) of the likelihood), the product  $b_1 \hat{\theta}_1^{2\nu}$  will plausibly be an efficient estimator of  $b_0 \theta_0^{2\nu}$ . Second, it is known that, at least in the no-nugget case, the moment  $\sum Z(\delta j)^2$  yields a successful estimating-equation in the case  $\nu = 1/2$  to estimate  $\theta_0$  when  $b_0 \theta_0^{2\nu}$  is known (see Kessler (2000)). So it is tempting to select  $b_1 := \hat{b}_{\text{EV}}$ , the empirical variance estimate of  $b_0$ . The second point is an admittedly weak justification of this method. However, notice that Yadrenko (Chapter 4 Section 3) has studied, for quite general, mean square continuous, stationary isotropic random fields, the properties of a continuous-time version of the empirical variance in the periodic case, i.e. the Lebesgue integral over sphere of  $Z(t)^2$ , and he has listed appealing properties of such estimates of the variance of the field. The theoretical results we give in this article, will provide in our context, a much stronger, and rather unexpected, justification, for  $\nu$  not too large (which is the case in numerous applications, see Stein (1999), Rasmussen et al. (2006), Gaetan et al. (2010)).

Of course, in our time-series setting, there now exist rather good implementations of ML; see Chen et al. (2006) for cases where the correlation is rather strong. However, our objective here is to provide first insights into the capability of the CGEM-EV method with the hope that they can be extended to more computationally complex settings. Indeed, this approach is not limited to observations on a one dimensional lattice, and is potentially not limited to regular grid (a weighted version, with Riemann-sum type coefficients, of the empirical variance should then be used instead) and to homogenous measurement errors: some successful applications for two-dimensional Matérn random fields, using a randomized-trace approximation to  $\text{tr}A_{b,\theta}$ , are described in Girard (2010). This approach might be, in principle, applied to other two-parameters models than the Matérn family (of course it is presently restricted to scalar  $\theta$ ). Encouraging experimental results are obtained in Girard (2010) for the common spherical autocovariance model in place of the Matérn model.

The rest of this article is structured as follows. Section 2 lists additional notations. In the asymptotic framework we adopt here (which may be thought as intermediate between the infill and increasing-domain frameworks) we first show, in Section 3, that, even when  $b$  is arbitrarily fixed to a wrong  $b_1$ , the CGEM estimating equation is a quite well-behaved estimating equation: it converges toward a monotonic equation whose the root  $\theta_1$  satisfies  $b_1 \theta_1^{2\nu} = b_0 \theta_0^{2\nu}$ . Next we study in

Section 4 what is sacrificed, as compared to ML, when plugging-in the simple  $\hat{b}_{\text{EV}}$  in the CGEM estimating equation. The performance-loss is classically quantified by the asymptotic mean squared errors of the two components of the CGEM-EV estimator as related to those of the ML estimator. We show that what is sacrificed, which depends on  $\nu$ , is quite reasonable provided  $\nu$  is not too large, independently of  $b_0$  and  $\theta_0$ . Indeed asymptotic full-efficiency for  $\theta$ , as well as for  $b$ , is reached as  $\nu$  decreases to  $1/2$  (the loss in efficiency being reasonable as long as  $\nu$  is not too large). Asymptotic full-efficiency for the microergodic -parameter is obtained for any  $\nu \geq 1/2$ . Proofs are outlined Section 5.

## 2. Further notations

Of course “time” could be replaced everywhere by “space”: we choose this vocabulary since we use in several places of the paper the classical time-series theory. Thus we assume everywhere without loss of generality that  $Z_\delta$  defined by  $Z_\delta(i) := Z(\delta i)$  is observed at times  $i = 1, 2, \dots, n$ . From the well known aliasing formula, the spectral density on  $(-\pi, \pi]$  of  $Z_\delta$  is thus  $f_{\nu, b, \theta}^\delta = b g_{\nu, \theta}^\delta$  with

$$g_{\nu, \theta}^\delta(\cdot) := \frac{1}{\delta} \sum_{k=-\infty}^{\infty} g_{\nu, \theta}^* \left( \frac{\cdot + 2k\pi}{\delta} \right) = \sum_{k=-\infty}^{\infty} g_{\nu, \alpha}^*(\cdot + 2k\pi) \quad \text{where } \alpha = \delta\theta, \quad (2.1)$$

the equality between the two sums resulting from the particular “variance-range” parameterisation of the Matérn family (1.1). We will also denote simply by  $g_{\nu, \alpha}(\cdot)$  the second sum in (2.1) in order that we keep in mind that  $g_{\nu, \theta}^\delta$  depends on  $\delta$  and  $\theta$  only through  $\alpha = \delta\theta$ . Simple closed expressions for  $g_{\nu, \theta}^\delta$  are available only when  $\nu - 1/2$  is a small integer, says 0, 1 or 2 : they then coincide with particular constrained ARMA spectral densities.

In order to simplify the following statements (and their proofs), let us define the “Wiener filter”  $a_{b, \theta}^\delta(\cdot)$  and its “unaliased” version  $a_{b, \alpha}^*(\cdot)$ , for  $\alpha = \delta\theta$ , by

$$a_{b, \theta}^\delta(\cdot) := \frac{b g_{\nu, \theta}^\delta(\cdot)}{b g_{\nu, \theta}^\delta(\cdot) + (2\pi)^{-1}}, \quad a_{b, \alpha}^*(\cdot) := \frac{b g_{\nu, \alpha}^*(\cdot)}{b g_{\nu, \alpha}^*(\cdot) + (2\pi)^{-1}}. \quad (2.2)$$

Indeed, as is well known from the “signal extraction” litterature,  $a_{b, \theta}^\delta$  is the spectral characteristic, or “frequency response”, of the “optimal convolution” (of series defined on  $\mathbb{Z}$ ) in the sense that this convolution would minimize the mean squared error of estimation for a “ $i = -\infty, \dots, +\infty$ ” version of the standard problem of extracting  $Z_\delta$  (i.e. one would observe  $Z_\delta$  plus an unknown standard white noise over  $\mathbb{Z}$ ; see e.g. Section 4.11 of Shumway and Stoffer 2006). Let us recall that  $A_{b, \theta} \mathbf{y}$  is also a “best estimate” in several senses, for extracting  $\mathbf{z}$  from the segment of observations  $\mathbf{y}$  (see e.g. Section 1.5 of Wahba 1990) so that it will not be surprising

to obtain relationships like  $\lim_{n \rightarrow +\infty} n^{-1} \text{tr} A_{b,\theta} = (2\pi)^{-1} \int_{-\pi}^{\pi} a_{b,\theta}^{\delta}(\lambda) d\lambda$  for any fixed  $\delta, b, \theta$ .

Another function which will play an important role in this article is the derivative of  $\log(g_{\nu,\theta}^{\delta}(\cdot))$  w.r.t.  $\theta$ , and its unaliased approximation (up to a factor  $\delta$ , and in a particular sense as we will see in the Proofs section):

$$h_{\nu,\theta}^{\delta} := \partial \log(g_{\nu,\theta}^{\delta}) / \partial \theta, \quad h_{\nu,\alpha}^* := \partial \log(g_{\nu,\alpha}^*) / \partial \alpha. \quad (2.3)$$

For any  $f : [-\pi, \pi] \rightarrow \mathbb{R}$ , s.t.  $\int_{-\pi}^{\pi} [a_{b,\theta}^{\delta}(\lambda)]^2 f(\lambda) d\lambda \neq 0$ , we define the weighted coefficient of variation of  $f$  by

$$J_{\delta,b,\theta}(f) := \frac{\frac{1}{f w} \int w \left[ f - \left( \frac{1}{f w} \int w f \right) \right]^2}{\left( \frac{1}{f w} \int w f \right)^2} = \frac{\frac{1}{f w} \int w f^2}{\left( \frac{1}{f w} \int w f \right)^2} - 1, \quad \text{where } w := (a_{b,\theta}^{\delta})^2.$$

Above and throughout this paper, “ $\int$ ” will denote integrals over  $[-\pi, \pi]$ . Omitting the indexes  $\delta$  and  $\nu$ , we will also use the notation  $g_0$  (resp.  $h_0$ ) for the function  $g_{\nu,\theta}^{\delta}$  (resp.  $h_{\nu,\theta}^{\delta}$ ) when  $\theta = \theta_0$  and also  $a_0 := a_{b_0,\theta_0}^{\delta}$ , and  $J_0(\cdot) := J_{\delta,b_0,\theta_0}(\cdot)$ .

### 3. Some asymptotic properties of the CGEM estimating equation

In this section, we establish some consistency and asymptotic identifiability properties enjoyed by the CGEM estimating equation in  $\theta$ . They are merely encouraging preliminary results. We expect that more complete results will be given elsewhere. To simplify our statement, we consider, the normalized version  $\mathbf{y}^T A_{b,\theta} (I_n - A_{b,\theta}) \mathbf{y} / \text{tr} A_{b,\theta} = 1$  which is, of course, an equivalent estimating-equation.

For any positive  $\delta, b, \theta, b_0, \theta_0$ , we can define the following weighted mean:

$$\psi(\delta, b, \theta, b_0, \theta_0) := \frac{1}{\int_{-\pi}^{\pi} a_{b,\theta}^{\delta}(\lambda) d\lambda} \int_{-\pi}^{\pi} [a_{b,\theta}^{\delta}(\lambda)]^2 \left( \frac{b_0 g_{\nu,\theta_0}^{\delta}(\lambda)}{b g_{\nu,\theta}^{\delta}(\lambda)} - 1 \right) d\lambda.$$

**Theorem 3.1.** *For any fixed  $b, \theta$ , we have the following convergence in probability:*

$$\frac{\mathbf{y}^T A_{b,\theta} (I_n - A_{b,\theta}) \mathbf{y}}{\text{tr} A_{b,\theta}} - 1 \rightarrow \psi(\delta, b, \theta, b_0, \theta_0) \quad \text{as } n \rightarrow \infty,$$

and  $\psi$  has a very simple small- $\delta$  equivalent:

$$\psi(\delta, b, \theta, b_0, \theta_0) \rightarrow 2\nu(2\nu + 1)^{-1} \left( \frac{b_0 \theta_0^{2\nu}}{b \theta^{2\nu}} - 1 \right) \quad \text{as } \delta \rightarrow 0.$$

The first part of this theorem is in fact not restricted to the Matérn family; this is a rather direct consequence of universal results of time-series theory about quadratic forms constructed from (product of power, possibly negative, of) Toeplitz matrices. In order to be allowed to apply such classical results (for instance, those stated by Azencott and Dacunha-Castelle (1986) and used in their analysis of maximum likelihood estimation, or its Whittle approximation), it is sufficient that  $(\theta, \lambda) \mapsto g_{\nu, \theta}^{\delta}(\lambda)$  be three-times continuously differentiable on  $\Theta \times [-\pi, \pi]$  where  $\Theta$  is a compact interval not containing 0; and this can be checked by applying the classical Weierstrass M-test.

The second part uses several approximations to integrals which can be obtained by the techniques of Section 5.

Theorem 3.1 thus says that the large- $n$  limit of the normalized CGEM equation has a small- $\delta$  equivalent which is a very simple monotonic function of  $b\theta^{2\nu}$ . Clearly if  $b$  is fixed at any value  $b_1$ , then the unique root  $\theta_1$  of this large- $n$ -small- $\delta$  equivalent equation will satisfy  $b_1\theta_1^{2\nu} = b_0\theta_0^{2\nu}$ . This gives some support to the first of the two underlying ideas (that is, choose  $\hat{\theta}_1$  adjusted so that  $(b_1, \hat{\theta}_1)$  cancels the  $b$ -derivative (1.3) of the likelihood) given in the Introduction.

#### 4. Mean squared error inefficiencies of CGEM-EV to ML for the variance, the range and the microergodic parameters

**4.1.** Let  $(\hat{b}_{\text{ML}}, \hat{\theta}_{\text{ML}})$  be a maximizer of the likelihood function on  $B \times \Theta$  where  $B$  and  $\Theta$  are compact intervals not containing 0 and such that  $(b_0, \theta_0)$  is in the interior of  $B \times \Theta$ . Then, since the classical identifiability and regularity conditions are well fulfilled, for any fixed  $\delta > 0$ , for our time-series Matérn model  $f_{\nu, b, \theta}^{\delta}$ , it is a well known result of times-series theory (Azencott and Dacunha-Castelle (1986)) that  $(\hat{b}_{\text{ML}}, \hat{\theta}_{\text{ML}})$  is a.s. consistent and satisfies:

$$n^{1/2} \left( \begin{bmatrix} \hat{b}_{\text{ML}} \\ \hat{\theta}_{\text{ML}} \end{bmatrix} - \begin{bmatrix} b_0 \\ \theta_0 \end{bmatrix} \right) \rightarrow_{\mathcal{D}} N \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, 4\pi \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \right) \quad \text{as } n \rightarrow \infty,$$

where

$$\begin{bmatrix} \sigma_1^2 \\ \sigma_{12} \\ \sigma_2^2 \end{bmatrix} = (f a_0^2 h_0)^{-2} J_0(h_0)^{-1} \begin{bmatrix} b_0^2 \int a_0^2 h_0^2 \\ -b_0 \int a_0^2 h_0 \\ \int a_0^2 \end{bmatrix}.$$

Note that the term  $(4\pi)^{-1}(f a_0^2 h_0)^2 J_0(h_0)$  can be checked to be the determinant ( $> 0$  since  $h_0(\cdot)$  cannot be a constant function) of the classical asymptotic information matrix (see Theorem 4.3 of Chapter XIII of Azencott and Dacunha-Castelle



(1986)) and that it has this expression only if  $\int a_0^2 h_0 \neq 0$  (otherwise  $J_0(h_0)$  is infinite).

Now the same regularity conditions as above, on  $g_{\nu, \theta}^\delta$ , are also sufficient to show (by the usual Taylor series argument) the part a) of the following theorem:

**Theorem 4.1.** *Let  $(\hat{b}_{\text{EV}}, \hat{\theta}_{\text{GEV}})$  be a consistent root of the CGEM-EV estimating equation (1.3). a) If  $\int a_0^2 h_0 \neq 0$  then*

$$n^{1/2} \left( \begin{bmatrix} \hat{b}_{\text{EV}} \\ \hat{\theta}_{\text{GEV}} \end{bmatrix} - \begin{bmatrix} b_0 \\ \theta_0 \end{bmatrix} \right) \rightarrow_{\mathcal{D}} N \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, 4\pi \begin{bmatrix} v_1 & v_{12} \\ v_{12} & v_2 \end{bmatrix} \right) \quad \text{as } n \rightarrow \infty,$$

where

$$\begin{cases} v_1 &= b_0^2 \int a_0^{-2} g_0^2 \\ v_{12} &= -b_0 J_0(g_0/a_0^2) (\int a_0^2 h_0)^{-1} \\ v_2 &= J_0(g_0/a_0^2) \int a_0^2 (\int a_0^2 h_0)^{-2} \end{cases} .$$

b) When  $\delta \rightarrow 0$  we have  $\int a_0^2 h_0 / \int a_0^2 \rightarrow 2\nu/\theta_0$ ; and defining by  $I_{\delta, b_0, \theta_0}^1 := v_1/\sigma_1^2$  (resp.  $I_{\delta, b_0, \theta_0}^2 := v_2/\sigma_2^2$ ) the asymptotic mean squared error inefficiency of CGEM-EV to ML for  $b_0$  (resp. for  $\theta_0$ ), these 2 inefficiencies have the following common limit

$$I_{\delta, b_0, \theta_0}^i \rightarrow \frac{\sqrt{\pi}}{2} \left( \frac{2\nu+1}{2\nu} \right)^2 \frac{\Gamma(\nu + \frac{1}{2})^2 \Gamma(2\nu + \frac{1}{2})}{\Gamma(\nu)^2 \Gamma(2\nu + 1)} =: \text{ineff}(\nu) \quad \text{as } \delta \rightarrow 0, \quad \text{for } i \in \{1, 2\}.$$

The first rather surprising fact, in part b) of this theorem, is that this large- $n$ -small- $\delta$  inefficiency limit is not function of the underlying  $b_0$ , nor of  $\theta_0$ . Secondly, Table 4.1 clearly demonstrates that the CGEM-EV estimates are asymptotically nearly efficient provided  $\nu$  is not too large. Asymptotic full-efficiency is reached for  $\nu$  close to 1/2.

$\nu$	1/2	1	3/2	2	5/2	3	7/2	4
$\lim_{\delta \rightarrow 0} I_{\delta, b_0, \theta_0}^i$	1	1.04093	$\frac{10}{9}$	1.18596	$\frac{63}{50}$	1.33174	$\frac{1716}{1225}$	1.46727

**Table 4.1.** Common limit of asymptotic MSE inefficiencies  $I_{\delta, b_0, \theta_0}^i, i \in \{1, 2\}$ , for “typical” values of  $\nu$

**4.2.** Let us now consider the estimation of the microergodic parameter  $c_0 = b_0 \theta_0^{2\nu}$ . By the classical delta-method, one can directly infer from the above joint asymptotic law of  $(\hat{b}_{\text{ML}}, \hat{\theta}_{\text{ML}})$  that

$$n^{1/2} \left( \hat{b}_{\text{ML}} \hat{\theta}_{\text{ML}}^{2\nu} - c_0 \right) \rightarrow_{\mathcal{D}} N(0, 4\pi\sigma_3^2)$$

where

$$\sigma_3^2 = c_0^2 (\int a_0^2 h_0)^{-2} J_0(h_0)^{-1} \int a_0^2 \left( h_0 - \frac{2\nu}{\theta_0} \right)^2.$$

We can now state a full-efficiency result for the CGEM-EV alternative:

**Theorem 4.2.** *Let  $\hat{c}_{\text{GEV}} := \hat{b}_{\text{EV}} \hat{\theta}_{\text{GEV}}^{2\nu}$  where  $\hat{b}_{\text{EV}}$  and  $\hat{\theta}_{\text{GEV}}$  are defined as in Theorem 4.1. We have : a) If  $\int a_0^2 h_0 \neq 0$  then*

$$n^{1/2} (\hat{c}_{\text{GEV}} - c_0) \rightarrow_{\mathcal{D}} N(0, 4\pi v_3) \quad \text{as } n \rightarrow \infty,$$

where

$$v_3 = \frac{c_0^2}{(\int a_0^2 h_0)^{-2} \int a_0^2} \left( J_0(g_0/a_0^2) \int a_0^2 \left( h_0 - \frac{2\nu}{\theta_0} \right)^2 + \left( \frac{2\nu}{\theta_0} \right)^2 \left( \int a_0^2 \right)^2 \right).$$

b) *For any small enough  $\delta > 0$ , the large- $n$  mean squared error inefficiency of CGEM-EV to ML for  $c_0$  is  $I_{\delta, b_0, \theta_0}^3 := v_3/\sigma_3^2$  and it holds that  $I_{\delta, b_0, \theta_0}^3 \rightarrow 1$  as  $\delta \rightarrow 0$ ; more precisely,*

$$\sigma_3^2 \sim \frac{c_0^2}{\int a_0^2} \quad \text{and} \quad v_3 \sim \frac{c_0^2}{\int a_0^2}, \quad \text{as } \delta \rightarrow 0.$$

This full-efficiency property for the microergodic parameter is not too surprising in the case of no measurement errors as soon as one takes for granted a certain “similarity” between the infill framework and the small- $\delta$  regime (some precise statements on this similarity can be found in Zhang and Zimmerman (2005)). Indeed, first, such asymptotic full-efficiency is known to typically hold under the infill framework using an arbitrarily fixed  $\theta_1$ , and, as discussed in the Introduction, should still hold with a random, or a data “adjusted”,  $\theta_1$ , plugged in  $\hat{b}_{\text{ML}}(\theta_1)\theta_1^{2\nu}$ ; second, also as discussed in the Introduction,  $\hat{c}_{\text{GEV}}$  is a particular instance of the latter. See Wang and Loh (2011) and the references therein for the case fixed  $\theta_1$ . Such a full-efficiency claim for a random  $\theta_1$  has been recently developed in Kaufman and Shaby (2011).

Let us remark that Kaufman and Shaby (2011) also demonstrates that, in fact, the microergodic parameter can be much better estimated in practice by appropriately estimating (generally by ML) the range parameter  $\theta$ . Thus the near equivalence between ML and CGEM-EV for estimating  $\theta$  (or  $b$ ) when  $\nu$  is not too large, stated in Theorem 4.1 and Table 4.1, should be perhaps considered as a more reassuring theoretical result than Theorem 4.2.

**4.3.** Concerning the case with known noise *and* signal variances, even if the assumption that  $b_0$  is known is rather restrictive, it may be worth to point out the

following limiting laws: denoting by  $\hat{\theta}_{\text{ML}_0}$  the ML estimate with  $b$  fixed at  $b_0$ , its asymptotic behavior is classically obtained :

$$n^{1/2} \left( \hat{\theta}_{\text{ML}_0} - \theta_0 \right) \rightarrow_{\mathcal{D}} N \left( 0, 4\pi \left( \int a_0^2 h_0^2 \right)^{-1} \right),$$

and denoting by  $\hat{\theta}_{\text{GE}_0}$  a consistent solution of the estimating-equation (1.3) with now  $b$  fixed at  $b_0$ , we have

**Theorem 4.3.** *If  $\int a_0^2 h_0 \neq 0$  then*

$$n^{1/2} \left( \hat{\theta}_{\text{GE}_0} - \theta_0 \right) \rightarrow_{\mathcal{D}} N \left( 0, 4\pi \int a_0^2 \left( \int a_0^2 h_0 \right)^{-2} \right)$$

and the ratio of this asymptotic variance to the one of  $\hat{\theta}_{\text{ML}_0}$ , says  $I_{\delta, b_0, \theta_0}^0$ , satisfies:

$$I_{\delta, b_0, \theta_0}^0 = 1 + J_0(h_0) \rightarrow 1 \quad \text{as } \delta \rightarrow 0.$$

Asymptotic (i.e. large- $n$ -small- $\delta$ ) full-efficiency is thus now enjoyed by the CGEM estimating function for *any* value of  $\nu \geq 1/2$ .

**4.4.** Consider now a second simplified setting: the case where the microergodic parameter  $c_0$  is assumed to be known. This assumption can be thought as being less restrictive than the one of Section 4.4. For instance, this is often assumed in studies of O.U. processes with  $\delta$  small (a framework often referred to as “high frequency” observations; and  $c_0$  is known as the diffusion coefficient). We believe that the main reason is that estimators of “quadratic variation” type are well known, at least in the no-measurement error setting, to produce rather good (although not always fully efficient) estimation of  $c_0$  even in the more general Matérn setting: see Anderes (2010) and references herein. In this setting, one can equivalently focus either on the estimation of  $b_0$  by  $\hat{b}_{\text{EV}}$  or that of  $\theta_0$  by  $\left( c_0 / \hat{b}_{\text{EV}} \right)^{\frac{1}{2\nu}}$ . We choose the former because the asymptotic limiting law of  $\hat{b}_{\text{EV}}$  is already displayed in Theorem 4.2. Now, by firstly re-parameterising the model  $f_{\nu, b, \theta}^\delta$  as a function only of  $b$  and secondly establishing the asymptotic Fisher information w.r.t.  $b$ , one directly obtains that the ML estimator, now denoted  $\hat{b}_{\text{ML}^0}$ , satisfies:

$$n^{1/2} \left( \hat{b}_{\text{ML}^0} - b_0 \right) \rightarrow_{\mathcal{D}} N \left( 0, 4\pi\sigma_4^2 \right)$$

where

$$\sigma_4^2 = b_0^2 \left( \frac{2\nu}{\theta_0} \right)^2 \left( \int a_0^2 \left( h_0 - \frac{2\nu}{\theta_0} \right)^2 \right)^{-1}.$$

We can now state a result of near-efficiency of  $\hat{b}_{\text{EV}}$  for not too large  $\nu$ :

**Theorem 4.4.** *The large- $n$  mean squared error inefficiency of  $\hat{b}_{\text{EV}}$  to the classical ML estimator of  $b_0$  when  $c_0$  is known, is  $I_{\delta, b_0, \theta_0}^4 := v_1/\sigma_4^2$ , and it holds that*

$$I_{\delta, b_0, \theta_0}^4 \rightarrow \text{ineff}(\nu) \quad \text{as } \delta \rightarrow 0,$$

where  $v_1$  and  $\text{ineff}(\nu)$  are defined in Theorem 4.1

For the particular  $\nu = 1/2$  this extends to the case with measurement errors the “small  $\delta$ ” efficient performance of  $\hat{b}_{\text{EV}}$  that is described in Kessler (2000).

**4.5.** Let us discuss the implications, possible limits, and possible extensions, of these results. Of course it would be interesting to study how fine and large the grid must be in order that the CGEM-EV statistical performances be comparable to ML.

One must keep in mind that, as  $\delta$  decreases to 0, larger data sizes are likely to be required to accurately approximate the actual law of any one of these estimates of  $\theta_0$  (or  $b_0$ ) by its asymptotic form. Indeed this is well known for the widely studied case  $\nu = 1/2$  (see e.g. Zhang and Zimmerman 2005 for a study of ML in this case; that paper also demonstrates that the normal approximation is much better for the microergodic parameter).

Notice that even if the results of this paper are limited to “small”  $\delta$ , however the CGEM-EV approach might be also “near efficient” for finite  $\delta$ : indeed such a good performance of  $\hat{b}_{\text{EV}}$  is demonstrated for a large range of  $\delta$  in the above mentioned study by Kessler (2000).

One may guess that the signal-to-noise ratio must also be “not too small”, even if its value does not matter asymptotically. Since the main improvements in numerical performance occur for multidimensional process (with randomized-traces used instead of the exact traces in (1.3)), a rather extensive Monte Carlo study has been recently done for two-dimensional random fields; we refer the reader to Girard (2010).

This approach might be extended in a simple way to the case of unknown variance, says  $\sigma_\varepsilon^2$ , of the measurement error (or unknown nugget effect). Indeed it is well known (Stein (1999, Section 6.2)) that the simple average of local squared

differences like  $(y_i - y_{i-1})^2$  would yield here a consistent estimate  $\hat{\sigma}_\varepsilon^2$ , and this is easily generalized to multidimensional processes which are mean square continuous. Notice that the matrix which now appears in the  $b$ -derivative (1.3) (the finite  $n$  “Wiener filter”) is still  $A_{b,\theta}$  except that  $b$  now denotes the signal-to-noise ratio such that  $\text{var}(Z) = b\sigma_\varepsilon^2$ . Then the whole CGEM-EV approach might be applied after having only replaced  $\mathbf{y}$  by its “standardized” version  $\mathbf{y}/\hat{\sigma}_\varepsilon$ . It would be thus very interesting to study whether similar near-efficiency results still hold or if a more elaborate estimate for  $\sigma_\varepsilon$  is required.

**4.6.** To finish, let us comment on the somewhat more natural  $\hat{b}_{\text{EV}}$ -based alternative to ML which would be akin to the hybrid estimate proposed by Zhang and Zimmerman (2007) : choose  $\hat{\theta}_{\text{H}}$  so as to maximize the likelihood function with  $\hat{b}_{\text{EV}}$  plugged-in. As to the computational aspects,  $\hat{\theta}_{\text{H}}$  might be much less easily computed than  $\hat{\theta}_{\text{GEEV}}$ , for example when an iterative solver is used for  $A_{b,\theta}\mathbf{y}$  (recall that the likelihood criterion requires a log-determinant instead of a trace). So it is worth to point out that again a neat result (which can be similarly obtained by the techniques used to prove Theorem 4.1) holds true here: the large- $n$  mean squared error of  $\hat{\theta}_{\text{H}}$  to the one of  $\hat{\theta}_{\text{GEEV}}$  tends to 1 as  $\delta \rightarrow 0$ .

## 5. Proofs

In the following,  $C$  denotes a constant that may change from line to line (that is,  $C$  will only possibly depend on  $\nu$  and on the positive lower and upper bounds for the candidate  $b$ 's and those for  $\theta$  (says  $\underline{b}, \bar{b}, \underline{\theta}, \bar{\theta}$ ). And  $O(\delta^{2\nu})$  denotes a term whose absolute value is bounded by  $C\delta^{2\nu}$ . Without loss of generality we assume that  $\delta \leq 1$ .

Firstly, it is easily checked from the definition of  $g_{\nu,\alpha}^*(\lambda)$ , that the un-aliased version of  $h_{\nu,\theta}^\delta$  has a very simple form:

$$h_{\nu,\alpha}^*(\lambda) = \alpha^{-1} \left( 2\nu - (2\nu + 1)G \left( \frac{\lambda}{\alpha} \right) \right), \quad \text{where} \quad G(\omega) = \frac{1}{1 + \omega^2}. \quad (5.1)$$

We first collect intermediary results in a few lemmas which might be of interest for other studies of this small- $\delta$  Matérn time-series model.

**Lemma 5.1.** *We have*

$$C \delta^{2\nu} \leq g_{\nu,\delta\theta}^*(\lambda) \leq g_{\nu,\theta}^\delta(\lambda) \leq g_{\nu,\delta\theta}^*(\lambda) + O(\delta^{2\nu}),$$

and

$$g_{\nu,\delta\theta}^*(\lambda) |h_{\nu,\theta}^\delta(\lambda) - \delta h_{\nu,\delta\theta}^*(\lambda)| = O(\delta^{2\nu}).$$

**Lemma 5.2.** *We have*

$$a_{b,\delta\theta}^*(\lambda) \leq a_{b,\theta}^\delta(\lambda) \leq a_{b,\delta\theta}^*(\lambda) + O(\delta^{2\nu}),$$

and

$$a_{b,\theta}^\delta(\lambda) \leq C g_{\nu,b\theta}^*(\lambda).$$

**Proof of these Lemmas.** From their definition, we have  $g_{\nu,\alpha}(\lambda) - g_{\nu,\alpha}^*(\lambda) = \sum_{k \neq 0} g_{\nu,\alpha}^*(\lambda + 2\pi k)$ . For any  $\lambda \in [-\pi, \pi]$ , for any  $k \geq 1$ , the monotonicity of  $g_{\nu,\alpha}^*$  implies  $g_{\nu,\alpha}^*(\lambda + 2\pi k) \leq g_{\nu,\alpha}^*(2\pi(k - 1/2)) \leq C \alpha^{2\nu} / (k - 1/2)^{2\nu+1}$ ; and summing these terms over  $k = 1, 2, \dots$ , thus gives a term  $O(\alpha^{2\nu})$ . This is shown similarly for the sum over  $k = -1, -2, \dots$ . We now prove the second part of Lemma 5.1. Omitting the index  $\nu$  and denoting  $\dot{g}_\alpha := \partial g_\alpha / \partial \alpha$ ,  $\dot{g}_\alpha^* := \partial g_\alpha^* / \partial \alpha$ , one can decompose  $\alpha g_\alpha^* (\delta^{-1} h_\theta^\delta - h_\alpha^*) = \alpha g_\alpha^* (\dot{g}_\alpha / g_\alpha - \dot{g}_\alpha^* / g_\alpha^*) = (g_\alpha^* - g_\alpha) (\alpha \dot{g}_\alpha / g_\alpha) + \alpha (\dot{g}_\alpha - \dot{g}_\alpha^*)$ . Observing that  $\alpha \dot{g}_\alpha^* = 2\nu g_{\nu,\alpha}^* + C g_{\nu+1,\alpha}^*$  yields  $\alpha (\dot{g}_\alpha - \dot{g}_\alpha^*) = O(\alpha^{2\nu})$ . Lastly, we can bound  $\alpha |\dot{g}_\alpha / g_\alpha| \leq \alpha |\dot{g}_\alpha / g_\alpha^*| \leq \alpha |\dot{g}_\alpha^* / g_\alpha^*| + \alpha |\dot{g}_\alpha - \dot{g}_\alpha^*| / g_\alpha^* \leq O(1) + \alpha |\dot{g}_\alpha - \dot{g}_\alpha^*| / C \alpha^{2\nu}$  from equation (5.1) and the first inequality of Lemma 5.1.

**Lemma 5.3.** *If  $\nu > 1/2$ , then for any continuous function  $F : \mathbb{R} \rightarrow \mathbb{R}_+$  such that  $0 < \int_{-\infty}^{\infty} F(x) dx < +\infty$ , we have, as  $\delta \rightarrow 0$*

$$\int a_{b,\theta}^\delta(\lambda) F\left(\frac{\lambda}{\delta\theta}\right) d\lambda \sim \delta\theta \int_{-\infty}^{\infty} F(x) dx, \quad \int [a_{b,\theta}^\delta(\lambda)]^2 F\left(\frac{\lambda}{\delta\theta}\right) d\lambda \sim \delta\theta \int_{-\infty}^{\infty} F(x) dx.$$

**Proof.** The first part of Lemma 5.2 and the boundedness of  $F$  permits us to replace  $a_{b,\theta}^\delta$  by the much more manageable  $a_{b,\alpha}^*$  (where  $\alpha = \delta\theta$ ) with an error  $O(\delta^{2\nu})$  which is enough when  $\nu > 1/2$ . Letting  $\bar{\lambda} = \alpha^{\epsilon + (2\nu/(2\nu+1))}$  where  $\epsilon$  is any arbitrarily small constant  $> 0$ , we see from the monotonicity of  $g_{\nu,\alpha}^*$ , that  $\inf_{\lambda \in [-\bar{\lambda}, \bar{\lambda}]} a_{b,\alpha}^*(\lambda) \geq \left(1 + (2\pi/b) [g_{\nu,\alpha}^*(\bar{\lambda})]^{-1}\right)^{-1}$  which tends to 1 since  $g_{\nu,\alpha}^*(\bar{\lambda}) \geq \alpha^{2\nu} / (2\bar{\lambda}^2)^{\nu+1/2}$  (for  $\alpha$  small enough) which tends to  $+\infty$ . And the same is true for  $[a_{b,\alpha}^*(\lambda)]^2$ . It remains to observe that both  $\int_{-\pi}^{\pi} F(\frac{\lambda}{\delta\theta}) d\lambda$  and  $\int_{-\bar{\lambda}}^{\bar{\lambda}} F(\frac{\lambda}{\delta\theta}) d\lambda$  are equivalent to the claimed term.

**Lemma 5.4.** *We have as  $\delta \rightarrow 0$*

$$\begin{aligned} \int a_{b,\theta}^\delta(\lambda) d\lambda &\sim (\delta\theta)^{\frac{2\nu}{2\nu+1}} (2\pi C_\nu b)^{\frac{1}{2\nu+1}} \int_{-\infty}^{\infty} (1+x^{2\nu+1})^{-1} dx, \\ \int [a_{b,\theta}^\delta(\lambda)]^2 d\lambda &\sim (\delta\theta)^{\frac{2\nu}{2\nu+1}} (2\pi C_\nu b)^{\frac{1}{2\nu+1}} \int_{-\infty}^{\infty} (1+x^{2\nu+1})^{-2} dx. \end{aligned}$$

**Proof.** By Lemma 5.2 we can replace, with enough accuracy, the filter by its un-aliased version in these integrals. Next the claimed equivalents can be obtained by a change of variable  $s = \lambda/\alpha^{\frac{2\nu}{2\nu+1}}$  and an application of the dominated convergence theorem.

We can now outline the proofs of Theorem 3.1 and Theorem 4.1, beginning by the second one.

**Proof of Theorem 4.1.** i) First it is directly seen that  $v_2/\sigma_2^2 = J_0(h_0)C_0$ . Let  $w(\lambda) := [a_{b,\theta}^\delta(\lambda)]^2$ . Consider first the constant  $c_0$ . Since  $1 + J_{\delta,b,\theta}(g_{\nu,\theta}^\delta/w)$  is the ratio of  $\int w \int [g_{\nu,\theta}^\delta(\lambda)]^2 / w(\lambda) d\lambda$ , where  $[g_{\nu,\theta}^\delta]^2 / w = [g_{\nu,\theta}^\delta + (2\pi b)^{-1}]^2$ , over  $[\int g_{\nu,\theta}^\delta d\lambda]^2$  ( $= 1$ ), it suffices to observe that

$$\left| \int g_{\nu,\alpha}^2 - \int g_{\nu,\alpha}^{*2} \right| \leq \int |g_{\nu,\alpha}^2 - g_{\nu,\alpha}^{*2}| \leq C\alpha^{2\nu} \int (g_{\nu,\alpha} + g_{\nu,\alpha}^*) \leq 2C\alpha^{2\nu}$$

and that  $\int g_{\nu,\alpha}^{*2} \sim \alpha^{-1}C_\nu^2 \int_{-\infty}^{\infty} (1+x^2)^{-2\nu-1} dx$  to see that this is also the dominant term of  $J_{\delta,b,\theta}(g_{\nu,\theta}^\delta/w) / \int w$ . Secondly, the equivalent of  $J_0(h_0)$  can be obtained as follows. The numerator of  $J_{\delta,b,\theta}(h_{\nu,\theta}^\delta)$  is

$$\frac{1}{\int w} \int w(h_{\nu,\theta}^\delta)^2 - \left| \frac{1}{\int w} \int w(h_{\nu,\theta}^\delta) \right|^2 = \frac{1}{\int w} \int w(h_{\nu,\theta}^\delta - 2\nu/\theta)^2 - \left| \frac{1}{\int w} \int w(h_{\nu,\theta}^\delta - 2\nu/\theta) \right|^2.$$

If we take for granted that the un-aliased  $\delta h_{\nu,\delta\theta}^*$  can be substituted for  $h_{\nu,\theta}^\delta$ , then the numerator is simplified using equation (5.1), and its equivalent is deduced from Lemmas 5.3 (case  $\nu > 1/2$ ) and 5.4 (with  $G_\alpha(\lambda) := G(\lambda/\alpha)$  defined from (5.1)) as follows:

$$\frac{1}{\int w} \int w |(2\nu+1)G_\alpha/\theta|^2 - \left| \frac{1}{\int w} \int w (2\nu+1)G_\alpha/\theta \right|^2 \sim ((2\nu+1)/\theta)^2 \frac{1}{\int w} \delta\theta \int_{-\infty}^{\infty} G(x)^2 dx.$$

Note that this can also be proved for  $\nu = 1/2$  by using the known close form of  $w$  available in this case. On the other hand, the denominator, also for the un-aliased version, is  $\left| (\int w)^{-1} \int w(\lambda)[2\nu - (2\nu+1)G(\lambda/\alpha)]/\theta \right|^2$  which tends to  $(2\nu/\theta)^2$  again by Lemmas 5.3 and 5.4. This proves the claimed result for the  $\theta$ -inefficiency provided the unaliased substitution was authorized. To see this, e.g. for  $\int w(h_{\nu,\theta}^\delta - 2\nu/\theta)^2$ , it suffices, to decompose  $(h_{\nu,\theta}^\delta - 2\nu/\theta)^2 - (\delta h_{\nu,\delta\theta}^* - 2\nu/\theta)^2 = (h_{\nu,\theta}^\delta - \delta h_{\nu,\delta\theta}^*)^2 + 2(\delta h_{\nu,\delta\theta}^* - 2\nu/\theta)(h_{\nu,\theta}^\delta - \delta h_{\nu,\delta\theta}^*)$  and to bound from Lemmas 5.1-5.3, for example for the second term:

$$\begin{aligned} \int w |h_{\nu,\theta}^\delta - \delta h_{\nu,\delta\theta}^*| G_\alpha &\leq C \int w^{1/2} g_{\nu,\delta\theta}^* |h_{\nu,\theta}^\delta - \delta h_{\nu,\delta\theta}^*| G_\alpha \\ &\leq O(\delta^{2\nu}) \int w^{1/2} G_\alpha = O(\delta^{2\nu+1}). \end{aligned}$$

ii) By a simple algebraic manipulation (and using that  $\int g_0 = 1$ ) we can write

$$\frac{v_1}{\sigma_1^2} = \frac{v_2}{\sigma_2^2} (1 + C_0^{-1}) \frac{1}{1 + J_0(h_0)}.$$

(as a side note, this implies  $I_{\delta, b_0, \theta_0}^1 < I_{\delta, b_0, \theta_0}^2$  since, of course,  $J_0(h_0)C_0 > 1$  by a Cauchy-Schwarz type inequality). Thus observing that  $\lim_{\delta \rightarrow 0} C_0 = +\infty$  and  $\lim_{\delta \rightarrow 0} J_0(h_0) = 0$  were, in fact, intermediary results of the paragraph i) above, we obtain the claimed result for the  $b$ -inefficiency.

**Proof of Theorem 3.1.** Again let  $w(\lambda) := [a_{b, \theta}^\delta(\lambda)]^2$  and let  $g := g_{\nu, \theta}^\delta$ . We easily see, from the expression of the equivalents of  $\int w^{1/2}$  and  $\int w$  of Lemma 5.4, that it is sufficient to prove that  $\int w(g_0/g) \sim (\theta_0^{2\nu}/\theta^{2\nu}) \int w$  as  $\delta \rightarrow 0$ . Let us first study this integral with the unaliased versions  $g^* := g_{\nu, \delta\theta}^*$  and  $g_0^* := g_{\nu, \delta\theta_0}^*$  in place of  $g$  and  $g_0$ . Starting from the definition (1.1) of  $g^*$ , we can write

$$\theta^{2\nu} g_0^*(\lambda) / (\theta_0^{2\nu} g^*(\lambda)) = (\theta^2 + (\lambda/\delta)^2)^{\nu+1/2} (\theta_0^2 + (\lambda/\delta)^2)^{-(\nu+1/2)} = 1 + F(\lambda/\delta)$$

with

$$F(\lambda) := (1 + (\theta^2 - \theta_0^2)/(\theta_0^2 + \lambda^2))^{\nu+1/2} - 1.$$

Note that  $F$  is integrable over  $\mathbb{R}$  and has the same sign as  $\theta^2 - \theta_0^2$ . By Lemma 5.3 and 5.4s, if  $\nu > 1/2$  then we obtain  $\int w(\lambda)F(\lambda/\delta)d\lambda = o(\int w)$  and thus  $\int w(\theta^{2\nu} g_0^*/(\theta_0^{2\nu} g^*)) \sim \int w$ . The same result for the case  $\nu = 1/2$  can be easily obtained (from the closed expression of  $w$  in this case). It remains to bound the effect of the aliasing:

$$\begin{aligned} \left| \int w(g_0^*/g^*) - \int w(g_0/g) \right| &\leq \int w |(g_0^*g - g_0g^*)/(g^*g)| \\ &\leq C \int w^{1/2} |(g_0^*g - g_0g^*)/g^*| \\ &\leq C \int w^{1/2} |g_0^* - g_0| + C \int w^{1/2} |(g - g^*)g_0^*/g^*| \end{aligned}$$

where the second inequality results from the obvious bound  $w^{1/2}/(bg) \leq 1$ . Now the first term of this sum is clearly  $O(\delta^{2\nu})$  from Lemma 5.1; this also holds for the second because of the boundedness of  $g_0^*/g^*$  which results from that of  $F(\lambda/\delta)$ .



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