On Recognizable Tree Languages Beyond the Borel Hierarchy

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Fundamenta Informaticae (2009) 1–19 1
IOS Press

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Abstract. We investigate the topological complexity of non Borel recognizable tree languages with regard to the difference hierarchy of analytic sets. We show that, for each integer $n \geq 1$, there is a $D_{\omega^n}(\mathbf{\Sigma}^1_1)$-complete tree language $\mathcal{L}_n$ accepted by a (non deterministic) Muller tree automaton. On the other hand, we prove that a tree language accepted by an unambiguous Büchi tree automaton must be Borel. Then we consider the game tree languages $W_{(\iota,\kappa)}$, for Mostowski-Rabin indices (ι, κ). We prove that the $D_{\omega^n}(\mathbf{\Sigma}^1_1)$-complete tree languages $\mathcal{L}_n$ are Wadge reducible to the game tree language $W_{(\iota,\kappa)}$ for $\kappa - \iota \geq 2$. In particular these languages $W_{(\iota,\kappa)}$ are not in any class $D_\alpha(\mathbf{\Sigma}^1_1)$ for $\alpha < \omega^\omega$.

Keywords: Infinite trees; tree automaton; regular tree language; Cantor topology: topological complexity; Borel hierarchy; difference hierarchy of analytic sets; complete sets; unambiguous tree automaton; game tree language.

1. Introduction

A way to study the complexity of languages of infinite words or infinite trees accepted by various kinds of automata is to study their topological complexity, and firstly to locate them with regard to the Borel and the projective hierarchies. It is well known that every ω-language accepted by a deterministic Büchi

automaton is a $\mathbf{\Pi}^0_2$-set. This implies that any ω-language accepted by a deterministic Muller automaton is a boolean combination of $\mathbf{\Pi}^0_2$-sets hence a $\mathbf{\Delta}^0_3$-set. [Tho90, Sta97, PP04]. But then it follows from Mc Naughton's Theorem, that all regular ω-languages, which are accepted by deterministic Muller automata, are also $\mathbf{\Delta}^0_3$-sets. The Borel hierarchy of regular ω-languages is then determined. Moreover Wagner determined a much more refined hierarchy on regular ω-languages, which is in fact the trace of the Wadge hierarchy on regular ω-languages, now called the Wagner hierarchy.

On the other hand, many questions remain open about the topological complexity of regular languages of infinite trees. We know that they can be much more complex than regular sets of infinite words. Skurczynski proved that for every integer $n \geq 1$, there are some $\mathbf{\Pi}^0_n$-complete and some $\mathbf{\Sigma}^0_n$-complete regular tree languages, [Sku93]. Notice that it is an open question to know whether there exist some regular sets of trees which are Borel sets of infinite rank. But there exist some regular sets of trees which are not Borel. Niwinski showed that there are some $\mathbf{\Sigma}^1_1$-complete regular sets of trees accepted by Büchi tree automata, and some $\mathbf{\Pi}^1_1$-complete regular sets of trees accepted by deterministic Muller tree automata, [Niw85]. Every set of trees accepted by a Büchi tree automaton is a $\mathbf{\Sigma}^1_1$-set and every set of trees accepted by a deterministic Muller tree automaton is a $\mathbf{\Pi}^1_1$-set. Niwinski and Walukiewicz proved that a tree language which is accepted by a deterministic Muller tree automaton is either in the class $\mathbf{\Pi}^0_3$ or $\mathbf{\Pi}^1_1$-complete, [NW03]. More recent results of Duparc and Murlak, on the Wadge hierarchy of recognizable tree languages, may be found in [Mur08, ADMN07].

It follows from the definition of acceptance by non deterministic Muller or Rabin automata and from Rabin's complementation Theorem that every regular set of trees is a $\mathbf{\Delta}^1_2$-set, see [Rab69, PP04, Tho90, LT94]. But there are only few known results on the complexity of non Borel regular tree languages. The second author gave examples of $D_{\omega^n}(\mathbf{\Sigma}^1_1)$-complete regular tree languages in [Sim92]. Arnold and Niwinski showed in [AN08] that the game tree languages $W_{(\iota,\kappa)}$ form a infinite hierarchy of non Borel regular sets of trees with regard to the Wadge reducibility.

In this paper, we investigate the topological complexity of non Borel recognizable tree languages with regard to the difference hierarchy of analytic sets. We show that, for each integer $n \geq 1$, there is a $D_{\omega^n}(\mathbf{\Sigma}^1_1)$-complete tree language $\mathcal{L}_n$ accepted by a (non deterministic) Muller tree automaton. On the other hand, we prove that non Borel recognizable tree languages accepted by Büchi tree automata have the maximum degree of ambiguity. In particular, a tree language recognized by an unambiguous Büchi tree automaton must be Borel. Then we consider the game tree languages $W_{(\iota,\kappa)}$, for Mostowski-Rabin indices (ι, κ). We prove that the $D_{\omega^n}(\mathbf{\Sigma}^1_1)$-complete tree languages $\mathcal{L}_n$ are Wadge reducible to the game tree language $W_{(\iota,\kappa)}$ for $\kappa - \iota \geq 2$. In particular, these languages $W_{(\iota,\kappa)}$ are not in any class $D_\alpha(\mathbf{\Sigma}^1_1)$ for $\alpha < \omega^\omega$.

The paper is organized as follows. In Section 2 we recall the notions of Büchi or Muller tree automata and of regular tree languages. The notions of topology, including the definition of the difference hierarchy of analytic sets, are recalled in Section 3. We show in Section 4 that there are $D_{\omega^n}(\mathbf{\Sigma}^1_1)$-complete tree languages $\mathcal{L}_n$ accepted by Muller tree automata. We consider the complexity of game tree languages in Section 5.

2. Recognizable tree languages

We recall now usual notations of formal language theory.
When Σ is a finite alphabet, a *non-empty finite word* over Σ is any sequence $x = a_1 \cdots a_k$, where $a_i \in \Sigma$ for $i = 1, \ldots, k$, and k is an integer ≥ 1. The *length* of x is k, denoted by $|x|$. The *empty word* has no letter and is denoted by λ; its length is 0. $\Sigma^\star$ is the *set of finite words* (including the empty word) over Σ.
A *finitary language* over the alphabet Σ is a subset of $\Sigma^\star$.

The *first infinite ordinal* is ω. An *ω-word* over Σ is an ω -sequence $a_1 \cdots a_n \cdots$, where for all integers $i \geq 1$, $a_i \in \Sigma$. When σ is an ω-word over Σ, we write $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)\cdots$, where for all i, $\sigma(i) \in \Sigma$, and $\sigma[n] = \sigma(1)\sigma(2)\cdots\sigma(n)$ for all $n \geq 1$ and $\sigma[0] = \lambda$.
The usual concatenation product of two finite words u and v is denoted $u \cdot v$ (and sometimes just uv). This product is extended to the product of a finite word u and an ω-word v: the infinite word $u \cdot v$ is then the ω-word such that:
$(u \cdot v)(k) = u(k)$ if $k \leq |u|$, and $(u \cdot v)(k) = v(k - |u|)$ if $k > |u|$.
The *prefix relation* is denoted $\sqsubseteq$: a finite word u is a *prefix* of a finite word v (respectively, an infinite word v), denoted $u \sqsubseteq v$, if and only if there exists a finite word w (respectively, an infinite word w), such that $v = u \cdot w$.
The *set of ω-words* over the alphabet Σ is denoted by Σ^ω. An *ω-language* over an alphabet Σ is a subset of Σ^ω.

We introduce now languages of infinite binary trees whose nodes are labelled in a finite alphabet Σ.
A node of an infinite binary tree is represented by a finite word over the alphabet $\{l, r\}$ where r means "right" and l means "left". Then an infinite binary tree whose nodes are labelled in Σ is identified with a function $t : \{l, r\}^\star \to \Sigma$. The set of infinite binary trees labelled in Σ will be denoted T^ω_Σ.

Let t be a tree. A branch B of t is a subset of the set of nodes of t which is linearly ordered by the tree partial order $\sqsubseteq$ and which is closed under prefix relation, i.e. if x and y are nodes of t such that $y \in B$ and $x \sqsubseteq y$ then $x \in B$.
A branch B of a tree is said to be maximal iff there is not any other branch of t which strictly contains B.

Let t be an infinite binary tree in T^ω_Σ. If B is a maximal branch of t, then this branch is infinite. Let $(u_i)_{i\geq 0}$ be the enumeration of the nodes in B which is strictly increasing for the prefix order.
The infinite sequence of labels of the nodes of such a maximal branch B, i.e. $t(u_0)t(u_1)\cdots t(u_n)\cdots$ is called a path. It is an ω-word over the alphabet Σ.

Let then $L \subseteq \Sigma^\omega$ be an ω-language over Σ. Then we denote $\exists \mathrm{Path}(L)$ the set of infinite trees t in T^ω_Σ such that t has (at least) one path in L.

We are now going to define tree automata and recognizable tree languages.

Definition 2.1. A (nondeterministic topdown) tree automaton is a quadruple $\mathcal{A} = (K, \Sigma, \Delta, q_0)$, where K is a finite set of states, Σ is a finite input alphabet, $q_0 \in K$ is the initial state and $\Delta \subseteq K \times \Sigma \times K \times K$

is the transition relation. The tree automaton $\mathcal{A}$ is said to be deterministic if the relation Δ is a functional one, i.e. if for each $(q,a) \in K \times \Sigma$ there is at most one pair of states (q', q'') such that $(q, a, q', q'') \in \Delta$. A run of the tree automaton $\mathcal{A}$ on an infinite binary tree $t \in T_\Sigma^\omega$ is a infinite binary tree $\rho \in T_K^\omega$ such that: (a) $\rho(\lambda) = q_0$ and (b) for each $u \in \{l,r\}^\star$, $(\rho(u), t(u), \rho(u.l), \rho(u.r)) \in \Delta$.

Definition 2.2. A Büchi (nondeterministic topdown) tree automaton is a 5-tuple $\mathcal{A} = (K, \Sigma, \Delta, q_0, F)$, where (K, Σ, Δ, q_0) is a tree automaton and $F \subseteq K$ is the set of accepting states.
A run ρ of the Büchi tree automaton $\mathcal{A}$ on an infinite binary tree $t \in T_\Sigma^\omega$ is said to be accepting if for each path of ρ there is some accepting state appearing infinitely often on this path.
The tree language $L(\mathcal{A})$ accepted by the Büchi tree automaton $\mathcal{A}$ is the set of infinite binary trees $t \in T_\Sigma^\omega$ such that there is (at least) one accepting run of $\mathcal{A}$ on t.

Definition 2.3. A Muller (nondeterministic topdown) tree automaton is a 5-tuple $\mathcal{A} = (K, \Sigma, \Delta, q_0, \mathcal{F})$, where (K, Σ, Δ, q_0) is a tree automaton and $\mathcal{F} \subseteq 2^K$ is the collection of designated state sets.
A run ρ of the Muller tree automaton $\mathcal{A}$ on an infinite binary tree $t \in T_\Sigma^\omega$ is said to be accepting if for each path p of ρ, the set of states appearing infinitely often on this path is in $\mathcal{F}$.
The tree language $L(\mathcal{A})$ accepted by the Muller tree automaton $\mathcal{A}$ is the set of infinite binary trees $t \in T_\Sigma^\omega$ such that there is (at least) one accepting run of $\mathcal{A}$ on t.
The class REG of regular, or recognizable, tree languages is the class of tree languages accepted by some Muller automaton.

Remark 2.4. Each tree language accepted by some (deterministic) Büchi automaton is also accepted by some (deterministic) Muller automaton. A tree language is accepted by a Muller tree automaton iff it is accepted by some Rabin tree automaton. We refer for instance to [Tho90, PP04] for the definition of Rabin tree automaton.

Example 2.5. Let $L \subseteq \Sigma^\omega$ be a regular ω-language (see [PP04] about regular ω-languages which are the ω-languages accepted by Büchi or Muller automata). Then the set $\exists \mathrm{Path}(L) \subseteq T_\Sigma^\omega$ is accepted by a Büchi tree automaton, hence also by a Muller tree automaton.
The set of infinite binary trees $t \in T_\Sigma^\omega$ having all their paths in L, denoted $\forall \mathrm{Path}(L)$, is accepted by a deterministic Muller tree automaton. It is in fact the complement of the set $\exists \mathrm{Path}(\Sigma^\omega - L)$.

3. Topology

We assume the reader to be familiar with basic notions of topology which may be found in [Mos80, LT94, Kec95, Sta97, PP04]. There is a natural metric on the set Σ^ω of infinite words over a finite alphabet Σ containing at least two letters which is called the *prefix metric* and defined as follows. For $u, v \in \Sigma^\omega$ and $u \neq v$ let $\delta(u,v) = 2^{-l_{\mathrm{pref}(u,v)}}$ where $l_{\mathrm{pref}(u,v)}$ is the first integer n such that the $(n+1)^{st}$ letter of u is different from the $(n+1)^{st}$ letter of v. This metric induces on Σ^ω the usual Cantor topology for which *open subsets* of Σ^ω are in the form $W \cdot \Sigma^\omega$, where $W \subseteq \Sigma^\star$. A set $L \subseteq \Sigma^\omega$ is a *closed set* iff its complement $\Sigma^\omega - L$ is an open set.

There is also a natural topology on the set T_Σ^ω [Mos80, LT94, Kec95]. It is defined by the following distance. Let t and s be two distinct infinite trees in T_Σ^ω. Then the distance between t and s is $\frac{1}{2^n}$ where n is the smallest integer such that $t(x) \neq s(x)$ for some word $x \in \{l, r\}^\star$ of length n.
The open sets are then in the form $T_0 \cdot T_\Sigma^\omega$ where T_0 is a set of finite labelled trees. $T_0 \cdot T_\Sigma^\omega$ is the set of infinite binary trees which extend some finite labelled binary tree $t_0 \in T_0$, t_0 is here a sort of prefix, an "initial subtree" of a tree in $t_0 \cdot T_\Sigma^\omega$.
It is well known that the set T_Σ^ω, equipped with this topology, is homeomorphic to the Cantor set 2^ω, hence also to the topological spaces Σ^ω, where Σ is an alphabet having at least two letters.

We now define the *Borel Hierarchy* of subsets of Σ^ω. It is defined similarly on the space T_Σ^ω.

Definition 3.1. For a non-null countable ordinal α, the classes $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ of the Borel Hierarchy on the topological space Σ^ω are defined as follows:
$\mathbf{\Sigma}^0_1$ is the class of open subsets of Σ^ω, $\mathbf{\Pi}^0_1$ is the class of closed subsets of Σ^ω,
and for any countable ordinal $\alpha \geq 2$:
$\mathbf{\Sigma}^0_\alpha$ is the class of countable unions of subsets of Σ^ω in $\bigcup_{\gamma<\alpha} \mathbf{\Pi}^0_\gamma$.
$\mathbf{\Pi}^0_\alpha$ is the class of countable intersections of subsets of Σ^ω in $\bigcup_{\gamma<\alpha} \mathbf{\Sigma}^0_\gamma$.

For a countable ordinal α, a subset of Σ^ω is a Borel set of *rank* α iff it is in $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha$ but not in $\bigcup_{\gamma<\alpha}(\mathbf{\Sigma}^0_\gamma \cup \mathbf{\Pi}^0_\gamma)$.

There exists another hierarchy beyond the Borel hierarchy, which is called the projective hierarchy. The classes $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$, for integers $n \geq 1$, of the projective hierarchy are obtained from the Borel hierarchy by successive applications of operations of projection and complementation. The first level of the projective hierarchy is formed by the class $\mathbf{\Sigma}^1_1$ of *analytic sets* and the class $\mathbf{\Pi}^1_1$ of *co-analytic sets* which are complements of analytic sets. In particular, the class of Borel subsets of Σ^ω is strictly included in the class $\mathbf{\Sigma}^1_1$ of *analytic sets* which are obtained by projection of Borel sets.

Definition 3.2. A subset A of Σ^ω is in the class $\mathbf{\Sigma}^1_1$ of **analytic** sets iff there exists another finite set Y and a Borel subset B of $(\Sigma \times Y)^\omega$ such that $x \in A \leftrightarrow \exists y \in Y^\omega$ such that $(x, y) \in B$, where (x, y) is the infinite word over the alphabet $\Sigma \times Y$ such that $(x, y)(i) = (x(i), y(i))$ for each integer $i \geq 1$.

Remark 3.3. In the above definition we could take B in the class $\mathbf{\Pi}^0_2$. Moreover analytic subsets of Σ^ω are the projections of $\mathbf{\Pi}^0_1$-subsets of $\Sigma^\omega \times \omega^\omega$, where ω^ω is the Baire space, [Mos80].

We now define the notion of Wadge reducibility via the reduction by continuous functions. Let X, Y be two finite alphabets. For $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, L is said to be Wadge reducible to L', denoted by $L \leq_W L'$, iff there exists a continuous function $f : X^\omega \to Y^\omega$, such that $L = f^{-1}(L')$.

We now define completeness with regard to reduction by continuous functions. For a countable ordinal $\alpha \geq 1$, and an integer $n \geq 1$, a set $F \subseteq \Sigma^\omega$ is said to be a $\mathbf{\Sigma}^0_\alpha$ (respectively, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$)-*complete set* iff for any set $E \subseteq Y^\omega$ (with Y a finite alphabet): $E \in \mathbf{\Sigma}^0_\alpha$ (respectively, $E \in \mathbf{\Pi}^0_\alpha$, $E \in \mathbf{\Sigma}^1_n$, $E \in \mathbf{\Pi}^1_n$) iff $E \leq_W F$. $\mathbf{\Sigma}^0_n$ (respectively $\mathbf{\Pi}^0_n$)-complete sets, with n an integer ≥ 1, are thoroughly characterized in [Sta86].

The Borel hierarchy and the projective hierarchy on T_Σ^ω are defined from open sets in the same manner as in the case of the topological space Σ^ω.

The ω-language $\mathcal{R} = (0^\star \cdot 1)^\omega$ is a well known example of $\mathbf{\Pi}_2^0$-complete subset of $\{0,1\}^\omega$. It is the set of ω-words over $\{0,1\}$ having infinitely many occurrences of the letter 1. Its complement $\{0,1\}^\omega - (0^\star \cdot 1)^\omega$ is a $\mathbf{\Sigma}_2^0$-complete subset of $\{0,1\}^\omega$.
The set of infinite trees in T_Σ^ω, where $\Sigma = \{0,1\}$, having at least one path in the ω-language $\mathcal{R} = (0^\star \cdot 1)^\omega$ is $\mathbf{\Sigma}_1^1$-complete. Its complement is the set of trees in T_Σ^ω having all their paths in $\{0,1\}^\omega - (0^\star \cdot 1)^\omega$; it is $\mathbf{\Pi}_1^1$-complete.

We now recall the notion of difference hierarchy of analytic sets. Let $\eta < \omega_1$ (where ω_1 is the first uncountable ordinal) be an ordinal and $(A_\theta)_{\theta<\eta}$ be an increasing sequence of subsets of some space X, then the set $D_\eta[(A_\theta)_{\theta<\eta}]$ is the set of elements $x \in X$ such that $x \in A_\theta \setminus \bigcup_{\theta'<\theta} A_{\theta'}$ for some $\theta < \eta$ whose parity is opposite to that of η. (Recall that a countable ordinal γ is said to be even iff it can be written in the form $\gamma = \alpha + n$, where α is a limit ordinal and n is an even non-negative integer; otherwise the ordinal γ is said to be odd; notice that all limit ordinals, like the ordinals ω^n, $n \geq 1$, or ω^ω, are even ordinals.)

We can now define the class of η-differences of analytic subsets of X, where $X = \Sigma^\omega$ or $X = T_\Sigma^\omega$.

$$D_\eta(\mathbf{\Sigma}_1^1) := \{ D_\eta[(A_\theta)_{\theta<\eta}] \mid \text{ for each ordinal } \theta < \eta \ \ A_\theta \text{ is a } \mathbf{\Sigma}_1^1\text{-set } \}$$

It is well known that the hierarchy of differences of analytic sets is strict, i.e. that for all countable ordinals $\alpha < \beta < \omega_1$, it holds that $D_\alpha(\mathbf{\Sigma}_1^1) \subset D_\beta(\mathbf{\Sigma}_1^1)$. This is considered as a folklore result of descriptive set theory which follows from the existence of universal sets for each class $D_\alpha(\mathbf{\Sigma}_1^1)$. Indeed we know first that the class $\mathbf{\Sigma}_1^1$ of analytic sets admits a universal set, see [Kec95, page 205]or [Mos80, page 43]. Then, using classical methods of descriptive set theory, one can show that, for each countable ordinal α, the class $D_\alpha(\mathbf{\Sigma}_1^1)$ admits also a universal set, see [Kan97, page 443]. This implies, as in the case of the Borel hierarchy in [Kec95, page 168], that the difference hierarchy of analytic sets is strict. As a universal set for the class $D_\alpha(\mathbf{\Sigma}_1^1)$ is also a $D_\alpha(\mathbf{\Sigma}_1^1)$-complete set for reduction by continuous functions, this implies also that there exists a $D_\alpha(\mathbf{\Sigma}_1^1)$-complete set.
Notice that in the sequel we shall only consider the classes $D_\alpha(\mathbf{\Sigma}_1^1)$, for ordinals $\alpha < \omega^\omega$, and that we shall reprove that there exists some $D_\alpha(\mathbf{\Sigma}_1^1)$-complete subsets of T_Σ^ω, giving examples which are regular sets of trees.

Another folklore result of descriptive set theory is that the union $\bigcup_{\alpha<\omega_1} D_\alpha(\mathbf{\Sigma}_1^1)$ represents only a small part of the class $\mathbf{\Delta}_2^1$. It is quoted for instance in [Ste82] or [Kan97, page 443]. (It is noticed in [Ste82] that the union $\bigcup_{\alpha<\omega_1} D_\alpha(\mathbf{\Sigma}_1^1)$ is strictly included in the class $\mathcal{A}(\mathbf{\Pi}_1^1)$ which is the closure of the class $\mathbf{\Pi}_1^1$ under Souslin's operation. The class $\mathcal{A}(\mathbf{\Pi}_1^1)$ is included in the class $\mathbf{\Delta}_2^1$ by [Mos80, 2.B.5 page 75]). Notice however that this result is not necessary in the sequel.

4. $D_{\omega^n}(\mathbf{\Sigma}^1_1)$-complete recognizable languages

It follows from the definition of the Büchi acceptance condition for infinite trees that each tree language recognized by a (non deterministic) Büchi tree automaton is an analytic set.
Niwinski showed that some Büchi recognized tree languages are actually $\mathbf{\Sigma}^1_1$-complete sets. An example is any tree language $T \subseteq T^\omega_\Sigma$ in the form $\exists \mathrm{Path}(L)$, where $L \subseteq \Sigma^\omega$ is a regular ω-language which is a $\mathbf{\Pi}^0_2$-complete subset of Σ^ω. In particular, the tree language $\mathcal{L} = \exists \mathrm{Path}(\mathcal{R})$, where $\mathcal{R} = (0^\star \cdot 1)^\omega$, is $\mathbf{\Sigma}^1_1$-complete hence non Borel [Niw85, PP04, Sim92].
Notice that its complement $\mathcal{L}^- = \forall \mathrm{Path}(\{0,1\}^\omega - (0^\star \cdot 1)^\omega)$ is a $\mathbf{\Pi}^1_1$-complete set. It cannot be accepted by any Büchi tree automaton because it is not a $\mathbf{\Sigma}^1_1$-set. On the other hand, it can be easily seen that it is accepted by a deterministic Muller tree automaton.

The tree languages $\mathcal{L}$ and $\mathcal{L}^-$ have been used by the second author in [Sim92] to give examples of $D_{\omega^n}(\mathbf{\Sigma}^1_1)$-complete recognizable tree languages, for integers $n \geq 1$. We now give first the construction of a $D_\omega(\mathbf{\Sigma}^1_1)$-complete set.

For a tree $t \in T^\omega_\Sigma$ and $u \in \{l,r\}^\star$, we shall denote $t_u : \{l,r\}^\star \to \Sigma$ the subtree defined by $t_u(v) = t(u \cdot v)$ for all $v \in \{l,r\}^\star$. It is in fact the subtree of t which is rooted in u.

Now we can define a $D_\omega(\mathbf{\Sigma}^1_1)$-complete tree language $\mathcal{L}_1$.

$\mathcal{L}_1 = \{t \in T^\omega_{\{0,1\}} \mid \exists n \geq 0 \ \ t_{l^n \cdot r} \in \mathcal{L}$ and $\min\{n \geq 0 \mid t_{l^n \cdot r} \in \mathcal{L}\}$ is odd $\}$.

Proposition 4.1. The tree language $\mathcal{L}_1$ is $D_\omega(\mathbf{\Sigma}^1_1)$-complete.

Proof. We first show that the language $\mathcal{L}_1$ is in the class $D_\omega(\mathbf{\Sigma}^1_1)$.
Consider firstly, for some integer $k \geq 0$, the set $T_k = \{t \in T^\omega_{\{0,1\}} \mid \ t_{l^k \cdot r} \in \mathcal{L}\}$. It is clear that this set is in the class $\mathbf{\Sigma}^1_1$ because the function $F_k : T^\omega_{\{0,1\}} \to T^\omega_{\{0,1\}}$ defined by $F_k(t) = t_{l^k \cdot r}$ is continuous and $T_k = F_k^{-1}(\mathcal{L})$ and the class $\mathbf{\Sigma}^1_1$ is closed under inverses of continuous functions.
Let now $H_n = \{t \in T^\omega_{\{0,1\}} \mid \exists k \leq n \ t_{l^k \cdot r} \in \mathcal{L}\}$. This set is also in the class $\mathbf{\Sigma}^1_1$ because the class $\mathbf{\Sigma}^1_1$ is closed under finite (and even countable) union and $H_n = \bigcup_{k \leq n} T_k$.
The sets H_n form an increasing sequence of $\mathbf{\Sigma}^1_1$-sets, and we can check that

$$\mathcal{L}_1 = D_\omega[(H_n)_{n<\omega}]$$

We now prove that $\mathcal{L}_1$ is $D_\omega(\mathbf{\Sigma}^1_1)$-*complete*.
Let $L \subseteq \Sigma^\omega$ be a $D_\omega(\mathbf{\Sigma}^1_1)$-subset of Σ^ω, where Σ is an alphabet having at least two letters. Then there is an increasing sequence $(A_n)_{n \in \omega}$ of $\mathbf{\Sigma}^1_1$-subsets of Σ^ω such that $L = D_\omega[(A_n)_{n<\omega}]$. On the other hand, we know that the tree language $\mathcal{L}$ is $\mathbf{\Sigma}^1_1$-complete. Thus for each integer $n \geq 0$ there exists a continuous function $f_n : \Sigma^\omega \to T^\omega_{\{0,1\}}$ such that $A_n = f_n^{-1}(\mathcal{L})$.
We now define a function $F : \Sigma^\omega \to T^\omega_{\{0,1\}}$ by : for all $x \in \Sigma^\omega$, for all integers $k \geq 0$, $F(x)(l^k) = 0$ and $F(x)_{l^k \cdot r} = f_k(x)$. It is clear that the function F is continuous because each function f_k is continuous.
We can now check that for every $x \in \Sigma^\omega$, x is in the set $L = D_\omega[(A_n)_{n<\omega}]$ iff there is an odd integer n such that $x \in A_n \setminus \bigcup_{k<n} A_k$ iff there is an odd integer n such that $f_n(x) \in \mathcal{L}$ and for all $k < n$

$f_k(x) \in \mathcal{L}^-$.
This means that $x \in L = D_\omega[(A_n)_{n<\omega}]$ iff $F(x) \in \mathcal{L}_1$.
Finally we have shown, using the reduction F, that $L = D_\omega[(A_n)_{n<\omega}] \leq_W \mathcal{L}_1$ and so the tree language $\mathcal{L}_1$ is $D_\omega(\mathbf{\Sigma}^1_1)$-complete. □

We can now generalize this construction to obtain some $D_{\omega^n}(\mathbf{\Sigma}^1_1)$-complete tree languages, for every integer $n \geq 1$.
Recall first that an ordinal α is strictly smaller than the ordinal ω^n, where $n \geq 2$ is an integer, if and only if it admits a Cantor Normal Form

$$\alpha = \omega^{n-1} \cdot a_{n-1} + \omega^{n-2} \cdot a_{n-2} + \ldots + \omega \cdot a_1 + a_0$$

where $a_{n-1}, a_{n-2}, \ldots, a_0$, are non-negative integers. In that case we shall denote $\mathrm{Ord}(a_{n-1}, a_{n-2}, \ldots, a_0) = \omega^{n-1} \cdot a_{n-1} + \omega^{n-2} \cdot a_{n-2} + \ldots + \omega \cdot a_1 + a_0$.
Recall also that if $\alpha = \mathrm{Ord}(a_{n-1}, a_{n-2}, \ldots, a_0)$ and $\beta = \mathrm{Ord}(b_{n-1}, b_{n-2}, \ldots, b_0)$, then $\alpha < \beta$ if and only if there is an integer k such that $0 \leq k \leq n-1$ and $a_j = b_j$ for $n-1 \geq j > k$ and $a_k < b_k$.

We now define the tree language $\mathcal{L}_n$, for $n \geq 2$, as the set of trees $t \in T^\omega_{\{0,1\}}$ for which there exist some integers $a_{n-1}, a_{n-2}, \ldots, a_0 \geq 0$ such that:

1. $t_{l^{a_{n-1}}.r.l^{a_{n-2}}.r \ldots l^{a_0}.r}$ is in $\mathcal{L}$ and the parity of $\mathrm{Ord}(a_{n-1}, a_{n-2}, \ldots, a_0)$ is odd,
2. If $\mathrm{Ord}(b_{n-1}, b_{n-2}, \ldots, b_0) < \mathrm{Ord}(a_{n-1}, a_{n-2}, \ldots, a_0)$ then the tree $t_{l^{b_{n-1}}.r.l^{b_{n-2}}.r \ldots l^{b_0}.r}$ is not in $\mathcal{L}$.

Proposition 4.2. For each integer $n \geq 2$, the tree language $\mathcal{L}_n$ is $D_{\omega^n}(\mathbf{\Sigma}^1_1)$-complete.

Proof. The proof is a simple generalization of the proof of Proposition 4.1. Notice that we have to use the closure of the class $\mathbf{\Sigma}^1_1$ under countable (and not only under finite) union. Details are here left to the reader. □

The tree languages $\mathcal{L}_n$ can not be accepted by any Büchi tree automaton because each tree language accepted by a (non deterministic) Büchi tree automaton is an analytic set and $D_{\omega^n}(\mathbf{\Sigma}^1_1)$-complete sets, for $n \geq 1$, are not in the class $\mathbf{\Sigma}^1_1$. We are going to see that the tree languages $\mathcal{L}_n$ are accepted by Muller tree automata.

We now recall the following result proved by Niwinski in [Niw85], see also for instance [PP04, Tho90].

Lemma 4.3. The language $\mathcal{L}^- = \forall \mathrm{Path}(\{0,1\}^\omega - (0^\star.1)^\omega)$ is a $\mathbf{\Pi}^1_1$-complete set accepted by a deterministic Muller tree automaton.

On the other hand, the tree language $\mathcal{L}$ is a $\mathbf{\Sigma}^1_1$-complete set. Thus it is not a $\mathbf{\Pi}^1_1$-set otherwise it would be in the class $\mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$ which is the class of Borel sets by Suslin's Theorem. But every tree language which is recognizable by a *deterministic* Muller tree automaton is a $\mathbf{\Pi}^1_1$-set therefore the tree language $\mathcal{L}$ can not be accepted by any *deterministic* Muller tree automaton. However we can now state the following result.

Lemma 4.4. The language $\mathcal{L}$ is a $\mathbf{\Sigma}^1_1$-complete set accepted by a non deterministic Büchi tree automaton, hence also by a non deterministic Muller tree automaton.

Proof. We recall informally how we can define a non-deterministic Büchi tree automaton $\mathcal{A}$ accepting the language $\mathcal{L}$. When reading a tree $t \in \mathcal{L}$, the automaton $\mathcal{A}$, using the non determinism, guesses an infinite branch of the tree. Then the automaton checks, using the Büchi acceptance condition, that the sequence of labels of nodes on this branch forms an ω-word in $(0^\star.1)^\omega$, i.e. contains an infinite number of letters 1. □

Lemma 4.5. For each integer $n \geq 1$, the language $\mathcal{L}_n$ is accepted by a (non deterministic) Muller tree automaton.

Proof. We first construct a non deterministic Muller tree automaton $\mathcal{A}_1$ accepting the language $\mathcal{L}_1$.

Recall that, for each tree $t \in \mathcal{L}_1$, there exists a least integer $n \geq 0$ such that $t_{l^n.r} \in \mathcal{L}$. This (odd) integer is defined *in a unique way*. One can now construct, from Muller tree automata $\mathcal{A}^-$ and $\mathcal{A}^+$ accepting the tree languages $\mathcal{L}^-$ and $\mathcal{L}$, a Muller tree automaton $\mathcal{A}_1$ accepting the tree language $\mathcal{L}_1$. Using the non-determinism, the automaton $\mathcal{A}_1$ will guess the (odd) integer $n \geq 0$ and then, using the behaviour of $\mathcal{A}^-$ and $\mathcal{A}^+$, it will check that $t_{l^n.r} \in \mathcal{L}$ and that, for every integer $k < n$, $t_{l^k.r} \notin \mathcal{L}$.

We now give the exact construction of the non deterministic Muller tree automaton $\mathcal{A}_1$.
Let $\Sigma = \{0, 1\}$ and $\mathcal{A}^- = (K, \Sigma, \Delta, q_0, \mathcal{F})$ be a (deterministic) Muller tree automaton accepting the tree language $\mathcal{L}^-$.
And let $\mathcal{A}^+ = (K', \Sigma, \Delta', q'_0, \mathcal{F}')$ be a (non deterministic) Muller tree automaton accepting the tree language $\mathcal{L}$. We assume that $K \cap K' = \emptyset$.
Then it is easy to see that the tree language $\mathcal{L}_1$ is accepted by the Muller tree automaton $\mathcal{A}_1 = (K^1, \Sigma, \Delta^1, q^1_0, \mathcal{F}^1)$, where
$K^1 = K \cup K' \cup \{q^1_0, q^1_1, q_f\}$,
$\Delta^1 = \Delta \cup \Delta' \cup \{(q^1_0, a, q^1_1, q_0), (q^1_1, a, q_f, q'_0), (q_f, a, q_f, q_f), (q^1_1, a, q^1_0, q_0) \mid a \in \{0, 1\}\}$,
$\mathcal{F}^1 = \mathcal{F} \cup \mathcal{F}' \cup \{q_f\}$.

For every integer $n > 1$, we can construct in a similar way a Muller tree automaton $\mathcal{A}_n$ accepting the tree language $\mathcal{L}_n$.

Recall that for each tree $t \in \mathcal{L}_n$ there exists a least ordinal $\alpha = \mathrm{Ord}(a_{n-1}, a_{n-2}, \ldots, a_0) < \omega^n$ such that $t_{l^{a_{n-1}}.r.l^{a_{n-2}}.r \ldots l^{a_0}.r}$ is in $\mathcal{L}$. This (odd) ordinal is defined *in a unique way*.
One can now construct, from the Muller tree automata $\mathcal{A}^-$ and $\mathcal{A}^+$ accepting the tree languages $\mathcal{L}^-$ and $\mathcal{L}$, a Muller tree automaton $\mathcal{A}_n$ accepting the tree language $\mathcal{L}_n$. Using the non-determinism, the automaton $\mathcal{A}_n$ will guess the (odd) ordinal $\alpha = \mathrm{Ord}(a_{n-1}, a_{n-2}, \ldots, a_0) < \omega^n$ and then, using the behaviour of $\mathcal{A}^-$ and $\mathcal{A}^+$, it will check that $t_{l^{a_{n-1}}.r.l^{a_{n-2}}.r \ldots l^{a_0}.r}$ is in $\mathcal{L}$ and that for each ordinal $\beta = \mathrm{Ord}(b_{n-1}, b_{n-2}, \ldots, b_0) < \mathrm{Ord}(a_{n-1}, a_{n-2}, \ldots, a_0)$ the tree language $t_{l^{b_{n-1}}.r.l^{b_{n-2}}.r \ldots l^{b_0}.r}$ is not in $\mathcal{L}$.
□

We can now summarize the above results in the following theorem.

Theorem 4.6. For each integer $n \geq 1$, the language $\mathcal{L}_n$ is a $D_{\omega^n}(\mathbf{\Sigma}^1_1)$-complete set accepted by a (non deterministic) Muller tree automaton.

Corollary 4.7. The class of tree languages recognized by Muller tree automata is not included into the boolean closure of the class of tree languages recognized by Büchi tree automata.

Proof. We know that every tree language recognized by a Büchi tree automaton is a $\mathbf{\Sigma}^1_1$-set. But a tree language which is a boolean combination of $\mathbf{\Sigma}^1_1$-sets is in the class $D_\omega(\mathbf{\Sigma}^1_1)$ which does not contain all tree languages recognized by (non deterministic) Muller tree automata. □

Remark 4.8. We have given above examples of $D_{\omega^n}(\mathbf{\Sigma}^1_1)$-complete tree languages accepted by Muller tree automata. In a similar way it is easy to construct, for each ordinal $\alpha < \omega^\omega$, a $D_\alpha(\mathbf{\Sigma}^1_1)$-complete tree language accepted by a Muller tree automaton. Each ordinal $\alpha < \omega^\omega$ may be written in the form $\alpha = \mathrm{Ord}(a_{n-1}, a_{n-2}, \ldots, a_0) < \omega^n$ for some integer $n \geq 1$ and where $a_{n-1}, a_{n-2}, \ldots, a_0$, are non-negative integers with $a_{n-1} \neq 0$.
The tree language $\mathcal{T}_\alpha$ is then the set of trees $t \in T^\omega_{\{0,1\}}$ for which there exist some integers $b_{n-1}, b_{n-2}, \ldots, b_0 \geq 0$ such that:

1. $\mathrm{Ord}(b_{n-1}, b_{n-2}, \ldots, b_0) < \mathrm{Ord}(a_{n-1}, a_{n-2}, \ldots, a_0)$.
2. $t_{l^{b_{n-1}}.r.l^{b_{n-2}}.r\cdots l^{b_0}.r}$ is in $\mathcal{L}$ and the parity of $\mathrm{Ord}(b_{n-1}, b_{n-2}, \ldots, b_0)$ is odd iff the parity of $\mathrm{Ord}(a_{n-1}, a_{n-2}, \ldots, a_0)$ is even.
3. If $\mathrm{Ord}(c_{n-1}, c_{n-2}, \ldots, c_0) < \mathrm{Ord}(b_{n-1}, b_{n-2}, \ldots, b_0)$ then the tree $t_{l^{c_{n-1}}.r.l^{c_{n-2}}.r\cdots l^{c_0}.r}$ is not in $\mathcal{L}$.

The tree language $\mathcal{T}_\alpha$ is $D_\alpha(\mathbf{\Sigma}^1_1)$-complete and it is accepted by a (non deterministic) Muller tree automaton.

The above results show that the topological complexity of tree languages recognized by *non deterministic* Muller tree automata is much greater than that of tree languages accepted by *deterministic* Muller tree automata.

Recall that a Büchi (respectively, Muller) tree automaton $\mathcal{A}$, reading trees labelled in the alphabet Σ, is said to be unambiguous if and only if each tree $t \in T^\omega_\Sigma$ admits at most one accepting run of $\mathcal{A}$.

A natural question is whether the tree languages $\mathcal{L}_n$ could be accepted by unambiguous Muller tree automata. A first step would be to prove that the tree language $\mathcal{L}$ is accepted by an unambiguous Muller tree automaton. But this is not possible. We have learned by personal communication from Damian Niwinski that the language $\mathcal{L}$ is inherently ambiguous, [Niw09].

We consider now the notion of ambiguity for Büchi tree automata and we shall prove in particular that a tree language accepted by an unambiguous Büchi tree automaton must be Borel. We shall indicate also why our methods do not work in the case of Muller automata.

We first recall some notations and a lemma proved in [FS03].
For two finite alphabets Σ and X, if $B \subseteq \Sigma^\omega \times X^\omega$ and $\alpha \in \Sigma^\omega$, we denote $B_\alpha = \{\beta \in X^\omega \mid (\alpha, \beta) \in B\}$ and $\mathrm{PROJ}_{\Sigma^\omega}(B) = \{\alpha \in \Sigma^\omega \mid B_\alpha \neq \emptyset\}$.
The cardinal of the continuum will be denoted by $2^{\aleph_0}$; it is also the cardinal of every set Σ^ω or T^ω_Σ, where Σ is an alphabet having at least two letters.

Lemma 4.9. ([FS03])
Let Σ and X be two finite alphabets having at least two letters and B be a Borel subset of $\Sigma^\omega \times X^\omega$ such that $\mathrm{PROJ}_{\Sigma^\omega}(B)$ is not a Borel subset of Σ^ω. Then there are $2^{\aleph_0}$ ω-words $\alpha \in \Sigma^\omega$ such that the section B_α has cardinality $2^{\aleph_0}$.

Proof. Let Σ and X be two finite alphabets having at least two letters and B be a Borel subset of $\Sigma^\omega \times X^\omega$ such that $\mathrm{PROJ}_{\Sigma^\omega}(B)$ is not Borel.

In a first step we prove that there are uncountably many $\alpha \in \Sigma^\omega$ such that the section B_α is uncountable.

Recall that by a Theorem of Lusin and Novikov, see [Kec95, page 123], if for all $\alpha \in \Sigma^\omega$, the section B_α of the Borel set B was countable, then $\mathrm{PROJ}_{\Sigma^\omega}(B)$ would be a Borel subset of Σ^ω.

Thus there exists at least one $\alpha \in \Sigma^\omega$ such that B_α is uncountable. In fact we have not only one α such that B_α is uncountable.

For $\alpha \in \Sigma^\omega$ we have $\{\alpha\} \times B_\alpha = B \cap [\{\alpha\} \times X^\omega]$. But $\{\alpha\} \times X^\omega$ is a closed hence Borel subset of $\Sigma^\omega \times X^\omega$ thus $\{\alpha\} \times B_\alpha$ is Borel as intersection of two Borel sets.

If there was only one $\alpha \in \Sigma^\omega$ such that B_α is uncountable, then $C = \{\alpha\} \times B_\alpha$ would be Borel so $D = B - C$ would be borel because the class of Borel sets is closed under boolean operations.
But all sections of D would be countable thus $\mathrm{PROJ}_{\Sigma^\omega}(D)$ would be Borel by Lusin and Novikov's Theorem. Then $\mathrm{PROJ}_{\Sigma^\omega}(B) = \{\alpha\} \cup \mathrm{PROJ}_{\Sigma^\omega}(D)$ would be also Borel as union of two Borel sets, and this would lead to a contradiction.

In a similar manner we can prove that the set $U = \{\alpha \in \Sigma^\omega \mid B_\alpha \text{ is uncountable } \}$ is uncountable, otherwise $U = \{\alpha_0, \alpha_1, \ldots \alpha_n, \ldots\}$ would be Borel as the countable union of the closed sets $\{\alpha_i\}$, $i \geq 0$.
For each $n \geq 0$ the set $\{\alpha_n\} \times B_{\alpha_n}$ would be Borel, and $C = \cup_{n \in \omega} \{\alpha_n\} \times B_{\alpha_n}$ would be Borel as a countable union of Borel sets. So $D = B - C$ would be borel too.
But all sections of D would be countable thus $\mathrm{PROJ}_{\Sigma^\omega}(D)$ would be Borel by Lusin and Novikov's Theorem. Then $\mathrm{PROJ}_{\Sigma^\omega}(B) = U \cup \mathrm{PROJ}_{\Sigma^\omega}(D)$ would be also Borel as union of two Borel sets, and this would lead to a contradiction.

So we have proved that the set $\{\alpha \in \Sigma^\omega \mid B_\alpha \text{ is uncountable } \}$ is uncountable.

On the other hand we know from another Theorem of Descriptive Set Theory that the set $\{\alpha \in \Sigma^\omega \mid B_\alpha \text{ is countable } \}$ is a $\mathbf{\Pi^1_1}$-subset of Σ^ω, see [Kec95, page 123]. Thus its complement $\{\alpha \in \Sigma^\omega \mid$

B_α is uncountable $\}$ is analytic. But by Suslin's Theorem an analytic subset of Σ^ω is either countable or has cardinality $2^{\aleph_0}$, [Kec95, p. 88]. Therefore the set $\{\alpha \in \Sigma^\omega \mid B_\alpha \text{ is uncountable } \}$ has cardinality $2^{\aleph_0}$.

Recall now that we have already seen that, for each $\alpha \in \Sigma^\omega$, the set $\{\alpha\} \times B_\alpha$ is Borel. Thus B_α itself is Borel and by Suslin's Theorem B_α is either countable or has cardinality $2^{\aleph_0}$. From this we deduce that $\{\alpha \in \Sigma^\omega \mid B_\alpha \text{ is uncountable } \} = \{\alpha \in \Sigma^\omega \mid B_\alpha \text{ has cardinality } 2^{\aleph_0}\}$ has cardinality $2^{\aleph_0}$. □

This Lemma was used in [FS03] to prove that analytic but non Borel context-free ω-languages have a maximum degree of ambiguity.

Theorem 4.10. ([FS03])
Let $L(\mathcal{A})$ be a context-free ω-language accepted by a Büchi pushdown automaton $\mathcal{A}$ such that $L(\mathcal{A})$ is an analytic but non Borel set. Then the set of ω-words, which have $2^{\aleph_0}$ accepting runs by $\mathcal{A}$, has cardinality $2^{\aleph_0}$.

Reasoning in a very similar way as in the proof of Theorem 4.10 in [FS03], we can now state that analytic but non Borel tree languages accepted by Büchi tree automata have a maximum degree of ambiguity.

If Σ is an alphabet having at least two letters, the topological space T_Σ^ω is homeomorphic to the topological space Σ^ω, so we can first state Lemma 4.9 in the following equivalent form.

Lemma 4.11. Let Σ and K be two finite alphabets having at least two letters and B be a Borel subset of $T_\Sigma^\omega \times T_K^\omega$ such that $\mathrm{PROJ}_{T_\Sigma^\omega}(B)$ is not a Borel subset of T_Σ^ω. Then there are $2^{\aleph_0}$ infinite trees $t \in T_\Sigma^\omega$ such that the section B_t has cardinality $2^{\aleph_0}$.

We can now state the following result.

Theorem 4.12. Let $L(\mathcal{A}) \subseteq T_\Sigma^\omega$ be a regular tree language accepted by a Büchi tree automaton $\mathcal{A}$ such that $L(\mathcal{A})$ is an analytic but non Borel set. Then the set of trees $t \in T_\Sigma^\omega$ which have $2^{\aleph_0}$ accepting runs by $\mathcal{A}$, has cardinality $2^{\aleph_0}$.

Proof. Let $\mathcal{A} = (K, \Sigma, \Delta, q_0, F)$ be a Büchi tree automaton accepting a non Borel tree language $L(\mathcal{A}) \subseteq T_\Sigma^\omega$, and let $R \subseteq T_\Sigma^\omega \times T_K^\omega$ be defined by :

$$R = \{(t, \rho) \mid t \in T_\Sigma^\omega \text{ and } \rho \in T_K^\omega \text{ is an accepting run of } \mathcal{A} \text{ on the tree } t\}.$$

The set R can be seen as a tree language over the product alphabet $\Sigma \times K$. Then it is easy to see that R is accepted by a *deterministic* Büchi tree automaton. But every tree language which is accepted by a *deterministic* Büchi tree automaton is a $\mathbf{\Pi}_2^0$-set, see [Mur05]. Thus the tree language R is a $\mathbf{\Pi}_2^0$-subset of the space $T_{(\Sigma\times K)^\omega}$ which is identified to the topological space $T_\Sigma^\omega \times T_K^\omega$. In particular, R is a Borel subset of $T_\Sigma^\omega \times T_K^\omega$. But by definition of R it turns out that $\mathrm{PROJ}_{T_\Sigma^\omega}(R) = L(\mathcal{A})$. Thus $\mathrm{PROJ}_{T_\Sigma^\omega}(R)$ is not Borel and Lemma 4.11 implies that there are $2^{\aleph_0}$ trees $t \in T_\Sigma^\omega$ such that R_t has cardinality $2^{\aleph_0}$. This means that these trees have $2^{\aleph_0}$ accepting runs by the Büchi tree automaton $\mathcal{A}$. □

Remark 4.13. The above proof is no longer valid if we replace "Büchi tree automaton" by "Muller tree automaton". Indeed if $L(\mathcal{A}) \subseteq T_\Sigma^\omega$ is a regular tree language accepted by a Muller tree automaton $\mathcal{A} = (K, \Sigma, \Delta, q_0, \mathcal{F})$, then the set $R \subseteq T_\Sigma^\omega \times T_K^\omega$ defined by :

$$R = \{(t, \rho) \mid t \in T_\Sigma^\omega \text{ and } \rho \in T_K^\omega \text{ is an accepting run of } \mathcal{A} \text{ on the tree } t\}.$$

is now accepted by a deterministic **Muller** tree automaton. Thus we can now only say that R is a $\mathbf{\Pi}_1^1$-set, and we cannot use the fact that R is Borel, which was crucial in the proof of Theorem 4.12.

In particular, Theorem 4.12 implies the following important result.

Corollary 4.14. Let $L(\mathcal{A}) \subseteq T_\Sigma^\omega$ be a regular tree language accepted by an unambiguous Büchi tree automaton. Then the tree language $L(\mathcal{A})$ is a Borel subset of T_Σ^ω.

Remark 4.15. The result given by Corollary 4.14 is weaker than the result given by Theorem 4.12. This weaker result can be proved by a simpler argument. We give now this proof which is also interesting.

Proof. Let $L(\mathcal{A}) \subseteq T_\Sigma^\omega$ be a regular tree language accepted by an unambiguous Büchi tree automaton $\mathcal{A} = (K, \Sigma, \Delta, q_0, F)$. Let R be defined as in the proof of Theorem 4.12 by:

$$R = \{(t, \rho) \mid t \in T_\Sigma^\omega \text{ and } \rho \in T_K^\omega \text{ is an accepting run of } \mathcal{A} \text{ on the tree } t\}.$$

The set R is accepted by a *deterministic* Büchi tree automaton so it is a $\mathbf{\Pi}_2^0$-subset of the space $T_{(\Sigma \times K)^\omega}$. Consider now the projection $\mathrm{PROJ}_{T_\Sigma^\omega} : T_\Sigma^\omega \times T_K^\omega \to T_\Sigma^\omega$ defined by $\mathrm{PROJ}_{T_\Sigma^\omega}(t, \rho) = t$ for all $(t, \rho) \in T_\Sigma^\omega \times T_K^\omega$. This projection is a continuous function and it is *injective* on the Borel set R because the automaton $\mathcal{A}$ is unambiguous. By a Theorem of Lusin and Souslin, see [Kec95, Theorem 15.1 page 89], the injective image of R by the continuous function $\mathrm{PROJ}_{T_\Sigma^\omega}$ is then Borel. Thus the tree language $L(\mathcal{A}) = \mathrm{PROJ}_{T_\Sigma^\omega}(R)$ is a Borel subset of T_Σ^ω. □

Remark 4.16. The above result given by Corollary 4.14 is of course false in the case of Muller automata because we already know an example of non Borel regular tree language accepted by a *deterministic hence unambiguous* Muller tree automaton. By Lemma 4.3, the tree language $\mathcal{L}^- = \forall \mathrm{Path}(\{0,1\}^\omega - (0^\star.1)^\omega)$ is a $\mathbf{\Pi}_1^1$-complete set accepted by a deterministic Muller tree automaton.

5. Game tree languages

Game tree languages are particular recognizable tree languages which are defined by the use of parity games. So we now recall the definition of these games, as introduced in [AN08, ADMN07].

A parity game is a game with perfect information between two players named Eve and Adam, as in [AN08, ADMN07].
The game is defined by a tuple $G = (V_\exists, V_\forall, \mathrm{Move}, p_0, \mathrm{rank})$. The sets $V_\exists$ and $V_\forall$ are disjoint sets of positions of Eve and Adam, respectively. We denote $V = V_\exists \cup V_\forall$ the set of positions. The relation $\mathrm{Move} \subseteq V \times V$ is the relation of possible moves. The initial position in a play is $p_0 \in V$. The ranking function is $\mathrm{rank} : V \to \omega$ and the number of values taken by this function is finite.
At the beginning of a play there is a token at the initial position p_0 where the play starts. The players

move the token according to the relation Move, always to a successor of the current position. The move is done by Eve if the current position is an element of $V_\exists$, otherwise Adam moves the token. This way the two players form a path in the graph (V, Move). If at some moment a player cannot move then she or he looses. Otherwise the two players construct an infinite path in the graph, $v_0, v_1, v_2, \ldots$ In this case Eve wins the play if $\limsup_{n\to\infty} \mathrm{rank}(v_n)$ is even, otherwise Adam wins the play.
Eve (respectively, Adam) wins the game G if she (respectively, he) has a winning strategy. It is well known that parity games are determined, i. e., that one of the players has a winning strategy. Moreover any position is winning for one of the players and she or he has a *positional* strategy from this position, see [GTW02] for more details.

We now recall the definition of game languages $W_{(\iota,\kappa)}$.
A Mostowski-Rabin index is a pair (ι, κ), where $\iota \in \{0, 1\}$ and $\iota \leq \kappa < \omega$. For such an index, we define the alphabet $\Sigma_{(\iota,\kappa)} = \{\exists, \forall\} \times \{\iota, \ldots, \kappa\}$.
For a letter $a \in \Sigma_{(\iota,\kappa)}$ we denote $a = (a_1, a_2)$, where $a_1 \in \{\exists, \forall\}$ and $a_2 \in \{\iota, \ldots, \kappa\}$.
For each tree $t \in T^\omega_{\Sigma_{(\iota,\kappa)}}$ we associate a parity game $G(t) = (V_\exists, V_\forall, \mathrm{Move}, p_0, \mathrm{rank})$, where

- $V_\exists = \{v \in \{l, r\}^\star \mid t(v)_1 = \exists\}$,
- $V_\forall = \{v \in \{l, r\}^\star \mid t(v)_1 = \forall\}$,
- $\mathrm{Move} = \{(w, wi) \mid w \in \{l, r\}^\star \text{ and } i \in \{l, r\}\}$,
- $p_0 = \lambda$ is the root of the tree,
- $\mathrm{rank}(v) = t(v)_2$, for each $v \in \{l, r\}^\star$.

The set $W_{(\iota,\kappa)} \subseteq T^\omega_{\Sigma_{(\iota,\kappa)}}$ is the set of infinite binary trees t labelled in the alphabet $\Sigma_{(\iota,\kappa)}$ such that Eve wins the associated game $G(t)$.

The recognizable tree language $W_{(\iota,\kappa)}$ is accepted by an alternating parity tree automaton of index (ι, κ). This notion will be useful in the sequel so we recall it now, as presented in [ADMN07].

Definition 5.1. An alternating parity tree automaton is a tuple $\mathcal{A} = (\Sigma, Q_\exists, Q_\forall, q_0, \delta, \mathrm{rank})$, where the set of states Q is partitioned in $Q_\exists$ and $Q_\forall$. The set $Q_\exists$ is the set of existential states and the set $Q_\forall$ is the set of universal states. The transition relation is $\delta \subseteq Q \times \Sigma \times \{l, r, \lambda\} \times Q$ and $\mathrm{rank} : Q \to \omega$ is the rank function. A tree $t \in T^\omega_\Sigma$ is accepted by the automaton $\mathcal{A}$ iff Eve has a winning strategy in the parity game $(Q_\exists \times \{l, r\}^\star, Q_\forall \times \{l, r\}^\star, (q_0, \lambda), \mathrm{Move}, \Omega)$, where $\mathrm{Move} = \{((p, v), (q, vd)) \mid v \in \mathrm{dom}(t),\ (p, t(v), d, q) \in \delta\}$ and $\Omega(q, v) = \mathrm{rank}(q)$.

Notice that it can be assumed without lost of generality that $\min \mathrm{rank}(Q)$ is equal to 0 or 1. The pair $(\min \mathrm{rank}(Q), \max \mathrm{rank}(Q))$ is called the Mostowski-Rabin index of the automaton.
It follows from [Rab69] that any alternating parity tree automaton can be simulated by a non deterministic Muller automaton, see also [GTW02].

There is a usual partial order on Mostowski-Rabin indices: $(\iota, \kappa) \sqsubseteq (\iota', \kappa')$ if either $\iota' \leq \iota$ and $\kappa \leq \kappa'$ (i.e. $\{\iota, \ldots, \kappa\} \subseteq \{\iota', \ldots, \kappa'\}$), or $\iota = 0, \iota' = 1$, and $\kappa + 2 \leq \kappa'$ (i.e. $\{\iota + 2, \ldots, \kappa + 2\} \subseteq \{\iota', \ldots, \kappa'\}$).
The indices $(1, n)$ and $(0, n - 1)$ are called dual and $\overline{(\iota, \kappa)}$ denotes the index dual to (ι, κ).

It is easy to see that each tree language $W_{(\iota,\kappa)}$ is accepted by an alternating parity tree automaton of index (ι, κ).

Moreover the set $W_{(\iota,\kappa)}$ is in some sense of the greatest possible topological complexity among tree languages accepted by alternating parity tree automata of index (ι, κ). This is expressed by the following lemma.

Lemma 5.2. (see [ADMN07])
If a set of trees T is recognized by an alternating parity tree automaton of index (ι, κ), then $T \leq_W W_{(\iota,\kappa)}$.

In order to use this result to get a lower bound on the topological complexity of the game tree languages $W_{(\iota,\kappa)}$, we first construct some alternating parity tree automata accepting the tree languages $\mathcal{L}$ and $\mathcal{L}^-$ defined in the preceding section.

Lemma 5.3. The tree language $\mathcal{L}$ is accepted by an alternating parity tree automaton of index $(1, 2)$.

Proof. Recall that $\mathcal{L} = \exists\mathrm{Path}(\mathcal{R})$, where $\mathcal{R} = (0^\star.1)^\omega$.
The tree language $\mathcal{L}$ is then accepted by the alternating parity tree automaton $\mathcal{A} = (\Sigma, Q_\exists, Q_\forall, q_0, \delta, \mathrm{rank})$, where
$\Sigma = \{0, 1\}$,
$Q_\exists = Q = \{q_0, q_1\}$,
$Q_\forall = \emptyset$,
$\delta = \{(q, 1, d, q_1), (q, 0, d, q_0) \mid q \in Q \text{ and } d \in \{l, r\}\}$,
$\mathrm{rank}(q_0) = 1$ and $\mathrm{rank}(q_1) = 2$. □

Notice that in the above automaton $\mathcal{A}$ all states are existential.

Lemma 5.4. The tree language $\mathcal{L}^-$ is accepted by an alternating parity tree automaton of index $(0, 1)$.

Proof. Recall that $\mathcal{L}^- = T^\omega_\Sigma - \mathcal{L} = \forall\mathrm{Path}(\{0, 1\}^\omega - (0^\star.1)^\omega)$.
The tree language $\mathcal{L}^-$ is then accepted by the alternating parity tree automaton $\mathcal{A}' = (\Sigma, Q'_\exists, Q'_\forall, q'_0, \delta', \mathrm{rank}')$, where
$\Sigma = \{0, 1\}$,
$Q'_\exists = \emptyset$,
$Q'_\forall = Q' = \{q'_0, q'_1\}$,
$\delta' = \{(q', 1, d, q'_1), (q', 0, d, q'_0) \mid q' \in Q' \text{ and } d \in \{l, r\}\}$,
$\mathrm{rank}'(q'_0) = 0$ and $\mathrm{rank}'(q'_1) = 1$.

□

Notice that in the above automaton $\mathcal{A}'$ all states are universal.

Remark 5.5. The $\mathbf{\Sigma}^1_1$-complete tree language $\mathcal{L}$ is accepted by an alternating parity tree automaton of index $(1, 2)$ and the $\mathbf{\Pi}^1_1$-complete tree language $\mathcal{L}^-$ is accepted by an alternating parity tree automaton of index $(0, 1)$. In fact for every tree language T accepted by an alternating parity tree automaton of index $(1, 2)$ (respectively, $(0, 1)$) it holds that T is in the class $\mathbf{\Sigma}^1_1$ (respectively, $\mathbf{\Pi}^1_1$), see [ADMN07, Theorem 3.6].

Recall now the definition of the $D_\omega(\mathbf{\Sigma}^1_1)$-complete tree language $\mathcal{L}_1$.

$\mathcal{L}_1 = \{t \in T^\omega_{\{0,1\}} \mid \exists n \geq 0 \ \ t_{l^n.r} \in \mathcal{L}$ and $\min\{n \geq 0 \mid t_{l^n.r} \in \mathcal{L}\}$ is odd $\}$.

We can now state the following result.

Lemma 5.6. The tree language $\mathcal{L}_1$ is accepted by an alternating parity tree automaton of index $(0, 2)$.

Proof. Let, as in the proofs of the two previous lemmas, $\mathcal{A} = (\Sigma, Q_\exists, Q_\forall, q_0, \delta, \mathrm{rank})$ be an alternating parity tree automaton of index $(1, 2)$ accepting the tree language $\mathcal{L} = \exists\mathrm{Path}(\mathcal{R})$, and $\mathcal{A}' = (\Sigma, Q'_\exists, Q'_\forall, q'_0, \delta', \mathrm{rank}')$ be an alternating parity tree automaton of index $(0, 1)$ accepting the tree language $\mathcal{L}^-$. We assume that $Q \cap Q' = \emptyset$, where $Q = Q_\exists \cup Q_\forall = Q_\exists$ and $Q' = Q'_\exists \cup Q'_\forall = Q'_\forall$.
It is then easy to see that the tree language $\mathcal{L}_1$ is accepted by the alternating parity tree automaton $\mathcal{A}^1 = (\Sigma, Q^1_\exists, Q^1_\forall, q^1_0, \delta^1, \mathrm{rank}^1)$, where
$\Sigma = \{0, 1\}$,
$Q^1_\exists = Q_\exists \cup Q'_\exists \cup \{q_\exists\} = Q_\exists \cup \{q_\exists\}$,
$Q^1_\forall = Q_\forall \cup Q'_\forall \cup \{q^1_0, q^1_1\} = Q'_\forall \cup \{q^1_0, q^1_1\}$,
$\delta^1 = \delta \cup \delta' \cup \{(q^1_0, a, l, q_\exists), (q^1_0, a, r, q'_0), (q_\exists, a, r, q_0), (q_\exists, a, \lambda, q^1_1), (q^1_1, a, r, q'_0), (q^1_1, a, l, q^1_0) \mid a \in \{0, 1\}\}$,
$\mathrm{rank}^1(q) = \mathrm{rank}(q)$ for $q \in Q$,
$\mathrm{rank}^1(q') = \mathrm{rank}'(q')$ for $q' \in Q'$,
$\mathrm{rank}^1(q^1_0) = 0$, $\mathrm{rank}^1(q^1_1) = 1$.

□

Notice that in the above construction of the alternating automaton $\mathcal{A}^1$ the universal states q^1_0, q^1_1 and the existential state $q_\exists$ are used to choose, when reading a tree $t \in \mathcal{L}_1$, the least integer n such that $t_{l^n.r} \in \mathcal{L}$ and to check that this integer is really the least (and odd) one with this property.
In a very similar manner, for each integer $n \geq 1$, we can define an alternating parity tree automaton $\mathcal{A}^n$ of index $(0, 2)$ accepting the language $\mathcal{L}_n$. The complete description would be tedious but the idea is that now the additional universal or existential states not in $Q \cup Q'$ are used to choose, for a given tree $t \in \mathcal{L}_n$, the least ordinal $\alpha = \omega^{n-1} \cdot a_{n-1} + \omega^{n-2} \cdot a_{n-2} + \ldots + \omega \cdot a_1 + a_0$ such that $t_{l^{a_{n-1}}.r.l^{a_{n-2}}.r \ldots l^{a_0}.r}$ is in $\mathcal{L}$ and to check that α is odd and that for any smaller ordinal $\beta = \mathrm{Ord}(b_{n-1}, b_{n-2}, \ldots, b_0) < \alpha$, the tree $t_{l^{b_{n-1}}.r.l^{b_{n-2}}.r \ldots l^{b_0}.r}$ is not in $\mathcal{L}$.

We can then state the following result.

Proposition 5.7. For each integer $n \geq 1$, the tree language $\mathcal{L}_n$ is accepted by an alternating parity tree automaton of index $(0, 2)$.

We can now infer from Theorem 4.6, Proposition 5.7, and Lemma 5.2, the following result.

Theorem 5.8. For each integer $n \geq 1$, the $D_{\omega^n}(\mathbf{\Sigma}^1_1)$-complete tree language $\mathcal{L}_n$ is Wadge reducible to the game tree language $W_{(0,2)}$, i.e. $\mathcal{L}_n \leq_W W_{(0,2)}$. In particular the language $W_{(0,2)}$ is not in any class $D_\alpha(\mathbf{\Sigma}^1_1)$ for $\alpha < \omega^\omega$.

On the other hand, Arnold and Niwinski proved in [AN08] that the game tree languages form a hierarchy with regard to the Wadge reducibility.

Theorem 5.9. ([AN08])
For all Mostowski-Rabin indices (ι, κ) and (ι', κ'), it holds that :

$$(\iota, \kappa) \sqsubseteq (\iota', \kappa') \quad \text{if and only if} \quad W_{(\iota,\kappa)} \leq_W W_{(\iota',\kappa')}$$

Then we can state the following result.

Theorem 5.10. For each integer $n \geq 1$ and each Mostowski-Rabin index (ι, κ) such that $(0, 2) \sqsubseteq (\iota, \kappa)$ or $(\iota, \kappa) = (1, 3) = \overline{(0, 2)}$, the $D_{\omega^n}(\mathbf{\Sigma}^1_1)$-complete tree language $\mathcal{L}_n$ is Wadge reducible to the game tree language $W_{(\iota,\kappa)}$, i.e. $\mathcal{L}_n \leq_W W_{(\iota,\kappa)}$. In particular the language $W_{(\iota,\kappa)}$ is not in any class $D_\alpha(\mathbf{\Sigma}^1_1)$ for $\alpha < \omega^\omega$.

Proof. The result follows directly from Theorems 5.8 and 5.9 in the case $(0, 2) \sqsubseteq (\iota, \kappa)$. What remains is the case of the index $(1, 3)$ which is the dual of the index $(0, 2)$. But it is proved in [AN08, Lemma 1] that $W_{\overline{(\iota,\kappa)}}$ coincide with $\overline{W_{(\iota,\kappa)}} = T^\omega_{\Sigma_{(\iota,\kappa)}} - W_{(\iota,\kappa)}$ up to renaming of symbols. On the other hand, we know from Theorem 5.8 that for each integer $n \geq 1$, the $D_{\omega^{n+1}}(\mathbf{\Sigma}^1_1)$-complete tree language $\mathcal{L}_{n+1}$ is Wadge reducible to the game tree language $W_{(0,2)}$, i.e. $\mathcal{L}_{n+1} \leq_W W_{(0,2)}$. This is easily seen to be equivalent to $T^\omega_{\{0,1\}} - \mathcal{L}_{n+1} \leq_W \overline{W_{(0,2)}}$, i.e. $T^\omega_{\{0,1\}} - \mathcal{L}_{n+1} \leq_W W_{(1,3)}$. But $\mathcal{L}_n$ is $D_{\omega^n}(\mathbf{\Sigma}^1_1)$-complete and $\mathcal{L}_{n+1}$ is $D_{\omega^{n+1}}(\mathbf{\Sigma}^1_1)$-complete so it follows from the properties of the difference hierarchy of analytic sets that $\mathcal{L}_n \leq_W T^\omega_{\{0,1\}} - \mathcal{L}_{n+1}$ and so $\mathcal{L}_n \leq_W W_{(1,3)}$ by transitivity of the relation $\leq_W$. □

6. Concluding remarks

We have got some new results on the topological complexity of non Borel recognizable tree languages with regard to the difference hierarchy of analytic sets. In particular, we have showed that the game tree language $W_{(0,2)}$ is not in any class $D_\alpha(\mathbf{\Sigma}^1_1)$ for $\alpha < \omega^\omega$. The great challenge in the study of the topological complexity of recognizable tree languages is to determine the Wadge hierarchy of tree languages accepted by *non deterministic* Muller or Rabin tree automata. Notice that the case of *deterministic* Muller or Rabin tree automata have been solved recently by Murlak, [Mur08].

It would be interesting to locate in a more precise way the game tree languages with regard to the difference hierarchy of analytic sets. We already know that $W_{(0,2)}$ is not in any class $D_\alpha(\mathbf{\Sigma}^1_1)$ for $\alpha < \omega^\omega$. Is there an ordinal α such that $W_{(0,2)}$ is in $D_\alpha(\mathbf{\Sigma}^1_1)$ and then what is the smallest such ordinal α? The same question may be asked for the other game tree languages $W_{(\iota,\kappa)}$. On the other hand, there are some sets in the class $\mathbf{\Delta}^1_2$ which does not belong to the σ-algebra generated by the analytic sets, see [Kec95, Exercise 37.8]. Could we expect that $W_{(0,2)}$ or another game tree language $W_{(\iota,\kappa)}$ is such an example?

Acknowledgements. We thank the anonymous referees for their very helpful comments which have led to a great improvement of our paper.

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