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► **To cite this version:**

A. Baranov, Isabelle Chalendar, Emmanuel Fricain, Javad Mashreghi, Dan Timotin. Bounded symbols and reproducing kernel thesis for truncated Toeplitz operators.. 2009. hal-00412228v1

HAL Id: hal-00412228

<https://hal.science/hal-00412228v1>

Preprint submitted on 1 Sep 2009 (v1), last revised 14 Nov 2009 (v2)

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BOUNDED SYMBOLS AND REPRODUCING KERNEL THESIS FOR TRUNCATED TOEPLITZ OPERATORS

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ABSTRACT. Compressions of Toeplitz operators to coinvariant subspaces of H^2 are called *truncated Toeplitz operators*. We study two questions related to these operators. The first, raised by Sarason, is whether boundedness of the operator implies the existence of a bounded symbol; the second is the reproducing kernel thesis. We show that in general the answer to the first question is negative, and we exhibit some classes of spaces for which the answers to both questions are positive.

1. INTRODUCTION

Truncated Toeplitz operators on model spaces have been formally introduced by Sarason in [16], although special cases have long ago appeared in literature, most notably as model operators for contractions with defect numbers one and their commutant. They represent a natural analogue of the classical Toeplitz and Hankel operators on the Hardy space. This is a new area of study, and it is remarkable that many natural questions remain still unsolved. As a basic reference for their main properties, [16] is invaluable.

Thus, being given a model space K_Θ (see Section 2 for precise definitions) and a function $\varphi \in L^2$, the truncated Toeplitz operator A_φ^Θ is defined on a dense subspace of K_Θ by the compression to K_Θ of multiplication by φ . The function φ is then called a symbol of the operator, and it is never uniquely defined.

In particular, if $\varphi \in L^\infty$, then A_φ^Θ is bounded. In view of well known facts about classical Toeplitz and Hankel operators, it is natural to ask whether the converse is true, that is, if a bounded truncated Toeplitz operator has necessarily a bounded symbol. This question has already been posed in [16], where it is noticed that it is already nontrivial for rank one operators. In the present paper we will provide a class of inner functions Θ for which there exist on K_Θ rank one truncated Toeplitz operators without bounded symbols. On

Date: September 2, 2009.

2000 Mathematics Subject Classification. 47B35, 47B32.

Key words and phrases. Toeplitz operators, reproducing kernel thesis, model spaces.

the other hand, we obtain positive results for some simple model spaces. Therefore the situation is quite different from the classical Toeplitz and Hankel operators.

The other natural question that we address is the reproducing kernel thesis for truncated Toeplitz operators. Recall that an operator on a reproducing kernel Hilbert space is said to satisfy the *Reproducing Kernel Thesis* (RKT) if its boundedness is determined by its behaviour on the reproducing kernels. This property has been studied for several classes of operators: Hankel operators on the Hardy space on the unit disc [4, 10, 18], Toeplitz operators on the Paley–Wiener space [17], semicommutators of Toeplitz operators [13], Hankel operators on the Bergman space [2], and Hankel operators on the Hardy space of the bidisk [9, 14]. It appears thus natural to ask the corresponding question for truncated Toeplitz operators. We will show that in this case it is more natural to suppose the boundedness of the operator on the reproducing kernels as well as on a related “dual” family and discuss its validity for certain model spaces.

The plan of the paper is the following. The next two sections contain preliminary material concerning model spaces and truncated Toeplitz operators. Section 4 discusses the main two problems we are concerned with: existence of a bounded symbols and the reproducing kernel thesis. Some counterexamples are presented in Section 5; in particular, Sarason’s question on the general existence of bounded symbols is answered in the negative. Section 6 exhibits some classes of model spaces for which the answers to both questions are positive. Finally, in Section 7 we present another class of well behaved truncated Toeplitz operators.

2. PRELIMINARIES

Recall that the Hardy space H^2 of the unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ is the Hilbert space of analytic functions $f(z) = \sum_{n \geq 0} a_n z^n$ defined in \mathbb{D} , such that $\sum_{n \geq 0} |a_n|^2 < \infty$. We denote also $H_0^2 = zH^2$. Alternatively, H^2 can be identified with a closed subspace of the Lebesgue space $L^2 = L^2(\mathbb{T})$ on the unit circle \mathbb{T} , by associating with each analytic function its radial limit. The algebra of bounded analytic functions on \mathbb{D} is denoted by H^∞ . Any $\varphi \in H^\infty$ acts as a multiplication operator on H^2 , that we will denote by T_φ .

Evaluations at points $\lambda \in \mathbb{D}$ are bounded functionals on H^2 and the corresponding reproducing kernel is $k_\lambda(z) = \frac{1}{1-\lambda z}$; thus, $f(\lambda) = \langle f, k_\lambda \rangle$. If $\varphi \in H^\infty$, then k_λ is an eigenvector for T_φ^* , and $T_\varphi^* k_\lambda = \overline{\varphi(\lambda)} k_\lambda$. By normalizing k_λ we obtain $h_\lambda = \frac{k_\lambda}{\|k_\lambda\|} = \sqrt{1-|\lambda|^2} k_\lambda$.

Suppose now Θ is an inner function. We define the corresponding *coinvariant subspace* generated by Θ (also called *model space*) by the formula $K_\Theta = H^2 \ominus \Theta H^2$; the orthogonal projection onto K_Θ is denoted by P_Θ . It is well known (see [13]) that $P_\Theta = P_+ - \Theta P_+ \bar{\Theta}$, where P_+ is the Riesz projection from L^2 onto H^2 . Since P_+ acts boundedly on L^p , $1 < p < \infty$, this formula shows that P_Θ can also be regarded as a bounded operator from L^p into $K_\Theta^p = H^p \cap \overline{\Theta z H^p}$, $1 < p < \infty$.

In K_Θ the reproducing kernel for a point $\lambda \in \mathbb{D}$ is the function

$$(2.1) \quad k_\lambda^\Theta(z) = P_\Theta k_\lambda = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \bar{\lambda}z}$$

and we denote by h_λ^Θ the normalized reproducing kernel,

$$(2.2) \quad h_\lambda^\Theta(z) = \sqrt{\frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2}} k_\lambda^\Theta(z).$$

Note that, according to (2.1), we have the orthogonal decomposition

$$(2.3) \quad k_\lambda = k_\lambda^\Theta + \overline{\Theta(\lambda)} k_\lambda.$$

We will use the antilinear isometry $J : L^2 \rightarrow L^2$, given by $J(f)(\zeta) = \bar{\zeta} \bar{f}(\zeta)$; it maps H^2 into $H_-^2 = L^2 \ominus H^2$ and conversely. More often will appear another antilinear isometry ($\omega = \Theta J$), whose main properties are summarized below.

Lemma 2.1. *Define, for $f \in L^2$, $\omega(f)(\zeta) = \bar{\zeta} \bar{f}(\zeta) \Theta(\zeta)$. Then:*

- (i) ω is antilinear, isometric, onto;
- (ii) $\omega^2 = \text{Id}$;
- (iii) $\omega P_\Theta = P_\Theta \omega$ (and therefore K_Θ reduces ω), $\omega(\Theta H^2) = H_-^2$ and $\omega(H_-^2) = \Theta H^2$;
- (iv) for all $f, g \in L^2$, $\langle \omega f, \omega g \rangle = \langle g, f \rangle$.

We define the difference quotient $\tilde{k}_\lambda^\Theta = \omega(k_\lambda^\Theta)$ and $\tilde{h}_\lambda^\Theta = \omega(h_\lambda^\Theta)$; thus

$$\tilde{k}_\lambda^\Theta(z) = \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda}, \quad \tilde{h}_\lambda^\Theta(z) = \sqrt{\frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2}} \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda}$$

In the sequel we will use the following simple lemma.

Lemma 2.2. *Suppose Θ_1, Θ_2 are two inner functions, $f_1 \in K_{\Theta_1}$, $f_2 \in K_{\Theta_2} \cap H^\infty$. Then $f_1 f_2, z f_1 f_2 \in K_{\Theta_1 \Theta_2}$.*

Proof. Obviously $zf_1f_2 \in H^2$. On the other side, $f_1 \in K_{\Theta_1}$ implies $f_1 = \Theta_1g_1$, with $g_1 \in H_-^2$, and similarly $f_2 = \Theta_2g_2$, $g_2 \in H_-^2 \cap L^\infty$. Thus $f_1f_2 \in \Theta_1\Theta_2\bar{z}H_-^2$. Therefore $zf_1f_2 \in H^2 \cap \Theta_1\Theta_2H_-^2 = K_{\Theta_1\Theta_2}$. The claim about f_1f_2 is an immediate consequence. \square

Lemma 2.3. *Suppose that θ and Θ are two inner functions such that θ^3 divides $z\Theta$. Then we have the followings:*

- (a) $K_{\theta^2} \subset K_\Theta$.
- (b) $\theta K_\theta \subset K_\Theta$.
- (c) *If $f \in H^\infty \cap \theta K_\theta$ and $\varphi \in K_\theta + \overline{K_\theta}$, then the functions φf and $\bar{\varphi}f$ belong to K_Θ .*

Proof. Since θ^3 divides $z\Theta$, there exists an inner function θ_1 such that $z\Theta = \theta^3\theta_1$. In particular it follows from this factorization that $\theta(0)\theta_1(0) = 0$, which implies that $\theta\theta_1H^2 \subset zH^2$.

(a): We have

$$K_{\theta^2} = H^2 \cap \theta^2 \overline{zH^2} = H^2 \cap \Theta z \overline{\Theta \theta^2 H^2} = H^2 \cap \Theta \overline{\theta\theta_1 H^2} \subset H^2 \cap \Theta \overline{zH^2} = K_\Theta,$$

because $\theta\theta_1H^2 \subset zH^2$.

(b): using $K_\theta = H^2 \cap \theta \overline{zH^2}$, we have

$$\theta K_\theta = \theta H^2 \cap \theta^2 \overline{zH^2} \subset H^2 \cap \theta^2 \overline{zH^2} = K_{\theta^2},$$

and it remains to apply (a).

(c): let $f = \theta f_1$ and $\varphi = \varphi_1 + \overline{\varphi_2}$, with $f_1 \in H^\infty \cap K_\theta$ and $\varphi_1, \varphi_2 \in K_\theta$. Since $\varphi_2 \in K_\theta$, we have $\varphi_2 = \theta \bar{z} \tilde{\varphi}_2$, with $\tilde{\varphi}_2 \in K_\theta$, which implies that

$$\varphi f = \theta f_1 (\varphi_1 + \overline{\varphi_2}) = \theta f_1 \varphi_1 + z f_1 \tilde{\varphi}_2.$$

But it follows from Lemma 2.2 that $z f_1 \tilde{\varphi}_2 \in K_{\theta^2}$ and by (a), we get that $z f_1 \tilde{\varphi}_2 \in K_\Theta$. So it remains to prove that $\theta f_1 \varphi_1 \in K_\Theta$. First we have obviously $\theta f_1 \varphi_1 \in H^2$. Moreover for every function $h \in H^2$, we have

$$\langle \theta f_1 \varphi_1, \Theta h \rangle = \langle z \theta f_1 \varphi_1, z \Theta h \rangle = \langle z \theta f_1 \varphi_1, \theta^3 \theta_1 h \rangle = \langle z f_1 \varphi_1, \theta^2 \theta_1 h \rangle = 0,$$

because using once more Lemma 2.2, we have $z f_1 \varphi_1 \in K_{\theta^2}$. That proves that $\theta f_1 \varphi_1 \in K_\Theta$. \square

Finally, it is useful to remember the connection with the ‘‘continuous’’ case. If $u(w) = \frac{w-i}{w+i}$, then u is a conformal homeomorphism of the Riemann sphere. It maps $-i$ to ∞ , ∞ to 1 , \mathbb{R} onto \mathbb{T} and \mathbb{C}_+ to \mathbb{D} (here $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$).

The operator

$$(\mathcal{U}f)(t) = \frac{1}{\sqrt{\pi}(t+i)} f(u(t))$$

maps $L^2(\mathbb{T})$ unitarily onto $L^2(\mathbb{R})$ and H^2 unitarily onto $H^2(\mathbb{C}_+)$, the Hardy space of the upper half-plane. The corresponding transformation for functions in L^∞ is

$$(2.4) \quad \tilde{\mathcal{U}}(\varphi) = \varphi \circ u;$$

it maps $L^\infty(\mathbb{T})$ isometrically onto $L^\infty(\mathbb{R})$, H^∞ isometrically onto $H^\infty(\mathbb{C}_+)$ and inner functions in \mathbb{D} into inner functions in \mathbb{C}_+ . Now if Θ is an inner function in \mathbb{D} , we have $\mathcal{U}K_\Theta = \mathbf{K}_\Theta$, where $\mathbf{K}_\Theta = H^2(\mathbb{C}_+) \ominus \Theta H^2(\mathbb{C}_+)$ and $\Theta = \Theta \circ u$. Moreover

$$(2.5) \quad \mathcal{U}h_\lambda^\Theta = c_\mu \mathbf{h}_\mu^\Theta \quad \text{and} \quad \mathcal{U}\tilde{h}_\lambda^\Theta = \overline{c_\mu} \tilde{\mathbf{h}}_\mu^\Theta,$$

where $\mu = u^{-1}(\lambda)$, $c_\mu = \frac{\bar{\mu}-i}{|\mu+i|}$ is a constant of modulus one and

$$\mathbf{h}_\mu^\Theta(\omega) = \frac{i}{\sqrt{\pi}} \sqrt{\frac{\operatorname{Im} \mu}{1 - |\Theta(\mu)|^2}} \frac{1 - \overline{\Theta(\mu)}\Theta(\omega)}{\omega - \bar{\mu}}, \quad \omega \in \mathbb{C}_+,$$

is the normalized reproducing kernel for \mathbf{K}_Θ and

$$\tilde{\mathbf{h}}_\mu^\Theta(\omega) = \frac{1}{i\sqrt{\pi}} \sqrt{\frac{\operatorname{Im} \mu}{1 - |\Theta(\mu)|^2}} \frac{\Theta(\omega) - \Theta(\mu)}{\omega - \mu}, \quad \omega \in \mathbb{C}_+,$$

is the normalized difference quotient in \mathbf{K}_Θ .

3. TRUNCATED TOEPLITZ OPERATORS

In [16], D. Sarason studied the class of *truncated Toeplitz operators* which are defined as the compression of Toeplitz operators to coinvariant subspaces of H^2 .

Let us begin with a few words about usual multiplication operators and their cognates, when the symbol is in L^2 . If $\varphi \in L^2$, then the formula $M_\varphi f = \varphi f$, $f \in L^\infty$, gives a densely defined operator. It is bounded if and only if φ is essentially bounded, and then $\|M_\varphi\| = \|\varphi\|_\infty$.

Toeplitz and Hankel operators with symbol φ are defined by the formulas $T_\varphi f = P_+ \varphi f$ and $H_\varphi f = P_- \varphi f$, where we suppose that $f \in H^\infty$ and $\varphi \in L^2$ and $P_- = I - P_+$ is the orthogonal projection onto H_-^2 . The operators T_φ and H_φ are therefore defined on a dense domain in H^2 ; they take values in H^2 and H_-^2 respectively. It is well-known that T_φ is

bounded if and only if $\varphi \in L^\infty$ (and $\|\varphi\|_\infty = \|T_\varphi\|$), while H_φ is bounded if and only if $P_-\varphi \in BMO$ (and $\|\varphi\|_{BMO}$ is equivalent to $\|H_\varphi\|$). Also, we have

$$(3.1) \quad T_\varphi^* = T_{\bar{\varphi}}, \quad H_\varphi^* = P_+ M_{\bar{\varphi}} P_-.$$

In [16], D. Sarason defines an analogous operator on K_Θ . Suppose $\varphi \in L^2$; the *truncated Toeplitz operator* A_φ^Θ will in general be a densely defined, possibly unbounded, operator on K_Θ . Its domain is $K_\Theta \cap H^\infty$, on which it acts by the formula

$$A_\varphi^\Theta(f) = P_\Theta \varphi f.$$

In particular, $K_\Theta \cap H^\infty$ contains all reproducing kernels k_λ^Θ and their linear combinations, and is therefore dense in K_Θ .

A useful formula is

$$(3.2) \quad \omega A_\varphi^\Theta \omega = A_{\bar{\varphi}}^\Theta = (A_\varphi^\Theta)^*.$$

We call φ a *symbol* of the operator A_φ^Θ . It is not unique; in [16], it is shown that $A_\varphi^\Theta = 0$ if and only if $\varphi \in \Theta H^2 + \overline{\Theta H^2}$. Let us denote $\mathfrak{S}_\Theta = L^2 \ominus (\Theta H^2 + \overline{\Theta H^2})$. Two spaces that contain \mathfrak{S}_Θ up to a subspace of dimension at most 1 admit a direct description. First, since

$$L^2 = \Theta H^2 \oplus \overline{\Theta H_0^2} \oplus K_\Theta \oplus \overline{\bar{z} K_\Theta},$$

it follows that $\mathfrak{S}_\Theta \subset K_\Theta \oplus \overline{\bar{z} K_\Theta}$; more precisely, if Q_Θ is the orthogonal projection onto $K_\Theta \oplus \overline{\bar{z} K_\Theta}$ then

$$(3.3) \quad K_\Theta \oplus \overline{\bar{z} K_\Theta} = \mathfrak{S}_\Theta + \mathbb{C} Q_\Theta(\bar{\Theta}).$$

Secondly (see [16, Section 3]), $\mathfrak{S}_\Theta \subset K_\Theta + \overline{K_\Theta}$. Each truncated Toeplitz operator has a symbol φ of the form $\varphi = \varphi_+ + \overline{\varphi_-}$ with $\varphi_\pm \in K_\Theta$; any other such decomposition corresponds to $\varphi_+ + ck_0^\Theta$, $\varphi_- - \bar{c}k_0^\Theta$ for some $c \in \mathbb{C}$. In particular, φ_\pm are uniquely determined if we fix (arbitrarily) the value of one of them in a point of \mathbb{D} .

Hankel operators have a unique symbol in H_-^2 , and this can be recaptured simply from the operator: it is the image of the constant function 1. It is interesting to obtain a similar direct formula for the symbol of a truncated Toeplitz operator. As noted above, this is unique if we assume, for instance that $\varphi = \varphi_+ + \overline{\varphi_-}$, with $\varphi_\pm \in K_\Theta$ and $\varphi_-(0) = 0$. We

can then recapture φ from its action on k_λ^Θ and \tilde{k}_λ^Θ . Indeed, one can check that

$$(3.4) \quad \begin{aligned} A_\varphi^\Theta k_0^\Theta &= \varphi_+ - \overline{\Theta(0)\varphi_-}, \\ A_\varphi^\Theta \tilde{k}_0^\Theta &= \omega \left(\varphi_- + \overline{\varphi_+(0)} - \overline{\Theta(0)\varphi_+} \right). \end{aligned}$$

From the first equation we obtain $\varphi_+(0) = \langle A_\varphi^\Theta k_0^\Theta, k_0^\Theta \rangle$. Then (3.4) imply, for any $\lambda \in \mathbb{D}$,

$$\begin{aligned} \varphi_+(\lambda) - \overline{\Theta(0)\Theta(\lambda)\varphi_-(\lambda)} &= \langle A_\varphi^\Theta k_0^\Theta, k_\lambda^\Theta \rangle, \\ \overline{\varphi_-(\lambda)} - \Theta(0)\overline{\Theta(\lambda)\varphi_+(\lambda)} &= \langle \tilde{k}_\lambda^\Theta, A_\varphi^\Theta \tilde{k}_0^\Theta \rangle - \langle A_\varphi^\Theta k_0^\Theta, k_0^\Theta \rangle. \end{aligned}$$

This is a linear system in $\varphi_+(\lambda)$ and $\overline{\varphi_-(\lambda)}$, whose determinant is $1 - |\Theta(0)\Theta(\lambda)|^2 > 0$; therefore, φ_\pm can be made explicit in terms of the products in the right hand side.

Note, however, that A_φ^Θ is completely determined by its action on reproducing kernels, so one should be able to recapture the values of the symbol only from $A_\varphi^\Theta k_\lambda^\Theta$. The precise formulas are more complicated, but we will have the occasion to use them below, and so we include the relevant computations.

Fix a point $\mu \in \mathbb{D}$ where $\Theta(\mu) \neq 0$. Choose the unique decomposition $\varphi = \varphi_+ + \overline{\varphi_-}$ with $\varphi_-(\mu) = 0$, and denote $\psi_+ = \omega(\varphi_+)$.

A careful repeated application of the formula $P_\Theta = P_+ - \Theta P_+ \bar{\Theta}$ yields

$$\omega(A_\varphi^\Theta(k_\lambda^\Theta)) = (I - \lambda S^*)^{-1} \psi_+ + \varphi_-(\lambda) (I - \lambda S^*)^{-1} S^* \Theta - \Theta(\lambda) (I - \lambda S^*)^{-1} S^* \varphi_-,$$

or

$$(3.5) \quad (I - \lambda S^*) \omega(A_\varphi^\Theta(k_\lambda^\Theta)) = \psi_+ + \varphi_-(\lambda) S^* \Theta - \Theta(\lambda) S^* \varphi_-.$$

If we take $\lambda = \mu$, we obtain (remembering that $\varphi_-(\mu) = 0$)

$$(3.6) \quad \psi_+ = (I - \mu S^*) \omega(A_\varphi^\Theta(k_\mu^\Theta)) + \Theta(\mu) S^* \varphi_-.$$

Denote, for simplicity,

$$F_{\lambda,\mu} = (I - \lambda S^*) \omega(A_\varphi^\Theta(k_\lambda^\Theta)) - (I - \mu S^*) \omega(A_\varphi^\Theta(k_\mu^\Theta));$$

then plugging (3.6) into (3.5) yields

$$\varphi_-(\lambda) S^* \Theta + (\Theta(\mu) - \Theta(\lambda)) S^* \varphi_- = F_{\lambda,\mu}.$$

It can easily be checked that $(S - \mu)(I - \mu S^*)^{-1} S^* f = f - f(\mu)$ for all $f \in H^2$; therefore, applying $(S - \mu)(I - \mu S^*)^{-1}$ and remembering that $\varphi_-(\mu) = 0$, we obtain

$$(3.7) \quad \varphi_-(\lambda) (\Theta - \Theta(\mu)) + (\Theta(\mu) - \Theta(\lambda)) \varphi_- = (S - \mu)(I - \mu S^*)^{-1} F_{\lambda,\mu}.$$

Finally, we take the scalar product of both sides with k_μ^Θ and use $\Theta \perp K_\Theta$, $P_\Theta 1 = 1 - \overline{\Theta(0)}\Theta$, and again $\varphi_-(\mu) = 0$. Therefore

$$-\varphi_-(\lambda)\Theta(\mu)(1 - \overline{\Theta(0)}\Theta(\mu)) = \langle (S - \mu)(I - \mu S^*)^{-1}F_{\lambda,\mu}, k_\mu^\Theta \rangle,$$

or

$$(3.8) \quad \varphi_-(\lambda) = \frac{\langle (S - \mu)(I - \mu S^*)^{-1}F_{\lambda,\mu}, k_\mu^\Theta \rangle}{\Theta(\mu)(\overline{\Theta(0)}\Theta(\mu) - 1)}.$$

This formula gives the value of $\varphi_-(\lambda)$ for all $\lambda \in \mathbb{D}$. We have thus proved the following result.

Proposition 3.1. *Suppose $\varphi = \varphi_+ + \overline{\varphi_-}$ with $\varphi_\pm \in K_\Theta$ and $\varphi_-(0) = 0$. Then $\varphi_-(\lambda)$ is determined by (3.8), while $\varphi_+ = \omega(\psi_+)$, where ψ_+ is given by (3.6).*

We will denote by $\mathcal{T}(K_\Theta)$ the Banach space of all bounded truncated Toeplitz operators on K_Θ .

The following proposition yields a relation between truncated operators and usual Hankel operators.

Proposition 3.2. *With respect to the decompositions $H_-^2 = \bar{\Theta}K_\Theta \oplus \bar{\Theta}H_-^2$, $H^2 = K_\Theta \oplus \Theta H^2$, the operator $H_\Theta^* H_{\bar{\Theta}\varphi} H_\Theta^* : H_-^2 \rightarrow H_2$ has the matrix*

$$(3.9) \quad \begin{pmatrix} A_\varphi^\Theta M_\Theta & 0 \\ 0 & 0 \end{pmatrix}$$

Proof. If $f \in \bar{\Theta}H_-^2$, then $H_\Theta^* f = 0$. If $f \in \bar{\Theta}K_\Theta$, then, according to (3.1), $H_\Theta^* f = \Theta f \in K_\Theta$. Since $P_\Theta = P_+ M_\Theta P_- M_{\bar{\Theta}}$, it follows that, for $f \in K_\Theta$,

$$A_\varphi^\Theta f = P_\Theta M_\varphi f = P_+ M_\Theta P_- M_{\bar{\Theta}} M_\varphi f = H_\Theta^* H_{\bar{\Theta}\varphi}^* f,$$

and therefore, if $f \in \bar{\Theta}K_\Theta$, then $A_\varphi^\Theta \Theta f = H_\Theta^* H_{\bar{\Theta}\varphi} H_\Theta^* f$ as required. \square

The non-zero entry in (3.9) consists in the isometry $M_\Theta : \bar{\Theta}K_\Theta \rightarrow K_\Theta$, followed by A_φ^Θ acting on K_Θ . There is therefore a close connection between properties of A_φ^Θ and properties of the corresponding product of three Hankel operators.

Remark 3.3. Truncated Toeplitz operators can be defined also on model spaces included in $H^2(\mathbb{C}_+)$, that is, $\mathbf{K}_\Theta = H^2(\mathbb{C}_+) \ominus \Theta H^2(\mathbb{C}_+)$ for Θ an inner function on \mathbb{C}_+ . We start then with a symbol $\varphi \in (t + i)L^2(\mathbb{R})$ (which contains $L^\infty(\mathbb{R})$) and define (for f

in $\mathbf{K}_\Theta \cap (z+i)^{-1}H^\infty(\mathbb{C}_+)$, a dense subspace of \mathbf{K}_Θ) the truncated Toeplitz operator $\mathbf{A}_\varphi^\Theta f = P_{\mathbf{K}_\Theta} \varphi f$. Let \mathbf{A} be a linear operator on \mathbf{K}_Θ . Then \mathbf{A} is a truncated Toeplitz operator on \mathbf{K}_Θ if and only if $A = \mathcal{U}^* \mathbf{A} \mathcal{U}$ is a truncated Toeplitz operator on K_Θ , and φ is a symbol for \mathbf{A} if and only if $\psi := \varphi \circ u^{-1}$ is a symbol for A . It follows that \mathbf{A} is bounded (or has a bounded symbol) if and only if A is bounded (respectively, has a bounded symbol). Moreover we easily deduce from (2.5) that

$$\|\mathbf{A}_\varphi^\Theta h_\mu^\Theta\|_2 = \|A_\psi^\Theta h_\lambda^\Theta\|_2 \quad \text{and} \quad \|\mathbf{A}_\varphi^\Theta \tilde{h}_\mu^\Theta\|_2 = \|A_\psi^\Theta \tilde{h}_\lambda^\Theta\|_2,$$

for every $\mu \in \mathbb{C}_+$ and $\lambda = u(\mu)$. Finally, $\mathbf{A}_\varphi^\Theta = 0$ if and only if $\varphi \in (t+i) \left(\Theta H^2(\mathbb{C}_+) \oplus \overline{\Theta H^2(\mathbb{C}_+)} \right)$ (note that the sum is in this case orthogonal, since $H^2(\mathbb{C}_+) \perp \overline{H^2(\mathbb{C}_+)}$).

4. EXISTENCE OF BOUNDED SYMBOLS AND THE REPRODUCING KERNEL THESIS

A truncated Toeplitz operator is bounded if it has a symbol in L^∞ . In [16], Sarason asked whether the converse is true.

Question 1. *Does every bounded truncated Toeplitz operator on K_Θ possess an L^∞ symbol?*

One should expect the answer to depend on the function Θ , and indeed we show below that it is the case. Note also that from the open mapping theorem it follows that, if, for a certain inner function Θ any operator in $\mathcal{T}(K_\Theta)$ has a bounded symbol, then there exists a constant C such that for any $A \in \mathcal{T}(K_\Theta)$ one can find $\varphi \in L^\infty$ with $\|\varphi\|_\infty \leq C\|A\|$, such that $A = A_\varphi^\Theta$.

A second natural question that may be asked about truncated Toeplitz operators is the reproducing kernel thesis (RKT). The functions h_λ^Θ are in L^∞ for all $\lambda \in \mathbb{D}$, so we apply them A_φ^Θ for any $\lambda \in \mathbb{D}$; if A_φ^Θ is bounded then obviously $\|A_\varphi^\Theta h_\lambda^\Theta\| \leq \|A_\varphi^\Theta\|$. Define then for $T \in \mathcal{L}(K_\Theta)$

$$(4.1) \quad \rho(T) := \sup_{\lambda \in \mathbb{D}} \|Th_\lambda^\Theta\|.$$

The following question is then natural:

Question 2. *(Reproducing kernel thesis for truncated Toeplitz operators): let Θ be an inner function and $\varphi \in L^2$. Assume that $\rho(A_\varphi^\Theta) < +\infty$. Is A_φ^Θ bounded on K_Θ ?*

We will see in Section 5 that the answer to this question is in general negative. As we will see below, it is more natural to restate the RKT by including in the hypothesis also

the functions \tilde{h}_λ^Θ . Define for $T \in \mathcal{L}(K_\Theta)$

$$\rho'(T) = \max\left\{\sup_{\lambda \in \mathbb{D}} \|Th_\lambda^\Theta\|, \sup_{\lambda \in \mathbb{D}} \|T\tilde{h}_\lambda^\Theta\|\right\}.$$

Question 3. *Let Θ be an inner function and $\varphi \in L^2$. Assume that $\rho'(A_\varphi^\Theta) < \infty$. Is A_φ bounded on K_Θ ?*

Now we will see that it is easy to deal with analytic or antianalytic symbols. The next proposition is a straightforward consequence of Bonsall's theorem [4] and the commutant lifting theorem. A part of this proposition (more precisely the equivalence between (i) and (ii)) has already been noticed in [16].

Proposition 4.1. *Let $\varphi \in H^2$ and let A_φ^Θ be a truncated Toeplitz operator. Then the following assertions are equivalent:*

- (i) A_φ^Θ has a bounded symbol.
- (ii) A_φ^Θ is bounded.
- (iii) $\rho(A_\varphi^\Theta) < +\infty$.

More precisely there exists a constant $C > 0$ such that any truncated Toeplitz operator A_φ^Θ has a bounded symbol φ_0 with $\|\varphi_0\|_\infty \leq C\rho(A_\varphi^\Theta)$.

Proof. It is immediate that (i) \implies (ii) \implies (iii). The implication (ii) \implies (i) has already noted in [16]; indeed if $\varphi \in H^2$ and A_φ^Θ is bounded, then A_φ^Θ commutes with $S_\Theta := A_z^\Theta$ and then, by a corollary of the commutant lifting theorem, A_φ^Θ has an H^∞ symbol with norm equal to the norm of A_φ^Θ .

So it remains to prove that there exists a constant $C > 0$ such that $\|A_\varphi^\Theta\| \leq C\rho(A_\varphi^\Theta)$. If $f \in K_\Theta \cap H^\infty$, then $\varphi f \in H^2$. Therefore $P_\Theta(\varphi f) = \Theta P_-(\bar{\Theta}\varphi f)$, or, in other words,

$$A_\varphi^\Theta(f) = \Theta H_{\bar{\Theta}\varphi} f.$$

On the other hand, $H^2 \ominus K_\Theta \subset \ker H_{\bar{\Theta}\varphi}$. Therefore, with respect to the decompositions $H^2 = K_\Theta \oplus \Theta H^2$, $H_-^2 = \bar{\Theta}K_\Theta \oplus \bar{\Theta}H_-^2$, one can write

$$(4.2) \quad H_{\bar{\Theta}\varphi} = \begin{pmatrix} \bar{\Theta}A_\varphi^\Theta & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that A_φ^Θ is bounded if and only if $H_{\bar{\Theta}\varphi}$ is. By Bonsall's Theorem ([4]), there exists a universal constant C (independent of φ) such that the boundedness of $H_{\bar{\Theta}\varphi}$ is equivalent

to $\sup_{\lambda \in \mathbb{D}} \|H_{\bar{\Theta}\varphi} h_\lambda\| < \infty$, and

$$\|H_{\bar{\Theta}\varphi}\| \leq C \sup_{\lambda \in \mathbb{D}} \|H_{\bar{\Theta}\varphi} h_\lambda\|.$$

But, again by (4.2),

$$H_{\bar{\Theta}\varphi} h_\lambda = \bar{\Theta} A_\varphi^\Theta P_\Theta h_\lambda = \bar{\Theta} (1 - |\Theta(\lambda)|^2)^{1/2} A_\varphi^\Theta h_\lambda^\Theta,$$

and thus $\sup_{\lambda \in \mathbb{D}} \|H_{\bar{\Theta}\varphi} h_\lambda\| \leq \sup_{\lambda \in \mathbb{D}} \|A_\varphi^\Theta h_\lambda^\Theta\|$. The proposition is proved. \square

A similar result is valid for antianalytic symbols.

Proposition 4.2. *Let $\varphi \in \overline{H^2}$ and let A_φ^Θ be a truncated Toeplitz operator. Then the following assertions are equivalent:*

- (i) A_φ^Θ has a bounded symbol.
- (ii) A_φ^Θ is bounded.
- (iii) $\sup_{\lambda \in \mathbb{D}} \|A_\varphi^\Theta \tilde{h}_\lambda^\Theta\| < +\infty$.

More precisely there exists a constant $C > 0$ such that any truncated Toeplitz operator A_φ^Θ has a bounded symbol φ_0 with $\|\varphi_0\|_\infty \leq C \sup_{\lambda \in \mathbb{D}} \|A_\varphi^\Theta \tilde{h}_\lambda^\Theta\|$.

Proof. Suppose $\varphi \in \overline{H^2}$. Since $\|A_\varphi^\Theta\| = \|(A_\varphi^\Theta)^*\| = \|A_{\bar{\varphi}}^\Theta\|$, and $\bar{\varphi} \in H^2$, we may apply Proposition 4.1 to $A_{\bar{\varphi}}^\Theta$ because by (3.2), we have

$$\sup_{\lambda \in \mathbb{D}} \|A_{\bar{\varphi}}^\Theta h_\lambda^\Theta\| = \sup_{\lambda \in \mathbb{D}} \|A_{\bar{\varphi}}^\Theta \omega h_\lambda^\Theta\| = \sup_{\lambda \in \mathbb{D}} \|A_\varphi^\Theta \tilde{h}_\lambda^\Theta\|.$$

\square

As we have seen, if φ is bounded, then obviously the truncated Toeplitz operator A_φ^Θ is bounded. We will see now that one can get a slightly more general result. It involves the so-called *Carleson curves* associated with an inner function. Recall that if Θ is an inner function and $\alpha \in (0, 1)$, then the system of Carleson curves Γ_α associated to Θ and α is the countable union of closed simple and rectifiable curves in $\text{clos } \mathbb{D}$ such that

- (1) the interior of curves in Γ_α are pairwise disjoint.
- (2) there is a constant $\eta(\alpha) > 0$ such that for every $z \in \Gamma_\alpha \cap \mathbb{D}$, we have

$$(4.3) \quad \eta(\alpha) \leq |\Theta(z)| \leq \alpha.$$

- (3) arclength $|dz|$ on Γ_α is a Carleson measure.

(4) for every function $\varphi \in H^1$, we have

$$(4.4) \quad \int_{\mathbb{T}} \frac{\varphi(z)}{\Theta(z)} dz = \int_{\Gamma_\alpha} \frac{\varphi(z)}{\Theta(z)} dz$$

We will use this construction of curves to give a sufficient condition on the symbol $\varphi \in L^2$ which gives a bounded truncated Toeplitz operator (with a bounded symbol).

Proposition 4.3. *Let $\varphi \in H^2$ and assume that $|\varphi||dz|$ is a Carleson measure on Γ_α . Then A_φ^Θ is bounded on K_Θ and it has a bounded symbol.*

Proof. Let $f, g \in K_\Theta$ and assume further that $f \in H^\infty$. Then we have

$$\langle A_\varphi^\Theta f, g \rangle = \langle \varphi f, g \rangle = \int_{\mathcal{T}} \varphi(z) f(z) \overline{g(z)} dz.$$

Since $g \in K_\Theta$, we can write (on \mathcal{T}), $g(z) = \overline{z} h(z) \Theta(z)$, with $h \in K_\Theta$. Therefore

$$\langle A_\varphi^\Theta f, g \rangle = \int_{\mathcal{T}} \frac{z \varphi(z) f(z) h(z)}{\Theta(z)} dz.$$

But $z f(z) \varphi(z) h(z) \in H^1$ and using (4.4), we can write

$$\langle A_\varphi^\Theta f, g \rangle = \int_{\Gamma_\alpha} \frac{z \varphi(z) f(z) h(z)}{\Theta(z)} dz.$$

Therefore, according to (4.3), we have

$$|\langle A_\varphi^\Theta f, g \rangle| \leq \int_{\Gamma_\alpha} \frac{|z \varphi(z) f(z) h(z)|}{|\Theta(z)|} |dz| \leq \frac{1}{\eta(\alpha)} \int_{\Gamma_\alpha} |f(z)| |h(z)| |\varphi||dz|.$$

Hence by Cauchy-Schwarz's inequality and using the fact that $|\varphi||dz|$ is a Carleson measure on Γ_α , we have

$$|\langle A_\varphi^\Theta f, g \rangle| \leq C \frac{1}{\eta(\alpha)} \|f\|_2 \|g\|_2.$$

Finally, we get that A_φ^Θ is bounded. Since φ is analytic it follows from Proposition 4.1 that A_φ^Θ has a bounded symbol. \square

Corollary 4.4. *Let $\varphi = \varphi_1 + \overline{\varphi_2}$, with $\varphi_i \in H^2$, $i = 1, 2$. Assume that $|\varphi_i||dz|$ are Carleson measures on Γ_α for $i = 1, 2$. Then A_φ^Θ is bounded and has a bounded symbol.*

Proof. Using Proposition 4.3, we get immediately that $A_{\varphi_i}^\Theta$ is bounded and has a bounded symbol $\widetilde{\varphi}_i$, for $i = 1, 2$. Therefore, $A_{\overline{\varphi_2}}^\Theta = (A_{\varphi_2}^\Theta)^*$ is also bounded and has a bounded symbol $\overline{\widetilde{\varphi_2}}$. Hence we get that $A_\varphi^\Theta = A_{\varphi_1}^\Theta + A_{\overline{\varphi_2}}^\Theta$ is bounded and it has a bounded symbol, say $\widetilde{\varphi}_1 + \overline{\widetilde{\varphi_2}}$. \square

Remark 4.5. According to (3), we know that $|dz|$ is a Carleson measure on Γ_α . Therefore, Proposition 4.3 can be applied if φ is bounded on Γ_α and Corollary 4.4 can be applied if φ_1, φ_2 are bounded on Γ_α .

In Section 5, we will show that the answer to Question 1 and 2 may be negative. Question 3 remains in general open. In Section 6, we will give some examples of spaces K_Θ on which the answers to Questions 1 and 3 are positive.

5. COUNTEREXAMPLES

It is known [1] that, if Θ has an angular derivative in the sense of Caratheodory at $\zeta \in \mathbb{T}$, then evaluation in ζ is continuous on K_Θ , and the corresponding reproducing kernel is $k_\zeta^\Theta \in K_\Theta$. In [16, Section 5] it is shown that in this case the selfadjoint operator $k_\zeta^\Theta \otimes k_\zeta^\Theta$ is a truncated Toeplitz operator.

Lemma 5.1. *If Θ has an angular derivative in the sense of Caratheodory at $\zeta \in \mathbb{T}$, then $\varphi_\zeta = \Theta \bar{z} k_\zeta^{\Theta^2} \in K_\Theta \oplus \bar{z} \overline{K_\Theta}$ is a symbol for $k_\zeta^\Theta \otimes k_\zeta^\Theta$.*

Proof. Note first that if Θ has an angular derivative in the sense of Caratheodory at ζ , then the same is true about Θ^2 . Then $k_\zeta^{\Theta^2} \in K_{\Theta^2}$, whence it follows easily that $\Theta \bar{z} k_\zeta^{\Theta^2} \in K_\Theta \oplus \bar{z} \overline{K_\Theta}$.

Take $g, h \in K_\Theta$, and, moreover, $g \in L^\infty$. Then

$$\langle A_{\varphi_\zeta}^\Theta g, h \rangle = \langle \varphi_\zeta g, h \rangle = \int \Theta \bar{z} k_\zeta^{\Theta^2} g \bar{h}.$$

But $\Theta \bar{z} h = \omega(h) \in K_\Theta$, $g \in K_\Theta \cap L^\infty$, and so $g \Theta \bar{z} h \in K_{\Theta^2}$. Therefore

$$\begin{aligned} \int \Theta \bar{z} k_\zeta^{\Theta^2} g \bar{h} &= \langle g \Theta \bar{z} h, k_\zeta^{\Theta^2} \rangle = g(\zeta) \Theta(\zeta) \overline{\zeta h(\zeta)} = \langle g, k_\zeta^\Theta \rangle \langle \omega(h), k_\zeta^\Theta \rangle \\ &= \langle g, k_\zeta^\Theta \rangle \overline{\langle h, \omega(k_\zeta^\Theta) \rangle} = \langle g, k_\zeta^\Theta \rangle \overline{\langle h, k_\zeta^\Theta \rangle} = \langle (k_\zeta^\Theta \otimes k_\zeta^\Theta) g, h \rangle, \end{aligned}$$

where we have used the fact that $\omega(k_\zeta^\Theta) = k_\zeta^\Theta$. \square

The construction of bounded truncated Toeplitz operators that have no bounded symbol is based on the next lemma.

Lemma 5.2. *Suppose $\varphi \in K_\Theta \oplus \bar{z} \overline{K_\Theta}$ and A_φ^Θ is bounded. If, for some $p > 2$, $\varphi \notin L^p$, then A_φ^Θ has no symbol in L^p . In particular, A_φ^Θ has no symbol in L^∞ .*

Proof. Remember that Q_Θ is the orthogonal projection from L^2 to $K_\Theta \oplus \overline{zK_\Theta}$. Since the Riesz projection is bounded on L^p for $p > 1$, the same is true about Q_Θ .

Suppose $A_\varphi^\Theta = A_\psi^\Theta$ with $\psi \in L^p$. According to (3.3), $Q_\Theta(\varphi - \psi) = aQ_\Theta(\bar{\Theta})$. Thus

$$\varphi = Q_\Theta\varphi = Q_\Theta\psi + aQ_\Theta(\bar{\Theta}) \in L^p,$$

which contradicts the hypothesis. \square

Lemmas 5.1 and 5.2 imply then a sufficient condition for the existence of the desired counterexamples.

Theorem 5.3. *Suppose that Θ is an inner function which has an angular derivative in $\zeta \in \mathbb{T}$, but such that $k_\zeta^\Theta \notin L^p$ for some $p > 2$. Then $k_\zeta^\Theta \otimes k_\zeta^\Theta$ is a bounded Toeplitz operator with no bounded symbol.*

Proof. Apply Lemmas 5.1 and 5.2 to $\varphi = \varphi_\zeta = \Theta \overline{z k_\zeta^{\Theta^2}}$ (note that $k_\zeta^{\Theta^2} \in L^p$ if and only if $k_\zeta^\Theta \in L^p$). \square

It remains to give concrete examples for the condition in the statement of the theorem. In [1] and [7] precise conditions are given for the inclusion of k_ζ^Θ into L^p (for $p > 1$); namely, if (a_k) are the zeros of Θ in \mathbb{D} and σ is the singular measure on \mathbb{T} corresponding to the singular part of Θ , then $k_\zeta^\Theta \in L^p$ if and only if

$$(5.1) \quad \sum_k \frac{1 - |a_k|}{|\zeta - a_k|^p} + \int_{\mathbb{T}} \frac{d\sigma(z)}{|\zeta - z|^p} < \infty.$$

Thus the condition in the statement of the theorem is satisfied for a point $\zeta \in \mathbb{T}$ such that (5.1) is true for $p = 2$ but not for some strictly larger value of p . It is now easy to give concrete examples, as, for instance:

(1) a Blaschke product with zeros a_k accumulating to the point 1, and such that

$$\sum_k \frac{1 - |a_k|}{|1 - a_k|^2} < \infty, \quad \sum_k \frac{1 - |a_k|}{|1 - a_k|^p} = \infty \quad \text{for some } p > 2;$$

(2) a singular function $\sigma = \sum_k c_k \delta_{\zeta_k}$ with $\sum_k c_k < \infty$, $\zeta_k \rightarrow 1$, and

$$\sum_k \frac{c_k}{|1 - \zeta_k|^2} < \infty, \quad \sum_k \frac{c_k}{|1 - \zeta_k|^p} = \infty \quad \text{for some } p > 2.$$

As discussed above, the question of the existence of bounded symbols for bounded truncated Toeplitz operators has been asked in [16], in general as well as specifically for rank one operators of type $k_\zeta \otimes k_\zeta$. Thus the counterexamples above answer also that question.

A related question raised in [16] remains open. Let μ be a positive measure on the closed unit disk $\text{clos } \mathbb{D}$ such that the support of the singular part of μ is contained in $\mathbb{T} \setminus \sigma(\Theta)$, where $\sigma(\Theta)$ is the spectrum of the inner function Θ . Then we say that μ is a *Carleson measure* for K_Θ^2 if there is a constant $c > 0$ such that

$$(5.2) \quad \int_{\mathbb{T}} |f|^2 d\mu \leq c \|f\|_2^2, \quad f \in K_\Theta^2.$$

It is easy to see (and had already been noticed in [6]) that (5.2) is equivalent to the boundedness of the operator A_μ^Θ defined by the formula

$$\langle A_\mu^\Theta f, g \rangle = \int_{\mathbb{T}} f \bar{g} d\mu,$$

and it is shown in [16] that A_μ^Θ is a truncated Toeplitz operator. The natural question whether every operator in $\mathcal{T}(K_\Theta)$ is of this form is not answered by our counterexample, since (as already noticed in [16]) if Θ has an angular derivative in the sense of Caratheodory at $\zeta \in \mathbb{T}$, then δ_ζ is a Carleson measure for K_Θ and $k_\zeta^\Theta \otimes k_\zeta^\Theta = A_{\delta_\zeta \text{ eta}}$.

We pass now to the reproducing kernel thesis and give the negative answer to Question 2. The next example shows why in general it is necessary to consider ρ' rather than ρ .

Example 5.4. Suppose Θ is a singular inner function and $s \in [0, 1)$. Then

$$\begin{aligned} A_{\Theta^s}^\Theta k_\lambda^\Theta &= P_\Theta \left(\frac{\bar{\Theta}^s - \overline{\Theta(\lambda)} \Theta^{1-s}}{1 - \bar{\lambda}z} \right) \\ &= P_\Theta \left(\frac{\bar{\Theta}^s - \overline{\Theta(\lambda)}^s + \overline{\Theta(\lambda)}^s (1 - \overline{\Theta(\lambda)}^{1-s} \Theta^{1-s})}{1 - \bar{\lambda}z} \right) \\ &= P_\Theta \left(\bar{z} \frac{\bar{\Theta}^s - \overline{\Theta^s(\lambda)}}{\bar{z} - \bar{\lambda}} \right) + \overline{\Theta(\lambda)}^s P_\Theta \left(\frac{1 - \overline{\Theta(\lambda)}^{1-s} \Theta^{1-s}}{1 - \bar{\lambda}z} \right). \end{aligned}$$

The first term is in $\bar{z} \overline{H^2}$, which is orthogonal to K_Θ , while the second is contained in $K_{\Theta^{1-s}} \subset K_\Theta$. Therefore we have

$$A_{\Theta^s}^\Theta k_\lambda^\Theta = \overline{\Theta(\lambda)}^s \frac{1 - \overline{\Theta(\lambda)}^{1-s} \Theta^{1-s}}{1 - \bar{\lambda}z}$$

and

$$\|A_{\Theta^s}^\Theta k_\lambda^\Theta\|^2 = |\Theta(\lambda)|^{2s} \frac{1 - |\Theta(\lambda)|^{2-2s}}{1 - |\lambda|^2}, \quad \|A_{\Theta^s}^\Theta h_\lambda^\Theta\|^2 = \frac{|\Theta(\lambda)|^{2s} (1 - |\Theta(\lambda)|^{2-2s})}{1 - |\Theta(\lambda)|^2}.$$

It is easy to see that $\sup_{x \in [0,1]} \frac{y^s - y}{1-y} \leq 1 - s \rightarrow 0$ when $s \rightarrow 1$, and therefore

$$\sup_{\lambda \in \mathbb{D}} \|A_{\Theta^s}^\Theta h_\lambda^\Theta\|^2 \rightarrow 0 \text{ for } s \rightarrow 1.$$

On the other hand, $\Theta^s K_{\Theta^{1-s}} \subset K_\Theta$ and $\bar{\Theta}^s(\Theta^s K_{\Theta^{1-s}}) = K_{\Theta^{1-s}} \subset K_\Theta$; therefore $A_{\Theta^s}^\Theta$ acts isometrically on $\Theta^s K_{\Theta^{1-s}}$, so it has norm 1. Thus there is no constant M such that

$$\|A_\varphi^\Theta\| \leq M \sup_{\lambda \in \mathbb{D}} \rho(A_\varphi^\Theta)$$

for all φ .

It seems natural to deduce that in the previous example we may actually have a truncated Toeplitz operator which is uniformly bounded on reproducing kernels but not bounded. This is indeed true, by an abstract argument based on Proposition 3.1. For an inner function Θ and any (not necessarily bounded) linear operator T on K_Θ whose domain contains all reproducing kernels, define

$$\rho(T) := \sup_{\lambda \in \mathbb{D}} \|Th_\lambda^\Theta\|.$$

Obviously ρ is a norm, and $\rho(T) \leq \|T\|$.

Proposition 5.5. *Suppose that for any (not necessarily bounded) truncated Toeplitz operator A on K_Θ the inequality $\rho(A) < \infty$ implies A bounded. Then $\mathcal{T}(K_\Theta)$ is complete with respect to ρ , and ρ is equivalent to the operator norm on $\mathcal{T}(K_\Theta)$.*

Proof. Fix $\mu \in \mathbb{D}$ such that $\Theta(\mu) \neq 0$. If a sequence $A_{\varphi_n}^\Theta$ is ρ -Cauchy, then it is in particular convergent on reproducing kernels. Suppose all φ_n are written as $\varphi_n = \varphi_{n,+} + \overline{\varphi_{n,-}}$, with $\varphi_{n,+}, \varphi_{n,-} \in K_\Theta$, and $\varphi_{n,-}(\mu) = 0$. It follows from Proposition 3.1 that $\varphi_n(\lambda)$ is convergent for any $\lambda \in \mathbb{D}$ to a value that we will denote $\varphi(\lambda)$; by Fatou's lemma, $\varphi \in L^2$. If we consider the truncated Toeplitz operator A_φ^Θ , then $A_{\varphi_n}^\Theta k_\lambda^\Theta \rightarrow A_\varphi^\Theta k_\lambda^\Theta$ for all $\lambda \in \mathbb{D}$, whence $\rho(A_{\varphi_n}^\Theta) < \infty$. It is easy to see then that $A_{\varphi_n}^\Theta \rightarrow A_\varphi^\Theta$ in the ρ -norm.

Thus $\mathcal{T}(K_\Theta)$ is indeed complete with respect to the ρ -norm. The equivalence of the norms is then a consequence of the open mapping theorem. \square

Proposition 5.5 implies that, if Θ is a singular inner function, then there exist truncated Toeplitz operators with $\rho(T)$ finite, but T unbounded.

6. POSITIVE RESULTS

There are essentially two cases in which one can give positive answers to Questions 1 and 3. There are similarities between them: in both one obtains a convenient decomposition of the symbol in three parts: one analytic, one coanalytic, and one that is neither analytic nor coanalytic, but very well controlled.

6.1. A general result. As we have seen in Proposition 4.1 and 4.2, the answers to Questions 1 and 3. are positive for classes of truncated Toeplitz operators corresponding to analytic and co-analytic symbols. We complete these propositions with a different boundedness result, which covers certain cases when the symbol is neither analytic nor coanalytic. The proof is based on an idea of [8].

Theorem 6.1. *Suppose θ is an inner function such that θ^3 divides $z\Theta$ and Θ divides θ^4 . If $\varphi \in K_\theta + \overline{K_\theta}$ then $\|\varphi\|_\infty \leq 2\rho(A_\varphi^\Theta)$.*

Proof. Using Lemma 2.3, if $f \in L^\infty \cap \theta K_\theta$, then $f \in K_\Theta$ and $\bar{\varphi}f \in K_\Theta$; thus $A_\varphi^\Theta f = \bar{\varphi}f$. If we write $f = \theta f_1$, $\varphi_1 = \theta \bar{\varphi}$, then $\varphi_1 \in H^2$, $f_1 \in K_\theta$, and $\varphi_1 f_1 = \bar{\varphi}f = A_\varphi^\Theta f \in K_\Theta$. Therefore

$$\begin{aligned} |\varphi_1(\lambda)f_1(\lambda)| &= |\langle \varphi_1 f_1, k_\lambda^\Theta \rangle| = |\langle \theta f_1, \varphi k_\lambda^\Theta \rangle| = |\langle \theta f_1, A_\varphi^\Theta k_\lambda^\Theta \rangle| \\ &\leq \|f_1\| \|A_\varphi^\Theta k_\lambda^\Theta\| \leq \|f_1\| \|k_\lambda^\Theta\| \rho(A_\varphi^\Theta), \end{aligned}$$

where we used the fact that $\theta f_1 \in K_\Theta$.

For a fixed $\lambda \in \mathbb{D}$,

$$\sup_{\substack{f_1 \in K_\theta \cap L^\infty \\ \|f_1\| \leq 1}} |f_1(\lambda)| = \sup_{\substack{f_1 \in K_\theta \cap L^\infty \\ \|f_1\| \leq 1}} |\langle f_1, k_\lambda^\theta \rangle| = \|k_\lambda^\theta\|,$$

and thus

$$|\varphi_1(\lambda)| \leq \rho(A_\varphi^\Theta) \frac{\|k_\lambda^\Theta\|}{\|k_\lambda^\theta\|} = \rho(A_\varphi^\Theta) \frac{(1 - |\Theta(\lambda)|^2)^{1/2}}{(1 - |\theta(\lambda)|^2)^{1/2}}.$$

If Θ divides θ^4 , then $|\Theta(\lambda)| \geq |\theta(\lambda)|^4$, and therefore

$$1 - |\Theta(\lambda)|^2 \leq 1 - |\theta(\lambda)|^8 \leq 4(1 - |\theta(\lambda)|^2).$$

It follows that $|\varphi_1(\lambda)| \leq 2\rho(A_\varphi^\Theta)$ for all $\lambda \in \mathbb{D}$, and thus $\|\varphi_1\|_\infty \leq 2\rho(A_\varphi^\Theta)$. The proof is finished by noting that $\|\varphi\|_\infty = \|\varphi_1\|_\infty$. \square

As a consequence, we obtain a general result for the existence of bounded symbols and reproducing kernels thesis.

Corollary 6.2. *Let Θ be an inner function, and θ another inner function with $\theta^3|z\Theta$ and $\Theta|\theta^4$. Suppose there are constants $C_i > 0$, $i = 1, 2, 3$ such that any $\varphi \in L^2$ can be written as $\varphi = \varphi_1 + \varphi_2 + \varphi_3$, with:*

- (a) $\varphi_1 \in K_\theta + \overline{K_\theta}$, $\varphi_2 \in H^2$, and $\varphi_3 \in \overline{H^2}$;
- (b) $\rho'(A_{\varphi_i}^\Theta) \leq C_i \rho'(A_\varphi^\Theta)$ for $i = 1, 2, 3$.

Then the following are equivalent:

- (i) A_φ^Θ has a bounded symbol.
- (ii) A_φ^Θ is bounded.
- (iii) $\rho'(A_\varphi^\Theta) < +\infty$.

More precisely, there exists a constant $C > 0$ such that any truncated Toeplitz operator A_φ^Θ has a symbol φ_0 with $\|\varphi_0\|_\infty \leq C \rho'(A_\varphi^\Theta)$.

There are of course many decompositions of φ as in (a); the difficulty consists in finding one that satisfies (b).

Proof. It is immediate that (i) \implies (ii) \implies (iii), so it remains to prove (iii) \implies (i). Since $\rho'(A_{\varphi_i}^\Theta) < +\infty$, $i = 2, 3$, Proposition 4.1 and 4.2 imply that $A_{\varphi_i}^\Theta$ have bounded symbols $\tilde{\varphi}_i$ with $\|\tilde{\varphi}_i\| \leq \tilde{C} \rho'(A_{\varphi_i}^\Theta) \leq \tilde{C} C_i \rho'(A_\varphi^\Theta)$. As for φ , we can apply Theorem 6.1 which gives that φ is bounded with $\|\varphi_1\|_\infty \leq 2\rho(A_{\varphi_1}^\Theta) \leq 2C_1 \rho'(A_\varphi^\Theta)$. Finally A_φ^Θ has the bounded symbol $\varphi_0 = \varphi_1 + \tilde{\varphi}_2 + \tilde{\varphi}_3$ whose norm is at most $(2C_1 + \tilde{C}(C_2 + C_3))\rho'(A_\varphi^\Theta)$. □

6.2. Classical Toeplitz matrices. Suppose $\Theta(z) = z^N$; the space K_Θ is then an N -dimensional space with orthonormal basis formed by monomials, and truncated Toeplitz operators have a (usual) Toeplitz matrix with respect of this basis. Of course every truncated Toeplitz operator has a bounded symbol; it is however interesting that there exists a universal estimate of this bound. The question had been raised in [16, Section 7]; the positive answer had actually been already independently obtained in [3] and [12]. The following result is stronger, giving a universal estimate in terms of the action on the reproducing kernels.

Theorem 6.3. *Suppose $\Theta(z) = z^N$. There exists a constant $C > 0$ such that any truncated Toeplitz operator A_φ^Θ has a symbol $\varphi_0 \in L^\infty$ such that $\|\varphi_0\|_\infty \leq C \rho'(A_\varphi^\Theta)$.*

Proof. Consider a smooth function η_k on \mathbb{T} , and the convolution (on \mathbb{T}) $\varphi_k = \eta_k * \varphi$, that is

$$\varphi_k(e^{is}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \eta_k(e^{it}) \varphi(e^{i(s-t)}) dt.$$

We have then $\hat{\varphi}_k(n) = \hat{\eta}_k(n) \hat{\varphi}(n)$, $n \in \mathbb{Z}$.

The map τ_t defined by $\tau_t : f(z) \mapsto f(e^{it}z)$ is a unitary on K_{Θ} and straightforward computations show that

$$(6.1) \quad \tau_t h_{\lambda}^{\Theta} = h_{e^{-it}\lambda}^{\Theta} \quad \text{and} \quad \tau_t \tilde{h}_{\lambda}^{\Theta} = e^{i(N-1)t} \tilde{h}_{e^{-it}\lambda}^{\Theta},$$

for every $\lambda \in \mathbb{D}$. By Fubini's Theorem and a change of variables we have

$$\langle A_{\varphi_k}^{\Theta} f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \eta_k(e^{it}) \langle A_{\varphi}^{\Theta} \tau_t(f), \tau_t(g) \rangle dt,$$

for every $f, g \in K_{\Theta}$. That implies that

$$\|A_{\varphi_k}^{\Theta} h_{\lambda}^{\Theta}\| = \sup_{\substack{g \in K_{\Theta} \\ \|g\|_2 \leq 1}} |\langle A_{\varphi_k}^{\Theta} h_{\lambda}^{\Theta}, g \rangle| \leq \sup_{\substack{g \in K_{\Theta} \\ \|g\|_2 \leq 1}} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\eta_k(e^{it})| |\langle A_{\varphi}^{\Theta} \tau_t(h_{\lambda}^{\Theta}), \tau_t(g) \rangle| dt,$$

and using (6.1), we obtain

$$\|A_{\varphi_k}^{\Theta} h_{\lambda}^{\Theta}\| \leq \|\eta_k\|_1 \rho(A_{\varphi}^{\Theta}) \leq \|\eta_k\|_1 \rho'(A_{\varphi}^{\Theta}).$$

A similar argument shows also that

$$\|A_{\varphi_k}^{\Theta} \tilde{h}_{\lambda}^{\Theta}\| \leq \|\eta_k\|_1 \rho'(A_{\varphi}^{\Theta})$$

and we obtain that

$$(6.2) \quad \rho'(A_{\varphi_k}^{\Theta}) \leq \|\eta_k\|_1 \rho'(A_{\varphi}^{\Theta}).$$

Now consider the Fejer kernel F_m , defined by the formula $\hat{F}_m(n) = 1 - \frac{|n|}{m}$ for $|n| \leq m$ and $\hat{F}_m(n) = 0$ otherwise. It is well known that $\|F_m\|_1 = 1$ for all $m \in \mathbb{N}$. If we take $M = \lfloor \frac{N+1}{3} \rfloor$ and define η_i ($i = 1, 2, 3$) by

$$\eta_1 = F_M, \quad \eta_2 = 2e^{i2Mt} F_{2M} - e^{i2Mt} F_M, \quad \eta_3 = \bar{\eta}_2,$$

then $\hat{\eta}_2(n) = 0$ for $n < 0$, $\hat{\eta}_3(n) = 0$ for $n > 0$, $\hat{\eta}_1(n) + \hat{\eta}_2(n) + \hat{\eta}_3(n) = 1$ for $|n| \leq N$, and $\|\eta_1\|_1 = 1$, $\|\eta_i\|_1 \leq 3$ for $i = 2, 3$. If we denote $\varphi_i = \eta_i * \varphi$, then $\varphi = \varphi_1 + \varphi_2 + \varphi_3$, $\varphi_1 \in K_{z^M} + \overline{K_{z^M}}$, φ_2 is analytic and φ_3 is coanalytic. Moreover z^{3M} divides z^{N+1} and z^N divides z^{4M} . According to (6.2), we can apply Corollary 6.2 to obtain that there exists a universal constant $C > 0$ such that A_{φ}^{Θ} has a bounded symbol φ_0 with $\|\varphi_0\|_{\infty} \leq C \rho'(A_{\varphi}^{\Theta})$.

□

In particular, it follows from Theorem 6.3 that any (classical) Toeplitz matrix $A_\varphi^{z^N}$ has a symbol φ_0 such that $\|\varphi_0\|_\infty \leq (6C + 2)\|A_\varphi^{z^N}\|$. The similar statement is proved with the better estimates $\|\varphi_0\|_\infty \leq 4\|A_\varphi^{z^N}\|$ in [3] and $\|\varphi_0\|_\infty \leq 3\|A_\varphi^{z^N}\|$ in [12].

We can obtain a slightly more general result (in the choice of the function Θ).

Corollary 6.4. *Suppose $\Theta = b_\alpha^N$, with $b_\alpha(z) = \frac{\alpha-z}{1-\bar{\alpha}z}$ a Blaschke factor. There exists a constant $C > 0$ such that any truncated Toeplitz operator A_φ^Θ has a symbol $\varphi_0 \in L^\infty$ such that $\|\varphi_0\|_\infty \leq C\rho'(A_\varphi^\Theta)$.*

Proof. The mapping U defined by

$$(U(f))(z) := \frac{(1 - |\alpha|^2)^{1/2}}{1 - \bar{\alpha}z} f(b_\alpha(z)), \quad z \in \mathbb{D}, f \in H^2,$$

is unitary on H^2 and one easily check that $UP_{z^N} = P_\Theta U$. In particular, it implies that $U(K_{z^N}) = K_\Theta$ and straightforward computations show that

$$(6.3) \quad Uh_\lambda^{z^N} = c_\lambda h_{b_\alpha(\lambda)}^\Theta \quad \text{and} \quad U\tilde{h}_\lambda^{z^N} = -\bar{c}_\lambda \tilde{h}_{b_\alpha(\lambda)}^\Theta,$$

for every $\lambda \in \mathbb{D}$, where $c_\lambda := \frac{|1-\bar{\lambda}\alpha|}{1-\lambda\bar{\alpha}}$ is a constant of modulus one.

Suppose A_φ^Θ is a bounded truncated Toeplitz operator; if $\Phi = \varphi \circ b_\alpha$, then the relation $UP_{z^N} = P_\Theta U$ yields $A_\Phi^{z^N} = U^* A_\varphi^\Theta U$. Thus, using (6.3), we obtain

$$\|A_\Phi^{z^N} h_\lambda^{z^N}\| = \|U^* A_\varphi^\Theta U h_\lambda^{z^N}\| = \|A_\varphi^\Theta h_{b_\alpha(\lambda)}^\Theta\|$$

and

$$\|A_\Phi^{z^N} \tilde{h}_\lambda^{z^N}\| = \|U^* A_\varphi^\Theta U \tilde{h}_\lambda^{z^N}\| = \|A_\varphi^\Theta \tilde{h}_{b_\alpha(\lambda)}^\Theta\|,$$

which implies that

$$(6.4) \quad \rho'(A_\Phi^{z^N}) = \rho'(A_\varphi^\Theta).$$

Now it remains to apply Theorem 6.3 to complete the proof.

□

6.3. Elementary singular inner functions. Let us now take $\Theta(z) = \exp(\frac{z+1}{z-1})$. A positive answer to Questions 1 and 3 is a consequence of results obtained by Rochberg [15] and Smith [17] on the Paley–Wiener space. We sketch the proof for completeness, without entering into details.

Theorem 6.5. *If $\Theta(z) = \exp(\frac{z+1}{z-1})$ and A_φ^Θ is a truncated Toeplitz operator, then the following are equivalent:*

- (i) A_φ^Θ has a bounded symbol.
- (ii) A_φ^Θ is bounded.
- (iii) $\rho'(A_\varphi^\Theta) < \infty$.

More precisely, there exists a constant $C > 0$ such that any truncated Toeplitz operator A_φ^Θ has a symbol φ_0 with $\|\varphi_0\|_\infty \leq C\rho'(A_\varphi^\Theta)$.

Proof. By Remark 3.3 it is enough to prove the corresponding result for the space \mathbf{K}_Θ , where $\Theta(w) = e^{iw}$, and ρ' is the analogue of ρ' for operators on \mathbf{K}_Θ . If \mathcal{F} denotes the Fourier transform on \mathbb{R} , then $\mathbf{K}_\Theta = \mathcal{F}^{-1}(L^2([0, 1]))$, and we may suppose that the symbol $\varphi \in (t+i)\mathcal{F}^{-1}(L^2([-1, 1]))$.

For a rapidly decreasing function η on \mathbb{R} , define

$$(6.5) \quad \Psi(s) = \int_{\mathbb{R}} \eta(t)\varphi(s-t) dt.$$

We have then $\hat{\Psi} = \hat{\eta}\hat{\varphi}$ and $\rho'(\mathbf{A}_\Psi^\Theta) \leq \|\eta\|_1 \cdot \rho'(\mathbf{A}_\varphi^\Theta)$.

Take now ψ_i , $i = 1, 2, 3$, such that $\text{supp } \hat{\psi}_1 \subset [-1/3, 1/3]$, $\text{supp } \hat{\psi}_2 \subset [0, 2]$, $\text{supp } \hat{\psi}_3 \subset [-2, 0]$, and $\hat{\psi}_1 + \hat{\psi}_2 + \hat{\psi}_3 = 1$ on $[-1, 1]$. If we define φ_i by replacing η with ψ_i in (6.5), then there is a constant $C_1 > 0$ such that $\rho'(\mathbf{A}_{\varphi_i}^\Theta) \leq C_1\rho'(\mathbf{A}_\varphi^\Theta)$ for $i = 1, 2, 3$.

On the other hand, $\varphi = \varphi_1 + \varphi_2 + \varphi_3$, $\varphi_1 \in \mathbf{K}_{\Theta^{1/3}} + \overline{\mathbf{K}_{\Theta^{1/3}}}$, φ_2 is analytic, φ_3 is antianalytic. We may then apply the analogue of Corollary 6.2 for the upper half-plane which completes the proof. □

One can see easily that a similar result is valid for any elementary singular function $\Theta(z) = \exp\left(\frac{z+\zeta}{z-\zeta}\right)$, for $\zeta \in \mathbb{T}$.

7. TRUNCATED TOEPLITZ OPERATORS WITH POSITIVE SYMBOLS

As noted at the end of Section 5, if $\varphi \in L^2$ is a positive function, then A_φ^Θ is bounded if and only if φdm is a Carleson measure for K_Θ . As a consequence mainly of results of

Cohn [5, 6], one can say more for positive symbols φ for a special class of model spaces. Recall that Θ is said to satisfy the *connected level set condition* (and we write $\Theta \in (CLS)$) if there is $\varepsilon \in (0, 1)$ such that the level set

$$\Omega(\Theta, \varepsilon) := \{z \in \mathbb{D} : |\Theta(z)| < \varepsilon\}$$

is connected.

Theorem 7.1. *Let Θ be an inner function such that $\Theta \in (CLS)$. If φ is a positive function in L^2 , then the following conditions are equivalent:*

- (1) A_φ^Θ is a bounded operator on K_Θ^2 .
- (2) $\sup_{\lambda \in \mathbb{D}} \|A_\varphi^\Theta h_\lambda^\Theta\| < +\infty$.
- (3) $\sup_{\lambda \in \mathbb{D}} |\langle A_\varphi^\Theta h_\lambda^\Theta, h_\lambda^\Theta \rangle_2| < +\infty$.
- (4) A_φ^Θ has a bounded symbol.

Proof. The implications (4) \implies (1) \implies (2) \implies (3) are obvious.

We have

$$(7.1) \quad \int_{\mathbb{T}} \varphi |h_\lambda^\Theta|^2 dm = \langle \varphi h_\lambda^\Theta, h_\lambda^\Theta \rangle_2 = \langle P_\Theta \varphi h_\lambda^\Theta, h_\lambda^\Theta \rangle_2 = \langle A_\varphi^\Theta h_\lambda^\Theta, h_\lambda^\Theta \rangle_2.$$

It is shown in [5] that for $\Theta \in (CLS)$ a positive μ satisfies $\sup_{\lambda \in \mathbb{D}} \|h_\lambda^\Theta\|_{L^2(\mu)} < \infty$ if and only if it is a Carleson measure for K_Θ . Thus (3) implies that φdm is a Carleson measure for K_Θ , which has been noted above to be equivalent to A_φ^Θ bounded; so (1) \iff (3).

On the other hand, it is proved in [6] that if A_φ^Θ is bounded, then there are functions $v \in L^\infty(\mathbb{T})$ and $h \in H^2$ such that $\varphi = \operatorname{Re}(v + \Theta h)$. Write then

$$\varphi = \operatorname{Re} v + \frac{1}{2}(\Theta h + \bar{\Theta} \bar{h}),$$

which implies that $\varphi - \operatorname{Re} v \in \Theta H^2 + \overline{\Theta H^2}$. Therefore $A_u^\Theta = A_{\operatorname{Re} v}^\Theta$ and $\operatorname{Re} v \in L^\infty(\mathbb{T})$. Thus the last remaining implication (1) \implies (4) is proved. \square

Remark 7.2. In [11] Nazarov and Volberg construct a counterexample concerning Carleson measures for K_Θ , which in our context can be reformulated as providing an inner Θ and a positive function $\varphi \in L^2$ such that

$$\sup_{\lambda \in \mathbb{D}} |\langle A_\varphi^\Theta h_\lambda^\Theta, h_\lambda^\Theta \rangle| < \infty,$$

but A_φ^Θ is not bounded. This condition is obviously weaker than (2) in the statement of Theorem 7.1.

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