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# RECTANGULAR $R$-TRANSFORM AT THE LIMIT OF RECTANGULAR SPHERICAL INTEGRALS

FLORENT BENAYCH-GEORGES

Abstract. In this paper, proving the analogue, for rectangular or square non symmetric real matrices, of a result that Guionnet and Maïda proved for symmetric matrices in [GM05], we connect rectangular free probability theory and spherical integrals. More specifically, we study the limit, as $n, m$ tend to infinity, of $\frac{1}{n}\log\mathbb{E}\{\exp[\sqrt{nm}\theta X_n]\}$, where $X_n$ is an entry of $U_nM_nV_m$, $\theta\in\mathbb{R}$, $M_n$ is a certain $n\times m$ deterministic matrix and $U_n, V_m$ are independent uniform random orthogonal matrices with respective sizes $n\times n$, $m\times m$. We prove that when the operator norm of $M_n$ is bounded and the singular law of $M_n$ converges to a probability measure $\mu$, for $\theta$ small enough, this limit actually exists and can be expressed with the rectangular $R$-transform of $\mu$. This gives an interpretation of this transform, which linearizes the rectangular free convolution, as the limit of a sequence of logarithms of Laplace transforms.

## Introduction

In this article, we study the limit, as $n, m$ tend to infinity in such a way that $n/m$ tends to a limit $\lambda\in[0,1]$, of

$$I_n(\theta)=\frac{1}{n}\log\mathbb{E}\{\exp[\sqrt{nm}\theta\operatorname{Tr}(E_nU_nM_nV_m)]\},$$

where $\theta\in\mathbb{R}$, $M_n$ is a certain $n\times m$ deterministic matrix, $U_n, V_m$ are independent uniform random orthogonal matrices with respective sizes $n\times n$, $m\times m$ and $E_n$ is an $m\times n$ elementary matrix (i.e. a matrix which entries are all zero, except one of them, which is equal to one).

The departure point of this study is the work of Collins, Zinn-Justin, Zuber, Guionnet, Maïda, Śniady, Mingo and Speicher who proved, in the papers [ZZ03, C03, GM05, CS06, CS07, CMSS07], that under various hypotheses on some $n\times n$ matrices $A_n$ and $B_n$ and a positive exponent $\alpha$, the asymptotics of

$$\frac{1}{n^\alpha}\log\mathbb{E}\{\exp[n\theta\operatorname{Tr}(B_nU_nA_nU_n^*)]\}$$

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are related to free probability theory. For example, it has been proved [GM05, Th. 1.2] that if the spectral law[1] of the symmetric matrix $A_n$ converges to a compactly supported probability measure $\mu$ and $B_n = \mathrm{Diag}(1, 0, \ldots, 0)$, then for $\theta$ small enough,

$$\frac{1}{n} \log \mathbb{E}\{\exp[n\theta \operatorname{Tr}(B_n U_n A_n U_n^*)]\} \underset{n\to\infty}{\longrightarrow} \frac{1}{2} \int_0^{\frac{\theta}{2}} R_\mu(t) \mathrm{d}t, \tag{1}$$

where $R_\mu$ is the so-called *R-transform* of $\mu$. The $R$-transform is an integral transform of probability measures on $\mathbb{R}$. Its main property is that it linearizes the additive free convolution $\boxplus$, the binary operation on probability measures on $\mathbb{R}$ which can be roughly defined by the fact that for $A$, $B$ large symmetric random matrices with spectral laws $\mu_A, \mu_B$ and $U$ a uniform orthogonal random matrix independent of $A$ and $B$, the spectral law of $A + UBU^*$ is approximately $\mu_A \boxplus \mu_B$: the free convolution $\boxplus$ can be thought as the analogue, for the spectral laws of certain symmetric random matrices, as the classical convolution for real random variables. Since for all probability measures $\mu, \nu$ on $\mathbb{R}$, we have

$$R_{\mu\boxplus\nu}(t) = R_\mu(t) + R_\nu(t) \qquad \text{(for } t \text{ in a neighborhood of zero)}, \tag{2}$$

in this analogy, the $R$-transform plays the role of the logarithm of the Laplace transform, and (1) gives a concrete sense to this analogy: the $R$-transform (more specifically its anti-derivative, which also satisfies (2)), is the limit of a certain sequence of Laplace transforms.

Let us now describe the content of our paper. For each $\lambda \in [0, 1]$, another free convolution, denoted by $\boxplus_\lambda$ and called the *rectangular free convolution with ratio* $\lambda$, defined in [B09a], does the same job as $\boxplus$ for the singular laws[2] of bi-orthogonaly rectangular $n \times m$ random matrices which dimensions $n, m$ tend to infinity in such a way that $n/m$ tends to $\lambda$: for $A, B$ some $n \times m$ real matrices, with $n, m$ large such that $n/m \simeq \lambda$, with singular laws $\nu_A, \nu_B$ and $U, V$ uniform orthogonal random matrices independent of $A$ and $B$, the singular law of $A + UBV$ is approximately $\nu_A \boxplus_\lambda \nu_B$ (see the introduction of [B09b] for a more precise definition of $\boxplus_\lambda$). Like the $R$-transform for $\boxplus$ and the logarithm of the Laplace transform for the classical convolution, an integral transform linearizes $\boxplus_\lambda$. It is called *the rectangular R-transform with ratio* $\lambda$ and is denoted by $C^{(\lambda)}$: for all probability measures $\mu, \nu$ on $\mathbb{R}$, we have

$$C^{(\lambda)}_{\mu\boxplus_\lambda\nu}(t) = C^{(\lambda)}_\mu(t) + C^{(\lambda)}_\nu(t) \qquad \text{(for } t \text{ in a neighborhood of zero)}. \tag{3}$$

The main result of the paper gives an interpretation of the rectangular $R$-transform (more specifically its anti-derivative, which also satisfies (3)) as the limit of a sequence of Laplace transforms: we prove that if the singular law of $M_n$ tends to a probability measure $\mu$ and $n/m$ tends to a limit $\lambda \in [0, 1]$ as $n, m$ tend to infinity, then for $\theta$ small enough,

$$\frac{1}{n} \log \mathbb{E}\{\exp[\sqrt{nm}\theta \operatorname{Tr}(E_n U_n M_n V_m)]\} \underset{n\to\infty}{\longrightarrow} \int_0^\theta \frac{C^{(\lambda)}_\mu(t^2)}{t} \mathrm{d}t. \tag{4}$$

[1]The *spectral law* of a matrix is the uniform law on its eigenvalues, counted with multiplicity.

[2]The *singular law* of a matrix is the uniform law on its singular values, counted with multiplicity.

Let us mention that free probability theory has initially, in the last three decades, been built in the area of operator algebras and that concrete relations between *free* and *classical* probability theory, like the ones of (1) and (4), are not that common.

Let us also mention that expectations of the exponential of traces of polynomials of constant matrices and uniform orthogonal random matrices, which have been extensively studied in physics and also other areas, like information theory, are often called *spherical integrals*. See e.g. [Z97, GZ02, G09] and the references above for the case of square matrices and [SW03, GT08] for the case of rectangular matrices.

The paper is organized as follows. In Section 1, we state the main result of the paper, Theorem 1.2, and discuss it. In Section 2, we recall the precise definition of the rectangular $R$-transform and prove a result of continuity of the map $(\lambda, \mu) \longmapsto C_\mu^{(\lambda)}$. At last, Section 3 is devoted to the proof of Theorem 1.2, inspired by the proof of [GM05, Th. 1.2].

## 1. Main result

Let us consider, for all $n \geq 1$, an integer $m_n \geq n$ such that, as $n$ tends to infinity, $n/m_n$ tends to a limit $\lambda \in [0,1]$ and an $n \times m_n$ nonrandom matrix $M_n$ which operator norm[3] is strictly bounded, uniformly in $n$, by a constant $K$ and such that, as $n$ tends to infinity, the singular law of $M_n$ converges weakly[4] to a probability measure that we shall denote by $\mu$. Let us define, for $\theta \in \mathbb{R}$,

$$I_n(\theta) = \frac{1}{n} \log \mathbb{E}\{\exp[\sqrt{nm_n}\theta \operatorname{Tr}(E_n U_n M_n V_n)]\},$$

where $U_n, V_n$ are independent uniform random orthogonal matrices with respective sizes $n \times n$, $m_n \times m_n$ and $E_n$ denotes an $m_n \times n$ matrix which entries are all zero, except one, which is equal to one.

In the case where $\lambda = 0$, we also suppose that there is $\alpha < 2$ such that

$$\text{for } n \text{ large enough}, \quad m_n \leq n^\alpha. \tag{5}$$

**Remark 1.1.** $I_n(\theta)$ can also be considered as the Laplace transform of a certain scalar product estimated at a pair of vectors constituted of a random vector of the unit sphere of $\mathbb{R}^n$ and of the projection, on $\mathbb{R}^n$, of a random vector of the unit sphere of $\mathbb{R}^{m_n}$. Indeed, let us denote the singular values of $M_n$ by $\mu_{n,1}, \ldots, \mu_{n,n}$ and introduce some orthogonal matrices $P_n, Q_n$ with respective sizes $n \times n$, $m_n \times m_n$ (see [HJ85]) such that

$$M_n = P_n \begin{bmatrix} \mu_{n,1} & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & \mu_{n,n} & 0 & \cdots & 0 \end{bmatrix} Q_n.$$

Let also, for each $n$, $(i_n, j_n)$ be the index of the non-null entry of $E_n$. Then the $j_n$th row (resp. $i_n$th column) $u_n = (u_{n,1}, \ldots, u_{n,n})$ (resp. $v_n = (v_{n,1}, \ldots, v_{n,m_n})^t$) of $U_n P_n$ (resp.

[3]The norms used on $\mathbb{R}^n$ and $\mathbb{R}^{m_n}$ are the canonical euclidian norms.

[4]Recall that a sequence $\mu_n$ of probability measures on $[-K, K]$ *converges weakly* to a limit $\mu$ if for each continuous function $g$ on $[-K, K]$, $\int g(t)\mathrm{d}\mu_n(t)$ tends to $\int g(t)\mathrm{d}\mu(t)$. This convergence is then uniform on any set of functions which is uniformly bounded and uniformly Lipschitz.

$Q_nV_n$) is uniformly distributed on the unit sphere of $\mathbb{R}^n$ (resp. $\mathbb{R}^{m_n}$) and one has

$$I_n(\theta) = \frac{1}{n}\log\mathbb{E}\{\exp[\sqrt{nm_n}\theta\sum_{k=1}^{n} u_{n,k}\mu_{n,k}v_{n,k}]\}. \tag{6}$$

The main result of the article is the following one.

**Theorem 1.2.** *The function $I_n$ converges uniformly on every compact subset of $(-K^{-1}, K^{-1})$ to the function*

$$I(\theta) = \int_0^\theta \frac{C_\mu^{(\lambda)}(t^2)}{t}\mathrm{d}t,$$

*where $C_\mu^{(\lambda)}$ denotes the rectangular $R$-transform of $\mu$ with ratio $\lambda$ (its definition is recalled in Section 2 bellow).*

**Remark 1.3.** Note that the function $C_\mu^{(\lambda)}$ is analytic on $(-K^{-2}, K^{-2})$ and vanishes at zero, so $I$ is actually well defined and analytic on $(-K^{-1}, K^{-1})$.

**Remark 1.4** (Cumulants point of view)**.** For $\lambda \in [0,1]$, the *rectangular free cumulants with ratio* $\lambda$ of $\mu$ have been defined in [B09a, Sect. 3.4] (see also [B07a, Sect. 2.2]): this is the sequence $(c_{2k}(\mu))_{k\geq 1}$ linked to the moments of $\mu$ by [B07b, Eq. (4.1)]. Recall also that for $X$ a bounded real random variable, the *classical cumulants* of $X$ are the numbers $\mathrm{Cl}_k(X)$ defined by the formula

$$\log\mathbb{E}(e^{zX}) = \sum_{k\geq 1}\frac{\mathrm{Cl}_k(X)}{k!}z^k.$$

Differentiating formally the convergence $I_n(\theta) \longrightarrow I(\theta)$, one would get the following "classical cumulants interpretation" of the rectangular free cumulants with ratio $\lambda$: for all positive integers $k$,

$$c_{2k}(\mu) = \lim_{n\to\infty}\frac{(nm_n)^k}{n}\frac{\mathrm{Cl}_{2k}(\mathrm{Tr}(E_nU_nM_nV_n))}{(2k-1)!}.$$

This formula can be considered as a "rectangular analogue" of [C03, Th. 4.7].

**Remark 1.5.** If $M_n$ is also chosen at random, independently of $U_n$ and $V_n$, it can easily be seen that Theorem 1.2 is not true in general anymore. However, it seems that, using the same kind of arguments as in the proof of [GM05, Th. 1.5], Theorem 1.2 can be generalized for $M_n$ chosen at random by the formula $M_n = A_n + P_nB_nQ_n$, with $A_n, B_n$ deterministic having limit singular laws and $P_n, Q_n$ uniform orthogonal random matrices with respective sizes $n$ and $m_n$. Such a generalization would give a new proof of (3).

Let us recall that the *$R$-transform*[5] of a probability measure $\nu$ is the function

$$R_\nu(z) = G_\nu^{-1}(z) - \frac{1}{z}, \qquad \text{for } G_\mu(z) = \int\frac{\mathrm{d}\mu(t)}{z-t}.$$

[5]There are two conventions regarding the $R$-transform. The one we use is the one used in the analytic approach to freeness [HP00, AGZ09], which is not exactly the one used in the combinatorial approach [NS06]: $R_\nu^{\text{combinatorics}}(z) = zR_\nu^{\text{analysis}}(z)$.

In the particular cases where the matrices $M_n$ are square or "asymptotically flat", i.e. when $\lambda = 1$ or $\lambda = 0$, one gets the following corollary. Let $\mu_s$ be the symmetrization of $\mu$, defined by $\mu_s(A) = \frac{\mu(A)+\mu(-A)}{2}$ for all Borel subset $A$ of $\mathbb{R}$, and $\mu^2$ be the push-forward of $\mu$ by the function $t \mapsto t^2$.

**Corollary 1.6.** *In the particular case where $\lambda = 1$ (resp. $\lambda = 0$), the limit $I$ of $I_n$ can be expressed via the $R$-transform of $\mu_s$ (resp. $\mu^2$) in the following way*

$$I(\theta) = \int_0^\theta R_{\mu_s}(t)\mathrm{d}t, \qquad (\textit{resp. } I(\theta) = \int_0^\theta tR_{\mu^2}(t^2)\mathrm{d}t.)$$

*Proof.* It suffices to prove that $C_\mu^{(1)}(t^2) = tR_{\mu_s}(t)$ and that $C_\mu^{(0)}(t) = tR_{\mu^2}(t)$. The second equation can be found in [B09a, Lem. 3.2 or Sect. 3.6]. The first equation follows from the fact that for all $\lambda$, $C_\mu^{(\lambda)} = C_{\mu_s}^{(\lambda)}$ and from the fact that for all symmetric probability measure $\nu$, by [B09a, Sect. 3.6], $C_\nu^{(1)}(z^2) = zR_\nu(z)$. □

## 2. Preliminaries about the rectangular $R$-transform

Let $\mu$ be a probability measure on the real line which support is contained in $[-K, K]$, with $K > 0$ (we do not suppose $\mu$ to be symmetric, how it was the case in the initial definition of the rectangular $R$-transform). Let us define the function

$$M_{\mu^2}(z) = \int_{t\in\mathbb{R}} \frac{t^2 z}{1 - t^2 z}\mathrm{d}\mu(t) = \int_{t\in\mathbb{R}} \frac{1}{1 - t^2 z}\mathrm{d}\mu(t) - 1 \qquad (z \in [0, K^{-2})).$$

It can easily be proved that $M_{\mu^2}$ is nonnegative and non decreasing on $[0, K^{-2})$. Let us define, for $\lambda \in [0, 1]$, $T^{(\lambda)}(z) = (\lambda z + 1)(z + 1)$, and

$$H_\mu^{(\lambda)}(z) = zT^{(\lambda)}(M_{\mu^2}(z)).$$

Then $H_\mu^{(\lambda)}$ defines an increasing analytic diffeomorphism[6] from $[0, K^{-2})$ onto the (possibly unbounded) interval $[0, \lim_{z\uparrow K^{-2}} H_\mu^{(\lambda)}(z))$ such that

$$H_\mu^{(\lambda)}(0) = 0, \qquad \partial_z H_\mu^{(\lambda)}(0) = 1, \qquad H_\mu^{(\lambda)}(z) \geq z, \qquad \lim_{z\uparrow K^{-2}} H_\mu^{(\lambda)}(z) \geq K^{-2}.$$

We denote its inverse by $H_\mu^{(\lambda)^{-1}}$. Moreover, $T^{(\lambda)}$ defines an analytic increasing diffeomorphism from $[-1, +\infty)$ to $[0, +\infty)$, thus on can define the *rectangular $R$-transform with ratio $\lambda$* of $\mu$:

$$C_\mu^{(\lambda)}(z) = T^{(\lambda)^{-1}}\left(\frac{z}{H_\mu^{(\lambda)^{-1}}(z)}\right) \text{ for } z \neq 0, \quad \text{and} \quad C_\mu^{(\lambda)}(0) = 0, \tag{7}$$

which is analytic and non negative on the interval $[0, \lim_{z\uparrow K^{-2}} H_\mu^{(\lambda)}(z))$ (which always contains $[0, K^{-2})$).

[6]In this paper, for $I, J$ intervals of $\mathbb{R}$, we shall call an *analytic function on $I$* (resp. *analytic diffeomorphism from $I$ to $J$*) a function on $I$ (resp. a diffeomorphism from $I$ to $J$) which extends analytically to an open subset of $\mathbb{C}$ containing $I$.

By Theorems 3.8 and 3.12 of [B09a], the rectangular $R$-transform characterizes symmetric measures, and for all pair $\mu_1, \mu_2$ of compactly supported symmetric probability measures, $\mu_1 \boxplus_\lambda \mu_2$ is characterized by the fact that in a neighborhood of zero,

$$C^{(\lambda)}_{\mu_1 \boxplus_\lambda \mu_2}(z) = C^{(\lambda)}_{\mu_1}(z) + C^{(\lambda)}_{\mu_2}(z).$$

The following theorem states the continuity of the mapping $(\lambda, \mu) \longmapsto C^{(\lambda)}_\mu$ in a way which is quite different from the one of Theorem 3.11 of [B09a] (where $\lambda$ was fixed).

**Theorem 2.1.** *Fix $K > 0$, let $\mu_n$ be a sequence of probability measures on $[-K, K]$ which converges weakly to a limit $\mu$, and let $\lambda_n$ be a sequence of elements of $[0, 1]$ which converges to a limit $\lambda \in [0, 1]$. Then the sequence of functions $C^{(\lambda_n)}_{\mu_n}$ converges to $C^{(\lambda)}_\mu$ uniformly on every compact subset of $[0, K^{-2})$.*

*Proof.* Recall that $C^{(\lambda)}_\mu$ is defined by (7). Since, by Heine's Theorem, $(\lambda, z) \mapsto T^{(\lambda)^{-1}}(z)$ is uniformly continuous on every compact subset of $[0, 1] \times [0, +\infty)$, it suffices to prove that $\frac{z}{H^{(\lambda_n)^{-1}}_{\mu_n}(z)}$ converges to $\frac{z}{H^{(\lambda)^{-1}}_{\mu}(z)}$ uniformly on every compact subset of $[0, K^{-2})$.
<u>Claim a</u> : *For each compact subset $E$ of $\mathbb{C}\backslash[K^{-2}, +\infty)$, there is a constant $k_E$ such that for any law $\nu$ on $[-K, K]$, for any $c \in [0, 1]$, for any $z \in E$*

$$|T^{(c)}(M_{\nu^2}(z))| \leq k_E.$$

Indeed, for $z \in \mathbb{C}\backslash[K^{-2}, +\infty)$, for any law $\nu$ on $[-K, K]$, for any $c \in [0, 1]$,

$$T^{(c)}(M_{\nu^2}(z)) = \int_{(t,t') \in [-K,K]^2} \frac{1}{(1 - zt^2)(1 - zt'^2)} \mathrm{d}\nu(t) \mathrm{d}(c\nu + (1 - c)\delta_0)(t'),$$

thus $k_E = \max\{|1 - zt^2|^{-2} \,;\, |t| \leq K, z \in E\}$ is convenient.
<u>Claim b</u> : *The set of functions*

$$\{z \in [0, K^{-2}) \longmapsto \frac{z}{H^{(c)^{-1}}_\nu(z)} \,;\, \nu \text{ law on } [-K, K],\ c \in [0, 1]\}$$

*is relatively compact for the topology of uniform convergence on every compact subset of $[0, K^{-2})$.* Let us prove it. By Ascoli's Theorem, it suffices to prove that this family is uniformly bounded and uniformly Lipschitz on every compact subset of $[0, K^{-2})$. Let us fix $\nu$ a law on $[-K, K]$ and $c \in [0, 1]$. Note that we have

$$\frac{z}{H^{(c)^{-1}}_\nu(z)} = \frac{H^{(c)}_\nu(z)}{z} \circ H^{(c)^{-1}}_\nu(z), \quad \partial_z \frac{z}{H^{(c)^{-1}}_\nu(z)} = \frac{zH^{(c)'}_\nu(z) - H^{(c)}_\nu(z)}{z^2 H^{(c)'}_\nu(z)} \circ H^{(c)^{-1}}_\nu(z).$$

Since, moreover, for all $z \in [0, K^{-2})$, $H^{(c)^{-1}}_\nu(z) \leq z$ (indeed, for all $z \in [0, K^{-2})$, $H^{(c)}_\nu(z) \geq z$), it suffices to verify that the sets of functions

$$\{z \longmapsto \tfrac{H^{(c)}_\nu(z)}{z} \,;\, \nu \text{ law on } [-K, K],\ c \in [0, 1]\}$$
$$\text{and } \{z \longmapsto \tfrac{zH^{(c)'}_\nu(z) - H^{(c)}_\nu(z)}{z^2 H^{(c)'}_\nu(z)} \,;\, \nu \text{ law on } [-K, K],\ c \in [0, 1]\}$$

are uniformly bounded on every compact subset of $[0, K^{-2})$. The family of functions $\frac{H^{(c)}_\nu(z)}{z} = T^{(c)}(M_{\nu^2}(z))$, indexed by $\nu, c$, is a family of analytic functions on $\mathbb{C}\backslash[K^{-2}, +\infty)$

which is uniformly bounded on every compact subset of $\mathbb{C}\backslash[K^{-2},+\infty)$ (by Claim a). As a consequence, the family of the derivatives $\partial_z \frac{H_\nu^{(c)}(z)}{z}$ is also uniformly bounded on every compact subset of $\mathbb{C}\backslash[K^{-2},+\infty)$. Since

$$\frac{zH_\nu^{(c)\prime}(z)-H_\nu^{(c)}(z)}{z^2H_\nu^{(c)\prime}(z)}=\frac{1}{H_\nu^{(c)\prime}(z)}\partial_z\frac{H_\nu^{(c)}(z)}{z}$$

and $H_\nu^{(c)\prime}(z)\geq 1$ on $[0,K^{-2})$, Claim b is proved.

Hence one can suppose that $\frac{z}{H_{\mu_n}^{(\lambda_n)-1}(z)}$ converges to a function $f$ uniformly on every compact of $[0,K^{-2})$. Let us fix $z\in[0,K^{-2})$ and let us prove that $f(z)=\frac{z}{H_\mu^{(\lambda)-1}(z)}$. If $z=0$, it is clear (since all these functions are implicitly defined to map 0 to 1). Suppose that $z>0$. Note that $f(z)\neq 0$, because for all $n$, $\frac{z}{H_{\mu_n}^{(\lambda_n)-1}(z)}\geq 1$. Let us denote $l=\frac{z}{f(z)}$. It suffices to prove that $l=H_\mu^{(\lambda)-1}(z)$, i.e. that $H_\mu^{(\lambda)}(l)=z$. Since

$$H_\mu^{(\lambda)}(l)=\lim_{n\to\infty}H_\mu^{(\lambda)}(H_{\mu_n}^{(\lambda_n)-1}(z)),$$

it suffices to prove that $H_{\mu_n}^{(\lambda_n)}$ converges to $H_\mu^{(\lambda)}$ uniformly on every compact subset of $[0,K^{-2})$. But it is easily to see, using the second sentence of Footnote 4, that $M_{\mu_n^2}$ converges to $M_{\mu^2}$ uniformly on every compact subset of $[0,K^{-2})$ and then that $H_\mu^{(\lambda_n)}$ converges to $H_\mu^{(\lambda)}$ uniformly on every compact subset of $[0,K^{-2})$. The proof is complete. □

## 3. Proof of Theorem 1.2

3.1. **Preliminaries.** We shall use the following lemma several times in the paper. Let $\|\cdot\|$ denote the canonical euclidian norm on each $\mathbb{R}^d$.

**Lemma 3.1.** *Let $(G_i)_{i\geq 1}$ be a family of independent real random variables with standard Gaussian law. Let $T$ be fixed and let, for each $n$, $(\sigma_{n,1},\dots,\sigma_{n,n})\in[0,T]^n$ be such that*

$$\frac{1}{n}\sum_{i=1}^n\sigma_{n,i}^2=1. \tag{8}$$

*Let us define, for each $n$, $X_n=(\sigma_{n,1}G_1,\dots,\sigma_{n,n}G_n)$. Then for all $\kappa\in(0,\frac{1}{2})$,*

$$\mathbb{P}\{|\|X_n\|-\sqrt{n}|\leq n^{\frac{1}{2}-\kappa}\}\underset{n\to\infty}{\longrightarrow}1. \tag{9}$$

*If, moreover, the $\sigma_{n,i}$'s depend on a parameter $\theta$, the convergence of (9) is uniform in $\theta$ as long as the upper-bound $T$ is uniform in $\theta$.*

*Proof.* Note that by (8), the random variable $N_n:=\frac{\|X_n\|^2}{n}-1$ is centered. Moreover, $\mathrm{Var}(N_n)=\frac{\mathrm{Var}(G_1^2)}{n^2}\sum_{i=1}^n\sigma_{n,i}^4\leq\frac{T^4\,\mathrm{Var}(Z_1^2)}{n}$. It follows, by Tchebichev's inequality, that for all $\kappa\in(0,\frac{1}{2})$,

$$\mathbb{P}\{|N_n|\geq n^{-\kappa}\}\leq T^4\,\mathrm{Var}(G_1^2)n^{2\kappa-1}.$$

To deduce that for $n$ large enough,

$$\mathbb{P}\left\{\left|\frac{\|X_n\|}{\sqrt{n}}-1\right|\geq n^{-\kappa}\right\}\leq T^4\operatorname{Var}(G_1^2)n^{2\kappa-1},$$

it suffices to notice that the function $\sqrt{\cdot}$ is 1-Lipschitz on $[1/4,+\infty)$ and that $n^{-\kappa}\leq 3/4$ for $n$ large enough. □

**Lemma 3.2.** *Let $\mu$ be a probability measure which support is contained in $[-K,K]$, fix $\lambda\in[0,1]$, $\theta\in[0,K^{-1})$ and define $\gamma=C_\mu^{(\lambda)}(\theta^2)$. Then*

$$M_{\mu^2}\left(\frac{\theta^2}{T^{(\lambda)}(\gamma)}\right)=\gamma. \tag{10}$$

*Proof.* By the definition of $C_\mu^{(\lambda)}$ given in (7), $\frac{\theta^2}{H_\mu^{(\lambda)^{-1}}(\theta^2)}=T^{(\lambda)}(\gamma)$, hence $\frac{\theta^2}{T^{(\lambda)}(\gamma)}=H_\mu^{(\lambda)^{-1}}(\theta^2)$. Since $\gamma\geq 0$, $\frac{\theta^2}{T^{(\lambda)}(\gamma)}\in[0,K^{-2})$ and one can apply the function $H_\mu^{(\lambda)}$ on both sides. We get $H_\mu^{(\lambda)}\left(\frac{\theta^2}{T^{(\lambda)}(\gamma)}\right)=\theta^2$, i.e.

$$\frac{\theta^2}{T^{(\lambda)}(\gamma)}T^{(\lambda)}\left(M_{\mu^2}\left(\frac{\theta^2}{T^{(\lambda)}(\gamma)}\right)\right)=\theta^2.$$

It follows that $T^{(\lambda)}\left(M_{\mu^2}\left(\frac{\theta^2}{T^{(\lambda)}(\gamma)}\right)\right)=T^{(\lambda)}(\gamma)$. Since both $M_{\mu^2}\left(\frac{\theta^2}{T^{(\lambda)}(\gamma)}\right)$ and $\gamma$ are nonnegative real numbers, one gets (10). □

The following elementary lemma shall be used many times, so we state it clearly here.

**Lemma 3.3.** *Let $X_n$ be a sequence of nonnegative random variables, with positive expectations. Let $Z_n$ be a sequence of real random variables such that there exists deterministic constants $C,\eta>0$ such that for all $n$, $|Z_n|\leq Cn^{1-\eta}$. Then as $n$ tends to infinity,*

$$\frac{1}{n}\log\mathbb{E}(X_ne^{Z_n})=\frac{1}{n}\log\mathbb{E}(X_n)+o(1).$$

*Proof.* It suffices to notice that we have $X_ne^{-Cn^{1-\eta}}\leq X_ne^{Z_n}\leq X_ne^{Cn^{1-\eta}}$. □

**Notation for the proof of Theorem 1.2:** In the next sections, $o(1)$ shall denote any sequence of functions on $(-K^{-1},K^{-1})$ which converges to zero as $n$ tends to infinity, uniformly on every compact subset of $(-K^{-1},K^{-1})$. Also, we shall work with the notation introduced in Remark 1.1 and handle $I_n(\theta)$ via Formula (6).

3.2. **Proof of Theorem 1.2: a) Proof of Equation** (19)**.** Firstly, up to a replacement, for all $n,k$, of $\mu_{n,k}$ by $\mu_{n,k}+\min\{m_n^{-1/2},\frac{K-\mu_{n,k}}{2}\}$ (which does not change the hypotheses nor, according to Lemma 3.3, the conclusion), one can suppose that all $\mu_{n,k}$'s are positive.

For each $n$, let us define the function

$$f_n:((x_1,\dots,x_n),(y_1,\dots,y_{m_n}))\in\mathbb{R}^n\times\mathbb{R}^{m_n}\longmapsto\sum_{k=1}^n x_k\mu_{n,k}y_k.$$

Up to a change of the probability space which does not change the expectation, one can suppose that there are independent standard Gaussian random vectors $x_n, y_n$ of respectively $\mathbb{R}^n, \mathbb{R}^{m_n}$ such that

$$u_n = \frac{x_n}{\|x_n\|}, \qquad v_n = \frac{y_n}{\|y_n\|}.$$

Let us fix $\kappa \in (0, 1/2)$. If $\lambda > 0$, the precise choice of $\kappa \in (0, 1/2)$ is irrelevant, but if $\lambda = 0$, we choose $\kappa \in (\frac{\alpha-1}{2}, \frac{1}{2})$ ($\alpha$ is the one of (5)). Let us now define the set

$$A_n := \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{m_n} \,;\, \left|\|x\| - \sqrt{n}\right| \leq n^{\frac{1}{2}-\kappa}, \left|\|y\| - \sqrt{m_n}\right| \leq m_n^{\frac{1}{2}-\kappa}\right\}.$$

The event $\{(x_n, y_n) \in A_n\}$ is well known to be independent of $(u_n, v_n)$, thus

$$I_n(\theta) \;=\; \frac{1}{n}\log \mathbb{E}\left[\mathbb{1}_{A_n}(x_n, y_n)\exp(\sqrt{nm_n}\theta f_n(u_n, v_n))\right] - \frac{1}{n}\log \mathbb{P}(A_n).$$

Moreover, by Lemma 3.1, $\mathbb{P}\{(x_n, y_n) \in A_n\} \longrightarrow 1$ as $n \to \infty$, thus

$$I_n(\theta) = \frac{1}{n}\log \mathbb{E}\left[\mathbb{1}_{A_n}(x_n, y_n)\exp(\sqrt{nm_n}\theta f_n(u_n, v_n))\right] + o(1). \tag{11}$$

Moreover, note that on the event $\{(x_n, y_n) \in A_n\}$,

$$\begin{aligned} \sqrt{n} - n^{\frac{1}{2}-\kappa} \leq \;& \|x_n\| \;\leq \sqrt{n} + n^{\frac{1}{2}-\kappa} \\ \sqrt{m_n} - m_n^{\frac{1}{2}-\kappa} \leq \;& \|y_n\| \;\leq \sqrt{m_n} + m_n^{\frac{1}{2}-\kappa}, \end{aligned}$$

thus, since $m_n \geq n$,

$$\sqrt{nm_n} - 3\sqrt{m_n}n^{\frac{1}{2}-\kappa} \leq \|x_n\|\|y_n\| \leq \sqrt{nm_n} + 3\sqrt{m_n}n^{\frac{1}{2}-\kappa}. \tag{12}$$

If $\lambda > 0$, since $m_n/n$ is bounded, it follows that there is a deterministic constant $C$ independent of $n$ such that on on the event $\{(x_n, y_n) \in A_n\}$,

$$|\,\|x_n\|\|y_n\| - \sqrt{nm_n}\,| \leq Cn^{1-\kappa}. \tag{13}$$

If $\lambda = 0$, it follows from (12) and (5) that for $\eta = \frac{1}{2} + \kappa - \alpha$ (which is positive by definition of $\kappa$), for $n$ large enough,

$$|\,\|x_n\|\|y_n\| - \sqrt{nm_n}\,| \leq 3n^{1-\eta}. \tag{14}$$

Note that by (11),

$$I_n(\theta) = \frac{1}{n}\log \mathbb{E}[\mathbb{1}_{A_n}(x_n, y_n)\exp\{\theta f_n(x_n, y_n) + \frac{\theta f_n(x_n, y_n)}{\|x_n\|\|y_n\|}(\sqrt{nm_n} - \|x_n\|\|y_n\|)\}] + o(1),$$

and that for all $n, k$, $|\mu_{k,n}| \leq K$, which implies that $\left|\frac{f_n(x_n,y_n)}{\|x_n\|\|y_n\|}\right| \leq K$. Hence by Lemma 3.3 and (13) (or (14) if $\lambda = 0$),

$$I_n(\theta) = \frac{1}{n}\log \mathbb{E}\left[\mathbb{1}_{A_n}(x_n, y_n)\exp\left\{\theta f_n(x_n, y_n)\right\}\right] + o(1). \tag{15}$$

Note that on the event $\{(x_n, y_n) \in A_n\}$, we have

$$\begin{aligned} n - 2n^{1-\kappa} \leq n - 2n^{1-\kappa} + n^{1-2\kappa} \leq \;& \|x_n\|^2 \;\leq n + 2n^{1-\kappa} + n^{1-2\kappa} \leq n + 3n^{1-\kappa} \\ n - 2n^{1-\kappa} \leq n - 2nm_n^{-\kappa} + nm_n^{-2\kappa} \leq \;& \tfrac{n}{m_n}\|y_n\|^2 \;\leq n + 2nm_n^{-\kappa} + nm_n^{-2\kappa} \leq n + 3n^{1-\kappa}. \end{aligned}$$

thus for all $n$, on the event $\{(x_n, y_n) \in A_n\}$,

$$\big|\, \|x_n\|^2 - n \,\big| + \big|\, \frac{n}{m_n}\|y_n\|^2 - n \,\big| \leq 6n^{1-\kappa}. \tag{16}$$

Now, let us define, for each $n$,

$$\gamma_n(\theta) = C_{\mu_n}^{(\frac{n}{m_n})}(\theta^2) \qquad \text{for } \mu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{\mu_{n,k}}. \tag{17}$$

Note $\mu_n$ is the singular law of $M_n$, which tends to $\mu$. Hence by Theorem 2.1, we have

$$\gamma_n(\theta) \underset{n\to\infty}{\longrightarrow} C_{\mu}^{(\lambda)}(\theta^2) \qquad \text{uniformly on every compact subset of } (-K^{-1}, K^{-1}), \tag{18}$$

so by (16), for every such compact set $E$, there is a constant $Q_E$ such that for all $n$, for all $\theta \in E$, on the event $\{(x_n, y_n) \in A_n\}$, we have

$$|\gamma_n(\theta)(\frac{1}{2}\|x_n\|^2 + \frac{n}{2m_n}\|y_n\|^2 - n)| \leq Q_E n^{1-\kappa}.$$

Hence, by (15) and Lemma 3.3,

$$\begin{aligned} I_n(\theta) =& \frac{1}{n}\log \mathbb{E}\left[\mathbb{1}_{A_n}(x_n, y_n)\exp\left\{\theta f_n(x_n, y_n) - \gamma_n(\theta)\left(\frac{1}{2}\|x_n\|^2 + \frac{n}{2m_n}\|y_n\|^2 - n\right)\right\}\right] + o(1) \\ =& \gamma_n(\theta) + \frac{1}{n}\log \underbrace{\mathbb{E}\left[\mathbb{1}_{A_n}(x_n, y_n)\exp\left\{\theta f_n(x_n, y_n) - \gamma_n(\theta)\left(\frac{1}{2}\|x_n\|^2 + \frac{n}{2m_n}\|y_n\|^2\right)\right\}\right]}_{\text{denoted by } J_n(\theta)} + o(1). \end{aligned}$$

Thus, by (18),

$$I_n(\theta) = C_{\mu}^{(\lambda)}(\theta^2) + \frac{1}{n}\log J_n(\theta) + o(1). \tag{19}$$

3.3. **Proof of Theorem 1.2: b) Asymptotics of $J_n(\theta)$ and conclusion.** We have, assimilating the vectors of $\mathbb{R}^n$ and $\mathbb{R}^{m_n}$ with column-matrices,

$$J_n(\theta) = (2\pi)^{-\frac{n+m_n}{2}} \int_{x\in\mathbb{R}^n, y\in\mathbb{R}^{m_n}} \mathbb{1}_{A_n}(x, y)\exp\left\{-\frac{1}{2}\begin{bmatrix} x^t & y^t \end{bmatrix} T_n \begin{bmatrix} x \\ y \end{bmatrix}\right\} \mathrm{d}x\mathrm{d}y, \tag{20}$$

for

$$T_n := \begin{bmatrix} a_n(\theta)I_n & \Lambda_n(\theta) & 0_{n,m_n-n} \\ \Lambda_n(\theta) & b_n(\theta)I_n & 0_{n,m_n-n} \\ 0_{m_n-n,n} & 0_{m_n-n,n} & b_n(\theta)I_{m_n-n} \end{bmatrix},$$

where $a_n(\theta) = 1 + \gamma_n(\theta)$, $b_n(\theta) = 1 + \frac{n}{m_n}\gamma_n(\theta)$ and $\Lambda_n(\theta)$ is the diagonal $n \times n$ matrix with diagonal entries

$$\lambda_{n,1}(\theta) := -\theta\mu_{n,1}, \ldots, \lambda_{n,n}(\theta) := -\theta\mu_{n,n}.$$

**Notation:** In this section, in order to lighten the notation, we shall write $J_n$ for $J_n(\theta)$, $a_n$ for $a_n(\theta)$,...

**Lemma 3.4.** *Let us fix $n \geq 1$ and let $a, b$ be real numbers and $\Lambda$ an invertible diagonal real $n \times n$ matrix. Let us define, using the functional calculus formalism (thus assimilating $a$ and $aI_n, \dots$),*

$$\Delta = (b-a)^2 + 4\Lambda^2, \quad r^{\pm} = \frac{a+b \pm \sqrt{\Delta}}{2}, \quad f^{\pm} = \frac{1}{\sqrt{2\Delta \pm 2(b-a)\sqrt{\Delta}}}$$

*and*

$$T = \begin{bmatrix} a & \Lambda \\ \Lambda & b \end{bmatrix}, \ D = \begin{bmatrix} r^+ & 0 \\ 0 & r^- \end{bmatrix}, \ P = \begin{bmatrix} 2\Lambda f^+ & 2\Lambda f^- \\ (b-a)f^+ + \sqrt{\Delta} f^+ & (b-a)f^- - \sqrt{\Delta} f^- \end{bmatrix}.$$

*Then $P$ is an orthogonal matrix and we have $T = PDP^t$.*

*Proof.* One can easily verify that $P$ is orthogonal. Let us define

$$Q = \begin{bmatrix} 2\Lambda & 2\Lambda \\ (b-a) + \sqrt{\Delta} & (b-a) - \sqrt{\Delta} \end{bmatrix}, H = \begin{bmatrix} f^+ & 0 \\ 0 & f^- \end{bmatrix}.$$

Then $P = QH$. One can easily verify that $TQ = QD$. If follows that $TQH = QDH$. Since $HD = DH$, $TQH = QHD$, i.e. $TP = PD$, thus $T = PDP^t$. □

For $\theta \neq 0$, let us define $\Delta_n$, $r_n^{\pm}$, $f_n^{\pm}$ as in the lemma, using $\Lambda_n$ instead of $\Lambda$, $a_n$ instead of $a$ and $b_n$ instead of $b$. Let us define $P_n$ in the same way, extended to an $(n+m_n)\times(n+m_n)$ matrix by adding $I_{m_n-n}$ on the lower-right corner, i.e.

$$P_n = \begin{bmatrix} 2\Lambda_n f_n^+ & 2\Lambda_n f_n^- & 0 \\ (b_n - a_n)f_n^+ + \sqrt{\Delta_n} f_n^+ & (b_n - a_n)f_n^- - \sqrt{\Delta_n} f^- & 0 \\ 0 & 0 & I_{m_n-n} \end{bmatrix},$$

and $D_n$ extended to an $(n+m_n)\times(n+m_n)$ matrix by adding $b_n I_{m_n-n}$ on the lower-right corner, i.e.

$$D_n = \begin{bmatrix} r_n^+ & 0 & 0 \\ 0 & r_n^- & 0 \\ 0 & 0 & b_n I_{m_n-n} \end{bmatrix}.$$

For $\theta = 0$, we set $r_n^{\pm} = 1$, $P_n = D_n = I_{n+m_n}$.

Let us denote, for $X$ an $(n+m_n)\times(n+m_n)$ matrix, $X(A_n) = \{X \begin{bmatrix} x \\ y \end{bmatrix} ; (x,y) \in A_n\}$. Let us also introduce a standard Gaussian random column vector in $\mathbb{R}^{n+m_n}$, that we shall denote by

$$Z_n = (\underbrace{Z_{n,1}^+, \dots, Z_{n,n}^+}_{\text{denoted by } Z_n^+}, \underbrace{Z_{n,1}^-, \dots, Z_{n,n}^-}_{\text{denoted by } Z_n^-}, \underbrace{Z_{n,1}^0, \dots, Z_{n,m_n-n}^0}_{\text{denoted by } Z_n^0})^t.$$

We have, by (20) and Lemma 3.4,

$$J_n = (2\pi)^{-\frac{n+m_n}{2}} \int_{A_n} \exp\{-\frac{1}{2} \begin{bmatrix} x^t & y^t \end{bmatrix} P_n D_n P_n^t \begin{bmatrix} x \\ y \end{bmatrix}\} \mathrm{d}x\mathrm{d}y.$$

Thus, since $P_n$ is an orthogonal matrix,

$$J_n = (2\pi)^{-\frac{n+m_n}{2}} \int_{P_n^t(A_n)} \exp\{-\frac{1}{2} \begin{bmatrix} x^t & y^t \end{bmatrix} D_n \begin{bmatrix} x \\ y \end{bmatrix}\} \mathrm{d}x\mathrm{d}y.$$

Hence, by definition of $D_n$, we have

$$J_n = (2\pi)^{-\frac{n+m_n}{2}}[b_n^{m_n-n}\prod_{i=1}^n r_{n,i}^+ r_{n,i}^-]^{-1/2}\int_{\sqrt{D_n}P_n^t(A_n)}\exp\{-\frac{1}{2}(\|x\|^2+\|y\|^2)\}\mathrm{d}x\mathrm{d}y,$$

which, by definition of $Z_n$, can be written

$$J_n = [b_n^{m_n-n}\prod_{i=1}^n(a_nb_n-\lambda_{n,i}^2)]^{-1/2}\underbrace{\mathbb{P}\{Z_n\in\sqrt{D_n}P_n^t(A_n)\}}_{=\mathbb{P}\{P_nD_n^{-1/2}Z_n\in A_n\}} \tag{21}$$

<u>Claim a</u> : *The probability of the event $\{P_nD_n^{-1/2}Z_n\in A_n\}$ tends to one as $n$ tends to infinity, uniformly on every compact subset of $(-K^{-1},K^{-1})$ (remember indeed that the matrices $P_n$ and $D_n$ depend on $\theta$).*

Let us prove it. Let $X_n=(X_{n,1},\dots,X_{n,n})^t$ be the vector of the first $n$ coordinates of $P_nD_n^{-1/2}Z_n$ and $Y_n=(Y_{n,1},\dots,Y_{n,m_n})^t$ be the one of the $m_n$ last ones. By definition of $A_n$, we have

$$P_nD_n^{-1/2}Z_n\in A_n \iff \left|\|X_n\|-\sqrt{n}\right|\le n^{-\kappa} \text{ and } |\|Y_n\|-\sqrt{m_n}|\le m_n^{-\kappa}.$$

Thus to prove Claim a, if suffices to prove:
<u>Claim b</u> : *Both events $\{|\|X_n\|-\sqrt{n}|\le n^{-\kappa}\}$ and $\left\{\left|\|X_n\|-\sqrt{m_n}\right|\le m_n^{-\kappa}\right\}$ have probabilities tending to one as $n$ tends to infinity, uniformly on every compact subset of $(-K^{-1},K^{-1})$ (the random vectors $X_n$ and $Y_n$ depend indeed on $\theta$).*

Let us prove Claim b. For $\theta=0$, $X_n$ (resp. $Y_n$) is a standard Gaussian random vector of $\mathbb{R}^n$ (resp. $\mathbb{R}^{m_n}$). For $\theta\neq 0$, by the definitions of $P_n$ and $D_n$,

$$\begin{aligned} X_n &= 2\Lambda_n(f_n^+(r_n^+)^{-1/2}Z_n^+ + f_n^-(r_n^-)^{-1/2}Z_n^-),\\ Y_n &= \begin{bmatrix}(b_n-a_n+\sqrt{\Delta_n})f_n^+(r_n^+)^{-1/2}Z_n^+ + (b_n-a_n-\sqrt{\Delta_n})f_n^-(r_n^-)^{-1/2}Z_n^-\\ b_n^{-1/2}Z_n^0\end{bmatrix}.\end{aligned}$$

Thus for each $n$, $X_n$ (resp. $Y_n$) has the law of

$$(\sigma_{n,1}G_1,\dots,\sigma_{n,n}G_n) \qquad (\text{resp. } (\sigma'_{n,1}G_1,\dots,\sigma'_{n,m_n}G_{m_n})),$$

for $(G_i)_{i\ge1}$ a family of independent real random variables with standard Gaussian law and where for $\theta=0$, all $\sigma_{n,i}$'s and $\sigma'_{n,i}$'s are equal to 1 and for $\theta\neq0$, for each $i=1,\dots,n$,

$$\begin{aligned}\sigma_{n,i}^2 &= 4\lambda_{n,i}^2\frac{2}{(2\Delta_{n,i}+2(b_n-a_n)\sqrt{\Delta_{n,i}})(a_n+b_n+\sqrt{\Delta_{n,i}})}\\ &\qquad +4\lambda_{n,i}^2\frac{2}{(2\Delta_{n,i}-2(b_n-a_n)\sqrt{\Delta_{n,i}})(a_n+b_n-\sqrt{\Delta_{n,i}})},\end{aligned}$$

for each $i=n+1,\dots,m_n$, ${\sigma'_{n,i}}^2=b_n^{-1}$ and for each $i=1,\dots,n$,

$${\sigma'_{n,i}}^2=\frac{(b_n-a_n+\sqrt{\Delta_n})^2}{2(b_n-a_n)\sqrt{\Delta_n}+2\Delta_n}\frac{2}{a_n+b_n+\sqrt{\Delta_n}}+\frac{(\sqrt{\Delta_n}-(b_n-a_n))^2}{-2(b_n-a_n)\sqrt{\Delta_n}+2\Delta_n}\frac{2}{a_n+b_n-\sqrt{\Delta_n}}.$$

Hence by Lemma 3.1, to prove Claim b, it suffices to prove:

$$\forall \varepsilon > 0, \qquad \sup_{|\theta| \le K^{-1-\varepsilon}} \sup_{\substack{1 \le i \le n \\ n \ge 1}} \sigma_{n,i} < +\infty \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} \sigma_{n,i}^2 = 1, \tag{22}$$

$$\forall \varepsilon > 0, \qquad \sup_{|\theta| \le K^{-1-\varepsilon}} \sup_{\substack{1 \le i \le m_n \\ n \ge 1}} \sigma'_{n,i} < +\infty \quad \text{and} \quad \frac{1}{m_n} \sum_{i=1}^{m_n} {\sigma'_{n,i}}^2 = 1. \tag{23}$$

Note first that the second parts of (22) and (23) both hold when $\theta = 0$.

We have, for $\theta \neq 0$,

$$\begin{aligned}
\sigma_{n,i}^2 &= \frac{4\lambda_{n,i}^2}{\sqrt{\Delta_{n,i}}} \frac{1}{(\sqrt{\Delta_{n,i}} + b_n - a_n)(\sqrt{\Delta_{n,i}} + b_n + a_n)} \\
&\qquad + \frac{4\lambda_{n,i}^2}{\sqrt{\Delta_{n,i}}} \frac{1}{(a_n - (b_n - \sqrt{\Delta_{n,i}}))(a_n + b_n - \sqrt{\Delta_{n,i}})} \\
&= \frac{4\lambda_{n,i}^2}{\sqrt{\Delta_{n,i}}} \left\{ \frac{1}{(\sqrt{\Delta_{n,i}} + b_n)^2 - a_n^2} - \frac{1}{(b_n - \sqrt{\Delta_{n,i}})^2 - a_n^2} \right\} \\
&= \frac{4\lambda_{n,i}^2}{\sqrt{\Delta_{n,i}}} \left\{ \frac{1}{b_n^2 + \Delta_{n,i} - a_n^2 + 2b_n\sqrt{\Delta_{n,i}}} - \frac{1}{b_n^2 + \Delta_{n,i} - a_n^2 - 2b_n\sqrt{\Delta_{n,i}}} \right\} \\
&= \frac{4\lambda_{n,i}^2}{\sqrt{\Delta_{n,i}}} \frac{-4b_n\sqrt{\Delta_{n,i}}}{(b_n^2 + \Delta_{n,i} - a_n^2)^2 - 4b_n^2\Delta_{n,i}} = \frac{-16b_n\lambda_{n,i}^2}{(b_n^2 + \Delta_{n,i} - a_n^2)^2 - 4b_n^2\Delta_{n,i}}.
\end{aligned}$$

But (removing the indices)

$$\begin{aligned}
(b^2 + \Delta - a^2)^2 - 4b^2\Delta &= (2b^2 - 2ab + 4\lambda^2)^2 - 4b^2(b^2 - 2ab + a^2 + 4\lambda^2) \\
&= 16\lambda^4 - 16ab\lambda^2.
\end{aligned}$$

It follows, writing $\gamma_n$ for $\gamma_n(\theta)$, that

$$\sigma_{n,i}^2 = \frac{b_n}{a_n b_n - \lambda_{n,i}^2} = \frac{1}{1+\gamma_n} \times \frac{T^{(\frac{n}{m_n})}(\gamma_n)}{T^{(\frac{n}{m_n})}(\gamma_n) - \theta^2\mu_{n,i}^2} = \frac{1}{1+\gamma_n} \times \frac{1}{1 - \frac{\theta^2}{T^{(\frac{n}{m_n})}(\gamma_n)}\mu_{n,i}^2}. \tag{24}$$

By definition of $\gamma_n(\theta)$, we have $\gamma_n(\theta) \ge 0$, hence $T^{(\frac{n}{m_n})}(\gamma_n(\theta)) \ge 1$. Since for all $n, i$, $|\mu_{n,i}| \le K$, it follows that the first part of (22) holds. Moreover, by the definition of $\mu_n$ given in (17), we have

$$\frac{1}{n} \sum_{i=1}^{n} \sigma_{n,i}^2 = \frac{1}{1+\gamma_n(\theta)} M_{\mu_n^2}\left( \frac{\theta^2}{T^{(\frac{n}{m_n})}(\gamma_n(\theta))} \right) + \frac{1}{1+\gamma_n(\theta)}.$$

By (10), it follows that the second part of (22) also holds.

Let us now prove (23). When $\theta \neq 0$, for all $i \leq n$,

$$\begin{aligned}
{\sigma'_{n,i}}^2 &= \frac{(b_n - a_n + \sqrt{\Delta_n})^2}{2(b_n - a_n)\sqrt{\Delta_n} + 2\Delta_n} \frac{2}{a_n + b_n + \sqrt{\Delta_n}} + \frac{(\sqrt{\Delta_n} - (b_n - a_n))^2}{-2(b_n - a_n)\sqrt{\Delta_n} + 2\Delta_n} \frac{2}{a_n + b_n - \sqrt{\Delta_n}} \\
&= \frac{b_n - a_n + \sqrt{\Delta_n}}{\sqrt{\Delta_n}(a_n + b_n + \sqrt{\Delta_n})} + \frac{\sqrt{\Delta_n} - (b_n - a_n)}{\sqrt{\Delta_n}(a_n + b_n - \sqrt{\Delta_n})} \\
&= \frac{1}{\sqrt{\Delta_n}} \left\{ \frac{b_n - a_n + \sqrt{\Delta_n}}{a_n + b_n + \sqrt{\Delta_n}} + \frac{\sqrt{\Delta_n} - b_n + a_n}{a_n + b_n - \sqrt{\Delta_n}} \right\} \\
&= \frac{1}{\sqrt{\Delta_n}} \frac{(b_n - a_n + \sqrt{\Delta_n})(a_n + b_n - \sqrt{\Delta_n}) + (\sqrt{\Delta_n} - b_n + a_n)(a_n + b_n + \sqrt{\Delta_n})}{(a_n + b_n)^2 - \Delta_n} \\
&= \frac{1}{\sqrt{\Delta_n}} \frac{b_n^2 - (a_n - \sqrt{\Delta_n})^2 + (a_n + \sqrt{\Delta_n})^2 - b_n^2}{(a_n + b_n)^2 - \Delta_n} = \frac{4a_n}{(a_n + b_n)^2 - \Delta_n} = \frac{a_n}{a_n b_n - \lambda_{n,i}^2} \\
&= \frac{1}{1 + \frac{n}{m_n}\gamma_n(\theta)} \times \frac{1}{1 - \frac{\theta^2}{T^{(\frac{n}{m_n})}(\gamma_n(\theta))}\mu_{n,i}^2},
\end{aligned}$$

and for all $i = n+1, \ldots, m_n$, ${\sigma'_{n,i}}^2 = \frac{1}{1+\frac{n}{m_n}\gamma_n(\theta)}$. The first part of (23) holds for the same reasons as the first part of (22) above. Moreover, we have, writing $\gamma_n$ for $\gamma_n(\theta)$,

$$\frac{1}{m_n}\sum_{i=1}^{m_n} {\sigma'_{n,i}}^2 = \frac{n}{m_n(1+\frac{n}{m_n}\gamma_n)} M_{\mu_n^2}\left(\frac{\theta^2}{T^{(\frac{n}{m_n})}(\gamma_n)}\right) + \frac{n}{m_n(1+\frac{n}{m_n}\gamma_n)} + \frac{m_n - n}{m_n(1+\frac{n}{m_n}\gamma_n)}.$$

By (10), it follows that the second part of (23) also holds.

The proof of Claim b (hence of Claim a) is complete.

By (21) and Claim a, we have, still writing $\gamma_n$ for $\gamma_n(\theta)$,

$$\frac{1}{n}\log(J_n(\theta)) =$$

$$\frac{n - m_n}{2n}\log(1 + \frac{n}{m_n}\gamma_n) - \frac{\log(T^{(\frac{n}{m_n})}(\gamma_n))}{2} - \frac{1}{2}\int_{t\in[-K,K]} \log\{1 - \frac{\theta^2}{T^{(\frac{n}{m_n})}(\gamma_n)}t^2\}\mathrm{d}\mu_n(t) + o(1)$$

$$= -\frac{m_n}{2n}\log(1 + \frac{n}{m_n}\gamma_n) - \frac{\log(1+\gamma_n)}{2} - \frac{1}{2}\int_{t\in[-K,K]} \log\{1 - \frac{\theta^2}{T^{(\frac{n}{m_n})}(\gamma_n)}t^2\}\mathrm{d}\mu_n(t) + o(1).$$

By hypothesis, $\mu_n$, which is defined in (17), converges weakly to $\mu$. Using (18) and the second sentence of Footnote 4, one easily sees that, writing $\gamma$ for $C_\mu^{(\lambda)}(\theta^2)$, we have

$$\frac{1}{n}\log(J_n(\theta)) = -\frac{1}{2\lambda}\log(1+\lambda\gamma) - \frac{\log(1+\gamma)}{2} - \frac{1}{2}\int_{t\in[-K,K]} \log\{1 - \frac{\theta^2}{T^{(\lambda)}(\gamma)}t^2\}\mathrm{d}\mu(t) + o(1),$$

where in the case where $\lambda = 0$, $\frac{1}{2\lambda}\log(1+\lambda\gamma)$ has to be understood as $\frac{\gamma}{2}$. By (19), one gets, still writing $\gamma$ for $C_\mu^{(\lambda)}(\theta^2)$,

$$I_n(\theta) = \underbrace{\gamma - \frac{1}{2\lambda}\log(1+\lambda\gamma) - \frac{\log(1+\gamma)}{2} - \frac{1}{2}\int_{t\in[-K,K]} \log\{1 - \frac{\theta^2}{T^{(\lambda)}(\gamma)}t^2\}\mathrm{d}\mu(t)}_{\text{denoted by } f(\theta)} + o(1).$$

$I(0) = f(0) = 0$ (indeed, by (7), $C_\mu^{(\lambda)}(0) = 0$). So to conclude the proof of Theorem 1.2, it suffices to verify that $I$ and $f$ have the same derivatives on $(-K^{-1}, K^{-1})$, which can easily be done using (10) again.

## References


[AGZ09] G. Anderson, A. Guionnet, O. Zeitouni *An introduction to random matrices.* To appear in Cambridge University Press, preliminary version available on the web.

[B07a] F. Benaych-Georges *Rectangular random matrices, related free entropy and free Fisher's information.* 2007, to appear in Journal of Operator Theory.

[B07b] F. Benaych-Georges *Infinitely divisible distributions for rectangular free convolution: classification and matricial interpretation.* Probab. Theory and Related Fields 139, 1–2, 2007, 143–189.

[B09a] F. Benaych-Georges *Rectangular random matrices, related convolution* Probability Theory and Related Fields Volume 144, issue 3 (2009), 471–515.

[B09b] F. Benaych-Georges *On a surprising relation between the Marchenko-Pastur law, rectangular and square free convolutions*, to appear in Annales de l'IHP Prob. Stats.

[C03] B. Collins *Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral and free probability.* Int. Math. Res. Not. 17 (2003), 953–982.

[CS06] B. Collins, P. Śniady *Integration with respect to the Haar measure on unitary, orthogonal and symplectic group* Comm. Math. Phys. 264 (2006), no. 3, 773–795.

[CS07] B. Collins, P. Śniady *New scaling of Itzykson-Zuber integrals* Annales de l'IHP Prob. Stats. 43, 2 (2007), 139–146.

[CMSS07] B. Collins, J.A. Mingo, P. Śniady, R. Speicher, *Second order freeness and fluctuations of random matrices. III. Higher order freeness and free cumulants.* Doc. Math. 12 (2007), 1–70.

[GT08] A. Ghaderipoor, C. Tellambura *Generalization of some integrals over unitary matrices by character expansion of groups* J. Math. Physics, 49 073519 (2008).

[GZ02] A. Guionnet, O. Zeitouni *Large deviations asymptotics for spherical integrals* J. Funct. Anal. 188, 2 (2002), 461–515.

[GM05] A. Guionnet, M. Maïda *A Fourier view on the R-transform and related asymptotics of spherical integrals* J. Funct. Anal. 222, 2, (2005), 435–490.

[G09] A. Guionnet *Large random matrices: lectures on macroscopic asymptotics* École d'Été de probabilités de Saint-Flour XXXVI–2006, Lect. Notes Math. 1957 (2009).

[HP00] F. Hiai, D. Petz *The semicircle law, free random variables, and entropy*, Amer. Math. Soc., Mathematical Surveys and Monographs Volume 77, 2000

[HJ85] R.A. Horn, C.R. Johnson *Matrix analysis* Cambridge University Press, Cambridge, 1985

[NS06] A. Nica, R. Speicher *Lectures on the combinatorics of free probability.* London Mathematical Society Lecture Note Series, 335. Cambridge University Press, Cambridge, 2006.

[SW03] B. Schlittgen, T. Wettig, *Generalizations of some integrals over the unitary group* J. Phys. A, 36, 3195 (2003).

[ZZ03] P. Zinn-Justin, J.-B. Zuber *On some integrals over the U(N) unitary group and their large N limit.* J. Phys. A 36, 12 (2003) 3173–3193.

[Z97] A. Zvonkin *Matrix integrals and map enumeration: an accessible introduction.* Math. Comput. Modelling 26, 8–10 (1997), 281–304. Combinatorics and physics (Marseilles, 1995).



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