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# SPHERICAL GRADIENT MANIFOLDS

by

Christian Miebach & Henrik Stötzel

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**Abstract.** — We study the action of a real-reductive group  $G = K \exp(\mathfrak{p})$  on real-analytic submanifold  $X$  of a Kähler manifold  $Z$ . We suppose that the action of  $G$  extends holomorphically to an action of the complexified group  $G^{\mathbb{C}}$  such that the action of a maximal Hamiltonian subgroup is Hamiltonian. The moment map  $\mu$  induces a gradient map  $\mu_{\mathfrak{p}} : X \rightarrow \mathfrak{p}$ . We show that  $\mu_{\mathfrak{p}}$  almost separates the  $K$ -orbits if and only if a minimal parabolic subgroup of  $G$  has an open orbit. This generalizes Brion's characterization of spherical Kähler manifolds with moment maps.

**Résumé.** — Nous étudions l'action d'un groupe réel-réductif  $G = K \exp(\mathfrak{p})$  sur une sous-variété réel-analytique  $X$  d'une variété kählérienne  $Z$ . Nous supposons que l'action de  $G$  peut être prolongée à une action holomorphe du groupe complexifié  $G^{\mathbb{C}}$  telle que l'action d'un sous-groupe maximal compact de  $G^{\mathbb{C}}$  soit hamiltonienne. L'application moment  $\mu$  induit une application gradient  $\mu_{\mathfrak{p}} : X \rightarrow \mathfrak{p}$ . Nous montrons que  $\mu_{\mathfrak{p}}$  sépare les orbites de  $K$  si et seulement si un sous-groupe minimal parabolique de  $G$  possède une orbite ouverte dans  $X$ . Ce résultat généralise la caractérisation de Brion des variétés kählériennes sphériques qui admettent une application moment.

## 1. Introduction

Let  $U^{\mathbb{C}}$  be a complex-reductive Lie group with compact real form  $U$  and let  $Z$  be a Kähler manifold on which  $U^{\mathbb{C}}$  acts holomorphically such that  $U$  acts by Kähler isometries. Assume furthermore that the  $U$ -action on  $Z$  is Hamiltonian, i. e. that there exists a  $U$ -equivariant moment map  $\mu : Z \rightarrow \mathfrak{u}^*$  where  $\mathfrak{u}$  denotes the Lie algebra of  $U$ .

In the special case that  $Z$  is compact it is shown in [Bri87] (see also [HW90]) that  $\mu$  separates the  $U$ -orbits if and only if  $Z$  is a spherical  $U^{\mathbb{C}}$ -manifold, which means that a Borel subgroup of  $U^{\mathbb{C}}$  has an open orbit in  $Z$ . Note that  $\mu$  separates the  $U$ -orbits if and only if it induces an injective map  $Z/U \hookrightarrow \mathfrak{u}/U$ . Moreover, this is equivalent to the property that the  $U$ -action on  $Z$  is coisotropic.

In this paper we generalize Brion's result to actions of real-reductive groups on real-analytic manifolds which moreover are not assumed to be compact. More precisely, we consider a closed subgroup  $G$  of  $U^{\mathbb{C}}$  which is compatible with the Cartan decomposition  $U^{\mathbb{C}} = U \exp(i\mathfrak{u})$ . This means that  $G = K \exp(\mathfrak{p})$  where  $K := G \cap U$  and  $\mathfrak{p}$  is an  $\text{Ad}(K)$ -invariant subspace of

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ii. Let  $X$  be a  $G$ -invariant real-analytic submanifold of  $Z$ . By restriction, the moment map  $\mu$  induces a  $K$ -equivariant gradient map  $\mu_{\mathfrak{p}}: X \rightarrow (\mathfrak{ip})^*$ .

There are two main differences between the complex and the real situation: Even if  $X$  is connected an open  $G$ -orbit in  $X$  does not have to be dense and in general the fibers of  $\mu_{\mathfrak{p}}$  are not connected. Therefore one cannot expect  $\mu_{\mathfrak{p}}$  to separate the  $K$ -orbits globally in  $X$ . We say that  $\mu_{\mathfrak{p}}$  locally almost separates the  $K$ -orbits if there exists a  $K$ -invariant open subset  $\Omega$  of  $X$  such that  $K \cdot x$  is open in  $\mu_{\mathfrak{p}}^{-1}(K \cdot \mu_{\mathfrak{p}}(x))$  for all  $x \in \Omega$ . Geometrically this means that the induced map  $\Omega/K \rightarrow \mathfrak{p}/K$  has discrete fibers. If  $\Omega = X$ , we say that  $\mu_{\mathfrak{p}}$  almost separates the  $K$ -orbits in  $X$ .

We suppose throughout this article that  $X/G$  is connected. Now we can state our main result.

**Theorem 1.** — *The following are equivalent.*

1. *The gradient map  $\mu_{\mathfrak{p}}$  locally almost separates the  $K$ -orbits.*
2. *The gradient map  $\mu_{\mathfrak{p}}$  almost separates the  $K$ -orbits in  $X$ .*
3. *The minimal parabolic subgroup  $Q_0$  of  $G$  has an open orbit in  $X$ .*

Hence, Theorem 1 gives a sufficient condition on the  $G$ -action for  $\mu_{\mathfrak{p}}$  to induce a map  $X/K \rightarrow \mathfrak{p}/K$  whose fibers are discrete, while on the other hand the gradient map yields a criterion for  $X$  to be spherical. Moreover we see that sphericity is independent of the particular choice of  $\mu_{\mathfrak{p}}$ , i. e. if one gradient map for the  $G$ -action on  $X$  generically separates the  $K$ -orbits in  $X$ , then this is true for every gradient map.

Let us outline the main ideas of the proof. First we observe that  $X$  contains an open  $Q_0$ -orbit if and only if  $(G/Q_0) \times X$  contains an open  $G$ -orbit with respect to the diagonal action of  $G$ . The gradient map  $\mu_{\mathfrak{p}}$  on  $X$  induces a gradient map  $\tilde{\mu}_{\mathfrak{p}}$  on  $(G/Q_0) \times X$ . Now we are in a situation where we can apply the methods introduced in [HS07b]. These allow us to show that open  $G$ -orbits correspond to isolated minimal  $K$ -orbits of the norm squared of  $\tilde{\mu}_{\mathfrak{p}}$ . In order to relate the property that  $\mu_{\mathfrak{p}}$  locally almost separates the  $K$ -orbits to the existence of an isolated minimal  $K$ -orbit, we need the following result. We consider the restriction  $\mu_{\mathfrak{p}}|_{K \cdot x}: K \cdot x \rightarrow K \cdot \mu_{\mathfrak{p}}(x)$  which is a smooth fiber bundle with fiber  $K_{\mu_{\mathfrak{p}}(x)}/K_x$ . In the special case  $G = K^{\mathbb{C}}$  it is proven in [GS84] that for generic  $x$  the fiber  $K_{\mu_{\mathfrak{p}}(x)}/K_x$  is a torus. As a generalization we prove the following proposition, which also allows us to extend the notion of “ $K$ -spherical” defined in [HW90] to actions of real-reductive groups.

**Proposition 2.** — *Let  $x \in X$  be generic and choose a maximal Abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p}$  containing  $\mu_{\mathfrak{p}}(x)$ . Then the orbits of the centralizer  $\mathcal{Z}_K(\mathfrak{a})$  of  $\mathfrak{a}$  in  $K$  are open in  $K_{\mu_{\mathfrak{p}}(x)}/K_x$ .*

These arguments yield the existence of an open  $Q_0$ -orbit under the assumption that  $\mu_{\mathfrak{p}}$  locally almost separates the  $K$ -orbits. For the other direction we apply the shifting technique for gradient maps.

Notice that our proof of Brion’s theorem is different from the ones in [Bri87] and [HW90]. In particular, for every generic element  $x \in X$  we construct a minimal parabolic subgroup  $Q_0$  of  $G$  such that  $Q_0 \cdot x$  is open in  $X$ .

At present we do not know whether a spherical  $G$ -gradient manifold does only contain a finite number of  $G$ - and  $Q_0$ -orbits (which is true in the complex-algebraic situation). These and other natural open questions will be addressed in future works.

## 2. Gradient manifolds

In this section we review the necessary background on  $G$ -gradient manifolds and gradient maps. We then define what it means that a gradient map locally almost separates the orbits

of a maximal compact subgroup of  $G$  and discuss several examples where this can be shown to be true.

**2.1. The gradient map.** — Here we recall the definition of the gradient map. For a detailed discussion we refer the reader to [HS07b].

Let  $U$  be a compact Lie group and  $U^{\mathbb{C}}$  its universal complexification (see [Ho65]). We assume that  $Z$  is a Kähler manifold with a holomorphic action of  $U^{\mathbb{C}}$  such that the Kähler form is invariant under the action of the compact real form  $U$  of  $U^{\mathbb{C}}$ . We assume furthermore that the action of  $U$  is Hamiltonian, i. e. that there exists a moment map  $\mu: Z \rightarrow \mathfrak{u}^*$ , where  $\mathfrak{u}^*$  is the dual of the Lie algebra of  $U$ . We require  $\mu$  to be real-analytic and  $U$ -equivariant, where the action of  $U$  on  $\mathfrak{u}^*$  is the coadjoint action.

The complex reductive group  $U^{\mathbb{C}}$  admits a Cartan involution  $\theta: U^{\mathbb{C}} \rightarrow U^{\mathbb{C}}$  with fixed point set  $U$ . The  $-1$ -eigenspace of the induced Lie algebra involution equals  $\mathfrak{iu}$ . We have an induced Cartan decomposition, i. e. the map  $U \times \mathfrak{iu} \rightarrow U^{\mathbb{C}}$ ,  $(u, \xi) \mapsto u \exp(\xi)$  is a diffeomorphism. Let  $G$  be a  $\theta$ -stable closed real subgroup of  $U^{\mathbb{C}}$  with only finitely many connected components. Equivalently, we assume that  $G$  is a closed subgroup of  $U^{\mathbb{C}}$ , such that the Cartan decomposition restricts to a diffeomorphism  $K \times \mathfrak{p} \rightarrow G$ , where  $K := G \cap U$  and  $\mathfrak{p} := \mathfrak{g} \cap \mathfrak{iu}$ . In this paper such a group  $G = K \exp(\mathfrak{p})$  is called *real-reductive*. Note that  $U^{\mathbb{C}}$  itself is an example for such a subgroup  $G$  of  $U^{\mathbb{C}}$ .

Let  $X$  be a  $G$ -invariant real-analytic submanifold of  $Z$  such that  $X/G$  is connected. We identify  $\mathfrak{u}$  with  $\mathfrak{u}^*$  by a  $U$ -invariant inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{u}$ . Moreover we identify  $\mathfrak{u}$  and  $\mathfrak{iu}$  by multiplication with  $i$ . Then the moment map  $\mu: Z \rightarrow \mathfrak{u}^*$  restricts to a real-analytic map  $\mu_{\mathfrak{p}}: X \rightarrow \mathfrak{p}$  which is defined by  $\langle \mu_{\mathfrak{p}}(x), \xi \rangle = \mu(x)(-i\xi)$  for  $\xi \in \mathfrak{p}$ . We call  $\mu_{\mathfrak{p}}$  a  *$G$ -gradient map on  $X$*  and we say that  $X$  is a  *$G$ -gradient manifold*. Note that  $\mu_{\mathfrak{p}}$  is  $K$ -equivariant with respect to the adjoint action of  $K$  on  $\mathfrak{p}$ . In the special case  $G = U^{\mathbb{C}}$ , the gradient map coincides with the moment map up to the identification of  $\mathfrak{u}^*$  with  $\mathfrak{iu}$ .

In this paper, we consider real-analytic gradient maps which *locally almost separate the  $K$ -orbits*. By this, we mean that there exists a  $K$ -invariant open subset  $\Omega$  of  $X$  such that the following equivalent conditions are satisfied.

1.  $K \cdot x$  is open in  $\mu_{\mathfrak{p}}^{-1}(K \cdot \mu_{\mathfrak{p}}(x))$  for all  $x \in \Omega$ .
2.  $K_{\mu_{\mathfrak{p}}(x)} \cdot x$  is open in  $\mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x))$  for all  $x \in \Omega$ .
3. The induced map  $\bar{\mu}_{\mathfrak{p}}: \Omega/K \rightarrow \mathfrak{p}/K$  has discrete fibers.

If  $\Omega = X$ , we say that  $\mu_{\mathfrak{p}}$  *almost separates the  $K$ -orbits*. We will show later that the set  $\Omega$  on which  $\mu_{\mathfrak{p}}$  almost separates the  $K$ -orbits can always be chosen to be  $X$ , i. e.  $\mu_{\mathfrak{p}}$  separates locally almost the  $K$ -orbits if and only if  $\mu_{\mathfrak{p}}$  almost separates them. If  $\mu_{\mathfrak{p}}^{-1}(K \cdot \mu_{\mathfrak{p}}(x)) = K \cdot x$  for all  $x \in X$ , then we say that  $\mu_{\mathfrak{p}}$  *globally separates the  $K$ -orbits*.

**Lemma 2.1.** — *Suppose that  $\mu_{\mathfrak{p}}: X \rightarrow \mathfrak{p}$  locally almost separates the  $K$ -orbits. Then  $G$  has an open orbit in  $X$ .*

*Proof.* — By assumption there exists a  $K$ -invariant open subset  $\Omega \subset X$  such that  $\mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x))^0 \subset K \cdot x$  holds for all  $x \in \Omega$ . Since  $\mu_{\mathfrak{p}}$  is real-analytic, we find a point  $x \in \Omega$  such that  $\mu_{\mathfrak{p}}$  has maximal rank in  $x$ . We conclude from Lemma 5.1 in [HS07b] that  $(\mathfrak{p} \cdot x)^{\perp} = T_x \mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x)) \subset \mathfrak{k} \cdot x$  and thus obtain

$$T_x X = (\mathfrak{p} \cdot x) \oplus (\mathfrak{p} \cdot x)^{\perp} \subset (\mathfrak{p} \cdot x) + (\mathfrak{k} \cdot x) = \mathfrak{g} \cdot x,$$

which means that  $G \cdot x$  is open in  $X$ . □

**2.2. Examples.** — In general, it is very difficult to verify directly that a  $G$ -gradient map separates (locally almost) the  $K$ -orbits. In this subsection we give some examples of situations where this can be done.

*Example.* — The connected group  $G = K \exp(\mathfrak{p})$  acts on itself by left multiplication. The standard gradient map for this action is given by  $\mu_{\mathfrak{p}}: G \rightarrow \mathfrak{p}$ ,  $\mu_{\mathfrak{p}}(k \exp(\xi)) = \text{Ad}(k)\xi$ . Let  $x_0 = k_0 \exp(\xi_0) \in G$  be given. One checks directly that  $\mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x_0)) = x_0 K$ . Hence,  $\mu_{\mathfrak{p}}$  locally almost separates the  $K$ -orbits if and only if there exists a  $K$ -invariant open subset  $\Omega \subset G$  such that  $xK = Kx$  for all  $x \in \Omega$ . We claim that this is the case if and only if  $\mathfrak{p}^K = \mathfrak{p}$ .

Suppose that  $xK = Kx$  holds for all  $x$  in a  $K$ -invariant open subset  $\Omega \subset G$ . This means that the fixed point set  $(G/K)^K$  has non-empty interior. Since  $G/K$  is  $K$ -equivariantly diffeomorphic to  $\mathfrak{p}$  with the adjoint  $K$ -action, we see that  $\mathfrak{p}^K$  has non-empty interior and thus  $\mathfrak{p}^K = \mathfrak{p}$ .

Conversely, if  $\mathfrak{p}^K = \mathfrak{p}$ , then we have for every  $x = k \exp(\xi) \in G$  that  $Kx = K \exp(\xi) = \exp(\xi)K = xK$  holds.

*Example.* — We describe a class of totally real  $G$ -gradient manifolds where  $\mu_{\mathfrak{p}}$  locally almost separates the  $K$ -orbits.

Let  $(Z, \omega)$  be a Kähler manifold endowed with a holomorphic  $U^{\mathbb{C}}$ -action such that the  $U$ -action is Hamiltonian with moment map  $\mu: Z \rightarrow \mathfrak{u}^*$ . Suppose that the action is defined over  $\mathbb{R}$  in the following sense. There exists an antiholomorphic involutive automorphism  $\sigma: U^{\mathbb{C}} \rightarrow U^{\mathbb{C}}$  with  $\sigma\theta = \theta\sigma$  and there is an antiholomorphic involution  $\tau: Z \rightarrow Z$  with  $\tau^*\omega = -\omega$  and  $\tau(g \cdot z) = \sigma(g) \cdot \tau(z)$  for all  $g \in U^{\mathbb{C}}$  and all  $z \in Z$ . Consequently, the fixed point set  $X := Z^{\tau}$  is a Lagrangian submanifold of  $Z$  and the compatible real form  $G = K \exp(\mathfrak{p}) = (U^{\mathbb{C}})^{\sigma}$  acts on  $X$ . Let  $\mu_{\mathfrak{p}}: X \rightarrow \mathfrak{p}$  be the  $K$ -equivariant gradient map induced by  $\mu$ .

We claim that if  $\mu$  locally almost separates the  $U$ -orbits in  $Z$ , then  $\mu_{\mathfrak{p}}$  locally almost separates the  $K$ -orbits in  $X$ . This claim is a consequence of the following three observations:

1. If  $\mu$  locally almost separates the  $U$ -orbits, then  $\mu$  separates all the  $U$ -orbits in  $Z$  (see [HW90]).
2. Since  $X$  is Lagrangian, we see that  $\mu_{\mathfrak{k}}|_X \equiv 0$ , where  $\mu_{\mathfrak{k}}$  denotes the moment map for the  $K$ -action on  $Z$ . Note that under our identification we have  $\mu = \mu_{\mathfrak{k}} + \mu_{\mathfrak{p}}$ .
3. For every  $x \in X$  the orbit  $K \cdot x$  is open in  $(U \cdot x) \cap X$ .

Locally injective gradient maps separate locally almost the  $K$ -orbits. A class of  $G$ -gradient manifolds for which  $\mu_{\mathfrak{p}}$  is locally injective is described in the following example.

*Example.* — Let  $Z = U/K$  be a Hermitian symmetric space of the compact type, and let  $G = K \exp(\mathfrak{p})$  be a Hermitian real form of  $U^{\mathbb{C}}$ . Then  $Z$  is a  $G$ -gradient manifold and every gradient map  $\mu_{\mathfrak{p}}: Z \rightarrow \mathfrak{p}$  is locally injective. Consequently,  $\mu_{\mathfrak{p}}$  separates locally almost the  $K$ -orbits in  $Z$ .

We will elaborate a little bit on further properties of  $\mu_{\mathfrak{p}}: Z \rightarrow \mathfrak{p}$ . Let  $\tau: Z \rightarrow Z$  be the holomorphic symmetry which fixes the base point  $z_0 = eK$ . Then we have  $Z^{\tau} = \mu_{\mathfrak{p}}^{-1}(0)$ . Moreover, one can show that  $Z^{\tau}$  is a  $K$ -invariant closed complex submanifold of  $Z$  and that every  $K$ -orbit in  $Z^{\tau}$  is open in  $Z^{\tau}$ . Furthermore,  $K^{\mathbb{C}}$  acts on  $Z^{\tau}$  and we have  $K^{\mathbb{C}} \cdot z = K \cdot z$  if and only if  $z \in Z^{\tau}$  holds. Finally, note that  $\mu_{\mathfrak{k}}$  separates all  $K$ -orbits in  $Z$ .

### 3. Spherical gradient manifolds and coadjoint orbits

As we have remarked above it is very hard to verify directly if a given gradient map defined on  $X$  separates the  $K$ -orbits. The main result of this paper states that this is true if and only if  $X$  is a spherical gradient manifold. Hence, this is independent of the particular choice of a gradient map  $\mu_{\mathfrak{p}}$ .

In this section we give the definition of spherical gradient manifolds. For this we first review the definition of minimal parabolic subgroups. After that, we discuss the orbits of the adjoint  $K$ -action on  $\mathfrak{p}$  which are the right analogues of complex flag varieties.

We continue the notation of the previous section: Let  $G = K \exp(\mathfrak{p})$  be a closed compatible subgroup of  $U^{\mathbb{C}}$  and let  $X$  be a real-analytic  $G$ -gradient manifold with  $K$ -equivariant real-analytic gradient map  $\mu_{\mathfrak{p}}: X \rightarrow \mathfrak{p}$ .

**3.1. Minimal parabolic subgroups.** — For more details and complete proofs of the material presented here we refer the reader to Chapter VII in [Kna02].

Since  $G = K \exp(\mathfrak{p})$  is invariant under the Cartan involution  $\theta$  of  $U^{\mathbb{C}}$ , the same holds for its Lie algebra  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ . Consequently  $\mathfrak{g}$  is reductive, i. e.  $\mathfrak{g}$  is the direct sum of its center and of the semi-simple subalgebra  $[\mathfrak{g}, \mathfrak{g}]$ .

Let  $\mathfrak{a}$  be a maximal Abelian subalgebra of  $\mathfrak{p}$  and let  $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Lambda} \mathfrak{g}_{\lambda}$  be the associated restricted root space decomposition. The centralizer  $\mathfrak{g}_0$  of  $\mathfrak{a}$  in  $\mathfrak{g}$  is  $\theta$ -stable with decomposition  $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$  where  $\mathfrak{m} = \mathcal{Z}_{\mathfrak{k}}(\mathfrak{a})$ . On the group level we define  $M := \mathcal{Z}_K(\mathfrak{a})$ .

Let us fix a choice  $\Lambda^+$  of positive restricted roots. Then we obtain the nilpotent subalgebra  $\mathfrak{n} := \bigoplus_{\lambda \in \Lambda^+} \mathfrak{g}_{\lambda}$ . Let  $A$  and  $N$  be the analytic subgroups of  $G$  with Lie algebras  $\mathfrak{a}$  and  $\mathfrak{n}$ , respectively. Then  $AN \subset G$  is a simply-connected solvable closed subgroup of  $G$ , isomorphic to the semi-direct product  $A \ltimes N$ . One checks directly that  $M$  stabilizes each restricted root space  $\mathfrak{g}_{\lambda}$ ; together with the compactness of  $M$  this implies that  $Q_0 := MAN$  is a closed subgroup of  $G$ .

Every subgroup of  $G$  which is conjugate to  $Q_0 = MAN$  is called a *minimal parabolic subgroup*. A subgroup  $Q \subset G$  is called *parabolic* if it contains a minimal parabolic subgroup.

**Remark.** — The notion of parabolic subgroups of  $G$  is independent of the choices made during the construction of  $Q_0$ .

**Example.** — For  $\xi \in \mathfrak{p}$  the group  $Q := \{g \in G; \lim_{t \rightarrow -\infty} \exp(t\xi)g \exp(-t\xi) \text{ exists in } G\}$  is a parabolic subgroup of  $G$ . It is a minimal parabolic subgroup if and only if  $\xi$  is regular, i. e. if and only if  $K_{\xi} = M$ .

If the group  $G$  is complex-reductive and connected, then minimal parabolic subgroups of  $G$  are the same as Borel subgroups. This motivates the following

**Definition 3.1.** — We call the  $G$ -gradient manifold  $X$  *spherical* if a minimal parabolic subgroup of  $G$  has an open orbit in  $X$ .

Note that  $X$  is spherical if and only if  $Q_0 = MAN$  has an open orbit in  $X$ .

**Example.** — Let  $G$  be a real form of  $U^{\mathbb{C}}$  and let  $X \subset Z$  be a totally real  $G$ -stable submanifold with  $\dim_{\mathbb{R}} X = \dim_{\mathbb{C}} Z$ . If  $Z$  is  $U^{\mathbb{C}}$ -spherical, then  $X$  is  $G$ -spherical in the above sense. This can be seen as follows. Since  $Q_0^{\mathbb{C}}$  is a parabolic subgroup of  $U^{\mathbb{C}} = G^{\mathbb{C}}$  and since  $Z$  is spherical,  $Q_0^{\mathbb{C}}$  has an open orbit in  $Z$ . Since  $X$  is maximally totally real,  $X$  cannot be contained in the complement of the open  $Q_0^{\mathbb{C}}$ -orbit in  $Z$ , hence we find a point  $x \in X$  such that  $Q_0^{\mathbb{C}} \cdot x$  is open in  $Z$ . Moreover,  $Q_0 \cdot x$  is open in  $(Q_0^{\mathbb{C}} \cdot x) \cap X$ , which implies that  $X$  is spherical.

**Example.** — As a special case of the above example we note that weakly symmetric spaces are spherical gradient manifolds. More precisely, let  $G^{\mathbb{C}}$  be connected complex-reductive and let  $L^{\mathbb{C}}$  be a complex-reductive compatible subgroup of  $G^{\mathbb{C}}$ . Let  $G$  be a connected compatible real form of  $G^{\mathbb{C}}$  such that  $L := L^{\mathbb{C}} \cap G$  is a compact real form of  $L^{\mathbb{C}}$ . According to Theorem 3.11 in [St08] the homogeneous manifold  $X = G/L$  is a  $G$ -gradient manifold. By a result of Akhiezer and Vinberg ([AV99], compare also Chapter 12.6 in [Wo07])  $X = G/L$  is weakly symmetric if and only if the affine variety  $G^{\mathbb{C}}/L^{\mathbb{C}}$  is spherical. This implies that if  $X = G/L$  is weakly symmetric, then it is a spherical  $G$ -gradient manifold. The converse is false as the next example shows.

**Example.** — Let  $U$  be connected. A special case of Example 2.2 is the case that  $Z = U^{\mathbb{C}}$  and  $\tau = \sigma = \theta$ . Then we have  $G = X = U$ . Note that  $\mu_{\mathfrak{p}} \equiv 0$  separates the  $K$ -orbits in  $X$  since  $X$  is  $K$ -homogeneous while in general  $\mu$  does not separate the  $U$ -orbits in  $Z$ . Note also that  $Q_0 = G$  is the only minimal parabolic subgroup of  $G$  and that  $G$  itself is the only subgroup of  $G$  having an open orbit in  $X$ . This explains the necessity to consider minimal parabolic subgroups instead of maximal connected solvable subgroups (which are maximal tori in  $G$  in this example).

**3.2. Coadjoint orbits.** — A class of examples of gradient manifolds is given by coadjoint orbits (see [HS07c]). Let  $\alpha \in \mathfrak{u}^*$  and let  $Z = U \cdot \alpha$  be the coadjoint orbit of  $\alpha$ . Identifying  $\mathfrak{u}^*$  with  $\mathfrak{iu}$  as before,  $\alpha$  corresponds to an element  $\xi \in \mathfrak{iu}$  and  $Z$  corresponds to the orbit of  $\xi$  of the adjoint action of  $U$  on  $\mathfrak{iu}$ . Let  $P := \{g \in U^{\mathbb{C}}; \lim_{t \rightarrow -\infty} \exp(t\xi)g \exp(-t\xi) \text{ exists in } U^{\mathbb{C}}\}$  denote the parabolic subgroup of  $U^{\mathbb{C}}$  associated to  $\xi$ . Then the map  $Z \rightarrow U^{\mathbb{C}}/P$ ,  $u \cdot \xi \mapsto uP$ , is a real analytic isomorphism. In particular it defines a complex structure and a holomorphic  $U^{\mathbb{C}}$ -action on  $Z$ . The reader should be warned that this  $U^{\mathbb{C}}$ -action is not the adjoint action. The form  $\omega(\eta_Z(\alpha), \zeta_Z(\alpha)) = -\alpha([\eta, \zeta])$  defines a  $U$ -invariant Kähler form on  $Z = U \cdot \alpha$  such that the map  $\mu: Z \rightarrow \mathfrak{u}^*$ ,  $\mu(u \cdot \alpha) = -\text{Ad}(u)\alpha$ , is a moment map on  $Z$ . Identifying  $Z$  with  $U/U_{\xi}$  where  $U_{\xi}$  denotes the centralizer of  $\xi$  in  $U$ , the gradient map with respect to the action of  $U^{\mathbb{C}}$  on  $Z$  is given by  $\mu_{\mathfrak{iu}}: U/U_{\xi} \rightarrow \mathfrak{iu}$ ,  $uU_{\xi} \mapsto -\text{Ad}(u)\xi$ . The  $U^{\mathbb{C}}$ -action on  $U \cdot \xi \cong U^{\mathbb{C}}/P$  induces a  $G$ -action on  $U \cdot \xi$ .

**Proposition 3.2** ([HS07c]). — *If  $\xi \in \mathfrak{p}$ , then  $X := K \cdot \xi = G \cdot \xi$  is a Lagrangian submanifold of  $Z \cong U \cdot \xi$ .*

The  $G$ -isotropy at  $\xi$  is given by the parabolic subgroup  $Q := P \cap G$  of  $G$ , so  $G \cdot \xi$  is isomorphic to  $G/Q$  and to  $K/K_{\xi}$  if  $\xi \in \mathfrak{p}$ . Note also that  $G/Q$  is a compact  $G$ -invariant submanifold of  $U^{\mathbb{C}}/P$  and in particular a  $G$ -gradient manifold with gradient map  $\mu_{\mathfrak{p}}: K/K_{\xi} \rightarrow \mathfrak{p}$ ,  $\mu_{\mathfrak{p}}(kK_{\xi}) = -\text{Ad}(k)\xi$ .

**Example.** — Consider the action of  $G = \text{SL}(2, \mathbb{R})$  on projective space  $Z = \mathbb{P}_1(\mathbb{C})$  induced by the standard representation of  $G$  on  $\mathbb{C}^2$ . Note that  $G$  is a compatible subgroup of  $U^{\mathbb{C}} = \text{SL}(2, \mathbb{C})$  where  $U = \text{SU}(2)$ . Moreover,  $Z$  can be realized as the coadjoint orbit  $U^{\mathbb{C}}/B$  where  $B$  is the Borel subgroup  $B = \left\{ \begin{pmatrix} z & w \\ 0 & z^{-1} \end{pmatrix}; z \in \mathbb{C}^*, w \in \mathbb{C} \right\}$ . Then  $Z$  can be viewed as a 2-sphere in the 3-dimensional space  $\mathfrak{iu}$ . The gradient map  $\mu_{\mathfrak{p}}$  is the projection onto the 2-dimensional subspace  $\mathfrak{p}$  of  $\mathfrak{iu}$ . The action of  $K$  on  $\mathfrak{iu}$  is given by rotation around the axes perpendicular to  $\mathfrak{p}$ . We observe that  $\mu_{\mathfrak{p}}$  almost separates the  $K$ -orbits, but that it does not separate all  $K$ -orbits. This corresponds to the fact that there exist two open orbits with respect to the action of a minimal parabolic subgroup of  $G$ .

If  $G = U^{\mathbb{C}}$  is complex reductive and acts algebraically on a connected algebraic variety  $Z$ , then the fibers of the moment map  $\mu$  are connected ([HH96]). Also, if  $Z$  is spherical, then  $\mu$  globally separates the  $U$ -orbits. The example above shows that one cannot expect  $\mu_{\mathfrak{p}}$  to separate the  $K$ -orbits globally for actions of real-reductive groups due to the non-connectedness of the  $\mu_{\mathfrak{p}}$ -fibers. Moreover, in the complex case an open orbit of a Borel subgroup is unique and dense in  $Z$  while this is no longer true for real-reductive groups.

#### 4. The generic fibers of the restricted gradient map

By equivariance, the moment map  $\mu: Z \rightarrow \mathfrak{u}^*$  maps each orbit  $U \cdot z$  onto the orbit  $U \cdot \mu(z) \subset \mathfrak{u}^*$ . Moreover, the restriction  $\mu|_{U \cdot z}: U \cdot z \rightarrow U \cdot \mu(z)$  is a smooth fiber bundle with fiber  $U_{\mu(z)}/U_z$ . Theorem 26.5 in [GS84] states that generically these fibers are tori; in [HW90] this theorem is applied to characterize coisotropic  $U$ -actions.

In this section we generalize these results in our context. Let  $x \in X$  and let  $\mathfrak{a}$  be a maximal Abelian subspace of  $\mathfrak{p}$  with  $\mu_{\mathfrak{p}}(x) \in \mathfrak{a}$ . Our goal is to prove that generically the group  $M = \mathcal{Z}_K(\mathfrak{a})$  has an open orbit in the fiber  $K_{\mu_{\mathfrak{p}}(x)}/K_x$  of  $\mu_{\mathfrak{p}}: K \cdot x \rightarrow K \cdot \mu_{\mathfrak{p}}(x)$ . For this we first have to discuss the notion of generic elements in  $X$ .

**4.1. Generic elements.** — There are several natural definitions of generic elements  $x \in X$ . We could require that the  $K$ -orbit through  $x$  has maximal dimension, or that the  $K$ -orbit through  $\mu_{\mathfrak{p}}(x)$  has maximal dimension in  $\mu_{\mathfrak{p}}(X)$ , or that the rank of  $\mu_{\mathfrak{p}}$  in  $x$  is maximal. It will turn out that we need all three properties.

**Definition 4.1.** — The element  $x \in X$  is called generic if

1. the dimension of  $K \cdot x$  is maximal,
2. the rank of  $\mu_{\mathfrak{p}}$  in  $x$  is maximal, and
3. the dimension of  $K \cdot \mu_{\mathfrak{p}}(x)$  is maximal in  $\mu_{\mathfrak{p}}(X)$ .

We write  $X_{\text{gen}}$  for the set of generic elements in  $X$ .

**Remark.** — In the complex case we have  $\text{rk}_z \mu = \dim U \cdot z$ ; hence, condition (2) in Definition 4.1 is superfluous in this case.

For the following lemma we need the analyticity of  $\mu_{\mathfrak{p}}$  and of the  $K$ -action on  $X$ .

**Lemma 4.2.** — *The set  $X_{\text{gen}}$  is  $K$ -invariant, open and dense in  $X$ .*

*Proof.* — Since  $X/G$  is connected, the same is true for  $X/K$ . It is then a well-known consequence of the Slice Theorem that the set of points  $x \in X$  such that  $K \cdot x$  has maximal dimension is open and dense in  $X$  (see Theorem 3.1, Chapter IV in [Bre72]). Since  $\mu_{\mathfrak{p}}: X \rightarrow \mathfrak{p}$  is real-analytic, its maximal rank set is also open and dense. Hence,  $X' := \{x \in X; \dim K \cdot x, \text{rk}_x \mu_{\mathfrak{p}} \text{ maximal}\}$  is open and dense in  $X$ .

We prove the lemma by showing that  $X' \setminus X_{\text{gen}}$  is analytic in  $X'$ . Let  $x_0 \in X' \setminus X_{\text{gen}}$ . Since  $\mu_{\mathfrak{p}}$  has constant rank on  $X'$ , there are local analytic coordinates  $(x, U)$  around  $x_0$  in  $X$  and  $(y, V)$  around  $\mu_{\mathfrak{p}}(x_0)$  in  $\mu_{\mathfrak{p}}(X)$  in which  $\mu_{\mathfrak{p}}$  takes the form  $\mu_{\mathfrak{p}}(x_1, \dots, x_n) = (x_1, \dots, x_k)$ . Since  $\mu_{\mathfrak{p}}$  is  $K$ -equivariant,  $U$  and  $V$  may be chosen  $K$ -invariant. Since  $A := \{y \in V; \dim K \cdot y \text{ is not maximal in } V\}$  is analytic in  $V$ , we see that  $(X' \setminus X_{\text{gen}}) \cap U = \mu_{\mathfrak{p}}^{-1}(A)$  is analytic in  $U$ . Thus  $X' \setminus X_{\text{gen}}$  is locally analytic in  $X$  and since it is closed, it is analytic.  $\square$



**4.2. The  $M$ -action on  $\mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x))$ .** — In this subsection we discuss the restricted gradient map

$$\mu_{\mathfrak{p}}|_{K \cdot x}: K \cdot x \rightarrow K \cdot \mu_{\mathfrak{p}}(x).$$

Recall that this map is a smooth fiber bundle with fiber  $K_{\mu_{\mathfrak{p}}(x)}/K_x$ .

**Remark.** — Let  $\mathfrak{a}$  be a maximal Abelian subspace of  $\mathfrak{p}$ . Then we have  $M \subset K_{\mu_{\mathfrak{p}}(x)}$  for every  $x \in X$  with  $\mu_{\mathfrak{p}}(x) \in \mathfrak{a}$ . Note that every  $K$ -orbit in  $X$  intersects  $\mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$ .

We will need the following lemma which extends the corresponding result in [GS84].

**Lemma 4.3.** — *For every  $x \in X_{\text{gen}}$  we have  $[\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}] \subset \mathfrak{p}_x$ .*

*Proof.* — By definition of  $X_{\text{gen}}$  the set

$$E := \{(x, \xi, \eta) \in X_{\text{gen}} \times \mathfrak{k} \times \mathfrak{p}; \xi \in \mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \eta \in \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}\}$$

is a linear subbundle of the trivial bundle  $X_{\text{gen}} \times \mathfrak{k} \times \mathfrak{p} \rightarrow X_{\text{gen}}$ .

Let  $\xi \in \mathfrak{k}_{\mu_{\mathfrak{p}}(x)}$  and  $\eta \in \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}$ , and let  $x_t$  be a smooth curve in  $X_{\text{gen}}$  with  $x_0 = x$ . Since  $E \rightarrow X_{\text{gen}}$  is locally trivial, we find a smooth curve  $(x_t, \xi_t, \eta_t)$  in  $E$  with  $\xi_0 = \xi$  and  $\eta_0 = \eta$ . Since  $[\xi_t, \eta_t] \in \mathfrak{p}_{\mu_{\mathfrak{p}}(x_t)}$  for all  $t$  and since the inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{p}$  is induced by a  $U$ -invariant inner product on  $\mathfrak{u}$ , we conclude

$$\langle \mu_{\mathfrak{p}}(x_t), [\xi_t, \eta_t] \rangle = -\langle [\xi_t, \mu_{\mathfrak{p}}(x_t)], \eta_t \rangle = 0$$

for all  $t$ . Differentiating and evaluating at  $t = 0$  yields

$$\begin{aligned} 0 &= \langle (\mu_{\mathfrak{p}})_{*,x} \dot{x}_0, [\xi, \eta] \rangle + \langle \mu_{\mathfrak{p}}(x), [\dot{\xi}_0, \eta] \rangle + \langle \mu_{\mathfrak{p}}(x), [\xi, \dot{\eta}_0] \rangle \\ &= \langle (\mu_{\mathfrak{p}})_{*,x} \dot{x}_0, [\xi, \eta] \rangle - \langle [\eta, \mu_{\mathfrak{p}}(x)], \dot{\xi}_0 \rangle - \langle [\xi, \mu_{\mathfrak{p}}(x)], \dot{\eta}_0 \rangle \\ &= \langle (\mu_{\mathfrak{p}})_{*,x} \dot{x}_0, [\xi, \eta] \rangle = g_x([\xi, \eta]_X(x), \dot{x}_0). \end{aligned}$$

Since  $X_{\text{gen}}$  is open, every tangent vector  $v \in T_x X$  is of the form  $v = \dot{x}_0$  for some curve  $x_t$  which implies  $[\xi, \eta]_X(x) = 0$ , i. e.  $[\xi, \eta] \in \mathfrak{p}_x$ .  $\square$

Now we are in the position to prove

**Proposition 4.4.** — *Suppose  $x \in X_{\text{gen}} \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$ . Then the orbit  $M \cdot x$  is open in  $\mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x)) \cap (K \cdot x)$ .*

Let  $x \in X_{\text{gen}} \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$  be given. In order to prove Proposition 4.4 it suffices to show that the map  $\mathfrak{m} \rightarrow \mathfrak{k}_{\mu_{\mathfrak{p}}(x)}/\mathfrak{k}_x$  is surjective. For this we need some information about  $\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}$  and  $\mathfrak{k}_x$ ; the idea is of course to apply Lemma 4.3 which gives

$$[[\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}], [\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}]] \subset [\mathfrak{p}_x, \mathfrak{p}_x] \subset \mathfrak{k}_x.$$

Consequently we must determine  $\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}$ ,  $\mathfrak{p}_{\mu_{\mathfrak{p}}(x)}$  as well as their Lie brackets.

This is most conveniently done via the restricted root space decomposition  $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Lambda} \mathfrak{g}_{\lambda}$  with respect to the maximal Abelian subspace  $\mathfrak{a} \subset \mathfrak{p}$ . The centralizer  $\mathfrak{g}_0$  of  $\mathfrak{a}$  in  $\mathfrak{g}$  is stable under the Cartan involution  $\theta$  and decomposes as  $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$  where  $\mathfrak{m} = \text{Lie}(M)$ . For later use we note the following proposition which is proven in Chapter VI.5 of [Kna02].

**Proposition 4.5.** — *For each  $\lambda \in \Lambda$  we write  $\mathfrak{a}_{\lambda} \subset \mathfrak{a}$  for the subspace generated by the elements  $[\xi_{\lambda}, \theta(\xi_{\lambda})]$  where  $\xi_{\lambda} \in \mathfrak{g}_{\lambda}$ . Then  $\dim \mathfrak{a}_{\lambda} = 1$  and  $\lambda[\xi_{\lambda}, \theta(\xi_{\lambda})] \neq 0$  for every  $0 \neq \xi_{\lambda} \in \mathfrak{g}_{\lambda}$ .*

In order to prove Proposition 4.4 we will first describe the centralizers of  $\mu_{\mathfrak{p}}(x)$  in  $\mathfrak{k}$  and in  $\mathfrak{p}$ . For this we introduce the subset  $\Lambda(x) := \{\lambda \in \Lambda; \lambda(\mu_{\mathfrak{p}}(x)) = 0\} \subset \Lambda$ . We also write  $\Lambda^+(x) := \Lambda(x) \cap \Lambda^+$ .

**Remark.** — If  $\lambda \in \Lambda(x)$ , then  $-\lambda \in \Lambda(x)$ . If  $\lambda_1, \lambda_2 \in \Lambda(x)$  and  $\lambda_1 + \lambda_2 \in \Lambda$ , then  $\lambda_1 + \lambda_2 \in \Lambda(x)$ .

**Lemma 4.6.** — 1. The centralizer of  $\mu_{\mathfrak{p}}(x)$  in  $\mathfrak{g}$  is given by  $\mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Lambda(x)} \mathfrak{g}_\lambda$ .

2. We have  $\mathfrak{k}_{\mu_{\mathfrak{p}}(x)} = \mathfrak{m} \oplus \left\{ \sum_{\lambda \in \Lambda^+(x)} (\xi_\lambda + \theta(\xi_\lambda)); \xi_\lambda \in \mathfrak{g}_\lambda \right\}$ .

3. We have  $\mathfrak{p}_{\mu_{\mathfrak{p}}(x)} = \mathfrak{a} \oplus \left\{ \sum_{\lambda \in \Lambda^+(x)} (\xi_\lambda - \theta(\xi_\lambda)); \xi_\lambda \in \mathfrak{g}_\lambda \right\}$ .

*Proof.* — In order to prove the first claim let  $\xi = \xi_0 + \sum_{\lambda \in \Lambda} \xi_\lambda \in \mathfrak{g}$  and calculate

$$[\mu_{\mathfrak{p}}(x), \xi] = \sum_{\lambda \in \Lambda} \lambda(\mu_{\mathfrak{p}}(x)) \xi_\lambda.$$

Hence,  $\xi$  centralizes  $\mu_{\mathfrak{p}}(x)$  if and only if  $\xi_\lambda = 0$  for all  $\lambda \notin \Lambda(x)$ .

The other two claims follow from (1) together with the fact that  $\theta(\mathfrak{g}_\lambda) = \mathfrak{g}_{-\lambda}$  for all  $\lambda \in \Lambda$ .  $\square$

It remains to show that  $\left\{ \sum_{\lambda \in \Lambda^+(x)} (\xi_\lambda + \theta(\xi_\lambda)); \xi_\lambda \in \mathfrak{g}_\lambda \right\}$  is contained in  $\mathfrak{k}_x$  because then Lemma 4.6 implies that  $\mathfrak{m} \rightarrow \mathfrak{k}_{\mu_{\mathfrak{p}}(x)}/\mathfrak{k}_x$  is surjective which in turn proves Proposition 4.4.

**Lemma 4.7.** — We have  $\left\{ \sum_{\lambda \in \Lambda^+(x)} (\xi_\lambda + \theta(\xi_\lambda)); \xi_\lambda \in \mathfrak{g}_\lambda \right\} \subset \mathfrak{k}_x$ .

*Proof.* — We will prove this lemma in three steps.

In the first step we prove

$$\mathfrak{p}^x := \bigoplus_{\lambda \in \Lambda(x)} \mathfrak{a}_\lambda \oplus \left\{ \sum_{\lambda \in \Lambda^+(x)} (\xi_\lambda - \theta(\xi_\lambda)); \xi_\lambda \in \mathfrak{g}_\lambda \right\} \subset [\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}].$$

Let  $\lambda \in \Lambda^+(x)$  and  $\xi_\lambda \in \mathfrak{g}_\lambda$ . Then we have  $\xi_\lambda + \theta(\xi_\lambda) \in \mathfrak{k}_{\mu_{\mathfrak{p}}(x)}$ , and we may choose an element  $\eta \in \mathfrak{a}$  with  $\lambda(\eta) \neq 0$ . Because of

$$\xi_\lambda - \theta(\xi_\lambda) = -\frac{1}{\lambda(\eta)} [\xi_\lambda + \theta(\xi_\lambda), \eta] \in [\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}]$$

we obtain  $\left\{ \sum_{\lambda \in \Lambda^+(x)} (\xi_\lambda - \theta(\xi_\lambda)); \xi_\lambda \in \mathfrak{g}_\lambda \right\} \subset [\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}]$ .

Moreover,

$$[\xi_\lambda, \theta(\xi_\lambda)] = -\frac{1}{2} [\xi_\lambda + \theta(\xi_\lambda), \xi_\lambda - \theta(\xi_\lambda)] \in [\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}]$$

implies  $\mathfrak{a}_\lambda \subset [\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}]$ .

The second step consists in showing

$$\left\{ \sum_{\lambda \in \Lambda^+(x)} (\xi_\lambda + \theta(\xi_\lambda)); \xi_\lambda \in \mathfrak{g}_\lambda \right\} \subset [\mathfrak{p}^x, \mathfrak{p}^x].$$

To see this, let  $\lambda \in \Lambda^+(x)$  and  $0 \neq \xi_\lambda \in \mathfrak{g}_\lambda$  be arbitrary. Then we have  $\xi_\lambda - \theta(\xi_\lambda) \in \mathfrak{p}^x$  and  $[\xi_\lambda, \theta(\xi_\lambda)] \in \mathfrak{a}_\lambda$ . Moreover, Proposition 4.5 implies  $\lambda[\xi_\lambda, \theta(\xi_\lambda)] \neq 0$ , which gives

$$\xi_\lambda + \theta(\xi_\lambda) = \frac{1}{\lambda[\xi_\lambda, \theta(\xi_\lambda)]} [[\xi_\lambda, \theta(\xi_\lambda)], \xi_\lambda - \theta(\xi_\lambda)] \in [\mathfrak{p}^x, \mathfrak{p}^x].$$

In the last step we combine the results obtained so far with Lemma 4.3 and arrive at

$$\left\{ \sum_{\lambda \in \Lambda^+(x)} (\xi_\lambda + \theta(\xi_\lambda)); \xi_\lambda \in \mathfrak{g}_\lambda \right\} \subset [\mathfrak{p}^x, \mathfrak{p}^x] \subset [[\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}], [\mathfrak{k}_{\mu_{\mathfrak{p}}(x)}, \mathfrak{p}_{\mu_{\mathfrak{p}}(x)}]] \subset \mathfrak{k}_x,$$

which was to be shown.  $\square$

Hence, the proof of Proposition 4.4 is finished.

**4.3. An equivalent condition of the separation property.** — Proposition 4.4 allows us to formulate an equivalent condition for  $\mu_{\mathfrak{p}}$  to separate locally almost the  $K$ -orbits which generalizes the notion of  $K$ -spherical symplectic manifolds defined in [HW90].

**Proposition 4.8.** — *The gradient map  $\mu_{\mathfrak{p}}$  locally almost separates the  $K$ -orbits if and only if  $\dim(\mathfrak{p} \cdot x)^{\perp} = \dim M - \dim M_x$  for one (and then every)  $x \in X_{\text{gen}} \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$ .*

*Proof.* — Let us suppose first that  $\mu_{\mathfrak{p}}$  locally almost separates the  $K$ -orbits. By definition, this means that there is an open  $K$ -invariant subset  $\Omega \subset X$  such that  $\mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x))^0 = K_{\mu_{\mathfrak{p}}(x)}^0 \cdot x$  for all  $x \in \Omega$ .

Since  $X_{\text{gen}}$  is dense, we find an element  $x \in \Omega \cap X_{\text{gen}} \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$ . It follows from maximality of  $\text{rk}_x \mu_{\mathfrak{p}}$  that  $\mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x)) \cap X_{\text{gen}}$  is a closed submanifold of  $X_{\text{gen}}$ . By Lemma 5.1 in [HS07b], we obtain  $\dim \ker(\mu_{\mathfrak{p}})_{*,x} = \dim(\mathfrak{p} \cdot x)^{\perp}$ . Hence, we conclude  $\dim K_{\mu_{\mathfrak{p}}(x)}/K_x = \dim(\mathfrak{p} \cdot x)^{\perp}$ . Since by Proposition 4.4 the orbit  $M \cdot x$  is open in  $K_{\mu_{\mathfrak{p}}(x)} \cdot x$ , we finally obtain  $\dim(\mathfrak{p} \cdot x)^{\perp} = \dim M/M_x = \dim M - \dim M_x$  which was to be shown.

In order to prove the converse let  $x \in X_{\text{gen}} \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$  be given. Our assumption implies that  $\mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x))$  is a closed submanifold of  $X$  of dimension  $\dim(\mathfrak{p} \cdot x)^{\perp} = \dim M - \dim M_x$ . We conclude that  $M \cdot x$  and hence  $K_{\mu_{\mathfrak{p}}(x)} \cdot x$  are open in  $\mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x))$ . Therefore we have  $\mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x))^0 = K_{\mu_{\mathfrak{p}}(x)}^0 \cdot x$ , which means that  $\mu_{\mathfrak{p}}$  separates the  $K$ -orbits in  $X_{\text{gen}}$ .  $\square$

Let us note explicitly the following corollary of the proof of Proposition 4.8.

**Corollary 4.9.** — *If  $\mu_{\mathfrak{p}}$  locally almost separates the  $K$ -orbits in  $X$ , then it almost separates the  $K$ -orbits in the dense open set  $X_{\text{gen}}$ .*

Consequently, if  $\mu_{\mathfrak{p}}$  locally almost separates the  $K$ -orbits in  $X$ , then  $\mu_{\mathfrak{p}}$  induces a map  $X_{\text{gen}}/K \rightarrow \mathfrak{p}/K \cong \mathfrak{a}/W$  whose fibers are discrete.

## 5. Proof of the main theorem

In the first subsection we review the shifting technique for gradient maps which translates the problem of finding an open  $Q_0$ -orbit in  $X$  into the problem of finding an open  $G$ -orbit in the bigger gradient manifold  $X \times (K/M)$ . Since  $G$  is real-reductive, we may apply the techniques developed in [HS07b] to solve the second problem.

Afterwards, it remains to find an open  $G$ -orbit in  $X \times (K/M)$  under the assumption that  $\mu_{\mathfrak{p}}$  locally almost separates the  $K$ -orbits. This is done in two steps: First we construct a special gradient map  $\tilde{\mu}_{\mathfrak{p}}$  on  $X \times (K/M)$  for which the set of global minima of  $\|\tilde{\mu}_{\mathfrak{p}}\|^2$  can be controlled. This will then be essentially used in the proof of existence of an open  $Q_0$ -orbit.

In the final subsection we prove the remaining implication (3)  $\implies$  (2) in our main theorem: If the minimal parabolic subgroup  $Q_0$  has an open orbit in  $X$ , then  $\mu_{\mathfrak{p}}$  almost separates the  $K$ -orbits.

**5.1. The shifting technique.** — Since the minimal parabolic subgroup  $Q_0 = MAN$  is not compatible, we cannot apply the theory developed in [HS07b] in order to link the action of  $Q_0$  on  $X$  with the theory of gradient maps. Therefore, we reformulate the problem of finding an open  $Q_0$ -orbit in  $X$  as the problem of finding an open  $G$ -orbit in a larger manifold.

**Lemma 5.1.** — *Let  $Q$  be a parabolic subgroup of  $G$ . Then  $Q$  has an open orbit in  $X$  if and only if  $G$  has an open orbit in  $X \times (G/Q)$  with respect to the diagonal action.*

*Proof.* — Recall that the twisted product  $G \times_Q X$  is by definition the quotient space of  $G \times X$  by the  $Q$ -action  $q \cdot (g, x) := (gq^{-1}, q \cdot x)$ . We denote the element  $Q \cdot (g, x) \in G \times_Q X$  by  $[g, x]$ . Then  $G$  acts on  $G \times_Q X$  by  $g \cdot [h, x] := [gh, x]$ , and every  $G$ -orbit in  $G \times_Q X$  intersects  $X \cong \{[e, x]; x \in X\}$  in a  $Q$ -orbit. Thus, the inclusion  $X \hookrightarrow G \times_Q X$ ,  $x \mapsto [e, x]$ , induces a homeomorphism  $X/Q \cong (G \times_Q X)/G$ . In particular,  $Q$  has an open orbit in  $X$  if and only if  $G$  has an open orbit in  $G \times_Q X$ .

The claim follows now from the fact that the map  $G \times_Q X \rightarrow X \times (G/Q)$ ,  $[g, x] \mapsto (g \cdot x, gQ)$ , is a  $G$ -equivariant diffeomorphism with respect to the diagonal  $G$ -action on  $X \times (G/Q)$ . To see this, it is sufficient to note that its inverse map is given by  $(x, gQ) \mapsto [g, g^{-1} \cdot x]$ .  $\square$

The gradient map  $\mu_{\mathfrak{p}}$  on  $X$  induces in a natural way a gradient map on the product  $\tilde{X} := X \times (G/Q)$  as follows. First recall from Section 3.2 that  $G/Q$  is a  $G$ -invariant closed submanifold of an adjoint  $U$ -orbit of an element  $\gamma \in \mathfrak{p}$ . In particular  $G/Q$  is isomorphic to  $K/K_\gamma$  and is equipped with a gradient map  $kK_\gamma \mapsto -\text{Ad}(k)\xi$ . The gradient maps on  $X$  and on  $K/K_\gamma$  induce a gradient map  $\tilde{\mu}_{\mathfrak{p}}$  on  $\tilde{X}$ , which is given by the sum of those two gradient maps. Explicitly, we have

$$\tilde{\mu}_{\mathfrak{p}}(x, kK_\gamma) = \mu_{\mathfrak{p}}(x) - \text{Ad}(k)\gamma.$$

Note that the choice of  $\gamma \in \mathfrak{p}$  depends only on the isotropy  $K_\gamma$ . In particular, if  $Q$  is a minimal parabolic subgroup of  $G$ , or equivalently if  $K_\gamma$  equals the centralizer  $M$  of  $\mathfrak{a}$  in  $K$ , then for every regular  $\gamma \in \mathfrak{p}$ , the assignment  $(x, kM) \mapsto \mu_{\mathfrak{p}}(x) - \text{Ad}(k)\gamma$  defines a gradient map on  $\tilde{X}$ .

**5.2. The shifted gradient map.** — Our goal is to construct a gradient map on  $\tilde{X} = X \times (K/M)$  which enables us to control the minima of the associated function  $\|\tilde{\mu}_{\mathfrak{p}}\|^2$ .

Let  $\mathfrak{a}_+$  denote the closed Weyl chamber in  $\mathfrak{a}$  associated to our choice of positive restricted roots. We generalize an inequality in [HS07a] which is a consequence of Kostant's Convexity Theorem ([Kos73]).

**Lemma 5.2.** — *Let  $\gamma, \xi \in \mathfrak{a}_+$  and assume that  $\xi$  is regular. Then*

$$\|\text{Ad}(k)\gamma - \xi\| \geq \|\gamma - \xi\|$$

for all  $k \in K$ . The inequality is strict for all  $k \notin K_\gamma$ .

*Proof.* — The  $K$ -invariance of the inner product implies

$$\|\text{Ad}(k)\gamma - \xi\|^2 - \|\gamma - \xi\|^2 = -2 \cdot \langle \text{Ad}(k)\gamma - \gamma, \xi \rangle.$$

Let  $\pi_{\mathfrak{a}}$  denote the orthogonal projection of  $\mathfrak{p}$  onto  $\mathfrak{a}$ . Then  $\langle \text{Ad}(k)\gamma, \xi \rangle = \langle \pi_{\mathfrak{a}}(\text{Ad}(k)\gamma), \xi \rangle$  and  $\pi_{\mathfrak{a}}(\text{Ad}(k)\gamma)$  is contained in the convex hull of the orbit of the Weyl group  $W := \mathcal{N}_K(\mathfrak{a})/\mathcal{Z}_K(\mathfrak{a})$  through  $\xi$  ([Kos73]). Since  $K$  acts by unitary operators, we have  $\pi_{\mathfrak{a}}(\text{Ad}(k)\gamma) = \gamma$  if and only if  $k \in K_\gamma$ . Therefore it suffices to show that  $\langle \text{Ad}(w)\gamma - \gamma, \xi \rangle < 0$  for all  $w \in W$ ,  $w \notin W_\gamma$ .

Let  $\lambda$  be a simple restricted root and  $\sigma_\lambda$  the corresponding reflection. Then either  $\sigma_\lambda(\gamma) = \gamma$  or  $\sigma_\lambda(\gamma) - \gamma = c \cdot \lambda$  for some  $c < 0$ . Here we have identified  $\lambda \in \mathfrak{a}^*$  with its dual in  $\mathfrak{a}$ . Since  $\xi$  is regular, this implies  $\langle \sigma_\lambda(\gamma) - \gamma, \xi \rangle < 0$  if  $\sigma_\lambda \notin W_\gamma$ .

An arbitrary element  $w \in W$  is of the form  $w = \sigma_{\lambda_1} \circ \cdots \circ \sigma_{\lambda_k}$  for simple restricted roots  $\lambda_1, \dots, \lambda_k$ . Then

$$\begin{aligned} \text{Ad}(w)\gamma - \gamma &= (\sigma_{\lambda_1} \circ \cdots \circ \sigma_{\lambda_k}(\gamma) - \sigma_{\lambda_2} \circ \cdots \circ \sigma_{\lambda_k}(\gamma)) \\ &\quad + (\sigma_{\lambda_2} \circ \cdots \circ \sigma_{\lambda_k}(\gamma) - \sigma_{\lambda_3} \circ \cdots \circ \sigma_{\lambda_k}(\gamma)) \\ &\quad + \cdots + (\sigma_{\lambda_k}(\gamma) - \gamma) \end{aligned}$$

is a linear combination of simple restricted roots with negative coefficients and it equals 0 if and only if  $\sigma_{\lambda_j} \in \mathcal{W}_\gamma$  for all  $j$ . Again, since  $\xi$  is regular, this implies  $\langle \text{Ad}(w)\gamma - \gamma, \xi \rangle < 0$  for all  $w \in W$ ,  $w \notin W_\gamma$ .  $\square$

Since each  $K$ -orbit in  $\mathfrak{p}$  intersects  $\mathfrak{a}$  in an orbit of the Weyl group, each  $K$ -orbit  $K \cdot x$  in  $X$  contains an  $x_0$  with  $\mu_{\mathfrak{p}}(x_0) \in \mathfrak{a}_+$ . Recall that each  $\xi \in \mathfrak{a}_+$  defines a gradient map  $\tilde{\mu}_{\mathfrak{p}}: \tilde{X} \rightarrow \mathfrak{p}$ ,  $\tilde{\mu}_{\mathfrak{p}}(x, kM) = \mu_{\mathfrak{p}}(x) - \text{Ad}(k)\xi$ .

**Proposition 5.3.** — *Let  $x_0 \in X_{\text{gen}}$  with  $\mu_{\mathfrak{p}}(x_0) \in \mathfrak{a}_+$ . Then there exists a regular  $\xi \in \mathfrak{a}_+$ , such that*

1.  $(x_0, eM)$  is a global minimum of the function  $\|\tilde{\mu}_{\mathfrak{p}}\|^2$ .
2. If  $(x, kM) \in \tilde{X}$  is another global minimum of  $\|\tilde{\mu}_{\mathfrak{p}}\|^2$ , then  $\mu_{\mathfrak{p}}(x) = \text{Ad}(k)\mu_{\mathfrak{p}}(x_0)$ .

*Proof.* — If  $\mu_{\mathfrak{p}}(x_0)$  is regular, define  $\xi := \mu_{\mathfrak{p}}(x_0)$ . Then  $\|\tilde{\mu}_{\mathfrak{p}}(x_0, eM)\|^2 = 0$  and  $(x_0, eM)$  is a global minimum of  $\|\tilde{\mu}_{\mathfrak{p}}\|^2$ . If  $(x, kM)$  is another global minimum, we have  $\mu_{\mathfrak{p}}(x) - \text{Ad}(k)\xi = 0$  and the second claim follows.

Now assume that  $\gamma := \mu_{\mathfrak{p}}(x_0)$  is singular. Let  $\lambda_1, \dots, \lambda_k$  be those simple restricted roots vanishing at  $\gamma$ . Let  $\mathfrak{b} := \{\eta \in \mathfrak{a}; \lambda_1(\eta) = \dots = \lambda_k(\eta) = 0\}$  be the subspace of  $\mathfrak{a}$  where these roots vanish. Let  $\mathfrak{b}^\perp$  be the orthogonal complement of  $\mathfrak{b}$  in  $\mathfrak{a}$ . Since  $x_0$  is regular, the orbit  $K \cdot \gamma$  has maximal dimension in  $\mu_{\mathfrak{p}}(X)$ . Therefore  $\mu_{\mathfrak{p}}(X) \cap \mathfrak{a}$  is contained in the union of the finitely many subspaces of  $\mathfrak{a}$  where at least  $k$  simple restricted roots vanish. Choosing a regular element  $\xi \in \gamma + \mathfrak{b}^\perp$  which is sufficiently near  $\gamma$ , we can assure that  $\gamma$  is the unique point in  $\mu_{\mathfrak{p}}(X) \cap \mathfrak{a}_+$  with minimal distance to  $\xi$ .

Let  $(x, kM) \in \tilde{X}$  and let  $l \in K$  with  $\gamma' := \text{Ad}(l)\mu_{\mathfrak{p}}(k^{-1} \cdot x) \in \mathfrak{a}_+$ . With Lemma 5.2 and the definition of  $\xi$  we obtain

$$\begin{aligned} \|\tilde{\mu}_{\mathfrak{p}}(x, kM)\|^2 &= \|\mu_{\mathfrak{p}}(x) - \text{Ad}(k)\xi\|^2 = \|\mu_{\mathfrak{p}}(k^{-1} \cdot x) - \xi\|^2 \\ &\geq \|\gamma' - \xi\|^2 \geq \|\gamma - \xi\|^2 = \|\tilde{\mu}_{\mathfrak{p}}(x_0, eM)\|^2, \end{aligned}$$

so in particular  $(x_0, eM)$  is a global minimum of  $\|\tilde{\mu}_{\mathfrak{p}}\|^2$ . Equality holds if and only if  $\gamma' = \gamma$  and  $l \in K_{\gamma'} = K_\gamma$ . The latter condition gives  $\text{Ad}(k)\gamma = \mu_{\mathfrak{p}}(x)$ .  $\square$

In Lemma 5.1, we reformulated the property that a parabolic subgroup  $Q$  has an open orbit in  $X$  as a property on the  $G$ -action on the product  $X \times (G/Q)$ . Now, we translate the condition, that  $\mu_{\mathfrak{p}}$  locally almost separates the  $K$ -orbits to a suitable condition on the shifted gradient map  $\tilde{\mu}_{\mathfrak{p}}$  on the product  $X \times (G/Q)$ .

**Lemma 5.4.** — *Let  $\xi \in \mathfrak{a}$  and let  $\tilde{\mu}_{\mathfrak{p}}: \tilde{X} \rightarrow \mathfrak{p}$  be the associated gradient map. Let  $x_0 \in X$  with  $\mu_{\mathfrak{p}}(x_0) \in \mathfrak{a}_+$  and set  $\beta := \mu_{\mathfrak{p}}(x_0) - \xi = \tilde{\mu}_{\mathfrak{p}}(x_0, eM)$ . Then the inclusion  $\mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x_0)) \hookrightarrow \tilde{\mu}_{\mathfrak{p}}^{-1}(\beta)$ ,  $x \mapsto (x, eM)$ , induces an injective continuous map  $\Phi: \mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x_0))/M \rightarrow \tilde{\mu}_{\mathfrak{p}}^{-1}(\beta)/K_\beta$ . If  $\xi$  is chosen such that the conclusions of Proposition 5.3 are satisfied, then  $\Phi$  is a homeomorphism.*

*Proof.* — First note that the map  $\Phi: \mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x_0))/M \rightarrow \tilde{\mu}_{\mathfrak{p}}^{-1}(\beta)/K_{\beta}$  is well-defined since  $M$  is contained in  $K_{\beta}$  and  $K_{\mu_{\mathfrak{p}}(x_0)}$  and since  $\mu_{\mathfrak{p}}$  and  $\tilde{\mu}_{\mathfrak{p}}$  are  $K$ -equivariant.

For injectivity, let  $x, y \in \mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x_0))$  with  $K_{\beta} \cdot (x, eM) = K_{\beta} \cdot (y, eM)$ . The latter condition implies  $M \cdot x = M \cdot y$  since  $K_{\beta} \cap M = M$ . This shows injectivity.

Assume that  $x_0 \in \mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x_0))$  satisfies the conclusions of Proposition 5.3 and let  $(x, kM) \in \tilde{\mu}_{\mathfrak{p}}^{-1}(\beta)$ . Then  $(x, kM)$  is a global minimum of  $\|\tilde{\mu}_{\mathfrak{p}}\|^2$  which implies  $\mu_{\mathfrak{p}}(x) = \text{Ad}(k)\mu_{\mathfrak{p}}(x_0)$ . We conclude  $\beta = \tilde{\mu}_{\mathfrak{p}}(x, kM) = \mu_{\mathfrak{p}}(x) - \text{Ad}(k)\xi = \text{Ad}(k)(\mu_{\mathfrak{p}}(x_0) - \xi) = \text{Ad}(k)\beta$ . This proves  $k \in K_{\beta}$ . Consequently  $K_{\beta} \cdot (x, kM)$  intersects  $\mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x_0)) \times \{eM\}$  and surjectivity follows. Finally, the inclusion  $\mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x_0)) \hookrightarrow \tilde{\mu}_{\mathfrak{p}}^{-1}(\beta)$  is continuous and proper, so  $\Phi$  is continuous and proper which implies that it is a homeomorphism.  $\square$

**5.3. Existence of an open  $Q_0$ -orbit.** — Finally we are in the position to prove that  $Q_0$  has an open orbit in  $X$  given that  $\mu_{\mathfrak{p}}$  locally almost separates the  $K$ -orbits.

Let us fix a point  $x_0 \in X_{\text{gen}}$  such that  $\mu_{\mathfrak{p}}(x_0)$  lies in the closed Weyl chamber  $\mathfrak{a}_+$ . By virtue of Proposition 5.3 we find a regular element  $\xi \in \mathfrak{a}_+$  such that  $\tilde{\mu}_{\mathfrak{p}}: X \times (K/M) \rightarrow \mathfrak{p}$ ,  $(x, kM) \mapsto \mu_{\mathfrak{p}}(x) - \text{Ad}(k)\xi$ , is a  $G$ -gradient map and such that  $\tilde{x}_0 := (x_0, eM)$  is a global minimum of  $\|\tilde{\mu}_{\mathfrak{p}}\|^2$ . Let  $Q_0 = MAN$  be the minimal parabolic subgroup of  $G$  associated to  $\xi$ . Then we may identify  $K/M$  with  $G/Q_0$  as gradient manifolds. Let  $\beta := \mu_{\mathfrak{p}}(x_0) - \xi$ . By Lemma 5.4 the quotients  $\mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x_0))/M$  and  $\tilde{\mu}_{\mathfrak{p}}^{-1}(\beta)/K_{\beta}$  are homeomorphic. This implies that  $K_{\beta} \cdot \tilde{x}_0$  is open in  $\tilde{\mu}_{\mathfrak{p}}^{-1}(\beta)$ .

As we have already seen in the proof of Lemma 2.1, it suffices to prove  $(\mathfrak{p} \cdot \tilde{x}_0)^{\perp} \subset \mathfrak{k} \cdot \tilde{x}_0$ , for then the orbit  $G \cdot \tilde{x}_0$  is open in  $X \times (G/Q_0)$  which in turn implies that  $Q_0 \cdot x_0$  is open in  $X$ . For this we will show that  $\tilde{\mu}_{\mathfrak{p}}$  has maximal rank in  $\tilde{x}_0$  as follows. The image of  $T_{x_0}X \oplus T_{eM}K/M$  under  $(\tilde{\mu}_{\mathfrak{p}})_{*, \tilde{x}_0}$  coincides with  $(\mu_{\mathfrak{p}})_{*, x_0}(T_{x_0}X) + [\mathfrak{k}, \xi]$ . Since  $\xi$  is regular, we obtain

$$[\mathfrak{k}, \xi] = \left\{ \sum_{\lambda \in \Lambda^+} (\xi_{\lambda} - \theta(\xi_{\lambda})); \xi_{\lambda} \in \mathfrak{g}_{\lambda} \right\} = \mathfrak{a}^{\perp}.$$

We use the decomposition  $T_x X = (\mathfrak{k} \cdot x) \oplus (\mathfrak{k} \cdot x)^{\perp}$  and note that  $(\mu_{\mathfrak{p}})_{*, x}$  maps  $\mathfrak{k} \cdot x$  into  $\mathfrak{a}^{\perp}$  for all  $x$  in a neighborhood of  $x_0$ . Since moreover  $\mu_{\mathfrak{p}}$  locally almost separates the  $K$ -orbits, one would expect that  $(\mu_{\mathfrak{p}})_{*, x_0}$  maps a subspace of  $T_{x_0}X$  which is transversal to  $\mathfrak{k} \cdot x_0$  onto a subspace of  $\mathfrak{p}$  which is transversal to  $\mathfrak{a}^{\perp}$ . This is the content of the following

**Lemma 5.5.** — *Assume that  $\mu_{\mathfrak{p}}$  locally almost separates the  $K$ -orbits. For every  $x \in X_{\text{gen}} \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$  we have  $(\mu_{\mathfrak{p}})_{*, x}((\mathfrak{k} \cdot x)^{\perp}) \cap \mathfrak{a}^{\perp} = \{0\}$ .*

*Proof.* — Since  $x$  is generic, there exists an open neighborhood  $V \subset X$  of  $x$  such that the rank of  $\mu_{\mathfrak{p}}$  is constant on  $V$ . We conclude that  $V \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$  is a submanifold of  $V$  and that the image  $\mu_{\mathfrak{p}}(V \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a}))$  is an open subset of the linear subspace  $\mathfrak{b} := \bigcap_{\lambda \in \Lambda(x)} \ker(\lambda) \subset \mathfrak{a}$ . Moreover, we have  $\mu_{\mathfrak{p}}(V)$  is an open subset of  $K \cdot \mathfrak{b} \cong K \times_{K_{\mu_{\mathfrak{p}}(x)}} \mathfrak{b} = (K/K_{\mu_{\mathfrak{p}}(x)}) \times \mathfrak{b}$ .

Since  $\mu_{\mathfrak{p}}$  separates the  $K$ -orbits and since  $x$  is generic, we have  $\ker(\mu_{\mathfrak{p}})_{*, x} = (\mathfrak{p} \cdot x)^{\perp} \subset \mathfrak{k} \cdot x$  which implies that  $(\mu_{\mathfrak{p}})_{*, x}$  is injective on  $(\mathfrak{k} \cdot x)^{\perp}$ . Consequently,  $\mu_{\mathfrak{p}}$  induces an injective immersion  $V/K \rightarrow \mathfrak{b}$ , therefore  $(\mu_{\mathfrak{p}})_{*, x}$  maps  $(\mathfrak{k} \cdot x)^{\perp}$  bijectively onto  $\mathfrak{b}$ . Since  $\mathfrak{b} \cap \mathfrak{a}^{\perp} = \{0\}$ , the claim follows.  $\square$

We conclude from Lemma 5.5 that the image of  $(\tilde{\mu}_{\mathfrak{p}})_{*, \tilde{x}_0}$  is given by  $(\mu_{\mathfrak{p}})_{*, x_0}((\mathfrak{k} \cdot x_0)^{\perp}) \oplus \mathfrak{a}^{\perp}$ . Since  $x_0$  is generic, the dimension of  $(\mu_{\mathfrak{p}})_{*, x_0}((\mathfrak{k} \cdot x_0)^{\perp})$  is the same for all  $x$  in a neighborhood of  $x_0$ . Furthermore, every  $K$ -orbit in  $X \times (K/M)$  intersects  $X \times \{eM\}$ , thus the rank of  $\tilde{\mu}_{\mathfrak{p}}$

is constant in a neighborhood of  $\tilde{x}_0$ . Consequently, the rank of  $\tilde{\mu}_{\mathfrak{p}}$  must be maximal in  $\tilde{x}_0$ . Together with the fact that  $K_{\beta} \cdot \tilde{x}_0$  is open in  $\tilde{\mu}_{\mathfrak{p}}^{-1}(\beta)$  this yields

$$(\mathfrak{p} \cdot \tilde{x}_0)^{\perp} = T_{\tilde{x}_0} \tilde{\mu}_{\mathfrak{p}}^{-1}(\beta) = \mathfrak{k}_{\beta} \cdot \tilde{x}_0 \subset \mathfrak{k} \cdot \tilde{x}_0.$$

Therefore we obtain  $T_{\tilde{x}_0} \tilde{X} = \mathfrak{p} \cdot \tilde{x}_0 \oplus (\mathfrak{p} \cdot \tilde{x}_0)^{\perp} \subset \mathfrak{p} \cdot \tilde{x}_0 + \mathfrak{k} \cdot \tilde{x}_0$  which shows that  $G \cdot \tilde{x}_0$  is open in  $\tilde{X}$ .

This proves the implication (1)  $\implies$  (3) of our our main theorem and gives in addition a precise description of the set of open  $Q_0$ -orbits in  $X$ .

**Theorem 5.6.** — *Suppose that  $\mu_{\mathfrak{p}}$  locally almost separates the  $K$ -orbits. Let  $x_0 \in X_{\text{gen}} \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a}_+)$  be given, let  $\xi$  be the element from Proposition 5.3, and let  $Q_0$  be the minimal parabolic subgroup of  $G$  associated to  $\xi$ . Then  $Q_0 \cdot x_0$  is open in  $X$ .*

The same method of proof gives the following

**Proposition 5.7.** — *Suppose that  $\mu_{\mathfrak{p}}: X \rightarrow \mathfrak{p}$  locally almost separates the  $K$ -orbits. Let  $x \in X_{\text{gen}} \cap \mu_{\mathfrak{p}}^{-1}(\mathfrak{a})$  and let  $Q$  be the parabolic subgroup of  $G$  associated to  $\beta := \mu_{\mathfrak{p}}(x)$ . Then  $Q \cdot x$  is open in  $X$ .*

*Proof.* — In order to show that  $Q \cdot x$  is open in  $X$ , it suffices to show that  $G \cdot (x, eQ)$  is open in  $X \times (G/Q)$ . For this we note that  $G/Q \cong K/K_{\beta}$  as a  $K$ -manifold and that for the shifted gradient map  $\tilde{\mu}_{\mathfrak{p}}: X \times (K/K_{\beta}) \rightarrow \mathfrak{p}$ ,  $(x, kK_{\beta}) \mapsto \mu_{\mathfrak{p}}(x) - \text{Ad}(k)\beta$  the element  $(x, eK_{\beta})$  lies in  $\widetilde{\mathcal{M}}_{\mathfrak{p}}$ . Then the same arguments as above apply to show that  $G \cdot (x, eK_{\beta})$  is open.  $\square$

**5.4. Proof of (3)  $\implies$  (2).** — In this subsection we complete the proof of our main theorem by showing the remaining non-trivial implication.

**Proposition 5.8.** — *Suppose that  $Q_0$  has an open orbit in  $X$ . Then  $\mu_{\mathfrak{p}}$  almost separates the  $K$ -orbits.*

*Proof.* — Let  $x_0 \in X$  be given. We must show that  $K_{\mu_{\mathfrak{p}}(x_0)} \cdot x_0$  is open in  $\mu_{\mathfrak{p}}^{-1}(\mu_{\mathfrak{p}}(x_0))$ . Let  $\gamma := \mu_{\mathfrak{p}}(x_0)$  and let  $Q$  be the parabolic subgroup of  $G$  associated to  $\gamma$ . Recall that  $G/Q \cong K/K_{\gamma}$  is a  $G$ -gradient space with gradient map  $kK_{\gamma} \mapsto -\text{Ad}(k)\gamma$ . Consider the shifted gradient map  $\tilde{\mu}_{\mathfrak{p}}: X \times (K \cdot \gamma) \rightarrow \mathfrak{p}$ ,  $(x, kK_{\gamma}) \mapsto x - \text{Ad}(k)\gamma$ . Since the minimal parabolic subgroup  $Q_0$  has an open orbit in  $X$ , the same is true for  $Q$ . Hence  $G$  has an open orbit in  $X \times (K/K_{\gamma})$  by Lemma 5.1.

By definition of  $\gamma$ , we have  $\tilde{\mu}_{\mathfrak{p}}(x_0, \gamma) = 0$ . Consider the set of semistable points  $\mathcal{S}_G(\tilde{\mu}_{\mathfrak{p}}^{-1}(0)) = \{\tilde{x} \in \tilde{X}; \overline{G \cdot \tilde{x}} \cap \tilde{\mu}_{\mathfrak{p}}^{-1}(0) \neq \emptyset\}$ . It is open in  $\tilde{X}$  ([HS07c]) and contains  $(x_0, \gamma)$ .

By analyticity of the action, the union  $V$  of the open  $G$ -orbits in  $\mathcal{S}_G(\tilde{\mu}_{\mathfrak{p}}^{-1}(0))$  is dense in  $\mathcal{S}_G(\tilde{\mu}_{\mathfrak{p}}^{-1}(0))$ . We note also that the union of the open  $G$ -orbits is locally finite in  $\mathcal{S}_G(\tilde{\mu}_{\mathfrak{p}}^{-1}(0))$  which can be seen as follows. For every  $p \in \tilde{\mu}_{\mathfrak{p}}^{-1}(0)$  there exists a slice neighborhood  $G \cdot S \cong G \times_{G_x} S$  where  $G_x$  is a compatible subgroup of  $G$  and  $S$  can be viewed as an open neighborhood of 0 in a  $G_x$ -representation space. Since  $G_x$  has at most finitely many open orbits in this representation space, we conclude that only finitely many open  $G$ -orbits intersect the open set  $G \cdot S$  which shows that the union of the open  $G$ -orbits in  $\mathcal{S}_G(\tilde{\mu}_{\mathfrak{p}}^{-1}(0))$  is locally finite.

Let  $W$  be the union of open  $G$ -orbits which contain  $(x_0, \gamma)$  in their closure and let  $\overline{W}$  be the closure of  $W$  in  $\mathcal{S}_G(\tilde{\mu}_{\mathfrak{p}}^{-1}(0))$ . Then  $W$  consists of only finitely many open  $G$ -orbits and consequently  $\overline{W}$  contains an open neighborhood of  $(x_0, \gamma)$ . By Corollary 11.18 in [HS07b],  $\overline{W}$  intersects  $\tilde{\mu}_{\mathfrak{p}}^{-1}(0)$  in  $K \cdot (x_0, \gamma)$ . Therefore  $K \cdot (x_0, \gamma)$  is isolated in  $\tilde{\mu}_{\mathfrak{p}}^{-1}(0)$  which shows that the quotient  $\tilde{\mu}_{\mathfrak{p}}^{-1}(0)/K$  is discrete. Then  $\mu_{\mathfrak{p}}^{-1}(\gamma)/M$  is discrete by Lemma 5.4 which means

that the  $M$ -orbits in  $\mu_{\mathfrak{p}}^{-1}(\gamma)$  are open. But  $M < K^\gamma$  so the  $K^\gamma$ -orbits are open in  $\mu_{\mathfrak{p}}^{-1}(\gamma)$  as well.  $\square$

This completes the proof of Theorem 1.

**Corollary 5.9.** — *Let  $X$  be a spherical  $G$ -gradient manifold. Then every  $G$ -stable real-analytic submanifold  $Y$  of  $X$  is also spherical.*

*Proof.* — The claim follows from the facts that  $Y$  is a  $G$ -gradient manifold with respect to  $\mu_{\mathfrak{p}}|_Y$  and that  $\mu_{\mathfrak{p}}|_Y$  almost separates the  $K$ -orbits in  $Y$  since this is true for  $\mu_{\mathfrak{p}}$ .  $\square$

**Corollary 5.10.** — *If one  $G$ -gradient map locally almost separates the  $K$ -orbits in  $X$ , then every  $G$ -gradient map on  $X$  almost separates the  $K$ -orbits.*

## 6. Applications

**6.1. Homogeneous semi-stable spherical gradient manifolds.** — Let  $G = K \exp(\mathfrak{p})$  be connected real-reductive and let  $X$  be a spherical  $G$ -gradient manifold with gradient map  $\mu_{\mathfrak{p}}: X \rightarrow \mathfrak{p}$ . We have seen in Lemma 2.1 that  $G$  has an open orbit in  $X$ . In this subsection we consider the case that  $X = G/H$  is homogeneous. In addition, we suppose that  $X$  is semi-stable, i. e. that  $X = \mathcal{S}_G(\mathcal{M}_{\mathfrak{p}})$  holds. Consequently, we may assume that  $H$  is of the form  $H = K_H \exp(\mathfrak{p}_H)$  with  $K_H = K \cap H$  and  $\mathfrak{p}_H = \mathfrak{p} \cap \mathfrak{h}$ .

**Remark.** — The class of homogeneous semi-stable spherical gradient manifolds generalizes the class of homogeneous affine spherical varieties in the complex setting.

Let  $\mathfrak{p} = \mathfrak{p}_H \oplus \mathfrak{p}_H^\perp$  be a  $K_H$ -invariant decomposition; then we have the Mostow decomposition  $G/H \cong K \times_{K_H} \mathfrak{p}_H^\perp$  (see Theorem 9.3 in [HS07b] for a proof which uses gradient maps). Since  $X$  is spherical, we conclude from Theorem 1 that the Mostow gradient map  $\mu_{\mathfrak{p}}: G/H \cong K \times_{K_H} \mathfrak{p}_H^\perp \rightarrow \mathfrak{p}$ ,  $[k, \xi] \mapsto \text{Ad}(k)\xi$ , almost separates the  $K$ -orbits. In other words, the inclusion  $\mathfrak{p}_H^\perp \hookrightarrow \mathfrak{p}$  induces a map  $\mathfrak{p}_H^\perp/K_H \rightarrow \mathfrak{p}/K$  which has discrete fibers. This discussion proves the following

**Proposition 6.1.** — *Let  $X = G/H$  be a semi-stable homogeneous  $G$ -gradient manifold and suppose that  $H = K_H \exp(\mathfrak{p}_H)$  is compatible in  $G = K \exp(\mathfrak{p})$ . Then  $X$  is spherical if and only if the map  $\mathfrak{p}_H^\perp/K_H \rightarrow \mathfrak{p}/K$  induced by the inclusion  $\mathfrak{p}_H^\perp \hookrightarrow \mathfrak{p}$  has discrete fibers.*

**Example.** — For  $H = \{e\}$  we have  $K_H = \{e\}$  and  $\mathfrak{p}_H^\perp = \mathfrak{p}$ . Consequently,  $X = G$  is spherical if and only if the quotient map  $\mathfrak{p} \rightarrow \mathfrak{p}/K$  has discrete fibers, i. e. if and only if  $K$  acts trivially on  $\mathfrak{p}$ .

Finally, we show that reductive symmetric spaces are spherical. Recall that  $G/H$  is a reductive symmetric space if there is an involutive automorphism  $\tau$  on  $G$  such that  $(G^\tau)^0 \subset H \subset G^\tau$  holds. In this situation we may assume without loss of generality that  $\tau$  commutes with the Cartan involution  $\theta$ . Hence,  $H = K^\tau \exp(\mathfrak{p}^\tau)$  is compatible. In order to show that  $X = G/H$  is spherical, we must prove that  $\mathfrak{p}^{-\tau}/K^\tau \rightarrow \mathfrak{p}/K$  has discrete fibers. From  $[\mathfrak{p}^{-\tau}, \mathfrak{p}^{-\tau}] \subset \mathfrak{k}^\tau$  we conclude that every  $K^\tau$ -orbit in  $\mathfrak{p}^{-\tau}$  intersects a maximal Abelian subspace  $\mathfrak{a}_0 \subset \mathfrak{p}^{-\tau}$  in an orbit of the finite group  $W_0 := \mathcal{N}_{K^\tau}(\mathfrak{a}_0)/\mathcal{Z}_{K^\tau}(\mathfrak{a}_0)$ . Extending  $\mathfrak{a}_0$  to a maximal Abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p}$  we see that  $\mathfrak{p}^{-\tau}/K^\tau \cong \mathfrak{a}_0/W_0 \rightarrow \mathfrak{a}/W \cong \mathfrak{p}/K$  has indeed finite fibers. Therefore we have proven the following

**Proposition 6.2.** — *Let  $X = G/H$  be a semi-stable homogeneous gradient manifold. If  $H$  is a symmetric subgroup of  $G$ , then the Mostow gradient map  $\mu_{\mathfrak{p}}: X \rightarrow \mathfrak{p}$  has finite fibers.*



**6.2. Relation to multiplicity-free representations.** — Let  $X$  be a real-analytic  $G$ -gradient manifold. Then  $G$  acts linearly on the space  $\mathcal{C}^\omega(X)$  of complex-valued real-analytic functions on  $X$ . Since  $G$  is a compatible subgroup of some complex-reductive group  $U^\mathbb{C}$ , we observe that  $G$  embeds as a closed subgroup into its complexification  $G^\mathbb{C}$ . Moreover, if  $G$  contains no non-compact Abelian factors, then  $G^\mathbb{C}$  is complex-reductive.

**Proposition 6.3.** — *Suppose that  $G$  acts properly on  $X$  and that  $G^\mathbb{C}$  is complex-reductive. If the  $G$ -representation on  $\mathcal{C}^\omega(X)$  is multiplicity-free, then  $X$  is spherical.*

*Proof.* — As is proven in [Hei93], there exists a Stein  $G^\mathbb{C}$ -manifold  $X^\mathbb{C}$  such that  $X$  admits a  $G$ -equivariant embedding as a closed maximally totally real submanifold into  $X^\mathbb{C}$ . According to the example discussed in Section 2.2 it suffices to show that  $X^\mathbb{C}$  is  $G^\mathbb{C}$ -spherical.

In order to see this, note that the restriction mapping  $\mathcal{O}(X^\mathbb{C}) \rightarrow \mathcal{C}^\omega(X)$  is injective and  $G$ -equivariant. This implies that the  $G$ - (and hence also the  $G^\mathbb{C}$ -)representation on  $\mathcal{O}(X^\mathbb{C})$  is multiplicity-free. Therefore, Theorem 2 in [AH04] applies to show that  $X^\mathbb{C}$  is spherical which finishes the proof.  $\square$

**Remark.** — In Proposition 6.3 properness of the  $G$ -action on  $X$  is needed to guarantee the existence of the complexification  $X^\mathbb{C}$ . If  $X = G/H$  is homogeneous, then we may take  $X^\mathbb{C} := G^\mathbb{C}/H^\mathbb{C}$  and the same argument as above shows: If the  $G$ -representation on  $\mathcal{C}^\omega(G/H)$  is multiplicity-free, then  $G/H$  is spherical.

Even if we assume that  $G$  acts properly on  $X$ , the converse of Proposition 6.3 does not hold as the following example shows.

**Example.** — Let  $G = K$  be a compact Lie group acting by left multiplication on  $X = K$ . Then  $\mu_{\mathfrak{p}} \equiv 0$  separates the  $K$ -orbits in  $X$  but the  $K$ -representation on  $\mathcal{C}^\omega(K)$  is not multiplicity-free which can be deduced from the Peter-Weyl Theorem.

However, there is a special class of real-reductive Lie groups for which the proof of the complex multiplicity-freeness result generalizes to the real situation. A real-reductive Lie group  $G$  belongs to this class if the minimal parabolic subalgebras  $\mathfrak{q}_0 = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$  are solvable, i. e. if  $\mathfrak{m}$  is Abelian.

**Example.** — Among the classical semi-simple Lie groups this is the case e. g. for  $\mathrm{SL}(n, \mathbb{R})$ ,  $\mathrm{Sp}(n, \mathbb{R})$ ,  $\mathrm{SU}(p, p)$ ,  $\mathrm{SO}(p, p)$  and  $\mathrm{SO}(p, p + 1)$  (see Appendix C.3 in [Kna02]).

**Lemma 6.4.** — *Let  $X$  be a spherical  $G$ -gradient manifold. If the minimal parabolic subalgebras of  $\mathfrak{g}$  are solvable, then the  $G$ -representation on  $\mathcal{C}^\omega(X)$  is multiplicity-free.*

*Proof.* — We must show that  $\dim \mathrm{Hom}_G(V, \mathcal{C}^\omega(X)) \leq 1$  holds for every complex finite-dimensional irreducible  $G$ -module  $V$ . Let  $Q_0 = MAN$  be a minimal parabolic subgroup of  $G$  and let  $V$  be a complex finite-dimensional irreducible  $G$ -module. By Engel's Theorem the space  $V^N$  of  $N$ -invariant vectors has positive dimension. The restriction map induces a linear map

$$\mathrm{Hom}_G(V, \mathcal{C}^\omega(X)) \rightarrow \mathrm{Hom}_{MA}(V^N, \mathcal{C}^\omega(X)^N),$$

which is injective since  $V^N$  generates  $V$  as a  $G$ -module. Hence, it is enough to show  $\dim \mathrm{Hom}_{MA}(V^N, \mathcal{C}^\omega(X)^N) \leq 1$ . Let us assume the contrary. Then there are linearly independent functions  $f_1, f_2 \in \mathcal{C}^\omega(X)^N$  which transform under the same character of the Abelian group  $M^0A$ . Consequently, the quotient  $f_1/f_2$  is a real-analytic function defined on the dense open set  $\{f_2 \neq 0\}$  and invariant under  $Q_0^0 = M^0AN$ . Since this contradicts the assumption that  $Q_0$  has an open orbit in  $X$ , the proof is finished.  $\square$

**6.3. Open Borel-orbits are Stein.** — In this subsection we consider the holomorphic situation, i. e.  $G = U^{\mathbb{C}}$  is complex-reductive and acts holomorphically on the Kähler manifold  $Z$  such that the  $U$ -action is Hamiltonian with moment map  $\mu: Z \rightarrow \mathfrak{u}^*$ . In Section 5 we have given a new proof of the following result of Brion.

**Theorem 6.5.** — *The moment map  $\mu: Z \rightarrow \mathfrak{u}^*$  separates the  $U$ -orbits in  $Z$  if and only if  $Z$  is spherical, i. e. if a Borel subgroup  $B \subset G$  has an open orbit in  $Z$ .*

In this subsection we will show that our proof further implies that the open  $B$ -orbit in  $Z$  is Stein.

**Proposition 6.6.** — *If the moment map  $\mu: Z \rightarrow \mathfrak{u}^*$  separates the  $U$ -orbits in  $Z$ , then the open  $B$ -orbit in  $Z$  is Stein.*

*Proof.* — Let  $z \in Z$  be a generic element and let  $Q \subset G$  be the parabolic subgroup associated to  $\mu(z)$ . Consequently, the zero fiber of the shifted moment map on the Kähler manifold  $Z \times (G/Q)$  is non-empty. We may assume without loss of generality that the element  $(z, eQ) \in Z \times (G/Q)$  is contained in this zero fiber. By Proposition 5.7 the orbit  $G \cdot (z, eQ)$  is open in  $Z \times (G/Q)$  which in turn implies that  $Q \cdot z$  is open in  $Z$ . Moreover, since  $(z, eQ)$  lies in the zero fiber of a moment map, the isotropy  $G_{(z, eQ)} = G_z \cap Q = Q_z$  is complex-reductive which proves that  $Q \cdot z \cong Q/Q_z$  is Stein (see Theorem 5 in [MM60]). The open  $B$ -orbit in  $Z$  must be contained in  $Q \cdot z$  and is therefore holomorphically separable. Applying a result of Huckleberry and Oeljeklaus ([HO86]) we finally see that the open  $B$ -orbit is Stein.  $\square$

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