Branching law for finite subgroups of SL(3,C) and McKay correspondence

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McKay correspondence and the branching law for finite subgroups of $\mathbf{SL}_3\mathbb{C}$

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Abstract
Given Γ a finite subgroup of $\mathbf{SL}_3\mathbb{C}$, we determine how an arbitrary finite dimensional irreducible representation of $\mathbf{SL}_3\mathbb{C}$ decomposes under the action of Γ. To the subgroup Γ we attach a generalized Cartan matrix C_Γ. Then, inspired by B. Kostant, we decompose the Coxeter element of the Kac-Moody algebra attached to C_Γ as a product of reflections of a special form, thereby suggesting an algebraic form for the McKay correspondence in dimension 3.

1 Introduction

1.1 Framework and results

Let Γ be a finite subgroup of $\mathbf{SL}_3\mathbb{C}$. In this paper, we determine how the finite dimensional irreducible representations of $\mathbf{SL}_3\mathbb{C}$ decompose under the action of the subgroup Γ. These representations are indexed by $\mathbb{N}^2$. For $(m,\, n) \in \mathbb{N}^2$, let $V(m,\, n)$ denote the corresponding simple finite dimensional module. Let $\{\gamma_0, \ldots, \gamma_l\}$ be the set of irreducible characters of Γ. We determine the numbers $m_i(m,\, n)$ — the multiplicity of the character γ_i in the representation $V(m,\, n)$. For that effect we introduce the formal power series:

$$P_\Gamma(t,\, u)_i = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} m_i(m,\, n)t^m u^n.$$

We show that $m_i(t,\, u)$ is a rational function. We determine the rational functions which are obtained in that way for all the finite subgroups of $\mathbf{SL}_3\mathbb{C}$.

The proof uses an inversion of the recursion formula for the numbers $m_i(m,\, n)$. The recursion formula is obtained through the decomposition of the tensor product of $V(m,\, n)$ with the natural representation of $\mathbf{SL}_3\mathbb{C}$. The key observation which leads to this inversion is that a certain pair matrices are simultaneously diagonalizable. The eigenvalues of the matrices are values from the character table of the group Γ. This leads to the proof that the power series

$$P_\Gamma(t,\, u)_i = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} m_i(m,\, n)t^m u^n$$

is rational. The actual calculation of this rational function then reduces to matrix multiplication.

This method applies indeed to the $\mathbf{SL}_2\mathbb{C}$ case. It gives a complete (very short) proof of the results obtained by B. Kostant in [Kos85], [Kos06], and by Gonzalez-Sprinberg and Verdier in [GSV83], and leads to an explicit determination of all the above multiplicities for the finite subgroups of $\mathbf{SL}_2\mathbb{C}$.
Although the results for $\mathbf{SL}_2\mathbb{C}$ are not new, the explicit relation of the rational functions with the eigenvalues of the Cartan matrix attached to the finite subgroup of $\mathbf{SL}_2\mathbb{C}$ doesn't seem to appear in the literature. In [Kos85] this is established through the analysis of the orbit structure of the Coxeter element.

The construction of a minimal resolution of singularities of the orbifold $\mathbb{C}^3/\Gamma$ centralizes a lot of interest. It is related to the geometric McKay correspondence, cf. (for example) [BKR01], [GSV83]. In this

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framework Gonzalez-Spriberg-Verdier [GSV83] use the Poincaré series determined above in their explicit construction of minimal resolution for singularities for $V = \mathbb{C}^2/\Gamma$ when Γ is a finite subgroup of $\mathbf{SL}_2\mathbb{C}$. Following that approach the results of our calculation could be eventually used to construct explicit synthetic minimal resolution of singularities for orbifolds of the form $\mathbf{SL}_3\mathbb{C}/\Gamma$ where Γ is a finite subgroup of $\mathbf{SL}_3\mathbb{C}$. This might clarify the description of the exceptional fiber of the minimal resolution of $\mathbf{SL}_3\mathbb{C}/\Gamma$ (see [GNS04]).

An essential ingredient of the approach of B. Kostant in [Kos85] is the decomposition of a Coxeter element in the Weyl group attached to the Lie algebra corresponding to a subgroup Γ of $\mathbf{SL}_2\mathbb{C}$ through the McKay correspondence as a product of simple reflections belonging to mutually orthogonal sets of roots.
Inspired by this approach, we attach to each finite subgroup Γ of $\mathbf{SL}_3\mathbb{C}$ a generalized Cartan matrix C_Γ. We then factorize this matrix as a product of elements in the Weyl group of the Kac-Moody Lie algebra corresponding to C_Γ. These elements are products of simple reflections corresponding to roots in mutually orthogonal sets.

1.2 Organization of the paper

In **Section 2** we treat the $\mathbf{SL}_2\mathbb{C}$ case. We show that the formal power series of the multiplicities is a rational function by showing that it is an entry in a vector obtained as product of three matrices, two of which are scalar matrices the third one being a matrix with rational entries, by a scalar vector. We calculate the matrices for each finite subgroup of $\mathbf{SL}_3\mathbb{C}$. We give then the rational functions obtained.

In **Section 3** we apply the above method for the finite subgroups of $\mathbf{SL}_3\mathbb{C}$. Here we use the notations of [YY93] in which a classification of the finite subgroups of $\mathbf{SL}_3\mathbb{C}$ is presented.
Here again we prove the rationality of the formal power series of the multiplicities by showing that each such a series is an entry in the product of three matrices, two of them are scalar matrices and the third being a matrix with rational entries, with a scalar vector.
For each finite subgroup of $\mathbf{SL}_3\mathbb{C}$ we give the the matrices involved in the product. To each subgroup Γ we attach a generalized Cartan matrix C_Γ (McKay correspondence in dimension 3) we show its graph and its decomposition as a product of elements in the Weyl group of the Kac-Moody Lie algebra $\mathfrak{g}(C_\Gamma)$.

Then, for the series A, B, C ([YY93] notation) we give all the rational functions explicitly, As for the series D we give the results for some specific examples because the description of the matrices engaged, in full generality doesn't have a simply presentable form.

For the exceptional finite subgroups of $\mathbf{SL}_3\mathbb{C}$ the numerators of the rational functions tend to be very long and we give them explicitly only for the cases where they are reasonably presentable. In all the cases we give the denominators explicitly. This is done in **Section 4**.

2 Branching law for the finite subgroups of $\mathrm{SL}_2\mathbb{C}$

2.1 The formal power series of the multiplicities is a rational function

• Let Γ be a finite subgroup of $\mathbf{SL}_2\mathbb{C}$ and $\{\gamma_0, \ldots, \gamma_l\}$ the set of equivalence classes of irreducible finite dimensional complex representations of Γ, where γ_0 is the trivial representation. We denote by χ_j the character associated to γ_i.
Consider $\gamma : \Gamma \to \mathbf{SL}_2\mathbb{C}$, the natural $2-$dimensional representation. Its character is denoted by χ. We have then the decomposition $\gamma_j \otimes \gamma = \bigoplus_{i=0}^{l} a_{ij}\gamma_i$ for every $j \in [\![0, l]\!]$. This defines an $(l+1) \times (l+1)$ square matrix $A := (a_{ij})_{(i,j)\in[\![0,l]\!]^2}$.

• Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{sl}_2\mathbb{C}$ and ϖ_1 be the corresponding fundamental weight, and $V(n\varpi_1)$ be the simple $\mathfrak{sl}_2$-module of highest weight $n\varpi_1$. This give rise to an irreducible representation $\pi_n : \mathbf{SL}_2\mathbb{C} \longrightarrow V(n\varpi_1)$.

The restriction of π_n to the subgroup Γ, is a representation of Γ, and by complete reducibility, we have a decomposition $\pi_n|_\Gamma = \bigoplus_{i=0}^{l} m_i(n)\gamma_i$, where the $m_i(n)$'s are non negative integers. Let $\mathcal{E} := (e_0, \ldots, e_l)$ be the canonical basis of $\mathbb{C}^{l+1}$, and

$$v_n := \sum_{i=0}^{l} m_i(n) e_i \in \mathbb{C}^{l+1}.$$

As γ_0 is the trivial representation, we have $v_0 = e_0$. Let us consider the vector (with elements of $\mathbb{C}[\![t]\!]$ as coefficients)

$$P_\Gamma(t) := \sum_{n=0}^{\infty} v_n t^n \in (\mathbb{C}[\![t]\!])^{l+1},$$

and denote by $P_\Gamma(t)_j$ its $j-$th coordinate in the basis $\mathcal{E}$. The series $P_\Gamma(t)_0$ is the Poincaré series of the invariant ring. Note also that $P_\Gamma(t)$ can also be seen as a formal power series with coefficients in $\mathbb{C}^{l+1}$.
We proceed to calculate $P_\Gamma(t)$.

• We get by Clebsch-Gordan formula that : $\pi_n \otimes \pi_1 = \pi_{n+1} \oplus \pi_{n-1}$, so we have $Av_n = v_{n+1} + v_{n-1}$. From this we deduce the relation

$$(1 - tA + t^2) P_\Gamma(t) = v_0.$$

Let us denote by $\{C_0, \ldots, C_l\}$ the set of conjugacy classes of Γ, and for any $j \in [\![0, l]\!]$, let g_j be an element of C_j. So the character table of Γ is the matrix $T_\Gamma \in \mathbf{M}_{l+1}\mathbb{C}$ defined by $(T_\Gamma)_{i,j} := \chi_i(g_j)$.
For all the finite subgroups of $\mathbf{SL}_2\mathbb{C}$ we have that, T_Γ is invertible, and $\Lambda := T_\Gamma^{-1} A T_\Gamma$ is diagonal, with $\Lambda_{jj} = \overline{\chi(g_j)}$.
Set $\Theta := (\Lambda_{00}, \ldots, \Lambda_{ll})$. We deduce from the preceding formula that

$$T_\Gamma (1 - t\Lambda + t^2) T_\Gamma^{-1} P_\Gamma(t) = v_0.$$

Let us define the rational function

$$\begin{array}{rccc} f \ : & \mathbb{C}^2 & \to & \mathbb{C}(t) \\ & d & \mapsto & \dfrac{1}{1 - td + t^2}. \end{array}$$

Then

$$P_\Gamma(t) = T_\Gamma\, \Delta(t)\, T_\Gamma^{-1} v_0 = (T_\Gamma\, \Delta(t)\, T_\Gamma)\, (T_\Gamma^{-2} v_0),$$

where $\Delta(t) \in \mathbf{M}_{l+1}\mathbb{C}(t)$ is the diagonal matrix with coefficients in $\mathbb{C}(t)$, defined by $\Delta_{jj}(t) = f(\Lambda_{jj})$. Consequently, the coefficients of the vector $P_\Gamma(t)$ are rational fractions in t.
Hence we get:

Proposition 1
For each $i \in [\![0, l]\!]$, the formal power series $P_\Gamma(t)_i$ is a rational function.

2.2 The results for the finite subgroups of $\mathbf{SL}_2\mathbb{C}$

• The complete classification up to conjugation of all finite subgroups of $\mathbf{SL}_2\mathbb{C}$ is given in [Sp77]. It consists of two infinite series (types A, D) and three exceptional cases (types E_6, E_7, E_8).
We set $\zeta_j := e^{\frac{2i\pi}{j}}$. For $\sigma \in \mathfrak{S}_{[\![0, j-1]\!]}$, we then define the matrix $Q^\sigma := \left(\delta_k^{\sigma(l)}\right)_{(k,l) \in [\![0, j-1]\!]}$.

2.2.1 Type A — Cyclic groups

• Here, we take $\Gamma = \mathbb{Z}/j\mathbb{Z}$. The natural representation and the natural character of Γ are

$$\begin{array}{rccc} \gamma \ : & \mathbb{Z}/j\mathbb{Z} & \to & \mathbf{SL}_2\mathbb{C} \\ & \overline{k} & \mapsto & \begin{pmatrix} \zeta_j^k & 0 \\ 0 & \zeta_j^{-k} \end{pmatrix} \end{array} \quad \left| \quad \begin{array}{rccc} \chi \ : & \mathbb{Z}/j\mathbb{Z} & \to & \mathbf{SL}_2\mathbb{C} \\ & \overline{k} & \mapsto & \zeta_j^k + \zeta_j^{-k}. \end{array} \right.$$

The character table is the Vandermonde matrix $T_\Gamma = \left(\zeta_j^{kl}\right)_{(k,l)\in[\![0,\,j-1]\!]}$. Let σ be the permutation $\sigma \in \mathfrak{S}_{[\![0,\,j-1]\!]}$ defined by $\sigma(0) = 0$ and $\forall\ i \in [\![0,\,j-1]\!],\ \sigma(i) = j - i$. Then $T_\Gamma^2 = j\,Q^\sigma$, i.e. $T_\Gamma^{-1} = \frac{1}{j}\,T_\Gamma Q^\sigma$. The eigenvalues of A are the numbers $\overline{\chi(k)} = \zeta_j^k + \zeta_j^{-k}$, for $k \in [\![0,\,j-1]\!]$. Then

$$P_\Gamma(t)_i = \frac{1}{j}\ (T_\Gamma \Delta(t) T_\Gamma Q^\sigma)_{i0} = \frac{1}{j}\sum_{p=0}^{j-1} \frac{\zeta_j^{ip}}{(1-t\zeta_j^p)(1-t\zeta_j^{-p})}.$$

Note that $(1-t^j)(1-t^2)$ is a common denominator of all the terms of the preceding sum.

2.2.2 Type D — Binary dihedral groups

The binary dihedral group is the subgroup $\langle a_n,\, b\rangle$ of $\mathbf{SL}_2\mathbb{C}$, with

$$a_n := \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix},\ b := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The order of Γ is $4n$. The $n+3$ conjugacy classes of Γ are

Class	id	a_nb	b	a_n^n	a_n	a_n^2	a_n^3	$\cdots$	a_n^{n-1}
Cardinality	1	n	n	1	2	2	2	$\cdots$	2

The character table of Γ is

$$T_\Gamma := \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & i^n & -i^n & (-1)^n & -1 & 1 & \cdots & (-1)^{n-1} \\ 1 & -i^n & i^n & (-1)^n & -1 & 1 & \cdots & (-1)^{n-1} \\ 1 & -1 & -1 & 1 & 1 & 1 & \cdots & 1 \\ \hline 2 & 0 & 0 & -2 & \zeta_{2n}+\zeta_{2n}^{-1} & \zeta_{2n}^2+\zeta_{2n}^{-2} & \cdots & \zeta_{2n}^{n-1}+\zeta_{2n}^{-(n-1)} \\ 2 & 0 & 0 & 2 & \zeta_{2n}^2+\zeta_{2n}^{-2} & \zeta_{2n}^4+\zeta_{2n}^{-4} & \cdots & \zeta_{2n}^{2(n-1)}+\zeta_{2n}^{-2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 2 & 0 & 0 & (-1)^{n-1}2 & \zeta_{2n}^{n-1}+\zeta_{2n}^{-(n-1)} & \zeta_{2n}^{2(n-1)}+\zeta_{2n}^{-2(n-1)} & \cdots & \zeta_{2n}^{(n-1)(n-1)}+\zeta_{2n}^{-(n-1)(n-1)} \end{array}\right)$$

The natural character χ of Γ is given by $(\chi(g_0), \ldots, \chi(g_l)) = \overline{\Theta}$, with

$$\begin{aligned} \Theta \ = \ \overline{\Theta} \ &= \ [tr(id),\ tr(a_nb),\ tr(b),\ tr(a_n^n),\ tr(a_n),\ tr(a_n^2),\ tr(a_n^3), \ldots,\ tr(a_n^{n-1})] \\ &= \ [2,\ 0,\ 0,\ -2,\ \zeta_{2n}+\zeta_{2n}^{-1},\ \zeta_{2n}^2+\zeta_{2n}^{-2},\ \zeta_{2n}^3+\zeta_{2n}^{-3}, \ldots,\ \zeta_{2n}^{n-1}+\zeta_{2n}^{-(n-1)}]. \end{aligned}$$

Set $Diag\,(d_1,\ d_2,\ d_3,\ d_4,\ \delta_1,\ \delta_2, \ldots,\ \delta_{n-1}) := \Delta(t)$. We deduce the formula for the series $P_\Gamma(t)$ that :

$$\begin{aligned} P_\Gamma(t)_0 \ = \ & \frac{3n-1}{8n^2}\left(d_1+d_2+d_3+d_4+2\sum_{k=1}^{n-1}\delta_k\right) + (-1)^n\,\frac{n-1}{8n^2}\left(d_1+(-1)^n(d_2+d_3)+d_4+2\sum_{k=1}^{n-1}(-1)^k\delta_k\right) \\ & + \sum_{l=1}^{n-1}(-1)^l\,\frac{n-1}{4n^2}\left(d_1+(-1)^l(d_2+d_3)+d_4+\sum_{k=1}^{n-1}(\zeta_{2n}^{lk}+\zeta_{2n}^{-lk})\delta_k\right), \end{aligned}$$

$$\begin{aligned} P_\Gamma(t)_1 \ = \ & \frac{3n-1}{8n^2}\left(d_1+i^nd_2-i^nd_3+(-1)^nd_4+2\sum_{k=1}^{n-1}(-1)^k\delta_k\right) \\ & +(-1)^n\,\frac{n-1}{8n^2}\left(d_1+(-1)^n(i^nd_2-i^nd_3+d_4)+2\sum_{k=1}^{n-1}\delta_k\right) \\ & + \sum_{l=1}^{n-1}(-1)^l\,\frac{n-1}{4n^2}\left(d_1+(-1)^l(i^nd_2-i^nd_3)+(-1)^nd_4+\sum_{k=1}^{n-1}(-1)^k(\zeta_{2n}^{lk}+\zeta_{2n}^{-lk})\delta_k\right), \end{aligned}$$

and then $P_\Gamma(t)_3$ (resp. $P_\Gamma(t)_2$) is obtained by replacing in $P_\Gamma(t)_0$ (resp. $P_\Gamma(t)_1$) d_2 by $-d_2$ and d_3 by $-d_3$.
Finally, for $i \in [\![1,\, n-1]\!]$, we have

$$\begin{aligned} P_\Gamma(t)_{i+3} &= \frac{3n-1}{8n^2}\left(2d_1 + 2(-1)^i d_4 + 2\sum_{k=1}^{n-1}(\zeta_{2n}^{ki} + \zeta_{2n}^{-ki})\delta_k\right) \\ &\quad +(-1)^n \frac{n-1}{8n^2}\left(2d_1 + 2(-1)^i d_4 + 2\sum_{k=1}^{n-1}(-1)^k(\zeta_{2n}^{ik} + \zeta_{2n}^{-ik})\delta_k\right) \\ &\quad +\sum_{l=1}^{n-1}(-1)^l \frac{n-1}{4n^2}\left(2d_1 + 2(-1)^i d_4 + \sum_{k=1}^{n-1}(\zeta_{2n}^{ki} + \zeta_{2n}^{-ki})\delta_k(\zeta_{2n}^{kl} + \zeta_{2n}^{-kl})\right). \end{aligned}$$

2.3 Exceptional cases

2.3.1 Type E_6 — Binary tetrahedral group

The binary tetrahedral group is the subgroup $\langle a^2,\, b,\, c\rangle$ of $\mathbf{SL}_2\mathbb{C}$, with

$$a := \begin{pmatrix} \zeta_8 & 0 \\ 0 & \zeta_8^7 \end{pmatrix},\ b := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ c := \frac{1}{\sqrt{2}}\begin{pmatrix} \zeta_8^7 & \zeta_8^7 \\ \zeta_8^5 & \zeta_8 \end{pmatrix}.$$

The order of Γ is 24. The 7 conjugacy classes of Γ are

Class	id	$a^4 = -id$	b	c	c^2	$-c$	$-c^2$
Cardinality	1	1	6	4	4	4	4

The character table T_Γ of Γ and the matrix A are

$$T_\Gamma = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & j & j^2 & j & j^2 \\ 1 & 1 & 1 & j^2 & j & j^2 & j \\ 2 & -2 & 0 & 1 & -1 & -1 & 1 \\ 2 & -2 & 0 & j & -j^2 & -j & j^2 \\ 2 & -2 & 0 & j^2 & -j & -j^2 & j \\ 3 & 3 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},\ A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix},$$

and the eigenvalues are $\Theta = (2,\ -2,\ 0,\ 1,\ -1,\ -1,\ 1)$.

The series $P_\Gamma(t)_i = \frac{N_\Gamma(t)_i}{D_\Gamma(t)}$ are given by $D_\Gamma(t) = (1-t^6)(1-t^8)$, and

$$\begin{aligned} N_\Gamma(t)_0 &= t^{12}+1, \\ N_\Gamma(t)_1 &= t^8+t^4, \\ N_\Gamma(t)_2 &= t^8+t^4, \\ N_\Gamma(t)_3 &= t^{11}+t^7+t^5+t, \end{aligned} \qquad \begin{aligned} N_\Gamma(t)_4 &= t^9+t^7+t^5+t^3, \\ N_\Gamma(t)_5 &= t^9+t^7+t^5+t^3, \\ N_\Gamma(t)_6 &= t^{10}+t^8+2t^6+t^4+t^2. \end{aligned}$$

2.3.2 Type E_7 — Binary octahedral group

The binary octahedral group is the subgroup $\langle a,\, b,\, c\rangle$ of $\mathbf{SL}_2\mathbb{C}$, with $a,\ b,\ c$ defined as in the preceding section. The order of Γ is 48. The 8 conjugacy classes of Γ are

Class	id	$a^4 = -id$	ab	b	c^2	c	a	a^3
Cardinality	1	1	12	6	8	8	6	6

The character table T_Γ of Γ and the matrix A are

$$T_\Gamma = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 2 & 2 & 0 & 2 & -1 & -1 & 0 & 0 \\ 2 & -2 & 0 & 0 & -1 & 1 & \sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & -1 & 1 & -\sqrt{2} & \sqrt{2} \\ 3 & 3 & -1 & -1 & 0 & 0 & 1 & 1 \\ 3 & 3 & 1 & -1 & 0 & 0 & -1 & -1 \\ 4 & -4 & 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}, \; A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix},$$

and the eigenvalues are $\Theta = (2,\ -2,\ 0,\ 0,\ -1,\ 1,\ \sqrt{2},\ -\sqrt{2})$.

The series $P_\Gamma(t)_i = \frac{N_\Gamma(t)_i}{D_\Gamma(t)}$ are given by $D_\Gamma(t) = (1-t^8)(1-t^{12})$, and

$$\begin{array}{lcl|lcl} N_\Gamma(t)_0 &=& t^{18}+1, & N_\Gamma(t)_4 &=& t^{13}+t^{11}+t^7+t^5, \\ N_\Gamma(t)_1 &=& t^{12}+t^6, & N_\Gamma(t)_5 &=& t^{16}+t^{12}+t^{10}+t^8+t^6+t^2, \\ N_\Gamma(t)_2 &=& t^{14}+t^{10}+t^8+t^4, & N_\Gamma(t)_6 &=& t^{14}+t^{12}+t^{10}+t^8+t^6+t^4, \\ N_\Gamma(t)_3 &=& t^{17}+t^{11}+t^7+t, & N_\Gamma(t)_7 &=& t^{15}+t^{13}+t^{11}+2t^9+t^7+t^5+t^3. \end{array}$$

2.3.3 Type E_8 — Binary icosahedral group

The binary icosahedral group is the subgroup $\langle a,\, b,\, c\rangle$ of $\mathbf{SL}_2\mathbb{C}$, with

$$a := \begin{pmatrix} -\zeta_5^3 & 0 \\ 0 & -\zeta_5^2 \end{pmatrix},\; b := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\; c := \frac{1}{\zeta_5^2+\zeta_5^{-2}} \begin{pmatrix} \zeta_5+\zeta_5^{-1} & 1 \\ 1 & -\zeta_5-\zeta_5^{-1} \end{pmatrix}.$$

The order of Γ is 120. The 9 conjugacy classes of Γ are

Class	id	$b^2=-id$	a	a^2	a^3	a^4	abc	$(abc)^2$	b
Cardinality	1	1	12	12	12	12	20	20	30

The character table T_Γ of Γ and the matrix A are

$$T_\Gamma := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & -2 & \frac{1-\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} & 1 & -1 & 0 \\ 2 & -2 & \frac{1+\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} & 1 & -1 & 0 \\ 3 & 3 & \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & 0 & 0 & -1 \\ 3 & 3 & \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & 0 & 0 & -1 \\ 4 & 4 & -1 & -1 & -1 & -1 & 1 & 1 & 0 \\ 4 & -4 & 1 & -1 & 1 & -1 & -1 & 1 & 0 \\ 5 & 5 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 6 & -6 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}, \; A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix},$$

and the eigenvalues are $\Theta = \left(2,\ -2,\ \frac{1+\sqrt{5}}{2},\ \frac{-1+\sqrt{5}}{2},\ \frac{1-\sqrt{5}}{2},\ \frac{-1-\sqrt{5}}{2},\ 1,\ -1,\ 0\right)$.

The series $P_\Gamma(t)_i = \frac{N_\Gamma(t)_i}{D_\Gamma(t)}$ are given by $D_\Gamma(t) = (1-t^{12})(1-t^{20})$, and

$$\begin{array}{lcl|lcl} N_\Gamma(t)_0 &=& t^{30}+1, & N_\Gamma(t)_4 &=& t^{24}+t^{20}+t^{16}+t^{14}+t^{10}+t^6, \\ N_\Gamma(t)_1 &=& t^{23}+t^{17}+t^{13}+t^7, & N_\Gamma(t)_5 &=& t^{24}+t^{22}+t^{18}+t^{16}+t^{14}+t^{12}+t^8+t^6, \\ N_\Gamma(t)_2 &=& t^{29}+t^{19}+t^{11}+t, & N_\Gamma(t)_6 &=& t^{27}+t^{21}+T^{19}+t^{17}+t^{13}+t^{11}+t^9+t^3, \\ N_\Gamma(t)_3 &=& t^{28}+t^{20}+t^{18}+t^{12}+t^{10}+t^2, & & & \end{array}$$

$$\begin{aligned} N_\Gamma(t)_7 &= t^{26}+t^{22}+t^{20}+t^{18}+t^{16}+t^{14}+t^{12}+t^{10}+t^8+t^4, \\ N_\Gamma(t)_8 &= t^{25}+t^{23}+t^{21}+t^{19}+t^{17}+2t^{15}+t^{13}+t^{11}+t^9+t^7+t^5. \end{aligned}$$

3 Branching law for the finite subgroups of $\mathrm{SL}_3\mathbb{C}$

• Let Γ be a finite subgroup of $\mathbf{SL}_3\mathbb{C}$ and $\{\gamma_0, \ldots, \gamma_l\}$ the set of equivalence classes of irreducible finite dimensional complex representations of Γ, where γ_0 is the trivial representation. The character associated to γ_j is denoted by χ_j.
Consider $\gamma : \Gamma \to \mathbf{SL}_3\mathbb{C}$ the natural 3−dimensional representation, and γ^* its contragredient representation. The character of γ is denoted by χ. By complete reducibility we get the decompositions

$$\forall\ j \in [\![0,\, l]\!],\ \ \gamma_j \otimes \gamma = \bigoplus_{i=0}^{l} a_{ij}^{(1)} \gamma_i \ \text{ and } \ \gamma_j \otimes \gamma^* = \bigoplus_{i=0}^{l} a_{ij}^{(2)} \gamma_i.$$

This defines two square matrices $A^{(1)} := \left(a_{ij}^{(1)}\right)_{(i,j)\in[\![0,\,l]\!]^2}$ and $A^{(2)} := \left(a_{ij}^{(2)}\right)_{(i,j)\in[\![0,\,l]\!]^2}$ of $\mathbf{M}_{l+1}\mathbb{N}$.

• Let $\underline{\mathbf{h}}$ be a Cartan subalgebra of $\mathbf{sl_3}\mathbb{C}$ and let ϖ_1, ϖ_2 be the corresponding fundamental weights, and $V(m\varpi_1 + n\varpi_2)$ the simple $\mathbf{sl_3}\mathbb{C}$ module of highest weight $m\varpi_1 + n\varpi_2$ with $(m,\, n) \in \mathbb{N}^2$. Then we get an irreducible representation $\pi_{m,n} : \mathbf{SL}_3\mathbb{C} \to \mathbf{GL}(V(m\varpi_1 + n\varpi_2))$. The restriction of $\pi_{m,n}$ to the subgroup Γ is a representation of Γ, and by complete reducibility, we get the decomposition

$$\pi_{m,n}|_\Gamma = \bigoplus_{i=0}^{l} m_i(m,n)\gamma_i,$$

where the $m_i(m,n)$'s are non negative integers. Let $\mathcal{E} := (e_0, \ldots, e_l)$ be the canonical basis of $\mathbb{C}^{l+1}$, and

$$v_{m,n} := \sum_{i=0}^{l} m_i(m,n) e_i \in \mathbb{C}^{l+1}.$$

As γ_0 is the trivial representation, we have $v_{0,0} = e_0$. Let us consider the vector (with elements of $\mathbb{C}[\![t,\, u]\!]$ as coefficients)

$$P_\Gamma(t,\, u) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} v_{m,n} t^m u^n \in (\mathbb{C}[\![t,\, u]\!])^{l+1},$$

and denote by $P_\Gamma(t,\, u)_j$ its j−th coordinate in the basis $\mathcal{E}$. Note that $P_\Gamma(t,\, u)$ can also be seen as a formal power series with coefficients in $\mathbb{C}^{l+1}$. The aim of this article is to compute $P_\Gamma(t,\, u)$.

3.1 The formal power series of the multiplicies is a rational function

Here we establish some properties of the series $P_\Gamma(t,\, u)$, in order to give an explicit formula for it. The first proposition follows from the uniqueness of the decomposition of a representation as sum of irreducible representations.

Proposition 2
• $A^{(2)} = {}^tA^{(1)}$.
• *$A^{(1)}$ and $A^{(2)}$ commute, i.e. $A^{(1)}$ is a normal matrix.*

Since $A^{(1)}$ is normal, we know that it is diagonalizable with eigenvectors forming an orthogonal basis. Now we will diagonalize the matrix $A^{(1)}$ by using the character table of the group Γ. Let us denote by $\{C_0, \ldots, C_l\}$ the set of conjugacy classes of Γ, and for any $j \in [\![0,\, l]\!]$, let g_j be an element of C_j. So the character table of Γ is the matrix $T_\Gamma \in \mathbf{M}_{l+1}\mathbb{C}$ defined by $(T_\Gamma)_{i,j} := \chi_i(g_j)$.

Proposition 3
For $k \in [\![0, l]\!]$, set $w_k := (\chi_0(g_k), \ldots, \chi_l(g_k)) \in \mathbb{C}^{l+1}$. Then w_k is an eigenvector of $A^{(2)}$ associated to the eigenvalue $\chi(g_k)$. Similarly, w_k is an eigenvector of $A^{(1)}$ associated to the eigenvalue $\overline{\chi(g_k)}$.

We will see in the sequel that $\mathcal{W} := (w_0, \ldots, w_l)$ is always a basis of eigenvectors of $A^{(1)}$ and $A^{(2)}$, so that $T_\Gamma^{-1} A^{(1)} T_\Gamma$ and $T_\Gamma^{-1} A^{(2)} T_\Gamma$ are diagonal matrices.
Now, we make use of the Clebsch-Gordan formula

$$\pi_{1,0} \otimes \pi_{m,n} = \pi_{m+1,n} \oplus \pi_{m,n-1} \oplus \pi_{m-1,n+1}, \quad \pi_{0,1} \otimes \pi_{m,n} = \pi_{m,n+1} \oplus \pi_{m-1,n} \oplus \pi_{m+1,n-1}. \tag{1}$$

Proposition 4
The vectors $v_{m,n}$ satisfy the following recurrence relations

$$A^{(1)} v_{m,n} = v_{m+1,n} + v_{m,n-1} + v_{m-1,n+1},$$
$$A^{(2)} v_{m,n} = v_{m,n+1} + v_{m-1,n} + v_{m+1,n-1}.$$

Proof:
The definition of $v_{m,n}$ reads $v_{m,n} = \sum_{i=0}^{l} m_i(m,n) e_i$, thus $A^{(1)} v_{m,n} = \sum_{i=0}^{l} \left(\sum_{j=0}^{l} m_j(m,n) a_{ij}^{(1)} \right) e_i$.
Now $(\pi_{m,n} \otimes \pi_{1,0})|_\Gamma = \pi_{m,n}|_\Gamma \otimes \gamma = \sum_{j=0}^{l} m_j(m,n) \gamma_j \otimes \gamma = \sum_{i=0}^{l} \left(\sum_{j=0}^{l} m_j(m,n) a_{ij}^{(1)} \right) \gamma_i$,
and $\pi_{m+1,n}|_\Gamma + \pi_{m,n-1}|_\Gamma + \pi_{m-1,n+1}|_\Gamma = \sum_{i=0}^{l} \left(m_i(m+1,n) + m_i(m,n-1) + m_i(m-1,n+1) \right) \gamma_i$.
By uniqueness, $\sum_{j=0}^{l} m_j(m,n) a_{ij}^{(1)} = m_i(m+1,n) + m_i(m,n-1) + m_i(m-1,n+1)$. ■

Proposition 5
The series $P_\Gamma(t, u)$ satisfies the following relation

$$\left(1 - tA^{(1)} + t^2 A^{(2)} - t^3\right) \left(1 - uA^{(2)} + u^2 A^{(1)} - u^3\right) P_\Gamma(t, u) = (1 - tu) v_{0,0}.$$

Proof:
• Set $x := P_\Gamma(t, u)$. Set also $v_{m,-1} := 0$ and $v_{-1,n} := 0$ for $(m, n) \in \mathbb{N}$, such that, according to the Clebsch-Gordan formula, the formulae of the preceding corollary are still true for $(m, n) \in \mathbb{N}$. We have

$$tuA^{(1)} x = tu \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A^{(1)} v_{m,n} t^m u^n = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (v_{m+1,n} + v_{m,n-1} + v_{m-1,n+1}) t^{m+1} u^{n+1}.$$

$$\text{Now} \qquad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} v_{m+1,n} t^{m+1} u^{n+1} = u \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} v_{m,n} t^m u^n = ux - u \sum_{n=0}^{\infty} v_{0,n} u^n,$$

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} v_{m,n-1} t^{m+1} u^{n+1} = tu^2 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} v_{m,n-1} t^m u^{n-1} = tu^2 x,$$

$$\text{and} \quad \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} v_{m-1,n+1} t^{m+1} u^{n+1} = t^2 \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} v_{m-1,n+1} t^{m-1} u^{n+1} = t^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} v_{m,n+1} t^m u^{n+1}$$
$$= t^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} v_{m,n} t^m u^n - t^2 \sum_{m=0}^{\infty} v_{m,0} t^m = t^2 x - t^2 \sum_{m=0}^{\infty} v_{m,0} t^m.$$

$$\text{Therefore} \qquad tuA^{(1)} x = (u + tu^2 + t^2) x - u \sum_{n=0}^{\infty} v_{0,n} u^n - t^2 \sum_{m=0}^{\infty} v_{m,0} t^m. \tag{2}$$

$$\text{We proceed likewise to obtain} \quad tuA^{(2)} x = (t + tu^2 + u^2) x - t \sum_{m=0}^{\infty} v_{m,0} t^m - u^2 \sum_{n=0}^{\infty} v_{0,n} u^n. \tag{3}$$

• By using Equations (2) and (3), we have $tuA^{(2)}x - tu^2A^{(1)}x = t(1-u^3)x + (t^2u-t)\sum_{m=0}^{\infty} v_{m,0}t^m$,

$$\text{i.e.} \qquad \left(1 - uA^{(2)} + u^2A^{(1)} - u^3\right)x = (1-tu)\sum_{m=0}^{\infty} v_{m,0}t^m. \tag{4}$$

Besides $A^{(1)}v_{m,0} = v_{m+1,0} + v_{m-1,1}$, and $A^{(2)}v_{m-1,0} = v_{m-1,1} + v_{m-2,0}$, hence

$$A^{(1)}v_{m,0} = v_{m+1,0} + A^{(2)}v_{m-1,0} - v_{m-2,0}.$$

Set $y := \sum_{m=0}^{\infty} v_{m,0}t^m$. Then

$$\begin{aligned} tA^{(1)}y &= \sum_{m=0}^{\infty} v_{m+1,0}t^{m+1} + A^{(2)}\sum_{m=1}^{\infty} v_{m-1,0}t^{m+1} - \sum_{m=2}^{\infty} v_{m-2,0}t^{m+1} \\ &= \sum_{m=1}^{\infty} v_{m,0}t^m + t^2A^{(2)}\sum_{m=0}^{\infty} v_{m,0}t^m - t^3\sum_{m=0}^{\infty} v_{m,0}t^m \;=\; y - v_{0,0} + t^2A^{(2)}y - t^3y. \end{aligned}$$

$$\text{So} \qquad \left(1 - tA^{(1)} + t^2A^{(2)} - t^3\right)y = v_{0,0}. \tag{5}$$

Combining Eq. 4 and 5, we have $\left(1 - tA^{(1)} + t^2A^{(2)} - t^3\right)\left(1 - uA^{(2)} + u^2A^{(1)} - u^3\right)x = (1-tu)v_{0,0}$. ■

We may inverse the relation obtained in Proposition 5 and obtain an explicit expression[3] for $P_\Gamma(t,\, u)$ as well as an explicit formula for the vector $v_{m,n}$. But, for the explicit calculations of $P_\Gamma(t,\, u)$, we will use an other fundamental formula (we inverse complex numbers instead of matrices). We need the rational function f defined by

$$\begin{array}{rccl} f & : & \mathbb{C}^2 & \to & \mathbb{C}(t,\, u) \\ & & (d_1,\, d_2) & \mapsto & \dfrac{1-tu}{(1-td_1+t^2d_2-t^3)(1-ud_2+u^2d_1-u^3)}. \end{array}$$

The complete classification up to conjugation of all finite subgroups of $\mathbf{SL}_3\mathbb{C}$ is given in [YY93]. It consists in four infinite series (types A, B, C, D) and eight exceptional cases (types E, F, G, H, I, J, K, L). In all the cases, the character table T_Γ is invertible, and $\Lambda^{(1)} := T_\Gamma^{-1}\, A^{(1)}\, T_\Gamma$ and $\Lambda^{(2)} := T_\Gamma^{-1}\, A^{(2)}\, T_\Gamma$ are diagonal matrices, with $\Lambda^{(1)}_{jj} = \overline{\chi(g_j)}$ and $\Lambda^{(2)}_{jj} = \chi(g_j)$. According to Proposition 5, we may write

$$T_\Gamma\left(1 - t\Lambda^{(1)} + t^2\Lambda^{(2)} - t^3\right)\left(1 - u\Lambda^{(2)} + u^2\Lambda^{(1)} - u^3\right)T_\Gamma^{-1}P_\Gamma(t,\, u) = (1-tu)v_{0,0}.$$

We deduce that

$$P_\Gamma(t,\, u) = T_\Gamma\,\Delta(t,\, u)\,T_\Gamma^{-1}v_{0,0} = (T_\Gamma\,\Delta(t,\, u)\,T_\Gamma)\,(T_\Gamma^{-2}v_{0,0}), \tag{6}$$

where $\Delta(t,\, u) \in \mathbf{M}_{l+1}\mathbb{C}(t,\, u)$ is the diagonal matrix defined by $\Delta(t,\, u)_{jj} = f(\Lambda_{jj},\, \overline{\Lambda_{jj}}) = f(\overline{\chi(g_j)},\, \chi(g_j))$. Let $\Theta := (\Lambda^{(1)}_{00}, \ldots,\, \Lambda^{(1)}_{ll})$ be the list of eigenvalues of $A^{(1)}$.

As a corollary of the preceding formula we get:

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[3] $P_\Gamma(t,\, u) = (1-tu)\left(\sum_{p=0}^{\infty}\left(u^3 + uA^{(2)} - u^2A^{(1)}\right)^p\right)\left(\sum_{q=0}^{\infty}\left(t^3 + tA^{(1)} - t^2A^{(2)}\right)^q\right)v_{0,0}$.
For $z \in \mathbb{R}$, let $\lceil z \rceil$ be the smallest integer that is greater or equal to z, and set $\{r,\, s\} := \{1,\, 2\}$. For $m \in \mathbb{N}$, set

$$\alpha_m^{(r)} := \sum_{q=\lceil \frac{m}{3} \rceil}^{m}\left(\sum_{j=\lceil \frac{3q-m}{2} \rceil}^{\min(3q-m,\, q)} C_q^j C_j^{3q-m-j}(-1)^{3q-m}{A^{(r)}}^{3q-m-j}{A^{(s)}}^{2j-3q+m}\right)$$

Then $v_{m,n} = v_{0,0}$ if $m = n = 0$; $\alpha_n^{(2)}v_{0,0}$ if $m = 0,\ n \neq 0$; $\alpha_m^{(1)}v_{0,0}$ if $n = 0,\ m \neq 0$; $(\alpha_n^{(2)}\alpha_m^{(1)} - \alpha_{n-1}^{(2)}\alpha_{m-1}^{(1)})v_{0,0}$ otherwise.

Proposition 6
The coefficients of the vector $P_\Gamma(t,\, u)$ are rational fractions in t and u, hence the formal power series of the multiplicities is a rational function.

We will denote them by

$$P_\Gamma(t,\, u)_i := \frac{N_\Gamma(t,\, u)_i}{D_\Gamma(t,\, u)_i},\ i \in [\![0,\, l]\!]$$

where $N_\Gamma(t,\, u)_i$ and $D_\Gamma(t,\, u)_i$ are elements of $\mathbb{C}[t,\, u]$ that will be explicitly computed in the sequel. Finally, we introduce a generalized Cartan matrix that we will study for every finite subgroup of $\mathbf{SL}_3\mathbb{C}$.

Definition 7
For every finite subgroup of $\mathbf{SL}_3\mathbb{C}$, we define a generalized Cartan matrix by the following formula:

$$C_\Gamma := 2\, I - A^{(1)} - {}^t A^{(1)} + 2\, Diag(A^{(1)}).$$

For $k \in [\![0,\, l]\!]$, the matrix of the reflection s_k associated to the $k-$th root of $g(C_\Gamma)$ the Kac Moody algebra attached to C_Γ is defined by

$$(s_k)_{ij} = \delta_i^j - (C_\Gamma)_{k,j}\delta_i^k.$$

For each finite subgroup, we will give a decomposition of the set of simple reflections $\mathcal{S} = \{s_0, \ldots,\, s_l\}$ in p sets (with p minimal), i.e.

$$\mathcal{S} = S_0 \sqcup \cdots \sqcup S_{p-1},$$

such that roots corresponding to reflections in those sets form a partition of the set of simple roots to mutually orthogonal sets. We denote by τ_l the (commutative) product of the elements of S_l. Then we deduce the following decomposition of C_Γ:

$$C_\Gamma = p\, I - \sum_{k=0}^{p-1} \tau_k.$$

Remark 8
Along this section we will present matrices that have only -2, -1, 0, 1, 2 as entries. For a clearer exposition, we represent the non-zero entries by colored points. The correspondence is the following: dark grey $= -2$, light grey $= -1$, white $= 1$, black $= 2$, empty $= 0$.

3.2 Explicit results for the infinite series — Types A, B, C, D

3.2.1 The A Series

In this section, we consider Γ a finite diagonal abelian subgroup of $\mathbf{SL}_3\mathbb{C}$. Then Γ is isomorphic to a product of cyclic groups:

$$\Gamma \simeq \mathbb{Z}/j_1\mathbb{Z} \times \cdots \times \mathbb{Z}/j_k\mathbb{Z}.$$

If Γ is a finite subgroup of $\mathbf{SL}_r\mathbb{C}$, then Γ is a *small* subgroup of $\mathbf{GL}_r\mathbb{C}$, i.e. no element of Γ has an eigenvalue 1 of multiplicity $r-1$. In fact, if $g \in \Gamma$ has an eigenvalue 1 of multiplicity $r-1$, then the last eigenvalue of g is different from 1 and the determinant of g is also different from 1, which is impossible. Then, according to a lemma of [DHZ05] (p.13), Γ has at most $r-1$ generators. So, for a subgroup Γ of type A, we may assume that $k \leq 2$, i.e. we have two cases:
(A_1) $\Gamma \simeq \mathbb{Z}/j\mathbb{Z}$,
(A_2) $\Gamma \simeq \mathbb{Z}/j_1\mathbb{Z} \times \mathbb{Z}/j_2\mathbb{Z}$, with $j_1 \geq j_2 \geq 2$.

3.2.2 Type A_1

• Here, we take $\Gamma = \mathbb{Z}/j\mathbb{Z}$. The natural representation and the natural character of Γ are

$$\begin{array}{rcl|rcl} \gamma \ : \ \mathbb{Z}/j\mathbb{Z} & \to & \mathbf{SL}_3\mathbb{C} & \chi \ : \ \mathbb{Z}/j\mathbb{Z} & \to & \mathbf{SL}_3\mathbb{C} \\ \overline{k} & \mapsto & \begin{pmatrix} \zeta_j^k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_j^{-k} \end{pmatrix} & \overline{k} & \mapsto & 1+\zeta_j^k+\zeta_j^{-k}. \end{array}$$

The character table of Γ is $T_\Gamma = \left(\zeta_j^{kl}\right)_{(k,l)\in[\![0,\,j-1]\!]}$. Let $\sigma \in \mathfrak{S}_{[\![0,\,j-1]\!]}$ be the permutation defined by $\sigma(0) = 0$ and $\forall\ i \in [\![0,\,j-1]\!],\ \sigma(i) = j-i$. Then $T_\Gamma^{-1} = \frac{1}{j}\,T_\Gamma Q^\sigma$. The eigenvalues of $A^{(1)}$ are the numbers $\overline{\chi(k)} = 1+\zeta_j^k+\zeta_j^{-k}$, for $k \in [\![0,\,j-1]\!]$. According to Formula 6,

$$P_\Gamma(t,\,u)_i = \frac{1}{j}\ \left(T_\Gamma \Delta(t,\,u) T_\Gamma Q^\sigma\right)_{i0} = \frac{1}{j}\sum_{p=0}^{j-1} \frac{\zeta_j^{ip}(1-tu)}{(1-t)(1-t\zeta_j^p)(1-t\zeta_j^{-p})(1-u)(1-u\zeta_j^p)(1-u\zeta_j^{-p})}.$$

Note that $(1-t^j)(1-t^2)(1-u^j)(1-u^2)$ is a common denominator of all the terms of the preceding sum, which is independent of i.

• The matrix $A^{(1)} \in \mathbf{M}_j\mathbb{C}$ is

$$A^{(1)} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \text{ if } j = 2, \quad A^{(1)} = \begin{pmatrix} 1 & 1 & & 1 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & & 1 & 1 \end{pmatrix} \text{ if } j \geq 3.$$

Then the set of reflections $\mathcal{S}$ may be decomposed in two (resp. three) sets if j is even (resp. odd).
▷ If j is even, we have $\tau_0 = s_0 s_4 \dots s_{j-2}$, $\tau_1 = s_1 s_3 \dots s_{j-1}$, and $C_{A_1}(j) = 2I_{j-1} - (\tau_0 + \tau_1)$.
▷ If j is odd, we have $\tau_0 = s_2 s_4 \dots s_{j-1}$, $\tau_1 = s_1 s_3 \dots s_{j-2}$, $\tau_2 = s_0$, and $C_{A_1}(j) = 3I_j - (\tau_0 + \tau_1 + \tau_2)$.
The graph associated to $C_{A_1}(j)$ is a cyclic graph with j vertices and j edges.

3.2.3 Type A_2

• We now consider the case $\Gamma = \mathbb{Z}/j_1\mathbb{Z} \times \mathbb{Z}/j_2\mathbb{Z}$, with $j_1 \geq j_2 \geq 2$. The natural representation and the natural character of Γ are

$$\begin{array}{rcl|rcl} \gamma \ : \ \mathbb{Z}/j_1\mathbb{Z} \times \mathbb{Z}/j_2\mathbb{Z} & \to & \mathbf{SL}_3\mathbb{C} & \chi \ : \ \mathbb{Z}/j_1\mathbb{Z} \times \mathbb{Z}/j_2\mathbb{Z} & \to & \mathbf{SL}_3\mathbb{C} \\ (\overline{k_1},\,\overline{k_1}) & \mapsto & \begin{pmatrix} \zeta_{j_1}^{k_1} & 0 & 0 \\ 0 & \zeta_{j_2}^{k_2} & 0 \\ 0 & 0 & \zeta_{j_1}^{-k_1}\zeta_{j_2}^{-k_2} \end{pmatrix} & (\overline{k_1},\,\overline{k_2}) & \mapsto & \zeta_{j_1}^{k_1} + \zeta_{j_2}^{k_2} + \zeta_{j_1}^{-k_1}\zeta_{j_2}^{-k_2}. \end{array}$$

The irreducible characters of Γ are the elements of the form $\chi_1 \otimes \chi_2$, where χ_1 and χ_2 are irreducible characters of $\mathbb{Z}/j_1\mathbb{Z}$ and $\mathbb{Z}/j_2\mathbb{Z}$, i.e. the irreducible characters of Γ are, for $(l_1,\,l_2) \in [\![0,\,j_1-1]\!] \times [\![0,\,j_2-1]\!]$,

$$\begin{array}{rcl} \chi_{l_1,l_2} \ : \ \mathbb{Z}/j_1\mathbb{Z} \times \mathbb{Z}/j_2\mathbb{Z} & \to & \mathbf{SL}_3\mathbb{C} \\ (\overline{k_1},\,\overline{k_2}) & \mapsto & \zeta_{j_1}^{k_1 l_1}\zeta_{j_2}^{k_2 l_2} \end{array}.$$

For $k \in \{1,\,2\}$, let us denote by T_k the character table of the group $\mathbb{Z}/j_k\mathbb{Z}$. Then the character table of $\Gamma = \mathbb{Z}/j_1\mathbb{Z} \times \mathbb{Z}/j_2\mathbb{Z}$ is the Kronecker product[4] $T_\Gamma = T_1 \otimes T_2$. Let $\sigma_k \in \mathfrak{S}_{[\![0,\,j_k-1]\!]}$ be the permutation

[4]Recall that the Kronecker product of two matrices $A \in \mathbf{M}_m\mathbb{C}$ and $B \in \mathbf{M}_n\mathbb{C}$ is the block-matrix $A \otimes B \in \mathbf{M}_{mn}\mathbb{C}$ defined by the formula:

$$\forall\ (i,\,j) \in [\![1,\,m]\!],\ (A \otimes B)_{ij} = a_{ij}B.$$

An important property of the Kronecker product is the relation

$$\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B).$$

The equality $T_\Gamma = T_1 \otimes T_2$ is implied by this relation

defined by $\sigma_k(0) = 0$ and $\forall\ i \in [\![0,\ j_k - 1]\!],\ \sigma_k(i) = j_k - i$. We have

$$T_\Gamma^{-1} = (T_1 \otimes T_2)^{-1} = \frac{1}{j_1 j_2}(T_1 \otimes T_2)(Q^{\sigma_1} \otimes Q^{\sigma_2}) = \frac{1}{j_1 j_2}(T_1 Q^{\sigma_1}) \otimes (T_2 Q^{\sigma_2}).$$

The eigenvalues of $A^{(1)}$ are the numbers $\overline{\chi(k_1,\ k_2)} = \zeta_{j_1}^{-k_1} + \zeta_{j_2}^{-k_2} + \zeta_{j_1}^{k_1}\zeta_{j_2}^{k_2}$, for $(k_1,\ k_2) \in [\![0,\ j_1 - 1]\!] \times [\![0,\ j_2 - 1]\!]$.
Let us denote by $\Lambda^{(1)} := \mathrm{Diag}\left(\Lambda_0^{(1)}, \ldots,\ \Lambda_{j_1}^{(1)}\right)$ the diagonal block-matrix defined by

$$(\Lambda_{k_1}^{(1)})_{k_2 k_2} = \overline{\chi(k_1,\ k_2)} = \zeta_{j_1}^{-k_1} + \zeta_{j_2}^{-k_2} + \zeta_{j_1}^{k_1}\zeta_{j_2}^{k_2}.$$

According to Formula 6, for $(m,\ n) \in [\![0,\ j_1 - 1]\!] \times [\![0,\ j_2 - 1]\!]$, we have

$$\begin{aligned} & P_\Gamma(t,\ u)_{mj_2+n} \\ = \ & \frac{1}{j_1 j_2} \sum_{k=0}^{j_1-1} \sum_{l=0}^{j_2-1} \zeta_{j_1}^{mk}\zeta_{j_2}^{nl}(1 - tu)\left(1 - t(\zeta_{j_1}^{-k} + \zeta_{j_2}^{-l} + \zeta_{j_1}^{k}\zeta_{j_2}^{l}) + t^2(\zeta_{j_1}^{k} + \zeta_{j_2}^{l} + \zeta_{j_1}^{-k}\zeta_{j_2}^{-l}) - t^3\right)^{-1} \\ & \qquad \left(1 - u(\zeta_{j_1}^{k} + \zeta_{j_2}^{l} + \zeta_{j_1}^{-k}\zeta_{j_2}^{-l}) + u^2(\zeta_{j_1}^{-k} + \zeta_{j_2}^{-l} + \zeta_{j_1}^{k}\zeta_{j_2}^{l}) - u^3\right)^{-1}. \end{aligned}$$

• The matrix $A^{(1)}$ is a block-matrix with j_1^2 blocs of size j_2, and we have

$A^{(1)} = \mathbf{1}_{4,4} - I_4$, if $j_1 = j_2 = 2$, and $A^{(1)} = \begin{pmatrix} Q^{\rho_2} & {}^tQ^{\rho_2} & & I_{j_2} \\ I_{j_2} & \ddots & \ddots & \\ & \ddots & \ddots & {}^tQ^{\rho_2} \\ {}^tQ^{\rho_2} & & I_{j_2} & Q^{\rho_2} \end{pmatrix}$ if $j_1 \geq 2$, $Q^{\rho_k} := \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$.

So, we may write $A^{(1)} = I_{j_1} \otimes Q^{\rho_2} + Q^{\rho_1} \otimes I_{j_2} + {}^tQ^{\rho_1} \otimes {}^tQ^{\rho_2}$.
Note that $Diag(A^{(1)}) = 0$. Then $C_{A_2}(j_1,\ j_2) = I_{j_1} \otimes W + Q^{\tau_1} \otimes S + {}^tQ^{\tau_1} \otimes {}^tS$.

• Now, let us decompose the matrix $A^{(1)}$:

If $j_1 = j_2 = 2$, then the decomposition of $C_{A_2}(2,\ 2)$ is $C_{A_2}(2,\ 2) = 4\,I_4 - (s_0 + s_1 + s_2 + s_3)$.
Now, we assume that $j_1 \geq 3$. For $(i_1,\ i_2) \in [\![0,\ j_1 - 1]\!] \times [\![0,\ j_2 - 1]\!]$, let s_{i_1,i_2} be the reflection associated to the $(i_1 j_2 + i_2)-$th root. Then the set $\mathcal{S}$ may be decomposed into p sets where $p \in \{4,\ 6,\ 9\}$.
For $i_1 \in [\![0,\ j_1 - 1]\!]$, define $\widetilde{S_{i_1}} := \{s_{i_1,0}, \ldots,\ s_{i_1,j_2-1}\}$.
▷ If j_1 is odd, we set

$$\widehat{I_0} := \{0,\ 2, \ldots,\ j_1 - 3\},\ \widehat{I_1} := \{1,\ 3, \ldots,\ j_1 - 2\},\ \widehat{I_0} := \{j_1 - 1\}.$$

▷ If j_1 is even, we set

$$\widehat{I_0} := \{0,\ 2, \ldots,\ j_1 - 2\},\ \widehat{I_1} := \{1,\ 3, \ldots,\ j_1 - 1\}.$$

Then, the roots associated to the reflections of distinct $\widetilde{S_{i_1}}$'s for i_1 belonging to a same $\widehat{I_k}$ are orthogonal.
Now, we decompose each $\widetilde{S_{i_1}}$, i.e. $\widetilde{S_{i_1}} = \widehat{S_{i_1,0}} \sqcup \cdots \sqcup \widehat{S_{i_1,q-1}}$, such that $q \in \{2,\ 3\}$ and for every $k \in [\![0,\ q-1]\!]$, the roots associated to the reflections belonging to $\widehat{S_{i_1,k}}$ are orthogonal:
▷ If j_2 is odd, then $\widetilde{S_{i_1}} = \widehat{S_{i_1,0}} \sqcup \widehat{S_{i_1,1}} \sqcup \widehat{S_{i_1,2}}$, with

$$\widehat{S_{i_1,0}} = \{s_{i_1,0},\ s_{i_1,2}, \ldots,\ s_{i_1,j_2-3}\},\ \widehat{S_{i_1,1}} = \{s_{i_1,1},\ s_{i_1,3}, \ldots,\ s_{i_1,j_2-2}\},\ \widehat{S_{i_1,2}} = \{s_{i_1,j_2-1}\}.$$

▷ If j_2 is even, then $\widetilde{S_{i_1}} = \widehat{S_{i_1,0}} \sqcup \widehat{S_{i_1,1}}$, with

$$\widehat{S_{i_1,0}} = \{s_{i_1,0},\ s_{i_1,2}, \ldots,\ s_{i_1,j_2-2}\},\ \widehat{S_{i_1,1}} = \{s_{i_1,1},\ s_{i_1,3}, \ldots,\ s_{i_1,j_2-1}\}.$$

Finally, for $(k,\ l) \in \{0,\ 1,\ 2\}^2$, we set

$$S_{k,l} := \coprod_{i_1 \in \widehat{I_k}} \widehat{S_{i_1,l}},$$

and we denote by $p \in \{4,\, 6,\, 9\}$ the number of non-empty sets $S_{k,l}$, and by $\tau_{k,l}$ the commutative product of the reflections of $S_{k,l}$. Then, we have

$$C_{A_2}(j_1,\, j_2) := 2\, I_{j_1 j_2} - A^{(1)} - {}^t A^{(1)} + 2\, Diag(A^{(1)}) = p\, I_{j_1 j_2} - \sum_{(k,l)\in\{0,\,1,\,2\}^2} \tau_{k,l}.$$

• If $j_1 \geq 3$, the graph associated to Γ is a j_1-gone, such that every vertex of this j_1-gone is a j_2-gone, and every vertex of each j_2-gone is connected with exactly 2 vertices of both adjacent j_2-gones (for $j_1 = j_2 = 2$, see Remark 9).

Remark 9
In the cases $j_1 = 2$, $j_2 = 2$ and $j_1 = 3$, $j_2 = 2$, we obtain full matrices and complete graphs. Moreover the complete graphs with 4 and 6 vertices are the unique complete graphs that we can obtain for the type A_2 (the complete graphs with 2 and 3 vertices are the unique complete graphs that we can obtain for the type A_1).

Example 10
We consider the case where $j_1 = 6$ and $j_2 = 5$. Then the decomposition of the Cartan matrix is

$$C_{A_2}(6,\, 5) = 6\, I_{30} - (\tau_{0,0} + \tau_{0,1} + \tau_{0,2} + \tau_{1,0} + \tau_{1,1} + \tau_{1,2}),$$

where the matrix $C_{A_2}(6,\, 5)$, the $\tau_{i,j}$'s and the graph associated to $C_{A_2}(6,\, 5)$ are given by Figure 1.

$\tau_{0,0} := (s_{0,0}s_{0,2})(s_{2,0}s_{2,2})(s_{4,0}s_{4,2})$
$\tau_{0,1} := (s_{0,1}s_{0,3})(s_{2,1}s_{2,3})(s_{4,1}s_{4,3})$
$\tau_{0,2} := (s_{0,4})(s_{2,4})(s_{4,4})$
$\tau_{1,0} := (s_{1,0}s_{1,2})(s_{3,0}s_{3,2})(s_{5,0}s_{5,2}),$
$\tau_{1,1} := (s_{1,1}s_{1,3})(s_{3,1}s_{3,3})(s_{5,1}s_{5,3}),$
$\tau_{1,2} := (s_{1,4})(s_{3,4})(s_{5,4}).$

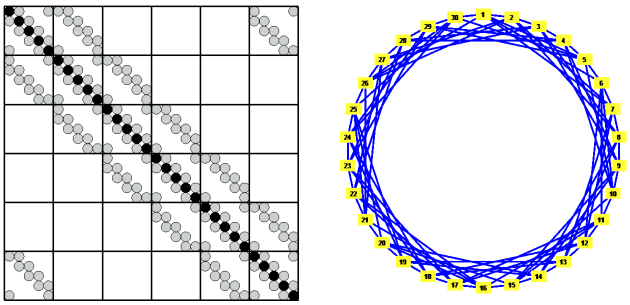


Figure 1: Matrix $C_{A_2}(6,\, 5)$ and corresponding graph.

3.3 The B series

In this section, we study the binary groups of $\mathbf{SL}_3\mathbb{C}$.
We give a general formula for the types BDa, BTa, BO and BI. In all these cases, the group Γ contains two normal subgroups Γ_1 and Γ_2 such that $\Gamma_1 \cap \Gamma_2 = \{id\}$, and $|\Gamma_1| \cdot |\Gamma_2| = |\Gamma|$, so that $\Gamma = \Gamma_2\Gamma_1$ and $\Gamma \simeq \Gamma_2 \times \Gamma_1$. The group Γ_1 is isomorphic to a binary group of $\mathbf{SL}_2\mathbb{C}$ and Γ_2 is isomorphic to $\mathbb{Z}/m\mathbb{Z}$. So, we can deduce the results for Γ of the results obtained for the group Γ_1.
If we denote by T_k the character table of the group Γ_k, the character table of the direct product $\Gamma = \Gamma_2 \times \Gamma_1$ is the Kronecker product $T_\Gamma = T_2 \otimes T_1$. The matrix T_2 is given in Section 2.2.1 (Type A — Cyclic groups), and the matrix T_1 is given in the section dealing with the corresponding binary group of $\mathbf{SL}_2\mathbb{C}$. We also have $T_\Gamma^{-1} = T_2^{-1} \otimes T_1^{-1} = \frac{1}{m}(T_2 Q^{\sigma_2}) \otimes T_1^{-1}$, where σ_2 is the permutation matrix defined by $\sigma_2(0) = 0$, and $\forall\ i \in [\![0,\, m-1]\!],\ \sigma_2(i) = m - i$.
Let us denote by h the number of conjugacy classes of Γ_1. The columns of T_Γ give a basis of eigenvectors and the eigenvalues of $A^{(1)}$ are the numbers $\overline{\chi_{i,j}}$, $(i,\, j) \in [\![0,\, m-1]\!] \times [\![0,\, h-1]\!]$ where $\chi_{i,j}$ is the value

of the natural character of Γ on the $(i,\,j)-$th conjugacy class.
Let us also denote by $\Lambda^{(1)}$ the diagonal block-matrix with $m\times m$ blocks of size $h\times h$ defined by

$$\Lambda^{(1)} := \mathrm{Diag}\left(\Lambda_0^{(1)},\ldots,\,\Lambda_{m-1}^{(1)}\right),\ (\Lambda_i^{(1)})_{jj} = \overline{\chi_{i,\,j}}.$$

According to Formula 6,

$$P_\Gamma(t,\,u) = T_\Gamma\Delta(t,\,u)T_\Gamma^{-1}v_{0,0} = \frac{1}{m}\,(T_2\otimes T_1)\Delta(t,\,u)((T_2Q^{\sigma_2})\otimes(T_1T_1^{-2}))v_{0,0},$$

where $\Delta(t,\,u)$ is the diagonal block-matrix defined by

$$\Delta(t,\,u) = \mathrm{Diag}\left(\Delta(t,\,u)^{(0)},\ldots,\,\Delta(t,\,u)^{(m-1)}\right),$$

with $\Delta(t,\,u)_{jj}^{(i)} = f(\overline{\chi_{i,j}},\,\chi_{i,j}) = \dfrac{1-tu}{(1-t\overline{\chi_{i,j}}+t^2\chi_{i,j}-t^3)(1-u\chi_{i,j}+u^2\overline{\chi_{i,j}}-u^3)} =: f_{ij}.$

The decomposition of the matrix $A^{(1)}$ and the description of the associated graph are made in the same way for the binary tetrahedral, octahedral and icosahedral groups: the results are collected in 3.3.5.

3.3.1 The BDa subseries — Binary dihedral groups

• For $(q,\,m)\in\mathbb{N}^2$, let ψ_{2q}, τ and ϕ_{2m} be the following elements of $\mathbf{SL}_3\mathbb{C}$:

$$\psi_{2q} = \begin{pmatrix} 1 & 0 & 0\\ 0 & \zeta_{2q} & 0\\ 0 & 0 & \zeta_{2q}^{-1}\end{pmatrix},\ \tau = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & i\\ 0 & i & 0\end{pmatrix},\ \phi_{2m} = \begin{pmatrix} \zeta_{2m}^{-2} & 0 & 0\\ 0 & \zeta_{2m} & 0\\ 0 & 0 & \zeta_{2m}\end{pmatrix}.$$

In this section[5], we assume that $1<q<n$, $n\wedge q=1$, and $m := n-q\equiv 1 \mod 2$, and we consider the subgroup $\Gamma := \langle\psi_{2q},\ \tau,\ \phi_{2m}\rangle$ of $\mathbf{SL}_3\mathbb{C}$. Note that $\phi_{2m} = \psi_{2q}^q\phi_m^{-\frac{m-1}{2}}$, so that $\Gamma = \langle\psi_{2q},\ \tau,\ \phi_m\rangle$. Set $\Gamma_1 := \langle\psi_{2q},\ \tau\rangle$ and $\Gamma_2 := \langle\phi_m\rangle \simeq \mathbb{Z}/m\mathbb{Z}$. Then $\Gamma\simeq\Gamma_2\times\Gamma_1$. With the notations used for $\mathbf{SL}_2\mathbb{C}$, ψ_{2q} (resp. τ) represents a_q (resp. b), where $\langle a_q,\,b\rangle$ is the binary dihedral subgroup of $\mathbf{SL}_2\mathbb{C}$.

The natural character of Γ is given by $\chi = (\chi_i)_{i=0\ldots m-1}$, with

$$\begin{aligned}
\chi_i &= [\chi_{i,0},\ \chi_{i,1},\ \chi_{i,2},\ \chi_{i,3},\ \chi_{i,4},\ \chi_{i,5},\ \chi_{i,6},\ldots,\ \chi_{i,q+2}]\\
&= \left[tr(\phi_m^i),\ tr(\phi_m^i\psi_{2q}\tau),\ tr(\phi_m^i\tau),\ tr(\phi_m^i\psi_{2q}^q),\ tr(\phi_m^i\psi_{2q}),\ tr(\phi_m^i\psi_{2q}^2),\ tr(\phi_m^i\psi_{2q}^3),\ldots,\ tr(\phi_m^i\psi_{2q}^{q-1})\right]\\
&= \Big[\zeta_m^{-2i}+2\zeta_m^i,\ \zeta_m^{-2i},\ \zeta_m^{-2i},\ \zeta_m^{-2i}-2\zeta_m^i,\ \zeta_m^{-2i}+\zeta_m^i(\zeta_{2q}+\zeta_{2q}^{-1}),\ \zeta_m^{-2i}+\zeta_m^i(\zeta_{2q}^2+\zeta_{2q}^{-2}),\\
&\quad \zeta_m^{-2i}+\zeta_m^i(\zeta_{2q}^3+\zeta_{2q}^{-3}),\ldots,\ \zeta_m^{-2i}+\zeta_m^i(\zeta_{2q}^{q-1}+\zeta_{2q}^{-(q-1)})\Big].
\end{aligned}$$

We deduce the following formula for the series $P_\Gamma(t,\,u)$:

$$\begin{aligned}
P_\Gamma(t,\,u)_{i_1(q+3)} = \frac{1}{m}\sum_{r=0}^{m-1}\zeta_m^{i_1r}\Bigg[&\frac{3q-1}{8q^2}\left(d_1^{(r)}+d_2^{(r)}+d_3^{(r)}+d_4^{(r)}+2\sum_{k=1}^{q-1}\delta_k^{(r)}\right)\\
&+(-1)^q\,\frac{q-1}{8q^2}\left(d_1^{(r)}+(-1)^q(d_2^{(r)}+d_3^{(r)})+d_4^{(r)}+2\sum_{k=1}^{q-1}(-1)^k\delta_k^{(r)}\right)\\
&+\sum_{l=1}^{q-1}(-1)^l\,\frac{q-1}{4q^2}\left(d_1^{(r)}+(-1)^l(d_2^{(r)}+d_3^{(r)})+d_4^{(r)}+\sum_{k=1}^{q-1}(\zeta_{2q}^{lk}+\zeta_{2q}^{-lk})\delta_k^{(r)}\right)\Bigg],
\end{aligned}$$

[5]The other case — the type BDb — is $1<q<n$, $n\wedge q=1$, and $m := n-q\equiv 0 \mod 2$. This group is not a direct product, the general expression for groupes in this subseries is unclear.

$$
\begin{aligned}
P_\Gamma(t,\,u)_{i_1(q+3)+1} \;=\; & \frac{1}{m}\sum_{r=0}^{m-1}\zeta_m^{i_1 r}\left[\frac{3q-1}{8q^2}\left(d_1^{(r)}+i^n d_2^{(r)}-i^n d_3^{(r)}+(-1)^n d_4^{(r)}+2\sum_{k=1}^{q-1}(-1)^k\delta_k^{(r)}\right)\right.\\
& +(-1)^q\,\frac{q-1}{8q^2}\left(d_1^{(r)}+(-1)^q(i^n d_2^{(r)}-i^n d_3^{(r)}+d_4^{(r)})+2\sum_{k=1}^{q-1}\delta_k^{(r)}\right)\\
& \left.+\sum_{l=1}^{q-1}(-1)^l\,\frac{q-1}{4q^2}\left(d_1^{(r)}+(-1)^l(i^n d_2^{(r)}-i^n d_3^{(r)})+(-1)^n d_4^{(r)}+\sum_{k=1}^{q-1}(-1)^k(\zeta_{2q}^{lk}+\zeta_{2q}^{-lk})\delta_k^{(r)}\right)\right],
\end{aligned}
$$

and $P_\Gamma(t)_{i_1(q+3)+3}$ (resp. $P_\Gamma(t)_{i_1(q+3)+2}$) is obtained by replacing in $P_\Gamma(t)_{i_1(q+3)}$ (resp. $P_\Gamma(t)_{i_1(q+3)+1}$) d_2 by $-d_2$ and d_3 by $-d_3$. Finally, for $i_2\in[\![1,\,q-1]\!]$, we have

$$
\begin{aligned}
P_\Gamma(t,\,u)_{i_1(q+3)+i_2+3} \;=\; & \frac{1}{m}\sum_{r=0}^{m-1}\zeta_m^{i_1 r}\left[\frac{3q-1}{8q^2}\left(2d_1^{(r)}+2(-1)^{i_2}d_4^{(r)}+2\sum_{k=1}^{q-1}(\zeta_{2q}^{ki_2}+\zeta_{2q}^{-ki_2})\delta_k^{(r)}\right)\right.\\
& +(-1)^q\,\frac{q-1}{8q^2}\left(2d_1^{(r)}+2(-1)^{i_2}d_4^{(r)}+2\sum_{k=1}^{q-1}(-1)^k(\zeta_{2q}^{i_2k}+\zeta_{2q}^{-i_2k})\delta_k^{(r)}\right)\\
& \left.+\sum_{l=1}^{q-1}(-1)^l\,\frac{q-1}{4q^2}\left(2d_1^{(r)}+2(-1)^{i_2}d_4^{(r)}+\sum_{k=1}^{q-1}(\zeta_{2q}^{ki_2}+\zeta_{2q}^{-ki_2})\delta_k^{(r)}(\zeta_{2q}^{kl}+\zeta_{2q}^{-kl})\right)\right].
\end{aligned}
$$

• We now make the matrix $A^{(1)}$ explicit: $A^{(1)}$ is a block-matrix with $m\times m$ blocks of size $(q+3)\times(q+3)$.
▷ If $m\geq 5$, then the matrices $A^{(1)}$ and $C_\Gamma:=2I-A^{(1)}-{}^tA^{(1)}$ are defined by

$$
A^{(1)}=\begin{pmatrix}
0 & & I & & B\\
B & \ddots & & \ddots & \\
 & \ddots & \ddots & & I\\
I & & \ddots & \ddots & \\
 & I & & B & 0
\end{pmatrix},\quad
C_\Gamma=\begin{pmatrix}
2I & -B & -I & & & -I & -B\\
-B & 2I & -B & -I & & & -I\\
-I & -B & 2I & -B & -I & & \\
 & -I & -B & \ddots & \ddots & \ddots & \\
 & & -I & \ddots & \ddots & -B & -I\\
-I & & & \ddots & -B & 2I & -B\\
-B & -I & & & -I & -B & 2I
\end{pmatrix},
$$

with

$$
B=\begin{pmatrix}
0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\1&1&1&1&0
\end{pmatrix},\ \text{if } q=2,\ \text{and } B=\left(\begin{array}{ccc|ccccc}
0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\\hline
0&0&0&0&1&0&0&0\\1&0&0&1&0&1&0&0\\0&0&0&0&1&0&\ddots&0\\0&0&0&0&0&\ddots&\ddots&1\\0&1&1&0&0&0&1&0
\end{array}\right)\ \text{if } q\geq 3
$$

▷ If $m=3$, then the matrices $A^{(1)}$ and $C_\Gamma:=2I-A^{(1)}-{}^tA^{(1)}$ are defined by

$$
A^{(1)}=\begin{pmatrix}0&0&B\\B&0&0\\0&B&0\end{pmatrix},\quad C_\Gamma=\begin{pmatrix}0&B&B\\B&0&B\\B&B&0\end{pmatrix},
$$

with

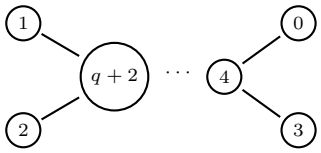
$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \text{ if } q = 2, \text{ and } B = \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \text{ if } q \geq 3.$$

▷ If $m = 1$, then the matrices $A^{(1)}$ and C_Γ are $A^{(1)} = B$ and $C_\Gamma := 2I - A^{(1)} - {}^tA^{(1)} + 2\,Diag(A^{(1)})$, with B defined as in the case $m = 3$.

• For $i_1 \in [\![0,\, m-1]\!]$ and $i_2 \in [\![0,\, q+2]\!]$, let s_{i_1,i_2} be the reflection associated to the $(i_1(q+3)+i_2)-$th root. Then the set $\mathcal{S}$ may be decomposed in p sets where $p \in \{2,\, 3,\, 4,\, 5\}$:
— If $m \geq 3$, then:
▷ If $m \equiv 0 \mod 3$, set $S_l := \{s_{3k+l,i_2} \,/\, (k,\, i_2) \in [\![0,\, \frac{m}{3}-1]\!] \times [\![0,\, q+2]\!]\}$ for $l \in [\![0,\, 2]\!]$. Then $\mathcal{S} = S_0 \sqcup S_1 \sqcup S_2$, $p = 3$, and $C_\Gamma = 3I - \tau_0 - \tau_1 - \tau_2$.
▷ If $m \equiv 1 \mod 3$, set $S_l := \{s_{3k+l,i_2} \,/\, (k,\, i_2) \in [\![0,\, \frac{m-1}{3} - 1]\!] \times [\![0,\, q+2]\!]\}$ for $l \in [\![0,\, 2]\!]$, and $S_3 := \{s_{m-1,i_2} \,/\, i_2 \in [\![0,\, q+2]\!]\}$. Then $\mathcal{S} = S_0 \sqcup S_1 \sqcup S_2 \sqcup S_3$, $p = 4$, and $C_\Gamma = 4I - \tau_0 - \tau_1 - \tau_2 - \tau_3$.
▷ If $m \equiv 2 \mod 3$, set $S_l := \{s_{3k+l,i_2} \,/\, (k,\, i_2) \in [\![0,\, \frac{m-2}{3} - 1]\!] \times [\![0,\, q+2]\!]\}$ for $l \in [\![0,\, 2]\!]$, and $S_3 := \{s_{m-2,i_2} \,/\, i_2 \in [\![0,\, q+2]\!]\}$, $S_4 := \{s_{m-1,i_2} \,/\, i_2 \in [\![0,\, q+2]\!]\}$. Then $\mathcal{S} = S_0 \sqcup S_1 \sqcup S_2 \sqcup S_3 \sqcup S_4$, $p = 5$, and $C_\Gamma = 5I - \tau_0 - \tau_1 - \tau_2 - \tau_3 - \tau_4$.

— If $m = 1$, then:
▷ If $q = 2$, set $S_0 := \{s_{0,0},\, s_{0,1},\, s_{0,2}\, s_{0,3}\}$ and $S_1 := \{s_{0,4}\}$. Then $\mathcal{S} = S_0 \sqcup S_1$, $p = 2$, and $C_\Gamma = 2I - \tau_0 - \tau_1$.
▷ If $q \geq 3$ and q is even, set $S_0 := \{s_{0,0},\, s_{0,1},\, s_{0,2}\}$ and $S_1 := \{s_{0,3}, \ldots,\, s_{0,\,q+1}\}$, $S_2 := \{s_{0,4}, \ldots,\, s_{0,\,q+2}\}$. Then $\mathcal{S} = S_0 \sqcup S_1 \sqcup S_2$, $p = 3$, and $C_\Gamma = 2I - \tau_0 - \tau_1 - \tau_1$.
▷ If $q \geq 3$ and q is odd, set $S_0 := \{s_{0,0},\, s_{0,1},\, s_{0,2}\}$ and $S_1 := \{s_{0,3}, \ldots,\, s_{0,\,q+2}\}$, $S_2 := \{s_{0,4}, \ldots,\, s_{0,\,q+1}\}$. Then $\mathcal{S} = S_0 \sqcup S_1 \sqcup S_2$, $p = 3$, and $C_\Gamma = 2I - \tau_0 - \tau_1 - \tau_1$.

• If $m = 1$, the graph associated to Γ is the following:



If $m \geq 2$, the graph associated to Γ consists in $q+3$ $m-$gones that are linked together.

3.3.2 The BTa subseries — Binary tetrahedral groups

Let $\psi_4,\ \tau,\ \eta,\ \phi_{2m}$ be the elements

$$\psi_4 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_4 & 0 \\ 0 & 0 & \zeta_4^{-1} \end{pmatrix},\ \tau := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix},\ \eta := \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \zeta_8^7 & \zeta_8^7 \\ 0 & \zeta_8^5 & \zeta_8 \end{pmatrix},\ \phi_{2m} := \begin{pmatrix} \zeta_{2m}^{-2} & 0 & 0 \\ 0 & \zeta_{2m} & 0 \\ 0 & 0 & \zeta_{2m} \end{pmatrix}.$$

In this section[6], we assume that $m \equiv 1 \text{ or } 5 \mod 6$, and we consider the subgroup $\Gamma := \langle \psi_4,\ \tau,\ \eta,\ \phi_{2m} \rangle$ of $\mathbf{SL}_3\mathbb{C}$. Note that $\phi_{2m} = \psi_4^2 \phi_m^{-\frac{m-1}{2}}$, so that $\Gamma := \langle \psi_4,\ \tau,\ \eta,\ \phi_m \rangle$. Set $\Gamma_1 := \langle \psi_4,\ \tau,\ \eta \rangle$ and $\Gamma_2 :=$

[6] The other case — the type BTb — is $m \equiv 3 \mod 6$. This group is not a direct product.

$\langle \phi_m \rangle \simeq \mathbb{Z}/m\mathbb{Z}$. Then we have $\Gamma \simeq \Gamma_2 \times \Gamma_1$.
With the notations of the binary tetrahedral subgroup of $\mathbf{SL}_2\mathbb{C}$, ψ_4 (resp. τ, η) represents a^2 (resp. b, c). $\Gamma_1 \simeq \langle a^2,\, b,\, c \rangle$ is the binary tetrahedral subgroup of $\mathbf{SL}_2\mathbb{C}$. Representatives to its 7 conjugacy classes are

$$\{id,\ a^4 = -id,\ b,\ c,\ c^2,\ -c,\ -c^2\},$$

and its character table is the matrix given in Section 2.3.1 (Type E_6 — Binary tetrahedral group).
The natural character of Γ is given by

$$\begin{array}{lcl|lcl} \chi_{i,0} = \chi(\phi_m^i) & = & \zeta_m^{-2i} + 2\zeta_m^i & \chi_{i,4} = \chi(\phi_m^i \eta^2) & = & \zeta_m^{-2i} - \zeta_m^i \\ \chi_{i,1} = \chi(\phi_m^i \psi_4^2) & = & \zeta_m^{-2i} - 2\zeta_m^i & \chi_{i,5} = \chi(\phi_m^i \psi_4^2 \eta) & = & \zeta_m^{-2i} - \zeta_m^i \\ \chi_{i,2} = \chi(\phi_m^i \tau) & = & \zeta_m^{-2i} & \chi_{i,6} = \chi(\phi_m^i \psi_4^2 \eta^2) & = & \zeta_m^{-2i} + \zeta_m^i \\ \chi_{i,3} = \chi(\phi_m^i \eta) & = & \zeta_m^{-2i} + \zeta_m^i & & & \end{array}$$

Finally, we obtain the series $P_\Gamma(t,\, u)$: for $p \in [\![0,\, m-1]\!]$, we get

$$P_\Gamma(t,\, u)_{7p} = \frac{1}{24\, m} \sum_{k=0}^{m-1} \zeta_m^{pk} (f_{k,0} + f_{k,1} + 6\, f_{k,2} + 4\, f_{k,3} + 4\, f_{k,4} + 4\, f_{k,5} + 4\, f_{k,6}),$$

$$P_\Gamma(t,\, u)_{7p+1} = \frac{1}{24\, m} \sum_{k=0}^{m-1} \zeta_m^{pk} (f_{k,0} + f_{k,1} + 6\, f_{k,2} + (4\, f_{k,3} + 4\, f_{k,5})\, j + (4\, f_{k,4} + 4\, f_{k,6})\, j^2),$$

$$P_\Gamma(t,\, u)_{7p+3} = \frac{1}{24\, m} \sum_{k=0}^{m-1} \zeta_m^{pk} (2\, f_{k,0} - 2\, f_{k,1} + 4\, f_{k,3} - 4\, f_{k,4} - 4\, f_{k,5} + 4\, f_{k,6}),$$

$$P_\Gamma(t,\, u)_{7p+4} = \frac{1}{24\, m} \sum_{k=0}^{m-1} \zeta_m^{pk} (2\, f_{k,0} - 2\, f_{k,1} + (4\, f_{k,3} - 4\, f_{k,5})\, j + (-4\, f_{k,4} + 4\, f_{k,6})\, j^2),$$

$$P_\Gamma(t,\, u)_{7p+6} = \frac{1}{24\, m} \sum_{k=0}^{m-1} \zeta_m^{pk} (3\, f_{k,0} + 3\, f_{k,1} - 6\, f_{k,2}),$$

and $P_\Gamma(t,\, u)_{7p+2}$ (resp. $P_\Gamma(t,\, u)_{7p+5}$) is obtained by exchanging j and j^2 in $P_\Gamma(t,\, u)_{7p+1}$ (resp. $P_\Gamma(t,\, u)_{7p+4}$).

3.3.3 The BO subseries — Binary octahedral groups

For $m \in \mathbb{N}$ such that $m \wedge 6 = 1$, let $\psi_8,\ \tau,\ \eta,\ \phi_{2m}$ be the elements

$$\psi_8 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_8 & 0 \\ 0 & 0 & \zeta_8^7 \end{pmatrix},\ \tau := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix},\ \eta := \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \zeta_8^7 & \zeta_8^7 \\ 0 & \zeta_8^5 & \zeta_8 \end{pmatrix},\ \phi_{2m} := \begin{pmatrix} \zeta_{2m}^{-2} & 0 & 0 \\ 0 & \zeta_{2m} & 0 \\ 0 & 0 & \zeta_{2m} \end{pmatrix},$$

and consider the subgroup $\Gamma = \langle \psi_8,\ \tau,\ \eta,\ \phi_{2m} \rangle$ of $\mathbf{SL}_3\mathbb{C}$. Note that $\phi_{2m} = \psi_8^4 \phi_m^{-\frac{m-1}{2}}$, so that $\Gamma := \langle \psi_8,\ \tau,\ \eta,\ \phi_m \rangle$. Set $\Gamma_1 := \langle \psi_8,\ \tau,\ \eta \rangle$ and $\Gamma_2 := \langle \phi_m \rangle \simeq \mathbb{Z}/m\mathbb{Z}$.
Then $\Gamma \simeq \Gamma_2 \times \Gamma_1$.
With the notations of the binary octahedral subgroup of $\mathbf{SL}_2\mathbb{C}$, ψ_8 (resp. τ, η) represents a (resp. b, c). $\Gamma_1 \simeq \langle a,\, b,\, c \rangle$ is the binary octahedral subgroup of $\mathbf{SL}_2\mathbb{C}$. Reperesentatives of its 8 conjugacy classes are

$$\{id,\ a^4 = -id,\ ab,\ b,\ c^2,\ c,\ a,\ a^3\},$$

and its character table is the matrix given in Section 2.3.2 (Type E_7 — Binary octahedral group).
The natural character of Γ is given by

$$\begin{array}{lcl|lcl} \chi_{i,0} = \chi(\phi_m^i) & = & \zeta_m^{-2i} + 2\zeta_m^i & \chi_{i,4} = \chi(\phi_m^i \eta^2) & = & \zeta_m^{-2i} - \zeta_m^i \\ \chi_{i,1} = \chi(\phi_m^i \psi_8^4) & = & \zeta_m^{-2i} - 2\zeta_m^i & \chi_{i,5} = \chi(\phi_m^i \eta) & = & \zeta_m^{-2i} + \zeta_m^i \\ \chi_{i,2} = \chi(\phi_m^i \psi_8 \tau) & = & \zeta_m^{-2i} & \chi_{i,6} = \chi(\phi_m^i \psi_8) & = & \zeta_m^{-2i} + \zeta_m^i \sqrt{2} \\ \chi_{i,3} = \chi(\phi_m^i \tau) & = & \zeta_m^{-2i} & \chi_{i,7} = \chi(\phi_m^i \psi_8^3) & = & \zeta_m^{-2i} - \zeta_m^i \sqrt{2} \end{array}$$

Finally, we obtain the series $P_\Gamma(t,\,u)$: for $p \in [\![0,\,m-1]\!]$, we have

$$P_\Gamma(t,\,u)_{8p} = \frac{1}{48\,m}\sum_{k=0}^{m-1}\zeta_m^{pk}(f_{k,0}+f_{k,1}+12\,f_{k,2}+6\,f_{k,3}+8\,f_{k,4}+8\,f_{k,5}+6\,f_{k,6}+6\,f_{k,7}),$$

$$P_\Gamma(t,\,u)_{8p+2} = \frac{1}{48\,m}\sum_{k=0}^{m-1}\zeta_m^{pk}(2\,f_{k,0}+2\,f_{k,1}+12\,f_{k,3}-8\,f_{k,4}-8\,f_{k,5}),$$

$$P_\Gamma(t,\,u)_{8p+3} = \frac{1}{48\,m}\sum_{k=0}^{m-1}\zeta_m^{pk}(2\,f_{k,0}-2\,f_{k,1}-8\,f_{k,4}+8\,f_{k,5}+6\sqrt{2}f_{k,6}-6\sqrt{2}f_{k,7}),$$

$$P_\Gamma(t,\,u)_{8p+5} = \frac{1}{48\,m}\sum_{k=0}^{m-1}\zeta_m^{pk}(3\,f_{k,0}+3\,f_{k,1}-12\,f_{k,2}-6\,f_{k,3}+6\,f_{k,6}+6\,f_{k,7}),$$

$$P_\Gamma(t,\,u)_{8p+7} = \frac{1}{48\,m}\sum_{k=0}^{m-1}\zeta_m^{pk}(4\,f_{k,0}-4\,f_{k,1}+8\,f_{k,4}-8\,f_{k,5}),$$

and $P_\Gamma(t,\,u)_{8p+1}$ (resp. $P_\Gamma(t,\,u)_{8p+4}$, $P_\Gamma(t,\,u)_{8p+6}$) is obtained by replacing $f_{k,2},\ f_{k,6},\ f_{k,7}$ by their opposite in $P_\Gamma(t,\,u)_{8p}$ (resp. $P_\Gamma(t,\,u)_{8p+3}$, $P_\Gamma(t,\,u)_{8p+5}$).

3.3.4 The BI subseries — Binary icosahedral groups

For $m \in \mathbb{N}$ such that $m \wedge 30 = 1$, let $\mu,\ \tau,\ \eta$ be the elements

$$\mu := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\zeta_5^3 & 0 \\ 0 & 0 & -\zeta_5^2 \end{pmatrix},\ \tau := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},\ \eta := \frac{1}{\zeta_5^2-\zeta_5^{-2}}\begin{pmatrix} \zeta_5^2-\zeta_5^{-2} & 0 & 0 \\ 0 & \zeta_5-\zeta_5^{-1} & 1 \\ 0 & 1 & -\zeta_5-\zeta_5^{-1} \end{pmatrix},$$

and consider the subgroup $\Gamma = \langle \mu,\ \tau,\ \eta,\ \phi_{2m}\rangle$ of $\mathbf{SL}_3\mathbb{C}$.
Note that $\phi_{2m} = \eta^2\phi_m^{-\frac{m-1}{2}}$, so that $\Gamma := \langle \mu,\ \tau,\ \eta,\ \phi_m\rangle$. Set $\Gamma_1 := \langle \mu,\ \tau,\ \eta\rangle$ and $\Gamma_2 := \langle \phi_m\rangle \simeq \mathbb{Z}/m\mathbb{Z}$. Then $\Gamma \simeq \Gamma_2 \times \Gamma_1$.
With the notations of the binary icosahedral subgroup of $\mathbf{SL}_2\mathbb{C}$, μ (resp. τ, η) represents a (resp. b, c). $\Gamma_1 \simeq \langle a,\,b,\,c\rangle$ is the binary icosahedral subgroup of $\mathbf{SL}_2\mathbb{C}$. Representatives of its 9 conjugacy classes are

$$\{id,\ b^2=-id,\ a,\ a^2,\ a^3,\ a^4,\ abc,\ (abc)^2,\ b\},$$

and its character table is given in Section 2.3.3 (Type E_8 — Binary icosahedral group).
The natural character of Γ is given by

$$\begin{array}{lllll|lllll}
\chi_{i,0} & = & \chi(\phi_m^i) & = & \zeta_m^{-2i}+2\zeta_m^i & \chi_{i,5} & = & \chi(\phi_m^i\mu^4) & = & \zeta_m^{-2i}-\frac{1+\sqrt{5}}{2}\,\zeta_m^i \\
\chi_{i,1} & = & \chi(\phi_m^i\tau^2) & = & \zeta_m^{-2i}-2\zeta_m^i & \chi_{i,6} & = & \chi(\phi_m^i\mu\tau\eta) & = & \zeta_m^{-2i}+\zeta_m^i \\
\chi_{i,2} & = & \chi(\phi_m^i\mu) & = & \zeta_m^{-2i}+\frac{1+\sqrt{5}}{2}\,\zeta_m^i & \chi_{i,7} & = & \chi(\phi_m^i(\mu\tau\eta)^2) & = & \zeta_m^{-2i}-\zeta_m^i \\
\chi_{i,3} & = & \chi(\phi_m^i\mu^2) & = & \zeta_m^{-2i}+\frac{-1+\sqrt{5}}{2}\,\zeta_m^i & \chi_{i,8} & = & \chi(\phi_m^i\tau) & = & \zeta_m^{-2i} \\
\chi_{i,4} & = & \chi(\phi_m^i\mu^3) & = & \zeta_m^{-2i}+\frac{1-\sqrt{5}}{2}\,\zeta_m^i & & & & &
\end{array}$$

Finally, we obtain the series $P_\Gamma(t,\,u)$: for $p \in [\![0,\,m-1]\!]$, we have

$$P_\Gamma(t,\,u)_{9p} = \frac{1}{120\,m}\sum_{k=0}^{m-1}\zeta_m^{pk}(f_{k,0}+f_{k,1}+12\,f_{k,2}+12\,f_{k,3}+12\,f_{k,4}+12\,f_{k,5}+20\,f_{k,6}+20\,f_{k,7}+30\,f_{k,8}),$$

$$\begin{aligned} P_\Gamma(t,\,u)_{9p+1} = \frac{1}{120\,m}\sum_{k=0}^{m-1}\zeta_m^{pk}&(2\,f_{k,0}-2\,f_{k,1}+(6-6\sqrt{5})f_{k,2}+(-6-6\sqrt{5})f_{k,3}+(6+6\sqrt{5})f_{k,4} \\ &+(-6+6\sqrt{5})f_{k,5}+20\,f_{k,6}-20\,f_{k,7}), \end{aligned}$$

$$P_\Gamma(t,\,u)_{9p+3} = \frac{1}{120\,m}\sum_{k=0}^{m-1}\zeta_m^{pk}(3\,f_{k,0}+3\,f_{k,1}+(6+6\sqrt{5})f_{k,2}+(6-6\sqrt{5})f_{k,3}+(6-6\sqrt{5})f_{k,4} +(6+6\sqrt{5})f_{k,5}-30\,f_{k,8}),$$

$$P_\Gamma(t,\,u)_{9p+5} = \frac{1}{120\,m}\sum_{k=0}^{m-1}\zeta_m^{pk}(4\,f_{k,0}+4\,f_{k,1}-12\,f_{k,2}-12\,f_{k,3}-12\,f_{k,4}-12\,f_{k,5}+20\,f_{k,6}+20\,f_{k,7})$$

$$P_\Gamma(t,\,u)_{9p+7} = \frac{1}{120\,m}\sum_{k=0}^{m-1}\zeta_m^{pk}(5\,f_{k,0}+5\,f_{k,1}-20\,f_{k,6}-20\,f_{k,7}+30\,f_{k,8})$$

$$P_\Gamma(t,\,u)_{9p+8} = \frac{1}{120\,m}\sum_{k=0}^{m-1}\zeta_m^{pk}(6\,f_{k,0}-6\,f_{k,1}-12\,f_{k,2}+12\,f_{k,3}-12\,f_{k,4}+12\,f_{k,5}),$$

and $P_\Gamma(t,\,u)_{9p+4}$ (resp. $P_\Gamma(t,\,u)_{9p+2}$) is obtained by replacing $6\sqrt{5}$ by its opposite in $P_\Gamma(t,\,u)_{9p+3}$ (resp. $P_\Gamma(t,\,u)_{9p+1}$), and $P_\Gamma(t,\,u)_{9p+6}$ is obtained by replacing $f_{k,1},\ f_{k,2},\ f_{k,4},\ f_{k,6}$ by their opposite in $P_\Gamma(t,\,u)_{9p+5}$.

3.3.5 Decomposition of C_Γ for the subseries BTa, BO, BI

• For the subseries BTa (resp. BO, BI), we set $n=7$ (resp. $n=8$, $n=9$). We now make the matrix $A^{(1)}$ explicit: $A^{(1)}$ is a block-matrix with $m\times m$ blocks of size $n\times n$.
▷ If $m\geq 5$, then the matrices $A^{(1)}$ and $C_\Gamma := 2I - A^{(1)} - {}^tA^{(1)}$ are defined by

$$A^{(1)}=\begin{pmatrix} 0 & & I & & B_n\\ B_n & \ddots & & \ddots & \\ & \ddots & \ddots & & I\\ I & & \ddots & \ddots & \\ & I & & B_n & 0\end{pmatrix},\quad C_\Gamma=\begin{pmatrix} 2I & -B_n & -I & & & -I & -B_n\\ -B_n & 2I & -B_n & -I & & & -I\\ -I & -B_n & 2I & -B_n & -I & & \\ & -I & -B_n & \ddots & \ddots & \ddots & \\ & & -I & \ddots & \ddots & -B_n & -I\\ & & & \ddots & -B_n & 2I & -B_n\\ -B_n & -I & & & -I & -B_n & 2I\end{pmatrix},$$

with

$$B_7=\begin{pmatrix} 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 1&0&0&0&0&0&1\\ 0&1&0&0&0&0&1\\ 0&0&1&0&0&0&1\\ 0&0&0&1&1&1&0\end{pmatrix},\ B_8=\begin{pmatrix} 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&1&0\\ 0&0&0&1&0&0&0&1\\ 0&0&0&0&1&0&0&1\\ 0&0&1&0&0&1&1&0\end{pmatrix},\ B_9=\begin{pmatrix} 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 1&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&1\\ 0&0&0&0&1&1&0&1&0\end{pmatrix}.$$

▷ If $m=1$, then the matrix $A^{(1)}$ is $A^{(1)} := B_n + I$, and C_Γ is defined by $C_\Gamma := 2I - A^{(1)} - {}^tA^{(1)} + 2\,Diag(A^{(1)})$, i.e. $C_\Gamma = 2I - 2B_n$.

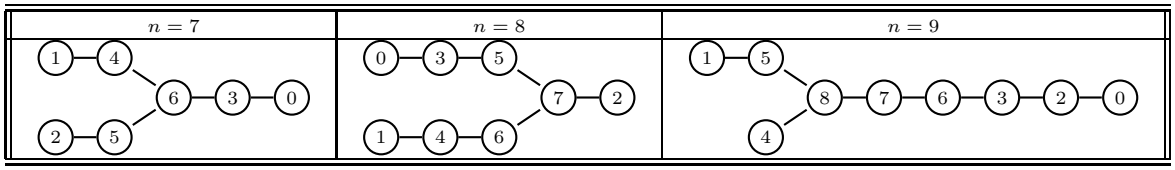
• The decomposition of C_Γ is the following: for $i_1\in[\![0,\,m-1]\!]$ and $i_2\in[\![0,\,n-1]\!]$, let s_{i_1,i_2} be the reflection associated to the $(ni_1+i_2)-$th root. Then the set $\mathcal{S}$ may be decomposed in p sets where $p\in\{2,\,4,\,5\}$.

— If $m\geq 5$, then:
▷ If $m\equiv 1 \mod 3$, set $S_l := \{s_{3k+l,i_2}\ /\ (k,\,i_2)\in[\![0,\,\frac{m-1}{3}-1]\!]\times[\![0,\,n-1]\!]\}$ for $l\in[\![0,\,2]\!]$, and

$S_3 := \{s_{m-1,i_2} \ / \ i_2 \in [\![0,\, n-1]\!]\}$. Then $S = S_0 \sqcup S_1 \sqcup S_2 \sqcup S_3$, $p = 4$, and $C_\Gamma = 4I - \tau_0 - \tau_1 - \tau_2 - \tau_3$.
▷ If $m \equiv 2 \mod 3$, set $S_l := \{s_{3k+l,i_2} \ / \ (k,\, i_2) \in [\![0,\, \frac{m-2}{3} - 1]\!] \times [\![0,\, n-1]\!]\}$ for $l \in [\![0,\, 2]\!]$, and $S_3 := \{s_{m-2,i_2} \ / \ i_2 \in [\![0,\, n-1]\!]\}$, $S_4 := \{s_{m-1,i_2} \ / \ i_2 \in [\![0,\, n-1]\!]\}$. Then $S = S_0 \sqcup S_1 \sqcup S_2 \sqcup S_3 \sqcup S_4$, $p = 5$, and $C_\Gamma = 5I - \tau_0 - \tau_1 - \tau_2 - \tau_3 - \tau_4$.

— If $m = 1$, then $S = S_0 \sqcup S_1$, $p = 2$, and $C_\Gamma = 2I - \tau_0 - \tau_1$, with
▷ If $n = 7$, $S_0 := \{s_{0,0},\, s_{0,1},\, s_{0,2},\, s_{0,6}\}$ and $S_1 := \{s_{0,3},\, s_{0,4},\, s_{0,5}\}$.
▷ If $n = 8$, $S_0 := \{s_{0,0},\, s_{0,1},\, s_{0,2},\, s_{0,5},\, s_{0,6}\}$ and $S_1 := \{s_{0,3},\, s_{0,4},\, s_{0,7}\}$.
▷ If $n = 9$, $S_0 := \{s_{0,0},\, s_{0,3},\, s_{0,4},\, s_{0,5},\, s_{0,7}\}$ and $S_1 := \{s_{0,1},\, s_{0,2},\, s_{0,6},\, s_{0,8}\}$.

• If $m = 1$, the graph associated to Γ is the following:



If $m \geq 5$, the graph associated to Γ is a graph of type BTa (resp. BO, BI), $m = 1$, such that every vertex is a $m-$gone.

3.4 The C series

Let $H \simeq \mathbb{Z}/j_1\mathbb{Z} \times \mathbb{Z}/j_2\mathbb{Z}$ be a group of the series A, with eventually $j_1 = 1$ or $j_2 = 1$, and consider the matrix $T := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, that is the matrix of the permutation $(1,\, 2,\, 3)$ of $\mathfrak{S}_3$. In this section, we study $\Gamma := \langle H,\, T\rangle$, the finite subgroup of $\mathbf{SL}_3\mathbb{C}$ generated by H and T. The subgroup N of Γ which consists of all the diagonal matrices of Γ is a normal subgroup of Γ. By using the Bezout theorem,

$$N = \left\{ g_{k_1,k_2} := \begin{pmatrix} \zeta_m^{k_1} & 0 & 0 \\ 0 & \zeta_m^{k_2} & 0 \\ 0 & 0 & \zeta_m^{-k_1-k_2} \end{pmatrix} \ / \ (k_1,\, k_2) \in [\![0,\, m-1]\!]^2 \right\}. \tag{7}$$

Moreover, we have $N \cap \langle T\rangle = \{id\}$ and $|N\langle T\rangle| = \frac{|N||\langle T\rangle|}{|N\cap\langle T\rangle|} = 3m^2 = |\Gamma|$. So, Γ is the semi-direct product

$$\Gamma \simeq N \rtimes \langle T\rangle \simeq (\mathbb{Z}/m\mathbb{Z})^2 \rtimes \langle T\rangle.$$

We will obtain all the irreducible characters of Γ by induction; we distinguish two cases corresponding to the two following subsections.

3.4.1 Series C — m non divisible by 3

• Set $n' := \frac{m^2-1}{3}$, so that $|N| = 3n' + 1$ and $|G| = 3m^2 = 3(3n' + 1)$. The conjugacy classes of Γ are:

Class	id	T	T^{-1}	$g \in N\backslash\{id\}$(n' classes)
Cardinality	1	m^2	m^2	3

For each element $g_{k_1,k_2} \in N\backslash\{id\}$, the conjugacy class of g is the set $\{g_{k_1,k_2},\, g_{k_2,-k_1-k_2},\, g_{-k_1-k_2,k_1}\}$. In order to obtain a transversal of $N\backslash\{id\}$, i.e. a set containing exactly one representant of each conjugacy class of $N\backslash\{id\}$, we represent the elements of $N\backslash\{id\}$ by points $(k_1,\, k_2)$ of $[\![0,\, m-1]\!]^2$.
So, we search a transversal for the set of elements of the form $(k_1,\, k_2)$, $(k_2,\, -k_1, -k_2 \mod m)$ and $(-k_1, -k_2 \mod m,\, k_1)$, with $(k_1,\, k_2) \in [\![0,\, m-1]\!]^2$.
A solution is the following: for a given conjugacy class, its three elements are on the edges of a triangle (see

Figure 2), with exactly one element on each edge of the triangle. Therefore we may take as transversal the set of all points that belong to the vertical edges minus the nearest point of the diagonal. More precisely, a transversal for $N\backslash\{id\}$ is the set E_{cc} defined by

$$\begin{aligned} & \{(0,\, k_2) \;/\; k_2 \in [\![1,\, m-1]\!]\} \\ \sqcup\; & \{(k_1,\, k_2) \;/\; k_1 \in [\![1,\, \lfloor \tfrac{m}{3} \rfloor]\!],\; k_2 \in [\![k_1,\, m-1-2k_1]\!]\} \\ \sqcup\; & \{(k_1,\, k_2) \;/\; k_1 \in [\![m-\lfloor \tfrac{m}{3} \rfloor,\, m-1]\!],\; k_2 \in [\![2(m-k_1)+1,\, k_1]\!]\}. \end{aligned}$$

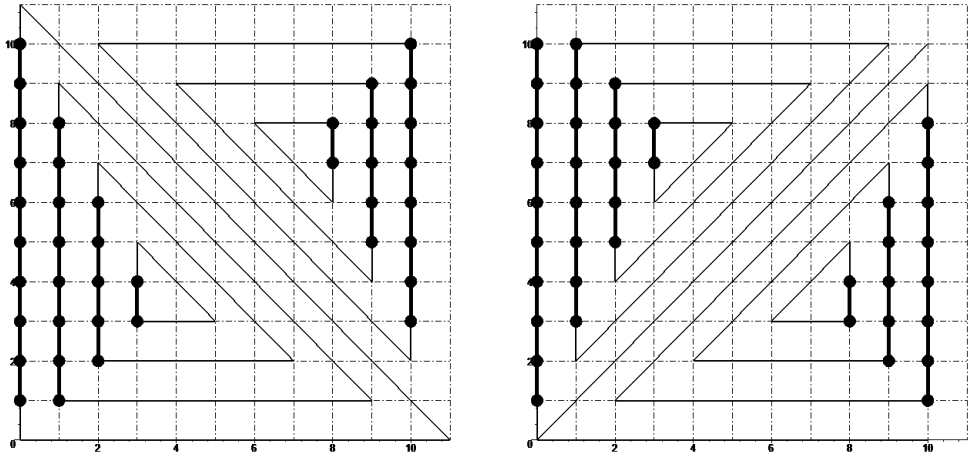


Figure 2: Conjugacy classes and irreducible characters for $m = 11$.

We choose the usual lexicographic order on E_{cc}, so that we can number its elements: $E_{cc} = \{c_1,\, c_2, \ldots,\, c_{\frac{m^2-1}{3}}\}$, with $c_i = (c_i^{(1)},\, c_i^{(2)}) \in [\![0,\, m-1]\!]^2$.

• The group Γ is generated by $R := g_{1,0}$ and T, which verify the relations $R^m = (RT)^3 = T^3 = id$. As m is not divisible by 3, the irreducible characters of degree 1 are $\chi^{0,l} : R \mapsto 1, T \mapsto j^l$ for $l \in [\![0,\, 2]\!]$. We have $[G : N] = 3$ with N abelian, so the possible degrees of the irreducible characters are 1, 2, 3. The irreducible characters $\chi_{l_1,\, l_2}$ induced by the irreducible characters of N are given by

Class	$[id]$	$[T]$	$[T^{-1}]$	$[g],\; g \in N\backslash\{id\}$
Value	3	0	0	$\zeta_m^{k_1 l_1 + k_2 l_2} + \zeta_m^{(-k_1-k_2) l_1 + k_1 l_2} + \zeta_m^{k_2 l_1 + (-k_1-k_2) l_2}$

The characters $\chi_{l_1,\, l_2}$ with $(l_1,\, l_2) \neq (0,\, 0)$ are represented by points $(l_1,\, l_2)$ of $[\![0,\, m-1]\!]^2$. The points that are associated to the same character are on a triangle or on a "trident ⩘", with exactly one point on each edge (see Figure 2). So the set of irreducible characters $\chi_{l_1,\, l_2}$ with $(l_1,\, l_2) \neq (0,\, 0)$ is obtained by taking the following set E_{ic} of indexes:

$$\begin{aligned} & \{(0,\, k_2) \;/\; k_2 \in [\![1,\, m-1]\!]\} \\ \sqcup\; & \{(k_1,\, k_2) \;/\; k_1 \in [\![1,\, \lfloor \tfrac{m}{3} \rfloor]\!],\; k_2 \in [\![2k_1+1,\, m-k_1]\!]\} \\ \sqcup\; & \{(k_1,\, k_2) \;/\; k_1 \in [\![m-\lfloor \tfrac{m}{3} \rfloor,\, m-1]\!],\; k_2 \in [\![m-k_1,\, 2k_1-m-1]\!]\}. \end{aligned}$$

We choose the usual lexicographic order on E_{ic}, so that we can number its elements: $E_{ic} = \{d_1,\, d_2, \ldots,\, d_{\frac{m^2-1}{3}}\}$,

with $d_i = (d_i^{(1)},\, d_i^{(2)}) \in [\![0,\, m-1]\!]^2$, and the character table T_Γ is

$$T_\Gamma = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & j & j^2 & 1 & \cdots & 1 \\ 1 & j^2 & j & 1 & \cdots & 1 \\ \hline 3 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & & C & \\ 3 & 0 & 0 & & & \end{array}\right),$$

where the general term of $C \in \mathbf{M}_{\frac{m^2-1}{3}}\mathbb{C}$ is

$$c_{i,\,j} := \zeta_m^{c_j^{(1)} d_i^{(1)} + c_j^{(2)} d_i^{(2)}} + \zeta_m^{(-c_j^{(1)} - c_j^{(2)}) d_i^{(1)} + c_j^{(1)} d_i^{(2)}} + \zeta_m^{c_j^{(2)} d_i^{(1)} + (-c_j^{(1)} - c_j^{(2)}) d_i^{(2)}},\ (i,\, j) \in [\![1,\, \frac{m^2-1}{3}]\!]^2.$$

• The values of the natural character χ of Γ are

Class	$[id]$	$[T]$	$[T^{-1}]$	$[g],\ g \in N\backslash\{id\}$
Value	3	0	0	$\zeta_m^{k_1} + \zeta_m^{k_2} + \zeta_m^{-k_1-k_2}$

Therefore the diagonal matrix $\Delta(t,\, u)$ is $Diag\left(\varepsilon_1,\, \varepsilon_2,\, \varepsilon_3,\, \widetilde{\Delta(t,\, u)}\right)$, with $\varepsilon_1 := f(3,\, 3)$, $\varepsilon_2 = \varepsilon_3 := f(0,\, 0)$, and the general term of $\widetilde{\Delta(t,\, u)} \in \mathbf{M}_{\frac{m^2-1}{3}}\mathbb{C}$ is

$$\gamma_j := f\left(\zeta_m^{-c_j^{(1)}} + \zeta_m^{-c_j^{(2)}} + \zeta_m^{c_j^{(1)} + c_j^{(2)}},\, \zeta_m^{c_j^{(1)}} + \zeta_m^{c_j^{(2)}} + \zeta_m^{-c_j^{(1)} - c_j^{(2)}}\right),\ j \in [\![1,\, \frac{m^2-1}{3}]\!].$$

Then, by setting

$$\Sigma := \sum_{p=1}^{\frac{m^2-1}{3}} \gamma_p,\ \Sigma_i := \sum_{j=1}^{\frac{m^2-1}{3}} \gamma_j \left(\zeta_m^{c_j^{(1)} d_i^{(1)} + c_j^{(2)} d_i^{(2)}} + \zeta_m^{(-c_j^{(1)} - c_j^{(2)}) d_i^{(1)} + c_j^{(1)} d_i^{(2)}} + \zeta_m^{c_j^{(2)} d_i^{(1)} + (-c_j^{(1)} - c_j^{(2)}) d_i^{(2)}}\right),$$

we obtain the formula for $P_\Gamma(t,\, u)$:

$$\begin{aligned} P_\Gamma(t,\, u)_0 &= \frac{5(m+1)(m-1)+1}{9m^4}(3\Sigma + \varepsilon_1 + \varepsilon_2 + \varepsilon_3) - \frac{2(m+1)(m-1)}{9m^2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3) \\ &\quad + \frac{2(m+1)(m-1)}{3m^4} \sum_{q=1}^{\frac{m^2-1}{3}} \left(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \sum_{p=1}^{\frac{m^2-1}{3}} \gamma_p c_{p,\,q}\right), \\ P_\Gamma(t,\, u)_1 &= \frac{5(m+1)(m-1)+1}{9m^4}(3\Sigma + \varepsilon_1 + j\varepsilon_2 + j^2\varepsilon_3) - \frac{2(m+1)(m-1)}{9m^2}(\varepsilon_1 - j\varepsilon_2 - j^2\varepsilon_3) \\ &\quad + \frac{2(m+1)(m-1)}{3m^4} \sum_{q=1}^{\frac{m^2-1}{3}} \left(\varepsilon_1 + j\varepsilon_2 + j^2\varepsilon_3 + \sum_{p=1}^{\frac{m^2-1}{3}} \gamma_p c_{p,\,q}\right), \end{aligned}$$

and $P_\Gamma(t,\, u)_2$ is obtained by exchanging j and j^2 in $P_\Gamma(t,\, u)_1$, and for $i \in [\![1,\, \frac{m^2-1}{3}]\!]$,

$$\begin{aligned} P_\Gamma(t,\, u)_{i+2} &= \frac{5(m+1)(m-1)+1}{9m^4}(3\Sigma_i + 3\varepsilon_1) - \frac{6(m+1)(m-1)}{9m^2}\varepsilon_1 \\ &\quad + \frac{2(m+1)(m-1)}{3m^4} \sum_{q=1}^{\frac{m^2-1}{3}} \left(3\varepsilon_1 + \sum_{p=1}^{\frac{m^2-1}{3}} \gamma_p c_{i,\,p} c_{p,\,q}\right). \end{aligned}$$

3.4.2 Series C — m divisible by 3

• Set $n' := \frac{m^2}{3}$, so that $|N| = 3n'$ and $|G| = 3m^2 = 3(3n')$. Set $a := R^{\frac{m}{3}} = Diag(j,\, j,\, j)$. The conjugacy classes of Γ are:

Class	id	a	a^2	T	T^{-1}	RT	RT^{-1}	R^2T	R^2T^{-1}	$g \in N\backslash\{id,\, a,\, a^2\}(n'-1$ classes)
Cardinality	1	1	1	$\frac{m^2}{3}$	$\frac{m^2}{3}$	$\frac{m^2}{3}$	$\frac{m^2}{3}$	$\frac{m^2}{3}$	$\frac{m^2}{3}$	3

For each element $g_{k_1,k_2} \in N\backslash\{id,\, a,\, a^2\}$, the conjugacy class of g is the set $\{g_{k_1,k_2},\, g_{k_2,-k_1-k_2},\, g_{-k_1-k_2,k_1}\}$. Its three elements are on the edges of a triangle, with exactly one element of each edge of the triangle. So, a transversal of $N\backslash\{id,\, a,\, a^2\}$ has the same form as in the case where 3 does not divide m, i.e. a transversal for $N\backslash\{id,\, a,\, a^2\}$ is the set E_{cc} defined by

$$\begin{aligned} & \{(0,\, k_2)\ /\ k_2 \in [\![1,\, m-1]\!]\} \\ \sqcup\ & \{(k_1,\, k_2)\ /\ k_1 \in [\![1,\, \tfrac{m}{3}-1]\!],\ k_2 \in [\![k_1,\, m-1-2k_1]\!]\} \\ \sqcup\ & \{(k_1,\, k_2)\ /\ k_1 \in [\![m-\tfrac{m}{3}+1,\, m-1]\!],\ k_2 \in [\![2(m-k_1)+1,\, k_1]\!]\}. \end{aligned}$$

• As m is divisible by 3, the irreducible characters of degree 1 of Γ are, for $(k,\, l) \in [\![0,\, 2]\!]^2$, the nine elements

$$\begin{aligned} \chi^{k,l} : R &\mapsto j^k \\ T &\mapsto j^l. \end{aligned}$$

As for the case where m is divisible by 3, the set of irreducible characters $\chi_{l_1,\,l_2}$ with $(l_1,\, l_2) \neq (0,\, 0)$ is obtained by taking the following set E_{ic} of indexes:

$$\begin{aligned} & \{(0,\, k_2)\ /\ k_2 \in [\![1,\, m-1]\!]\} \\ \sqcup\ & \{(k_1,\, k_2)\ /\ k_1 \in [\![1,\, \tfrac{m}{3}-1]\!],\ k_2 \in [\![2k_1+1,\, m-k_1]\!]\} \\ \sqcup\ & \{(k_1,\, k_2)\ /\ k_1 \in [\![m-\tfrac{m}{3}+1,\, m-1]\!],\ k_2 \in [\![m-k_1,\, 2k_1-m-1]\!]\}. \end{aligned}$$

We choose the usual lexicographic order on E_{cc} and E_{ic}, so that we can number its elements:

$$E_{cc} = \{c_1,\, c_2,\, \dots,\, c_{\frac{m^2}{3}-1}\},\quad E_{ic} = \{d_1,\, d_2,\, \dots,\, d_{\frac{m^2}{3}-1}\},$$

with $c_j = (c_j^{(1)},\, c_j^{(2)}) \in [\![0,\, m-1]\!]^2$, and $d_i = (d_i^{(1)},\, d_i^{(2)}) \in [\![0,\, m-1]\!]^2$, and we deduce the character table T_Γ of Γ:

$$T_\Gamma = \left(\begin{array}{ccc|ccc|ccc|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
1 & 1 & 1 & j & j^2 & j & j^2 & j & j^2 & 1 \\
1 & 1 & 1 & j^2 & j & j^2 & j & j^2 & j & \\
\hline
1 & 1 & 1 & 1 & 1 & j & j & j^2 & j^2 & \\
1 & 1 & 1 & j & j^2 & j^2 & 1 & 1 & j & j^{c_i^{(1)}-c_i^{(2)}} \\
1 & 1 & 1 & j^2 & j & 1 & j^2 & j & 1 & \\
\hline
1 & 1 & 1 & 1 & 1 & j^2 & j^2 & j & j & \\
1 & 1 & 1 & j & j^2 & 1 & j & j^2 & 1 & j^{2(c_i^{(1)}-c_i^{(2)})} \\
1 & 1 & 1 & j^2 & j & j & 1 & 1 & j^2 & \\
\hline
3 & J_1^{(1)} & J_1^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & C \\
3 & J_{\frac{m^2}{3}-1}^{(1)} & J_{\frac{m^2}{3}-1}^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right),$$

where $C \in \mathbf{M}_{\frac{m^2}{3}-1}\mathbb{C}$ is a block-matrix with general term

$$c_{i,\,j} := \zeta_m^{c_j^{(1)}d_i^{(1)}+c_j^{(2)}d_i^{(2)}} + \zeta_m^{(-c_j^{(1)}-c_j^{(2)})d_i^{(1)}+c_j^{(1)}d_i^{(2)}} + \zeta_m^{c_j^{(2)}d_i^{(1)}+(-c_j^{(1)}-c_j^{(2)})d_i^{(2)}},\ (i,\, j) \in [\![1,\, \frac{m^2}{3}-1]\!]^2,$$

$J_i^{(1)} := j^{d_i^{(1)}+d_i^{(2)}} + j^{-2d_i^{(1)}+d_i^{(2)}} + j^{d_i^{(1)}-2d_i^{(2)}}$, $J_i^{(2)} := j^{2d_i^{(1)}+2d_i^{(2)}} + j^{-d_i^{(1)}+2d_i^{(2)}} + j^{2d_i^{(1)}-d_i^{(2)}}$, $i \in [\![1,\, \frac{m^2}{3}-1]\!]^2$.

• The values of the natural character χ of Γ are

Class	id	a	a^2	T	T^{-1}	RT	RT^{-1}	R^2T	R^2T^{-1}	$g \in N\backslash\{id,\, a,\, a^2\}$
Value	3	$3j$	$3j^2$	0	0	0	0	0	0	$\zeta_m^{k_1}+\zeta_m^{k_2}+\zeta_m^{-k_1-k_2}$

Therefore the diagonal matrix $\Delta(t,\, u)$ is $Diag(\widetilde{\widetilde{\Delta(t,\, u)}},\, \widetilde{\Delta(t,\, u)})$, with

$$\widetilde{\widetilde{\Delta(t,\, u)}} = Diag(\beta_1,\, \beta_2,\, \beta_3,\, \underbrace{\beta_4, \ldots,\, \beta_4}_{6 \text{ terms}}) = Diag\left(f(3,\, 3),\, f(3j^2,\, 3j),\, f(3j,\, 3j^2),\, \underbrace{f(0,\, 0), \ldots,\, f(0,\, 0)}_{6 \text{ terms}}\right),$$

and the general term of $\widetilde{\Delta(t,\, u)} \in \mathbf{M}_{\frac{m^2}{3}-1}\mathbb{C}$ is

$$\gamma_j := f\left(\zeta_m^{-c_j^{(1)}} + \zeta_m^{-c_j^{(2)}} + \zeta_m^{c_j^{(1)}+c_j^{(2)}},\, \zeta_m^{c_j^{(1)}} + \zeta_m^{c_j^{(2)}} + \zeta_m^{-c_j^{(1)}-c_j^{(2)}}\right),\ j \in [\![1,\, \frac{m^2}{3}-1]\!].$$

For $(i,\, r) \in \{1,\, 2\} \times \{1,\, 2,\, 3\}$, and $(s,\, q) \in \{1,\, 2\} \times [\![1,\, \frac{m^2}{3}-1]\!]$, let us define

$$\Sigma^{(r)} := \sum_{p=1}^{\frac{m^2}{3}-1} \gamma_p j^{(r-1)(c_p^{(1)}-c_p^{(1)})},\ \Phi^{(i,r)} := \sum_{p=1}^{\frac{m^2}{3}-1} J_p^{(i)} \gamma_p j^{(r-1)(c_p^{(1)}-c_p^{(1)})},$$

$$\Sigma_q^{(s)} := \sum_{p=1}^{\frac{m^2}{3}-1} J_p^{(s)} \gamma_p \left(\zeta_m^{c_p^{(1)}d_q^{(1)}+c_p^{(2)}d_q^{(2)}} + \zeta_m^{(-c_p^{(1)}-c_p^{(2)})d_q^{(1)}+c_p^{(1)}d_q^{(2)}} + \zeta_m^{c_p^{(2)}d_q^{(1)}+(-c_p^{(1)}-c_p^{(2)})d_q^{(2)}}\right),$$

$$\xi := (\underbrace{-1,\, -1,\, 2}_{1},\, \underbrace{-1,\, -1,\, 2}_{2},\, \underbrace{-1,\, -1,\, 2}_{3}, \ldots,\, \underbrace{-1,\, -1,\, 2}_{\frac{\frac{m}{3}(\frac{m}{3}+1)}{2}-1},\, -1,\, -1,\, \underbrace{-1,\, -1,\, 2}_{},\, \underbrace{-1,\, -1,\, 2}_{}, \ldots,\, \underbrace{-1,\, -1,\, 2}_{}).$$

Then, we may give the expression of the series $P_\Gamma(t,\, u)$:

$$\begin{aligned}
P_\Gamma(t,\, u)_0 &= \frac{3(5(\frac{m}{3})^2-1)+1}{3m^4}(3\Sigma^{(1)} + \beta_1 + \beta_2 + \beta_3 + 6\beta_4) + \frac{6(\frac{m}{3})^2-2}{3m^4}(\Phi^{(11)} + \Phi^{(21)} + 2\beta_1 + 2\beta_2 + 2\beta_3 + 12\beta_4) \\
&\quad + \frac{3(\frac{m}{3})^2-1}{m^4} \sum_{q=1}^{\frac{m^2}{3}-1} \xi_q \left(\beta_1 + \beta_2 + \beta_3 + 3j^{c_q^{(1)}-c_q^{(2)}}\beta_4 + 3j^{2(c_q^{(1)}-c_q^{(2)})}\beta_4 + \sum_{p=1}^{\frac{m^2}{3}-1} \gamma_p c_{p,q}\right), \\
P_\Gamma(t,\, u)_1 &= \frac{3(5(\frac{m}{3})^2-1)+1}{3m^4}(3\Sigma^{(1)} + \beta_1 + \beta_2 + \beta_3 - 3\beta_4) + \frac{6(\frac{m}{3})^2-2}{3m^4}(\Phi^{(11)} + \Phi^{(21)} + 2\beta_1 + 2\beta_2 + 2\beta_3 - 6\beta_4) \\
&\quad + \frac{3(\frac{m}{3})^2-1}{m^4} \sum_{q=1}^{\frac{m^2}{3}-1} \xi_q \left(\beta_1 + \beta_2 + \beta_3 + (2j+j^2)j^{c_q^{(1)}-c_q^{(2)}}\beta_4 + (2j^2+j)j^{2(c_q^{(1)}-c_q^{(2)})}\beta_4 + \sum_{p=1}^{\frac{m^2}{3}-1} \gamma_p c_{p,q}\right), \\
P_\Gamma(t,\, u)_3 &= \frac{3(5(\frac{m}{3})^2-1)+1}{3m^4}(3\Sigma^{(2)} + \beta_1 + \beta_2 + \beta_3) + \frac{6(\frac{m}{3})^2-2}{3m^4}(\Phi^{(12)} + \Phi^{(22)} + 2\beta_1 + 2\beta_2 + 2\beta_3) \\
&\quad + \frac{3(\frac{m}{3})^2-1}{m^4} \sum_{q=1}^{\frac{m^2}{3}-1} \xi_q \left(\beta_1 + \beta_2 + \beta_3 + (2+j)j^{c_q^{(1)}-c_q^{(2)}}\beta_4 + (2j^2+j)j^{2(c_q^{(1)}-c_q^{(2)})}\beta_4 + \sum_{p=1}^{\frac{m^2}{3}-1} \gamma_p j^{c_p^{(1)}-c_p^{(2)}} c_{p,q}\right), \\
P_\Gamma(t,\, u)_5 &= \frac{3(5(\frac{m}{3})^2-1)+1}{3m^4}(3\Sigma^{(2)} + \beta_1 + \beta_2 + \beta_3) + \frac{6(\frac{m}{3})^2-2}{3m^4}(\Phi^{(12)} + \Phi^{(22)} + 2\beta_1 + 2\beta_2 + 2\beta_3) \\
&\quad + \frac{3(\frac{m}{3})^2-1}{m^4} \sum_{q=1}^{\frac{m^2}{3}-1} \xi_q \left(\beta_1 + \beta_2 + \beta_3 + \sum_{p=1}^{\frac{m^2}{3}-1} \gamma_p j^{c_p^{(1)}-c_p^{(2)}} c_{p,q}\right),
\end{aligned}$$

$$P_\Gamma(t,\,u)_6 \;=\; \frac{3(5(\frac{m}{3})^2-1)+1}{3m^4}(3\Sigma^{(3)}+\beta_1+\beta_2+\beta_3)+\frac{6(\frac{m}{3})^2-2}{3m^4}(\Phi^{(13)}+\Phi^{(23)}+2\beta_1+2\beta_2+2\beta_3)$$
$$+\frac{3(\frac{m}{3})^2-1}{m^4}\sum_{q=1}^{\frac{m^2}{3}-1}\xi_q\left(\beta_1+\beta_2+\beta_3+(2+j^2)j^{c_q^{(1)}-c_q^{(2)}}\beta_4+(2j+j^2)j^{2(c_q^{(1)}-c_q^{(2)})}\beta_4+\sum_{p=1}^{\frac{m^2}{3}-1}\gamma_p j^{2(c_p^{(1)}-c_p^{(2)})}c_{p,q}\right),$$

and $P_\Gamma(t,\,u)_2$ (resp. $P_\Gamma(t,\,u)_4$, $P_\Gamma(t,\,u)_8$) is obtained by exchanging the coefficients $2j+j^2$ and $2j^2+j$ (resp. $2+j$ and $2j^2+j$, $2+j^2$ and $2j+j^2$) in $P_\Gamma(t,\,u)_1$ (resp. $P_\Gamma(t,\,u)_3$, $P_\Gamma(t,\,u)_6$); $P_\Gamma(t,\,u)_7$ is obtained by replacing $\Phi^{(1i)}$ by $\Phi^{(2i)}$ and $j^{c_p^{(1)}-c_p^{(2)}}$ by $j^{2(c_p^{(1)}-c_p^{(2)})}$ in $P_\Gamma(t,\,u)_5$.
Finally, for $i\in[\![1,\,\frac{m^2}{3}-1]\!]$, we have

$$P_\Gamma(t,\,u)_{i+8} \;=\; \frac{3(5(\frac{m}{3})^2-1)+1}{3m^4}(3\beta_1+J_i^{(1)}\beta_2+J_i^{(2)}\beta_3+3\Sigma_i)+\frac{6(\frac{m}{3})^2-2}{3m^4}(6\beta_1+2J_i^{(1)}\beta_2+2J_i^{(2)}\beta_3+\widetilde{\Sigma_i^{(1)}}+\widetilde{\Sigma_i^{(2)}})$$
$$+\frac{3(\frac{m}{3})^2-1}{m^4}\sum_{q=1}^{\frac{m^2}{3}-1}\xi_q\left(3\beta_1+J_i^{(1)}\beta_2+J_i^{(2)}\beta_3+\sum_{p=1}^{\frac{m^2}{3}-1}\gamma_p c_{i,p}c_{p,q}\right).$$

3.4.3 Decomposition of C_Γ

We now make the matrix $A^{(1)}$ explicit: the form of the matrix $A^{(1)}$ is nearly the same in the case where m is divisible by 3 as in the other case. The main difference between these two cases is due to the fact that in the case where m is divisible by 3, there are 9 irreducible characters of degree 1 instead of 3.
Set $\kappa_m := \frac{m}{3}-1$ if 3 divides m, and $\kappa_m := \lfloor\frac{m}{3}\rfloor$ otherwise. The matrix $A^{(1)}$ is a block-matrix with $(2+2\kappa_m)\times(2+2\kappa_m)$ blocks: for example, if $m=16$, the matrices $A^{(1)}$ and C_Γ are matrices of size 88.

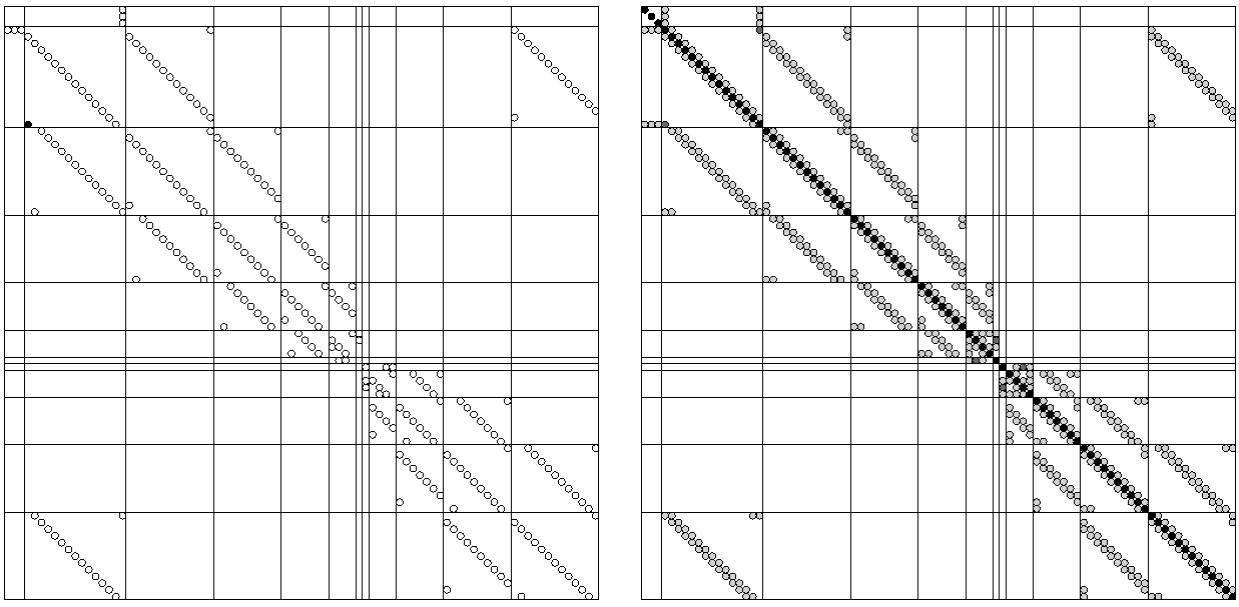


Figure 3: The matrices $A^{(1)}$ and C_Γ for $m=16$.

- If $m=2$, then $A^{(1)}=\begin{pmatrix}\mathbf{0}_{3,3} & \mathbf{1}_{3,1}\\ \mathbf{1}_{1,3} & 2\end{pmatrix}$, and $C_\Gamma = 3I-\tau_0-\tau_1-\tau_2$, with $\tau_0:=s_0s_2,\ \tau_1:=s_1,\ \tau_2:=s_3$.
- If $m=3$, then $A^{(1)}=\begin{pmatrix}\mathbf{0}_{9,9} & \mathbf{1}_{9,2}\\ \mathbf{1}_{2,9} & \widetilde{A}\end{pmatrix}$, $\widetilde{A}=\begin{pmatrix}0 & 0\\ 3 & 0\end{pmatrix}$, and $C_\Gamma = 2I-(s_0s_1\dots s_9)-(s_{10}s_{11})$.

• Now, we assume that $m \geq 4$. For $i_1 \in \{0\} \sqcup [\![1,\, \kappa_m]\!] \sqcup [\![m-\kappa_m,\, m-1]\!]$, we define the set $\widetilde{S_{i_1}}$ by:

$$\widetilde{S_{i_1}} := \begin{cases} \{s_{0,i_2} \ / \ i_2 \in [\![1,\, m-1]\!]\} & \text{if } i_1 = 0, \\ \{s_{i_1,i_2} \ / \ i_2 \in [\![i_1,\, m-1-2i_1]\!]\} & \text{if } i_1 \in [\![1,\, \kappa_m]\!], \\ \{s_{i_1,i_2} \ / \ i_2 \in [\![2(m-i_1)+1,\, i_1]\!]\} & \text{if } i_1 \in [\![m-\kappa_m,\, m-1]\!]. \end{cases}$$

Then, we distinguish two cases:

▷ If κ_m is odd, we set $\widehat{I_0} := \{0,\, 2,\, 4, \dots,\, \kappa_m - 1,\, m-\kappa_m,\, m-\kappa_m+2, \dots,\, m-3\}$,

$$\widehat{I_1} := \{1,\, 3,\, 5, \dots,\, \kappa_m,\, m-\kappa_m+1,\, m-\kappa_m+3, \dots,\, m-2\},\ \widehat{I_2} := \{m-1\}.$$

▷ If κ_m is even, we set $\widehat{I_0} := \{0,\, 2,\, 4, \dots,\, \kappa_m,\, m-\kappa_m+1,\, m-\kappa_m+3, \dots,\, m-3\}$,

$$\widehat{I_1} := \{1,\, 3,\, 5, \dots,\, \kappa_m - 1,\, m-\kappa_m,\, m-\kappa_m+2, \dots,\, m-2\},\ \widehat{I_2} := \{m-1\}.$$

Then, the roots associated to the reflections of distinct $\widetilde{S_{i_1}}$'s for i_1 belonging to a same $\widehat{I_k}$ are orthogonal. Now, we decompose each $\widetilde{S_{i_1}}$, i.e. $\widetilde{S_{i_1}} = \widehat{S_{i_1,0}} \sqcup \cdots \sqcup \widehat{S_{i_1,q-1}}$, such that $q \in \{1,\, 2,\, 3\}$ and for every $k \in [\![0,\, q-1]\!]$, the roots associated to the reflections belonging to $\widehat{S_{i_1,k}}$ are orthogonal:

▷ If $i_1 = 0$, then

◇ If $m-1$ is odd, then $\widetilde{S_0} = \widehat{S_{0,0}} \sqcup \widehat{S_{0,1}} \sqcup \widehat{S_{0,2}}$, with

$$\widehat{S_{0,0}} = \{s_{0,1},\, s_{0,3}, \dots,\, s_{0,m-3}\},\ \widehat{S_{0,1}} = \{s_{0,2},\, s_{0,4}, \dots,\, s_{0,m-2}\},\ \widehat{S_{0,2}} = \{s_{0,m-1}\}.$$

◇ If $m-1$ is even, then $\widetilde{S_0} = \widehat{S_{0,0}} \sqcup \widehat{S_{0,1}}$, with

$$\widehat{S_{0,0}} = \{s_{0,1},\, s_{0,3}, \dots,\, s_{0,m-2}\},\ \widehat{S_{0,1}} = \{s_{0,2},\, s_{0,4}, \dots,\, s_{0,m-1}\}.$$

▷ If $i_1 \in [\![1,\, \kappa_m]\!]$, then we have $\widetilde{S_{i_1}} = \widehat{S_{i_1,0}} \sqcup \widehat{S_{i_1,1}} \sqcup \widehat{S_{i_1,2}}$, with:

◇ If $m-3i_1$ is odd, then $\widehat{S_{i_1,0}} = \{s_{i_1,i_1},\, s_{i_1,i_1+2}, \dots,\, s_{i_1,m-2i_1-3}\}$,

$$\widehat{S_{i_1,1}} = \{s_{i_1,i_1+1},\, s_{i_1,i_1+3}, \dots,\, s_{i_1,m-2i_1-2}\},\ \widehat{S_{i_1,2}} = \{s_{i_1,m-2i_1-1}\}.$$

◇ If $m-3i_1$ is even, then $\widehat{S_{i_1,0}} = \{s_{i_1,i_1},\, s_{i_1,i_1+2}, \dots,\, s_{i_1,m-2i_1-4}\}$,

$$\widehat{S_{i_1,1}} = \{s_{i_1,i_1+1},\, s_{i_1,i_1+3}, \dots,\, s_{i_1,m-2i_1-1}\},\ \widehat{S_{i_1,2}} = \{s_{i_1,m-2i_1-2}\}.$$

▷ If $i_1 \in [\![m-\kappa_m,\, m-1]\!]$, then we have $\widetilde{S_{i_1}} = \widehat{S_{i_1,0}} \sqcup \widehat{S_{i_1,1}} \sqcup \widehat{S_{i_1,2}}$, with:

◇ If $3i_1-2m$ is odd, then $\widehat{S_{i_1,0}} = \{s_{i_1,2(m-i_1)+1},\, s_{i_1,2(m-i_1)+3}, \dots,\, s_{i_1,i_1-2}\}$,

$$\widehat{S_{i_1,1}} = \{s_{i_1,2(m-i_1)+2},\, s_{i_1,2(m-i_1)+4}, \dots,\, s_{i_1,i_1-1}\},\ \widehat{S_{i_1,2}} = \{s_{i_1,i_1}\}.$$

◇ If $3i_1-2m$ is even, then $\widehat{S_{i_1,0}} = \{s_{i_1,2(m-i_1)+1},\, s_{i_1,2(m-i_1)+3}, \dots,\, s_{i_1,i_1-1}\}$,

$$\widehat{S_{i_1,1}} = \{s_{i_1,2(m-i_1)+2},\, s_{i_1,2(m-i_1)+4}, \dots,\, s_{i_1,i_1-2}\},\ \widehat{S_{i_1,2}} = \{s_{i_1,i_1}\}.$$

Note that some sets $\widehat{S_{i_1,k}}$ can be empty for $k \in \{1,\, 2\}$.

Finally, we set $S_{k,l} := \coprod_{i_1 \in \widehat{I_k}} \widehat{S_{i_1,l}}$, for $(k,\, l) \in \{0,\, 1,\, 2\}^2 \backslash \{(2,\, 2)\}$, and $S_{2,2} := \left(\coprod_{i_1 \in \widehat{I_2}} \widehat{S_{i_1,2}}\right) \sqcup \{s_{-1,0}, \dots,\, s_{-1,r}\}$, with $r = 8$ if 3 divides m, and $r = 2$ otherwise. We denote by $p \in [\![1,\, 9]\!]$ the number of non-empty sets $S_{k,l}$, and by $\tau_{k,l}$ the commutative product of the reflections of $S_{k,l}$. Then, $C_\Gamma = p\, I - \sum_{(k,l) \in \{0,\, 1,\, 2\}^2} \tau_{k,l}$.

Example 11

For $m = 16$, we have the following decomposition:

$\tau_{0,0} = (s_{0,1}s_{0,3} \dots s_{0,13})(s_{2,2}s_{2,4} \dots s_{2,8})(s_{4,4})(s_{11,11})(s_{13,7}s_{13,9}s_{13,11})$
$\tau_{0,1} = (s_{0,2}s_{0,4} \dots s_{0,14})(s_{2,3}s_{2,5} \dots s_{2,11})(s_{4,5}s_{4,7})(s_{13,8}s_{13,10}s_{13,12})$
$\tau_{1,0} = (s_{1,1}s_{1,3} \dots s_{1,11})(s_{3,3}s_{3,5}s_{3,7})(s_{5,5})(s_{12,9}s_{12,11})(s_{14,5}s_{14,7} \dots s_{14,13})$
$\tau_{1,1} = (s_{1,2}s_{1,4} \dots s_{1,12})(s_{3,4}s_{3,6}s_{3,8})(s_{12,10})(s_{14,6}s_{14,8} \dots s_{14,12})$
$\tau_{2,2} = (s_{15,15})(s_{-1,0}s_{-1,1}s_{-1,2})$

$\tau_{0,2} = (s_{0,15})(s_{2,10})(s_{4,6})(s_{13,13})$
$\tau_{1,2} = (s_{1,13})(s_{3,9})(s_{12,12})(s_{14,14})$
$\tau_{2,0} = (s_{15,3}s_{15,5} \dots s_{15,13})$
$\tau_{2,1} = (s_{15,4}s_{15,6} \dots s_{15,14})$

3.5 The D series

A group of type D is generated by a group of type C and a matrix

$$Q := \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix},$$

with $abc = -1$. This group is not a direct product. We can't give a general formula for this group and we only give a simple example.

Example 12
Consider the group

$$\Gamma := \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle.$$

This group is isomorphic to the symmetric group $\mathfrak{S}_4$, so Γ has 5 conjugacy classes.
The series $P_\Gamma(t,\, u)$ verifies

$$\forall\ i \in [\![0,\, 4]\!],\ P_\Gamma(t,\, u)_i = (1-tu)\frac{N(t,\, u)_i}{D(t)D(u)},$$

with $D(t) = (t-1)^3(t^2+t+1)(t^2+1)(t+1)^2$, and

$$\begin{aligned}
N(t,\, u)_0 &= t^6u^6 + t^5u^5 - t^6u^3 - t^3u^6 + t^5u^3 + 2\,t^4u^4 + t^3u^5 + t^4u^3 + t^3u^4 + t^6 + t^5u + 2\,t^4u^2 + 4\,t^3u^3 \\
&\quad + 2\,t^2u^4 + tu^5 + u^6 + t^3u^2 + t^2u^3 + t^3u + 2\,t^2u^2 + tu^3 - t^3 - u^3 + tu + 1, \\
N(t,\, u)_1 &= t^6u^3 + t^5u^4 + t^4u^5 + t^3u^6 + t^5u^3 + t^4u^4 + t^3u^5 + t^5u^2 + t^4u^3 + t^3u^4 + t^2u^5 + t^4u^2 + t^2u^4 \\
&\quad + t^4u + t^3u^2 + t^2u^3 + tu^4 + t^3u + t^2u^2 + tu^3 + t^3 + t^2u + tu^2 + u^3, \\
N(t,\, u)_2 &= \left(t^4u^2 + t^3u^3 + t^2u^4 + t^3u^2 + t^2u^3 + t^3u + tu^3 + t^2u + tu^2 + t^2 + tu + u^2\right)\left(t^2+1\right)\left(u^2+1\right), \\
N(t,\, u)_3 &= \left(t^4u^2 + t^3u^3 + t^2u^4 + t^3u + tu^3 + t^2 + tu + u^2\right)\left(t^2+t+1\right)\left(u^2+u+1\right), \\
N(t,\, u)_4 &= (t^4u^3 + t^3u^4 - t^4u^2 - t^3u^3 - t^2u^4 + t^4u + 2\,t^3u^2 + 2\,t^2u^3 + tu^4 - t^3u - tu^3 + t^3 + 2\,t^2u \\
&\quad + 2\,tu^2 + u^3 - t^2 - tu - u^2 + t + u)\left(t^2+t+1\right)\left(u^2+u+1\right).
\end{aligned}$$

4 Exceptional subgroups of $\mathbf{SL}_3\mathbb{C}$ — Types $E,\ F,\ G,\ H,\ I,\ J,\ K,\ L$

For every exceptional subgroup of $\mathbf{SL}_3\mathbb{C}$, we begin by making the matrix $A^{(1)}$ explicit. Then we give a decomposition of $C_\Gamma := 2\,I - A^{(1)} - A^{(2)} + 2\,Diag(A^{(1)})$ as a sum of p elements, with $p \in \{3,\, 4\}$, so that $C_\Gamma = p\,I - (\tau_0 + \cdots + \tau_{p-1})$, and we give the graph associated to C_Γ. We also write the list Θ of eigenvalues of $A^{(1)}$.

Finally, we compute the sum of the series $P_\Gamma(t,\, u) = \dfrac{N_\Gamma(t,\, u)}{D_\Gamma(t,\, u)}$. In all the cases, the denominator is of the form $D_\Gamma(t,\, u)_i = \widetilde{D_\Gamma}(t)_i\widetilde{D_\Gamma}(u)_i$. Moreover, we will take the lowest common multiple $D_\Gamma(t)$ of the $\widetilde{D_\Gamma}(t)_i$'s in order that all the denominators are the same and have the form $D_\Gamma(t)D_\Gamma(u)$, i.e.

$$\forall\ i \in [\![0,\, l]\!],\ P_\Gamma(t,\, u)_i = (1-tu)\frac{M_\Gamma(t,\, u)_i}{D_\Gamma(t)D_\Gamma(u)}.$$

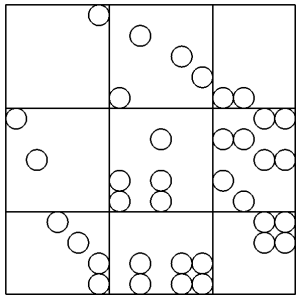
Because of the to big size of the numerators, only the denominator and the relations between the numerators are given in the text: all the numerators may be found on the web.
We also give the Poincaré series of the invariant ring $\widehat{P}_\Gamma(t) := P_\Gamma(t,\, 0)_0 = P_\Gamma(0,\, t)_0$.

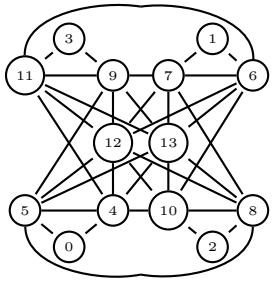
4.1 Type E

The group of type E is the group $\langle S,\ T,\ V\rangle$, with

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix},\ T := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},\ V := \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{pmatrix}.$$

Here $l+1 = 14$, $\mathrm{rank}(A^{(1)}) = 12$, $\Theta = (3,\ -\zeta_3,\ -\zeta_3^2,\ 0,\ 0,\ -1,\ \zeta_3,\ 1,\ \zeta_3^2,\ 1,\ \zeta_3,\ 3\,\zeta_3,\ \zeta_3^2,\ 3\,\zeta_3^2)$, $p = 3$, and $\tau_0 := s_0s_1s_2s_3s_{12}s_{13},\ \tau_1 := s_5s_7s_{10}s_{11},\ \tau_2 := s_4s_6s_8s_9$.





$$M_E(t,\ u)_2 = M_E(t,\ u)_3$$
$$M_E(t,\ u)_5 = M_E(u,\ t)_4$$
$$M_E(t,\ u)_7 = M_E(u,\ t)_6$$
$$M_E(t,\ u)_8 = M_E(t,\ u)_9$$
$$M_E(t,\ u)_{10} = M_E(t,\ u)_{11} = M_E(u,\ t)_8$$
$$M_E(t,\ u)_{12} = M_E(t,\ u)_{13}$$

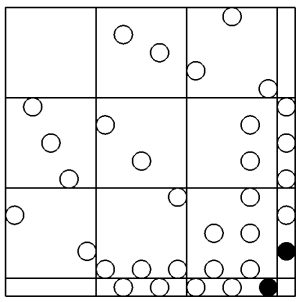
$$D_E(t) = (t-1)^3\left(t^2+t+1\right)^3\left(t^2+1\right)\left(t^4-t^2+1\right)(t+1)^2\left(t^2-t+1\right)^2$$
$$\widehat{P}_E(t) = \frac{-t^{18}+t^{15}-t^{12}-t^6+t^3-1}{(t-1)^3\left(t^2+t+1\right)^3\left(t^2+1\right)\left(t^4-t^2+1\right)(t+1)^2\left(t^2-t+1\right)^2}.$$

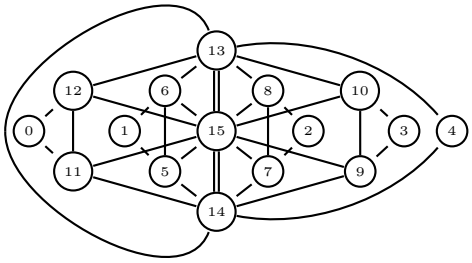
4.2 Type F

The group of type F is the group $\langle S,\ T,\ V,\ P\rangle$, with $S,\ T,\ V$ as for the type E, and

$$P := \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & \zeta_3^2 \\ 1 & \zeta_3 & \zeta_3 \\ \zeta_3 & 1 & \zeta_3 \end{pmatrix}.$$

Here $l+1 = 16$, $\mathrm{rank}(A^{(1)}) = 15$, $\Theta = (3,\ -\zeta_3,\ -\zeta_3^2,\ 0,\ -1,\ \zeta_3,\ 1,\ \zeta_3,\ 1,\ \zeta_3,\ 1,\ \zeta_3^2,\ \zeta_3^2, \zeta_3^2,\ 3\,\zeta_3,\ 3\,\zeta_3^2)$, $p = 3$, and $\tau_0 := s_0s_1s_2s_3s_4s_{15},\ \tau_1 := s_5s_7s_9s_{11}s_{13},\ \tau_2 := s_6s_8s_{10}s_{12}s_{14}$.





$$M_F(t,\ u)_1 = M_F(t,\ u)_2 = M_F(t,\ u)_3$$
$$M_F(t,\ u)_5 = M_F(t,\ u)_7 = M_F(t,\ u)_9$$
$$M_F(t,\ u)_{12} = M_F(u,\ t)_{11}$$
$$M_F(t,\ u)_{14} = M_F(u,\ t)_{13}$$
$$M_F(t,\ u)_6 = M_F(t,\ u)_8 = M_F(t,\ u)_{10}$$
$$= M_F(u,\ t)_5$$

$$D_F(t) = (t-1)^3\left(t^2+t+1\right)^3\left(t^2+1\right)\left(t^4-t^2+1\right)(t+1)^2\left(t^2-t+1\right)^2$$
$$\widehat{P}_F(t) = \frac{-t^{18}+t^{15}-t^9+t^3-1}{(t-1)^3\left(t^2+t+1\right)^3\left(t^2+1\right)\left(t^4-t^2+1\right)(t+1)^2\left(t^2-t+1\right)^2}.$$

4.3 Type G

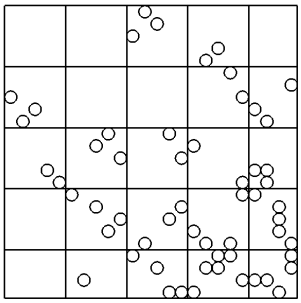
The group of type G is the group $\langle S,\ T,\ V,\ U\rangle$, with $S,\ T,\ V$ as for the type E, and

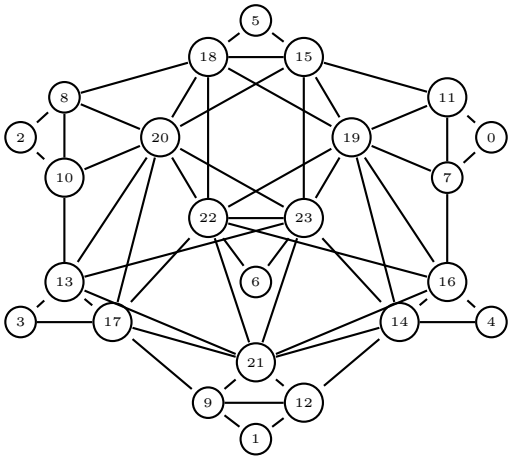
$$U := \begin{pmatrix} \zeta_9^2 & 0 & 0 \\ 0 & \zeta_9^2 & 0 \\ 0 & 0 & \zeta_9^2\zeta_3 \end{pmatrix}.$$

Here $l+1 = 24$, $\mathrm{rank}(A^{(1)}) = 21$,

$$\Theta = (3,\ -\zeta_3,\ -\zeta_3^2,\ 0,\ 0,\ -\zeta_9^7,\ -\zeta_9^4,\ \zeta_9^4+\zeta_9^7,\ 0,\ -\zeta_9^5,\ -\zeta_9^2, \zeta_9^2+\zeta_9^5,\ -1,\ \zeta_3,\ 1,\ \zeta_3^2,\ 3\,\zeta_3,\ 3\,\zeta_3^2,$$
$$-2\,\zeta_9^4-\zeta_9^7,\ \zeta_9^4-\zeta_9^7,\ -\zeta_9^2-2\,\zeta_9^5,\ -\zeta_9^2+\zeta_9^5,\ 2\,\zeta_9^2+\zeta_9^5,\ \zeta_9^4+2\,\zeta_9^7),$$

$p = 3$, and $\tau_0 := s_0s_1s_2s_3s_4s_5s_6s_{19}s_{20}s_{21},\ \tau_1 := s_7s_8s_9s_{13}s_{14}s_{15}s_{22},\ \tau_2 := s_{10}s_{11}s_{12}s_{16}s_{17}s_{18}s_{23}$.





$M_G(t,\ u)_2 = M_G(u,\ t)_1$
$M_G(t,\ u)_5 = M_G(u,\ t)_4$
$M_G(t,\ u)_{10} = M_G(u,\ t)_9$
$M_G(t,\ u)_{11} = M_G(u,\ t)_7$
$M_G(t,\ u)_{12} = M_G(u,\ t)_8$
$M_G(t,\ u)_{16} = M_G(u,\ t)_{15}$
$M_G(t,\ u)_{18} = M_G(u,\ t)_{14}$
$M_G(t,\ u)_{21} = M_G(u,\ t)_{20}$
$M_G(t,\ u)_{23} = M_G(u,\ t)_{22}$

$$D_G(t) = (t-1)^3\left(t^6-t^3+1\right)\left(t^2+t+1\right)^3(t+1)^2\left(t^2+1\right)\left(t^4-t^2+1\right)\left(t^2-t+1\right)^2\left(t^6+t^3+1\right)^2$$

$$\widehat{P}_G(t) = \frac{-t^{18}-t^{36}-1}{(t-1)^3\left(t^6-t^3+1\right)\left(t^2+t+1\right)^3(t+1)^2\left(t^2+1\right)\left(t^4-t^2+1\right)\left(t^2-t+1\right)^2\left(t^6+t^3+1\right)^2}$$

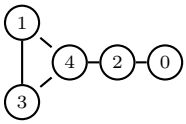
4.4 Type H

The group of type H, isomorphic to the alternating group $\mathfrak{A}_5$, is the group $\langle S,\ U,\ T\rangle$, with

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_5^4 & 0 \\ 0 & 0 & \zeta_5 \end{pmatrix},\ U := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},\ T := \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 1 & 1 \\ 2 & \zeta_5^2+\zeta_5^3 & \zeta_5+\zeta_5^4 \\ 2 & \zeta_5+\zeta_5^4 & \zeta_5^2+\zeta_5^3 \end{pmatrix}.$$

Here $l+1 = 5$, $\mathrm{rank}(A^{(1)}) = 4$, $A^{(1)}$ is symmetric, $\Theta = (3,\ -1,\ -\zeta_5^2-\zeta_5^3,\ -\zeta_5-\zeta_5^4,\ 0)$,
$p = 3$, and $\tau_0 := s_0s_3,\ \tau_1 := s_1s_2,\ \tau_2 := s_4$.

$$A^{(1)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$



$$D_H(t) = (t-1)^3\left(t^2+t+1\right)\left(t^4+t^3+t^2+t+1\right)(t+1)^2$$

$$\widehat{P}_H(t) = \frac{-t^8-t^7+t^5+t^4+t^3-t-1}{(t-1)^3(t^2+t+1)(t^4+t^3+t^2+t+1)(t+1)^2}$$

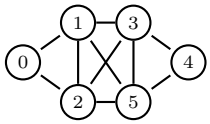
4.5 Type I

The group of type I is the group $\langle S,\ T,\ R\rangle$, with

$$S := \begin{pmatrix} \zeta_7 & 0 & 0 \\ 0 & \zeta_7^2 & 0 \\ 0 & 0 & \zeta_7^4 \end{pmatrix},\ T := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},\ R := \frac{i}{\sqrt{7}} \begin{pmatrix} \zeta_7^4 - \zeta_7^3 & \zeta_7^2 - \zeta_7^5 & \zeta_7 - \zeta_7^6 \\ \zeta_7^2 - \zeta_7^5 & \zeta_7 - \zeta_7^6 & \zeta_7^4 - \zeta_7^3 \\ \zeta_7 - \zeta_7^6 & \zeta_7^4 - \zeta_7^3 & \zeta_7^2 - \zeta_7^5 \end{pmatrix}.$$

Here $l+1 = 6$, $\mathrm{rank}(A^{(1)}) = 5$, $\Theta = (3,\ 0,\ 1,\ \zeta_7 + \zeta_7^2 + \zeta_7^4,\ \zeta_7^3 + \zeta_7^5 + \zeta_7^6,\ -1)$,
$p = 4$, and $\tau_0 := s_5 s_0,\ \tau_1 := s_1 s_4,\ \tau_2 := s_2,\ \tau_3 := s_3$.

$$A^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$



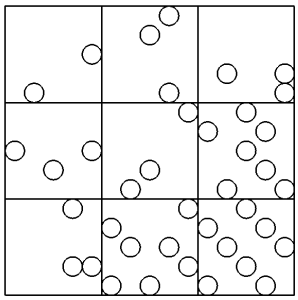
$$D_I(t) = (t-1)^3 \left(t^2+t+1\right) \left(t^2+1\right) (t+1)^2 \left(t^6+t^5+t^4+t^3+t^2+t+1\right)$$

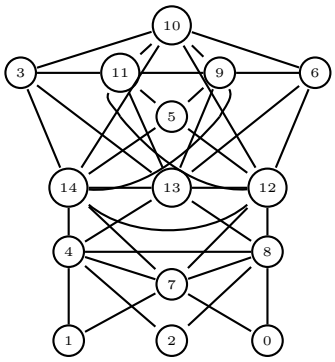
$$M_I(t,\ u)_2 = M_I(u,\ t)_1$$

$$\widehat{P}_I(t) = \frac{-t^{12} - t^{11} + t^9 + t^8 - t^6 + t^4 + t^3 - t - 1}{(t-1)^3 (t^2+t+1) (t^6+t^5+t^4+t^3+t^2+t+1) (t^2+1) (t+1)^2}$$

4.6 Type J

The group of type J is the group $\langle S,\ U,\ T,\ W\rangle$, with $S,\ U,\ T$ as for the type H, and $W := Diag(j,\ j,\ j)$. It is the direct product of the group of type H and the center of $\mathbf{SL}_3\mathbb{C}$. Here $l+1 = 15$, $\mathrm{rank}(A^{(1)}) = 12$,
$\Theta = (3,\ -1,\ -\zeta_5^2 - \zeta_5^3,\ -\zeta_5 - \zeta_5^4,\ -\zeta_3^2,\ -\zeta_3,\ 3\zeta_3,\ -\zeta_{15}^{11} - \zeta_{15}^{14},\ -\zeta_{15}^2 - \zeta_{15}^8,\ 3\zeta_3^2,\ -\zeta_{15}^7 - \zeta_{15}^{13},$
$-\zeta_{15} - \zeta_{15}^4,\ 0,\ 0,\ 0)$, $p = 3$, and $\tau_0 := s_2 s_5 s_7 s_{10} s_{13},\ \tau_1 := s_1 s_6 s_8 s_{11} s_{14},\ \tau_2 := s_0 s_3 s_4 s_9 s_{12}$.





$$M_J(t,\ u)_2 = M_J(u,\ t)_1$$
$$M_J(t,\ u)_6 = M_J(u,\ t)_5$$
$$M_J(t,\ u)_8 = M_J(u,\ t)_7$$
$$M_J(t,\ u)_{11} = M_J(u,\ t)_{10}$$
$$M_J(t,\ u)_{14} = M_J(u,\ t)_{13}$$

$$D_J(t) = \left(t^4+t^3+t^2+t+1\right) \left(t^8-t^7+t^5-t^4+t^3-t+1\right) (t+1)^2 \left(t^2-t+1\right)^2 (t-1)^3 \left(t^2+t+1\right)^3$$

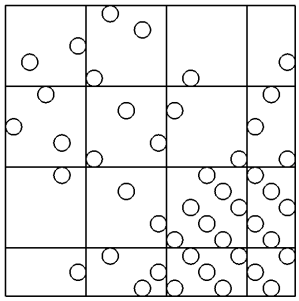
$$\widehat{P}_J(t) = \frac{-t^{24} - t^{12} - 1}{(t^4+t^3+t^2+t+1) (t^8-t^7+t^5-t^4+t^3-t+1) (t+1)^2 (t^2-t+1)^2 (t-1)^3 (t^2+t+1)^3}$$

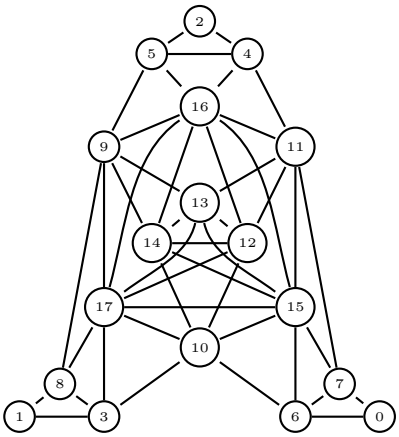
4.7 Type K

The group of type K is the group $\langle S,\ T,\ R,\ W\rangle$, with $S,\ T,\ R$ as for the type I, and $W := Diag(j,\ j,\ j)$. It is the direct product of the group of type I and the center of $\mathbf{SL}_3\mathbb{C}$. Here $l+1 = 18$, $\mathrm{rank}(A^{(1)}) = 15$,

$$\Theta = (3,\ 0,\ 0,\ 0,\ 3\zeta_3,\ 3\zeta_3^2,\ 1,\ \zeta_7 + \zeta_7^2 + \zeta_7^4,\ \zeta_7^3 + \zeta_7^5 + \zeta_7^6,\ -1,\ \zeta_3^2,\ \zeta_{21}^2 + \zeta_{21}^8 + \zeta_{21}^{11},\ \zeta_3,$$
$$\zeta_{21} + \zeta_{21}^4 + \zeta_{21}^{16},\ \zeta_{21}^5 + \zeta_{21}^{17} + \zeta_{21}^{20},\ \zeta_{21}^{10} + \zeta_{21}^{13} + \zeta_{21}^{19},\ -\zeta_3,\ -\zeta_3^2),$$

$p = 3$, and $\tau_0 := s_1s_5s_6s_{11}s_{14}s_{17},\ \tau_1 := s_0s_3s_4s_9s_{12}s_{15},\ \tau_2 := s_2s_7s_8s_{10}s_{13}s_{16}$.





$$M_K(t,\,u)_2 = M_K(u,\,t)_1$$
$$M_K(t,\,u)_4 = M_K(u,\,t)_3$$
$$M_K(t,\,u)_7 = M_K(u,\,t)_6$$
$$M_K(t,\,u)_8 = M_K(u,\,t)_5$$
$$M_K(t,\,u)_{11} = M_K(u,\,t)_{10}$$
$$M_K(t,\,u)_{14} = M_K(u,\,t)_{13}$$
$$M_K(t,\,u)_{17} = M_K(u,\,t)_{16}$$

$$D_K(t) = (t-1)^3\left(t^2+t+1\right)^3\left(t^2+1\right)\left(t^4-t^2+1\right)\left(t^6+t^5+t^4+t^3+t^2+t+1\right)$$
$$\left(t^{12}-t^{11}+t^9-t^8+t^6-t^4+t^3-t+1\right)(t+1)^2\left(t^2-t+1\right)^2$$

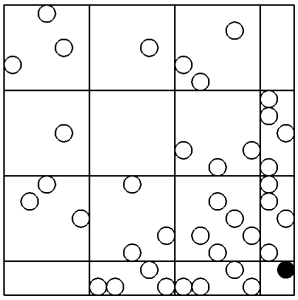
$$\widehat{P}_K(t) = \frac{-t^{36}-t^{18}-1}{(t-1)^3(t^2+t+1)^3(t^2+1)(t^4-t^2+1)(t^6+t^5+t^4+t^3+t^2+t+1)(t^{12}-t^{11}+t^9-t^8+t^6-t^4+t^3-t+1)(t+1)^2(t^2-t+1)^2}$$

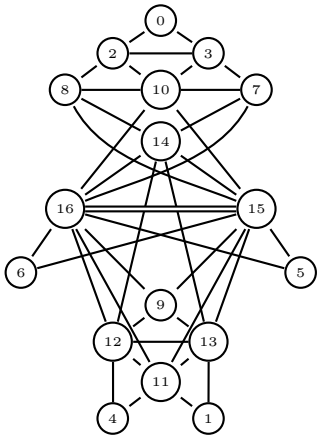
4.8 Type L

The group of type L is the group $\langle S,\ U,\ T,\ V\rangle$, with $S,\ U,\ T$ as for the type H, and

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_5^4 & 0 \\ 0 & 0 & \zeta_5 \end{pmatrix},\ U := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},\ T := \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 1 & 1 \\ 2 & s & t \\ 2 & t & s \end{pmatrix},\ V := \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & \lambda_1 & \lambda_1 \\ 2\lambda_2 & s & t \\ 2\lambda_2 & t & s \end{pmatrix},$$

where $s := \zeta_5^2+\zeta_5^3$, $t := \zeta_5+\zeta_5^4$, $\lambda_1 := \frac{-1+i\sqrt{15}}{4}$, and $\lambda_2 := \frac{-1-i\sqrt{15}}{4}$. Here $l+1 = 17$, $\operatorname{rank}(A^{(1)}) = 15$, $p = 3$, and $\tau_0 := s_0s_5s_6s_9s_{10}s_{11}s_{14},\ \tau_1 := s_1s_2s_7s_{12}s_{15},\ \tau_2 := s_3s_4s_8s_{13}s_{16}$.
$\Theta = (3,\ -\zeta_3,\ -\zeta_3^2,\ 3\zeta_3^2,\ 3\zeta_3,\ -1,\ \zeta_3^2,\ \zeta_3,\ 1,\ -\zeta_{15}^2-\zeta_{15}^8,\ -\zeta_{15}-\zeta_{15}^4, -\zeta_{15}^{11}-\zeta_{15}^{14},\ -\zeta_{15}^7-\zeta_{15}^{13},\ -\zeta_5^2-\zeta_5^3,$
$-\zeta_5-\zeta_5^4,\ 0,\ 0)$.





$$M_L(t,\,u)_3 = M_L(u,\,t)_2$$
$$M_L(t,\,u)_5 = M_L(t,\,u)_6$$
$$M_L(t,\,u)_8 = M_L(u,\,t)_7$$
$$M_L(t,\,u)_{13} = M_L(u,\,t)_{12}$$
$$M_L(t,\,u)_{16} = M_L(u,\,t)_{15}$$

$$D_L(t) = (t-1)^3\left(t^2+t+1\right)^3\left(t^2+1\right)\left(t^4-t^2+1\right)\left(t^4+t^3+t^2+t+1\right)$$
$$\left(t^8-t^7+t^5-t^4+t^3-t+1\right)(t+1)^2\left(t^2-t+1\right)^2$$

$$\widehat{P}_L(t) = \frac{-t^{30}+t^{15}-1}{(t-1)^3(t^2+t+1)^3(t^2+1)(t^4-t^2+1)(t^4+t^3+t^2+t+1)(t^8-t^7+t^5-t^4+t^3-t+1)(t+1)^2(t^2-t+1)^2}$$

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