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# High-gain observer with uniform in the initial condition finite time convergence

Vincent Andrieu<sup>\*</sup>      Laurent Praly<sup>†</sup>      Alessandro Astolfi<sup>‡</sup>

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## 1 Introduction

In this short note, we show how the framework introduced in [1] allows to obtain an observer with finite time convergence for globally Lipchitz upper triangular systems.

## 2 Finite time observer

we introduce an observer for systems of the form :

$$\dot{x} = \mathcal{S}x + Bu + \delta(x, t) \quad , \quad y = x_1 \quad , \quad (1)$$

where  $x = (x_1, \dots, x_n)$  is in  $\mathbb{R}^n$  and  $\delta : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is a continuous function globally Lipchitz in its first argument (uniformly in  $t$ ).

The domination approach has been used to design observer for systems of the form (1). This approach has been popularized by high-gain observer [2]. These observers are given as:

$$\dot{\hat{x}} = \mathcal{S}\hat{x} + Bu + \delta(\hat{x}, t) + L\mathfrak{L}^{-1}K(\hat{x}_1 - y) \quad (2)$$

where  $L$  is the extra high-gain parameter,  $\mathfrak{L} = \text{diag}(1, L^{-1}, L^{-2}, \dots, L^{1-n})$  and  $K$  is the output injection which have to be designed to ensure that the state of the error system:

$$\dot{\tilde{x}} = \mathcal{S}\tilde{x} + \delta(\tilde{x}, t) - \delta(\hat{x} - \tilde{x}, t) + L\mathfrak{L}^{-1}K(\tilde{x}_1) \quad (3)$$

converges to the origin.

The error system (3) has the structure of a chain of integrators disturbed by nonlinear terms which, assuming a global Lipschitz condition (as in [2]), is linearly bounded. In [2], the

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domination approach has been employed and a linear vector field  $K$  in the observer (2) was introduced to ensure global and asymptotic convergence of the error  $\tilde{x}$  toward the origin.

Recently, this approach has been extended in [4] (see also [3]) to a homogeneous vector field  $K$  with negative degree to allow semi-global and finite time estimation.

The homogeneous in the bi-limit vector field  $K$  obtained from [1, Section 3] allows us to get a global observer with finite-time estimation and with an estimation time uniform in the initial condition:

**Corollary 1 (Finite time observer)** *If for  $(x, \tilde{x})$  in  $\mathbb{R}^{2n}$ ,*

$$|\delta_i(x + \tilde{x}, t) - \delta_i(x, t)| \leq c \sum_{j=1}^i |\tilde{x}_j| \quad (4)$$

*where  $c$  is a positive real numbers, then there exist a continuous vector field  $K : \mathbb{R} \rightarrow \mathbb{R}^n$  and a real number  $L^* > 0$  such that for every  $L$  in  $[L^*, +\infty)$ , the estimate given by the system (2) converges to the state of system (1) in finite time uniformly in the initial condition, i.e., there exists a positive real number  $T$  such that for all initial state  $x_0$  in  $\mathbb{R}^n$ , initial estimate  $\hat{x}_0$  in  $\mathbb{R}^n$  and all locally bounded continuous function  $u : [0, T] \rightarrow \mathbb{R}$ , we get:*

$$x(T) = \hat{x}(T)$$

*where  $(x, \hat{x}) : \mathbb{R} \rightarrow \mathbb{R}^{2n}$  is a  $C^1$  functions solution of systems (1) and (2) such that  $x(0) = x_0$  and  $\hat{x}(0) = \hat{x}_0$ .*

**Proof :** To construct the vector field  $K$  we employ the homogeneous in the bi-limit framework and the procedure introduced in [1]. We introduce two real numbers  $\mathfrak{d}_0$  and  $\mathfrak{d}_\infty$  (the degree of the homogeneous in the bi-limit vector field  $K$ ) such that

$$-1 < \mathfrak{d}_0 < 0 < \mathfrak{d}_\infty < \frac{1}{n-1} . \quad (5)$$

As in [1], we introduce the associated weights vector  $r_0$  and  $r_\infty$  both in  $\mathbb{R}_+^n$  defined as

$$r_{b,n} = 1 , \quad r_{b,i} = r_{b,i+1} - \mathfrak{d}_b = 1 - \mathfrak{d}_b(n-i) , \quad (6)$$

where the letter "b" stand for "0" or " $\infty$ ". Following the procedure [1, Section 3], we obtain a homogeneous in the bi-limit vector field  $K : \mathbb{R} \rightarrow \mathbb{R}^n$  with associated triples  $(r_0, \mathfrak{d}_0, K_0)$  and  $(r_\infty, \mathfrak{d}_\infty, K_\infty)$  such that the origin of the systems with state  $z = (z_1, \dots, z_n)$  in  $\mathbb{R}^n$  :

$$\begin{aligned} \dot{z} &= \mathcal{S}z + K(z_1) , \\ \dot{z} &= \mathcal{S}z + K_0(z_1) , \\ \dot{z} &= \mathcal{S}z + K_\infty(z_1) , \end{aligned}$$

is globally and asymptotically stable. Hence, we can employ [1, Corollary 2.22] to get a positive real number  $c_G$  such that for all continuous function  $R : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ , satisfying

$$R_i(z, t) \leq c_G \left( \sum_{j=1}^i |z_j|^{\frac{1-\mathfrak{d}_0(n-i-1)}{1-\mathfrak{d}_0(n-j)}} + \sum_{j=1}^i |z_j|^{\frac{1-\mathfrak{d}_\infty(n-i-1)}{1-\mathfrak{d}_\infty(n-j)}} \right) \quad i = 1, \dots, n , \quad (7)$$

where  $R(z, t) = (R_1(z, t), \dots, R_n(z, t))$ , the origin of the system :

$$\dot{z} = \mathcal{S}z + K(z_1) + R(z, t) \quad (8)$$

is globally and asymptotically stable.

Note that since  $\mathfrak{d}_0 < 0 < \mathfrak{d}_\infty$ , it follows from Young's inequality that, given a continuous function  $R$  satisfying

$$R_i(z, t) \leq c_G \sum_{j=1}^i |z_j| .$$

then the  $R_i$ 's satisfy also the bound (7) and in this case, the origin of system (8) is globally and asymptotically stable.

We introduce now the scaled coordinates defined as :

$$e_i = L^{1-i} \tilde{x}_i \quad , \quad i = 1, \dots, n \quad , \quad (9)$$

where  $L$ , the high-gain parameter, is a positive real number which will be selected later. We can rewrite this change of coordinates in compact form as:

$$E = \mathfrak{L} \tilde{x} \quad , \quad \mathfrak{L} = \text{diag}(1, L^{-1}, L^{-2}, \dots, L^{1-n}) \quad .$$

We get along the trajectory of the error system (3) :

$$\dot{E} = L \left[ \mathcal{S}E + \Delta(L, \hat{x}, \tilde{x}, t) + K(e_1) \right] ,$$

where

$$\Delta(L, \hat{x}, \tilde{x}, t) = L^{-1} \mathcal{L} [\delta(\hat{x}, t) - \delta(\hat{x} - \tilde{x}, t)] \quad .$$

Moreover, due to (4), with  $L \geq 1$ , we get :

$$|\Delta_i(L, \hat{x}, \tilde{x}, t)| \leq L^{-i} c \sum_{j=1}^i |\tilde{x}_j| \leq L^{-1} c \sum_{j=1}^i |e_j|$$

Consequently with  $c_G$  defined in (7) and taking  $L^* > \frac{c}{c_G}$ , we get that, for all  $L$  in  $[L^*, +\infty)$ , the origin of the system:

$$\dot{E} = L \left[ \mathcal{S}E + \Delta(L, \tilde{x}, t) + K(e_1) \right] ,$$

is globally and asymptotically stable. Hence, the estimate  $\hat{x}$  converges toward the state  $x$ .

Moreover, the origin is also globally and asymptotically stable for the homogeneous approximations:

$$\begin{aligned} \dot{E} &= L \left( \mathcal{S}E + K_0(e_1) \right) , \\ \dot{E} &= L \left( \mathcal{S}E + K_\infty(e_1) \right) , \end{aligned}$$

and with (5), we can apply [1, Corollary 2.24] to obtain the result.  $\square$

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