A cohomology free description of eigencones in type A, B and C

Nicolas Ressayre

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Abstract

Let K be a connected compact Lie group. The triples $(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ of adjoint K-orbits such that $\mathcal{O}_1+\mathcal{O}_2+\mathcal{O}_3$ contains 0 are parametrized by a closed convex polyhedral cone. This cone is denoted $\Gamma(K)$ and called the eigencone of K. For K simple of type A, B or C we give an inductive cohomology free description of the minimal set of linear inequalities which characterizes $\Gamma(K)$.

1 Introduction

1.1 — Consider the following Horn problem: What can be said about the eigenvalues of a sum of two Hermitian matrices, in terms of the eigenvalues of the summands?

If A is a Hermitian n by n matrix, we will denote by $\lambda(A) = (\lambda_1 \geq \cdots \geq \lambda_n)$ its spectrum. Consider the following set:

$$\mathrm{Horn}_{\mathbb{R}}(n) = \{(\lambda(A), \lambda(B), \lambda(C)) \ : \ \begin{array}{l} A, B, C \text{ are 3 Hermitian matrices} \\ \text{s.t. } A + B + C = 0\}. \end{array}$$

It turns out that $\mathrm{Horn}_{\mathbb{R}}(n)$ is a closed convex polyhedral cone in $\mathbb{R}^{3n}$. We now want to explain the Horn conjecture which describes inductively a list of inequalities which characterizes this cone. Let $\mathcal{P}(r, n)$ denote the set of parts of $\{1, \cdots, n\}$ with r elements. Let $I = \{i_1 < \cdots < i_r\} \in \mathcal{P}(r, n)$. We set: $\lambda_I = (i_r - r, i_{r-1} - (r-1), \cdots, i_2 - 2, i_1 - 1)$. We will denote by 1^r the vector $(1, \cdots, 1)$ in $\mathbb{R}^r$.

Theorem 1 *Let (λ, μ, ν) be a triple of non increasing sequences of n real numbers. Then, $(\lambda, \mu, \nu) \in \mathrm{Horn}_{\mathbb{R}}(n)$ if and only if*

$$\sum_i \lambda_i + \sum_j \mu_j + \sum_k \nu_k = 0 \tag{1}$$

and for any $r = 1, \cdots, n-1$, *for any* $(I, J, K) \in \mathcal{P}(r, n)^3$ *such that*

$$(\lambda_I, \lambda_J, \lambda_K - (n-r)1^r) \in \mathrm{Horn}_{\mathbb{R}}(r), \tag{2}$$

we have:

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j + \sum_{k \in K} \nu_k \geq 0. \tag{3}$$

Note that if starting with a point in $\mathrm{Horn}_{\mathbb{R}}(r)$, one adds 1^r to one factor add -1^r to another one, one stays in $\mathrm{Horn}_{\mathbb{R}}(r)$. This remark implies that Condition 2 is symmetric in I, J and K.

In 1962, Horn [Hor62] conjectured Theorem 1. This conjecture was proved by combining works by Klyachko [Kly98] and Knutson-Tao [KT99] (see also [Ful00] for a survey). Despite the proof, the statement of Theorem 1 is as elementary as the Horn problem is. Note that I, J and K are sets of indexes in Inequality 3 whereas λ_I, λ_J and λ_K are eigenvalues of Hermitian matrices in Condition 2. This very curious remark certainly contributed to the success of the Horn conjecture.

As pointed out by C. Woodward, Theorem 1 has a weakness. Indeed, it gives redundant inequalities. To describe a minimal set of inequalities, we need to introduce some notation. Let $\mathbb{G}(r, n)$ be the Grassmann variety of r-dimensional subspaces of a $\mathbb{C}^n$. Consider its cohomology ring $\mathrm{H}^*(\mathbb{G}(r, n), \mathbb{Z})$. To any $I \in \mathcal{P}(r, n)$ is associated a Schubert class $\sigma_I \in \mathrm{H}^*(\mathbb{G}(r, n), \mathbb{Z})$. Let $[\mathrm{pt}] \in \mathrm{H}^{2r(n-r)}(\mathbb{G}(r, n), \mathbb{Z})$ denote the Poincaré dual class of the point. Belkale proved in [Bel01] the following:

Theorem 2 *Let* (λ, μ, ν) *be a triple of non increasing sequences of* n *real numbers. Then,* $(\lambda, \mu, \nu) \in \mathrm{Horn}_{\mathbb{R}}(n)$ *if and only if*

$$\sum_{i} \lambda_i + \sum_{j} \mu_j + \sum_{k} \nu_k = 0 \tag{4}$$

and for any $r = 1, \cdots, n-1$, *for any* $(I, J, K) \in \mathcal{P}(r, n)^3$ *such that*

$$\sigma_I . \sigma_J . \sigma_K = [\mathrm{pt}] \in \mathrm{H}^*(\mathbb{G}(r, n), \mathbb{Z}), \tag{5}$$

we have:

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j + \sum_{k \in K} \nu_k \geq 0. \tag{6}$$

The statement of Theorem 2 is not elementary, but as proved by Knutson-Tao-Woodward in [KTW04] it is optimal:

Theorem 3 *In Theorem 2, no inequality can be omitted.*

1.2 — In this work, we give an elementary (that is cohomology free) **inductive algorithm to decide if a given Littlewood-Richardson coefficient equals to one or not** (see Section 3.1). In other words, our algorithm decides if Condition 5 is fulfilled. The combination of this algorithm and Theorems 2 and 3 gives an inductive description of the minimal set of inequalities of $\mathrm{Horn}_{\mathbb{R}}(n)$. Note that our algorithm uses the Derksen-Weyman one (see [DW02]) like a black box.

1.3 — We now want to explain a generalization of the Horn problem. Let G be a reductive complex group and K be a maximal compact subgroup. Let $\mathfrak{k}$ denote its Lie algebra. We are interested in the following problem: what are the triples $(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ of adjoint orbits such that $\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3$ contains 0.

Let T be a maximal torus of G such that $T \cap K$ is a Cartan subgroup of K. Let $\mathfrak{t}$ denote its Lie algebras and $\mathfrak{t}^+$ be a fixed Weyl chamber of $\mathfrak{t}$. It turns out that the triples of orbits as above are parametrized by a closed convex polyhedral cone contained in $(\mathfrak{t}^+)^3$ (see Section 6.2 for details). We will denote by $\Gamma(K)$ this cone.

Using the Cartan-Killing form one can identify $\Gamma(U(n))$ with $\mathrm{Horn}_{\mathbb{R}}(n)$.

We now introduce notation to describe a minimal set of inequalities for $\Gamma(K)$.

Let α be a root of G and $\alpha^\vee$ denote the corresponding coroot. We will consider the standard maximal parabolic subgroup P_α associated to α. Consider the fundamental one parameter subgroup $\omega_{\alpha^\vee}$ of T. Let W denote the Weyl group of G. The Weyl group W_α of P_α is also the stabilizer of $\omega_{\alpha^\vee}$. We will denote by $\langle \cdot ; \cdot \rangle$ the natural paring between one parameter subgroups and characters of T.

Consider now the cohomology group $\mathrm{H}^*(G/P_\alpha, \mathbb{Z})$: it is freely generated by the Schubert classes σ_w parametrized by the cosets $w \in W/W_\alpha$. In [BK06], Belkale-Kumar defined a new product denoted $\odot_0$ on $\mathrm{H}^*(G/P_\alpha, \mathbb{Z})$. We can now state the main result of [BK06] which generalizes Theorem 2:

Theorem 4 *We assume that K is semisimple. Let $(\lambda, \mu, \nu) \in (\mathfrak{t}^+)^3$. Then, (λ, μ, ν) belongs to $\Gamma(K)$ if and only if for any simple root α and any triple of Schubert classes σ_u, σ_v and σ_w in $\mathrm{H}^*(G/P_\alpha, \mathbb{Z})$ such that*

$$\sigma_u \odot_0 \sigma_v \odot_0 \sigma_w = [\mathrm{pt}], \tag{7}$$

we have:

$$\langle u\omega_{\alpha^\vee}, \lambda\rangle + \langle v\omega_{\alpha^\vee}, \mu\rangle + \langle w\omega_{\alpha^\vee}, \nu\rangle \leq 0. \tag{8}$$

In [Res07], the following generalization of Theorem 3 is obtained:

Theorem 5 *In Theorem 4, no inequality can be omitted.*

1.4 — For K simple of type B or C, in Theorems 14 and 15 below, we prove that each Condition 7 is equivalent to the fact that two (ordinary !) Littlewood-Richardson coefficients are equal to one. The combination of Algorithm 3.1 and these results give a **cohomology free description of the minimal set of inequalities for** $\Gamma(K)$. Note that in [BK07], Belkale-Kumar gives a redundant cohomology free description of $\Gamma(K)$.

1.5 — The paper is organized as follows. In Section 2, we introduce basic material about the Littlewood-Richardson coefficients and the Horn cone. In Section 3, we state and prove our inductive algorithm to decide if a given Littlewood-Richardson coefficient equals to one or not. In Section 4, we introduce a parametrization of the Schubert classes of any complete rational homogeneous space and give some examples. In Section 5, we recall from [BK06] the notion of Levi-movability. In Section 6, we recall some results about the generalization of the Horn cone to any connected compact Lie group. In Sections 7 and 8, we prove our results about the cohomology of isotropic and odd orthogonal Grassmannians. In Section 9, we recall some useful results about quiver representations.

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2 The Horn cone

2.1 The Littlewood-Richardson coefficients

2.1.1 — Schubert Calculus. Let $\mathbb{G}(r, n)$ be the Grassmann variety of r-dimensional subspaces L of a fixed n-dimensional vector space V. Let $F_\bullet$: $\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = V$ be a complete flag of V.

If $a \leq b$, we will denote by $[a;b]$ the set of integers between a and b. Let $\mathcal{P}(r,n)$ denote the set of subsets of $[1;n]$ with r elements. For any $I = \{i_1 < \cdots < i_r\} \in \mathcal{P}(r,n)$, we define the Schubert variety $\Omega_I(F_\bullet)$ in $\mathbb{G}(r,n)$ by

$$\Omega_I(F_\bullet) = \{L \in \mathbb{G}(r,n) \,:\, \dim(L \cap F_{i_j}) \geq j \text{ for } 1 \leq j \leq r\}.$$

The Poincaré dual of the homology class of $\Omega_I(F_\bullet)$ does not depend on $F_\bullet$; it is denoted by σ_I. The σ_I's form a $\mathbb{Z}$-basis for the cohomology ring of $\mathbb{G}(r,n)$. The class associated to $[1;r]$ is the class of the point; it will be denoted by [pt]. It follows that for any subsets I, $J \in \mathcal{P}(r,n)$, there is a unique expression

$$\sigma_I.\sigma_J = \sum_{K \in \mathcal{P}(r,n)} c_{IJ}^K \sigma_K,$$

for integers c_{IJ}^K. We define $K^\vee$ by: $i \in K^\vee$ if and only if $n+1-i \in K$. Then, σ_K and $\sigma_{K^\vee}$ are Poincaré dual. So, if the sum of the codimensions of $\Omega_I(F_\bullet)$, $\Omega_J(F_\bullet)$ and $\Omega_K(F_\bullet)$ equals the dimension of $\mathbb{G}(r,n)$, we have

$$\sigma_I.\sigma_J.\sigma_K = c_{IJ}^{K^\vee}[\text{pt}].$$

We set

$$c_{IJK} := c_{IJ}^{K^\vee}.$$

Note that $c_{IJK} = c_{JIK} = c_{IKJ} = \ldots$

2.1.2 — Recall that the irreducible representations of $G = \mathrm{Gl}_r(\mathbb{C})$ are indexed by sequences $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r) \in \mathbb{Z}^r$. Let us denote Λ_r^+ the set of such sequences. We set $|\lambda| = \lambda_1 + \cdots + \lambda_r$. Denote the representation corresponding to λ by V_λ. For example, the representation V_{1^r} is the determinant representation of $\mathrm{Gl}_r(\mathbb{C})$. Define the Littlewood-Richardson coefficients $c_{\lambda\,\mu}^\nu \in \mathbb{N}$ by:

$$V_\lambda \otimes V_\mu = \sum_{\nu \in \Lambda_r^+} c_{\lambda\,\mu}^\nu V_\nu.$$

For $\nu = (\nu_1 \geq \cdots \geq \nu_r)$, we set: $\nu^\vee = (-\nu_r \geq \cdots \geq -\nu_1)$. Then, $V_{\nu^\vee}$ is the dual of V_ν. Finally, we set $c_{\lambda\mu\nu}^r = c_{\lambda\mu}^{\nu^\vee}$. Note that $c_{\lambda\mu\nu}^r$ is the dimension of the subspace $(V_\lambda \otimes V_\mu \otimes V_\nu)^G$ of G-invariant vectors in $V_\lambda \otimes V_\mu \otimes V_\nu$. Consider:

$$\mathrm{Horn}(r) := \{(\lambda,\mu,\nu) \in (\Lambda_r^+)^3 \,:\, c_{\lambda\mu\nu} \neq 0\}.$$

Note that $c^r\lambda\mu\nu$ depends on r, since $\nu^\vee$ does.

2.1.3 — We will use the standard correspondence between elements $I = \{i_1 < \cdots < i_r\}$ of $\mathcal{P}(r,n)$ and partitions $\lambda_I \in \Lambda_r^+$ such that $\lambda_1 \leq n-r$ and $\lambda_r \geq 0$. This correspondence is obtained by defining

$$\lambda_I = (i_r - r, i_{r-1} - (r-1), \cdots, i_2 - 2, i_1 - 1).$$

For I, J and K in $\mathcal{P}(r,n)$, Lesieur showed in 1947 (see [Les47]) that:

$$c_{IJ}^K = c_{\lambda_I \lambda_J}^{\lambda_K}.$$

Note that $\lambda_{I^\vee} = \lambda_I^\vee + (n-r)1^r$ and so, that

$$c_{IJK} = c_{\lambda_I \lambda_J \lambda_K - (n-r)1^r}^r.$$

The *type* of I is defined by:

$$\text{type}(I) = \{j = 1, \cdots, r-1 \,:\, i_{j+1} \neq i_j\}.$$

2.1.4 — We set:

$U(r,n) = \{(I,J,K) \in \mathcal{P}(r,n) \,:\, \sum_{i\in I} i + \sum_{j\in J} j + \sum_{k\in K} k = \frac{r(2n+r+3)}{2}\}$,
$T(r,n) = \{(I,J,K) \in \mathcal{P}(r,n) \,:\, c_{IJK} \neq 0\}$, and
$I(r,n) = \{(I,J,K) \in \mathcal{P}(r,n) \,:\, c_{IJK} = 1\}$.

A triple (I,J,K) belongs to $U(r,n)$ if and only if $\text{codim}\sigma_I + \text{codim}\sigma_J + \text{codim}\sigma_K = \dim \mathbb{G}(r,n)$. A triple (I,J,K) belongs to $T(r,n)$ (resp. $I(r,n)$) if and only if $\sigma_I.\sigma_J.\sigma_K$ is a non zero multiple of (resp. equal to) [pt]. In particular, we have $I(r,n) \subset T(r,n) \subset U(r,n)$.

The set of $(\lambda,\mu,\nu) \in (\Lambda_r^+)^3$ such that $c_{\lambda\mu\nu} = 1$ is denoted by $\tilde{I}(r)$. The image of $I(r,n)$ in $(\Lambda_r^+)^3$ by the map $(I,J,K) \mapsto (\lambda_I, \lambda_J, \lambda_K)$ will be denoted by $\tilde{I}(r,n)$. Obviously, $\tilde{I}(r) = \cup_{n\geq r} \tilde{I}(r,n)$.

2.2 The Horn cone

Let $I = \{i_1 < \cdots < i_k\} \in \mathcal{P}(k,r)$ and $\lambda \in \Lambda^+$. We set

$$\lambda^I = (\lambda_{i_1} \geq \cdots \geq \lambda_{i_k}).$$

In particular, $|\lambda^I| = \sum_{i\in I} \lambda_i$. Let $I,J,K \in \mathcal{P}(k,r)$. We define the "linear form" on $(\Lambda_r^+)^3$ by:

$$\varphi_{IJK}(\lambda,\mu,\nu) = |\lambda^I| + |\mu^J| + |\nu^K|.$$

Combining [Bel01] and [KT99], we obtain the following description of Horn(r):

Theorem 6 *Let* $(\lambda,\mu,\nu)\in(\Lambda_r^+)^3$. *The point* (λ,μ,ν) *belongs to* Horn(r) *if and only if*

$$|\lambda|+|\mu|+|\nu|=0,$$

and for any $k\in[1;r-1]$, *for any* $(I,J,K)\in I(k,r-k)$ *we have:*

$$\varphi_{IJK}(\lambda,\mu,\nu)\geq 0.$$

3 An algorithm

3.1 Description of the algorithm

Let $\{d_1<\cdots<d_s\}$ be a part of $[1;r-1]$. We will consider the following flag variety:

$$\mathcal{F}l_r(d_1,\cdots,d_s):=\{(V_1\subset\cdots\subset V_s)\}\subset\mathbb{G}(d_1,r)\times\cdots\times\mathbb{G}(d_s,r).$$

For $I\in\mathcal{P}(r,n)$, we will denote by I^c the complementary of I in $[1;n]$.

Let $(I,J,K)\in U(r,n)$. We now present an inductive algorithm to decide if $c_{IJK}=1$ (without computing c_{IJK} !). We assume that we know $I(k,m)$ for all $1\leq k<r$ and $m<n$.

(i) For $k=1,\cdots,r-1$ and $(I',\,J',\,K')\in I(k,r-k)$ do

(a) Compute $\phi=\varphi_{I'J'K'}(\lambda_I,\lambda_J,\lambda_K-(n-r)1^r)$.

(b) If $\phi<0$ then answer $(I,J,K)\notin I(r,n)$.

(c) If $\phi>0$ then drop Item (i)d.

(d) If $(\lambda_I^{I'},\lambda_J^{J'},\lambda_K^{K'}-(n-r)1^k)\in\tilde{I}(k,n-(r-k))$ and
$(\lambda_I^{I'^c},\lambda_J^{J'^c},\lambda_K^{K'^c}-(n-r)1^{r-k})\in\tilde{I}(r-k,n-k)$
then answer $(I,J,K)\in I(r,n)$
else answer $(I,J,K)\notin I(r,n)$.

(ii) Check if $\mathcal{F}l_n(\text{type}(I))\times\mathcal{F}l_n(\text{type}(J))\times\mathcal{F}l_n(\text{type}(K))$ is quasihomogeneous (using the algorithm shortly presented in Section 9 below).
If it is then answer $(I,J,K)\in I(r,n)$
else answer $(I,J,K)\notin I(r,n)$.

The proof of the algorithm need some preparation.

3.2 Modularity and GIT

3.2.1 — Non-standard GIT. Let G be a reductive group acting on an irreducible projective variety X. Let $\mathrm{Pic^G(X)}$ denote the group of G-linearized line bundles on X. For $\mathcal{L} \in \mathrm{Pic^G(X)}$, we denote by $\mathrm{H}^0(X, \mathcal{L})$ the G-module of regular sections of $\mathcal{L}$ and by $\mathrm{H}^0(X, \mathcal{L})^G$ the subspace of G-invariant sections. For any $\mathcal{L} \in \mathrm{Pic^G(X)}$, we set

$$X^{\mathrm{ss}}(\mathcal{L}) = \{x \in X \,:\, \exists n > 0 \text{ and } \sigma \in \mathrm{H}^0(X, \mathcal{L}^{\otimes n})^G \text{ s.t. } \sigma(x) \neq 0\}.$$

Note that this definition of $X^{\mathrm{ss}}(\mathcal{L})$ is like in [MFK94] if $\mathcal{L}$ is ample but not in general. We consider the following projective variety:

$$X^{\mathrm{ss}}(\mathcal{L})/\!/G := \mathrm{Proj} \bigoplus_{n \geq 0} \mathrm{H}^0(X, \mathcal{L}^{\otimes n})^G,$$

and the natural G-invariant morphism

$$\pi \,:\, X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow X^{\mathrm{ss}}(\mathcal{L})/\!/G.$$

If $\mathcal{L}$ is ample π is a good quotient.

3.2.2 — Let Y be an irreducible G-variety not necessarily projective. We denote by $\mathrm{mod}(Y, G)$ the minimal codimension of G-orbits in Y. Let us recall that X is projective.

Proposition 1 *We assume that X is smooth. The maximal of the dimensions of the varieties $X^{\mathrm{ss}}(\mathcal{L})/\!/G$ for $\mathcal{L} \in \mathrm{Pic^G(X)}$ is equal to* $\mathrm{mod}(Y, G)$.

Proof. Let $\mathcal{L} \in \mathrm{Pic^G(X)}$. We use notation of Paragraph 3.2.1. Since π is G-invariant, we have:

$$\dim(X^{\mathrm{ss}}(\mathcal{L})/\!/G) \leq \mathrm{mod}(X^{\mathrm{ss}}(\mathcal{L}), G) = \mathrm{mod}(X, G).$$

Conversely, set $m = \mathrm{mod}(X, G)$. It remains to construct $\mathcal{L}$ such that $\dim(X^{\mathrm{ss}}(\mathcal{L})/\!/G) \geq m$. It is well known that m is the transcendence degree of the field $\mathbb{C}(X)^G$ of G-invariant rational functions on X. Let $f_1, \cdots, f_m$ be algebraically independent elements of $\mathbb{C}(X)^G$. For each $i = 1, \cdots, m$, consider the two effective divisors D_i^0 and D_i^∞ such that $\mathrm{div}(f_i) = D_i^0 - D_i^\infty$. Consider the line bundle $\mathcal{L}_i = \mathcal{O}(D_i^0) = \mathcal{O}(D_i^\infty)$. Let σ_i^0 be a regular section of $\mathcal{L}_i$ such that $\mathrm{div}(\sigma_i^0) = D_i^0$. Since D_i^0 is G-stable, there exists a unique G-linearization of $\mathcal{L}_i$ such that σ_i^0 is G-invariant; we now consider $\mathcal{L}_i$ endowed with this linearization. There exists a unique section σ_i^∞ of $\mathcal{L}_i$ such that $f_i = \sigma_i^0/\sigma_i^\infty$; since f_i and σ_i^0 are G-invariant, so is σ_i^∞.

Set $\mathcal{L} = \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_m$. Consider the following sections of $\mathcal{L}$:

$$\tau_i = \sigma_1^\infty \otimes \cdots \otimes \sigma_{i-1}^\infty \otimes \sigma_i^0 \otimes \sigma_{i+1}^\infty \otimes \cdots \otimes \sigma_m^\infty \quad \forall i = 1, \cdots, m,$$
$$\tau_0 = \sigma_1^\infty \otimes \cdots \otimes \sigma_m^\infty.$$

Consider now the rational map

$$\begin{array}{rcl} \theta \, : \, X^{\mathrm{ss}}(\mathcal{L}) & \longrightarrow & \mathbb{P}^m \\ x & \longmapsto & [\tau_0(x) : \cdots : \tau_m(x)]. \end{array}$$

Since $f_1, \cdots, f_m$ are algebraically independent, θ is dominant. Moreover, θ factors by $\pi \, : \, X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow X^{\mathrm{ss}}(\mathcal{L})//G$. It follows that $\dim(X^{\mathrm{ss}}(\mathcal{L})//G) \geq m$. □

3.2.3 — We assume here that $\mathrm{Pic^G(X)}$ has finite rank and consider the rational vector space $\mathrm{Pic^G(X)_\mathbb{Q}} := \mathrm{Pic^G(X)} \otimes_\mathbb{Z} \mathbb{Q}$. Since $X^{\mathrm{ss}}(\mathcal{L}) = X^{\mathrm{ss}}(\mathcal{L}^{\otimes n})$ for any positive integer n, one can define $X^{\mathrm{ss}}(\mathcal{L})$ for any element in $\mathrm{Pic^G(X)_\mathbb{Q}}$. The set of ample line bundles in $\mathrm{Pic^G(X)}$ generated an open convex cone $\mathrm{Pic^G(X)^+_\mathbb{Q}}$ in $\mathrm{Pic^G(X)_\mathbb{Q}}$. The following cone was defined in [DH98] and will be called the *ample GIT-cone*:

$$\mathcal{AC}^G(X) := \{\mathcal{L} \in \mathrm{Pic^G(X)^+_\mathbb{Q}} \, : \, \mathrm{X^{ss}}(\mathcal{L}) \neq \emptyset\}.$$

Indeed, since the product of two non zero G-invariant sections of two line bundles is a non zero G-invariant section of the tensor product of the two line bundles, $\mathcal{AC}^G(X)$ is convex. The following result is certainly well-known and can be deduced from [Res08a, Proposition 1.1]:

Proposition 2 *The dimension of $X^{\mathrm{ss}}(\mathcal{L})//G$ does not depend on $\mathcal{L}$ in the relative interior of $\mathcal{AC}^G(X)$.*

3.2.4 — We now consider the case when X is a product of flag manifolds:

Lemma 1 *We assume that X is a product of flag manifolds for G such that $\mathcal{AC}^G(X)$ is non empty. For any $\mathcal{L}$ in the relative interior of $\mathcal{AC}^G(X)$, the dimension of $X^{\mathrm{ss}}(\mathcal{L})//G$ equals* $\mathrm{mod}(X, G)$.

Proof. Let $\mathcal{M} \in \mathrm{Pic^G(X)}$ such that $X^{\mathrm{ss}}(\mathcal{M})$ is not empty. By [Res07, Proposition], $\mathcal{M}$ belongs to the closure of $\mathcal{AC}^G(X)$. By [Res07, Proposition], there exists $\mathcal{L}$ in the relative interior of $\mathcal{AC}^G(X)$ such that $X^{\mathrm{ss}}(\mathcal{L}) \subset$

$X^{ss}(\mathcal{M})$. Corresponding to this inclusion we have a dominant (and so surjective) morphism $X^{ss}(\mathcal{L})/\!/G \longrightarrow X^{ss}(\mathcal{M})/\!/G$. In particular, we have:

$$\dim(X^{ss}(\mathcal{L})/\!/G) \geq \dim(X^{ss}(\mathcal{M})/\!/G).$$

With Proposition 2, this implies that for any $\mathcal{L}$ in the relative interior of $\mathcal{AC}^G(X)$, the dimension of $X^{ss}(\mathcal{L})/\!/G$ equals the maximal dimension of the varieties $X^{ss}(\mathcal{M})/\!/G$ for $\mathcal{M} \in \mathrm{Pic^G(X)}$. With Proposition 1, this implies the lemma. □

3.3 Properties of the LR-coefficients

3.3.1 — Saturation. Let $(\lambda, \mu, \nu) \in \Lambda_r^+$ and n be a positive integer. Knutson-Tao proved in [KT99] that:

$$c^r_{n\lambda\, n\mu\, n\nu} \neq 0 \Rightarrow c^r_{\lambda\mu\nu} \neq 0.$$

A geometric proof is given in [Bel06]. Note that this statement is a corollary (or a part) of Theorem 6.

3.3.2 — The Fulton conjecture. Let $(\lambda, \mu, \nu) \in \Lambda_r^+$ and n be a positive integer. Knutson-Tao proved in [KT99] the following Fulton conjecture:

$$c^r_{\lambda\mu\nu} = 1 \Rightarrow c^r_{n\lambda\, n\mu\, n\nu} = 1.$$

Geometric proofs of this result are given in [Bel07] or [Res08c].

3.3.3 — LR-coefficients on the boundary of Horn(r)**.** The following theorem has been proved independently in [KTT09] and [DW06].

Theorem 7 *Let $(\lambda, \mu, \nu) \in (\Lambda_r^+)^3$. Let $(I, J, K) \in I(k, r)$. We assume that $\varphi_{IJK}(\lambda, \mu, \nu) = 0$. Then,*

$$c^r_{\lambda\,\mu\,\nu} = c^k_{\lambda^I\,\mu^J\,\nu^K} \cdot c^{r-k}_{\lambda^{I^c}\,\mu^{J^c}\,\nu^{K^c}}.$$

3.4 Proof of the algorithm

Theorem 8 *The algorithm described in Section 3.1 decides if $c_{IJK} = 1$.*

Proof. If $\phi > 0$ then Theorem 6 implies that $c_{IJK} = 0$. In Step (i)d, ϕ is equal to 0. Then Theorem 7 express c_{IJK} like a product. Since a product of two non negative integers equals 1 if and only if each one equals 1; the algorithm works in this case.

We now consider Step (ii). If the algorithm enters in this step then for any $k = 1, \cdots, r-1$, for any $(I', J', K') \in I(k, r-k)$ we have:

$$\varphi_{I'J'K'}(\lambda_I, \lambda_J, \lambda_K - (n-r)1^r) > 0.$$

Let T and B be the usual maximal torus and Borel subgroup of GL_r. Then, λ_I corresponds to a character of T or B. The group B fixes a unique point in $\mathcal{F}l_r(\mathrm{type}(I)$ whose the stabilizer in G will be denoted by P_I. Moreover, λ_I extends to un ique character of P_I. Similarly, we can think about λ_J and $\lambda_K - (n-r)1^r$ like characters of P_J and P_K. Consider the GL_r^3-variety $X = \mathcal{F}l_r(\mathrm{type}(I)) \times \mathcal{F}l_r(\mathrm{type}(J)) \times \mathcal{F}l_r(\mathrm{type}(K))$. Let $\mathcal{L}$ be the GL_r^3-linearized line bundle on X associated to $(\lambda_I, \lambda_J, \lambda_K - (n-r)1^r)$ (see Paragraph 6.1.1 below for details). It is well known that $\mathcal{L}$ is ample and that $H^0(X, \mathcal{L}^{\otimes n}) = V^*_{n\lambda_I} \otimes V^*_{n\lambda_J} \otimes V^*_{n(\lambda_K-(n-r)1^r)}$, for any positive integer n.

Let $\bar{\mathcal{L}}$ be the GL_r-linearized line bundle on X obtained by restriction the action of GL_r^3 to the diagonal. Since each $\varphi_{I'J'K'}(\lambda_I, \lambda_J, \lambda_K - (n-r)1^r)$ is negative, Theorem 6 implies that $\bar{\mathcal{L}}$ belongs to the relative interior of $\mathcal{AC}^{\mathrm{Gl}_r}(X)$. Now, Lemma 1 implies that the dimension of $X^{\mathrm{ss}}(\bar{\mathcal{L}})/\!/\mathrm{GL}_r$ is $\mathrm{mod}(X, \mathrm{GL}_r)$.

On the other hand, saturation and Fulton conjecture imply that: $c_{IJK} = 1$ if and only if $X^{\mathrm{ss}}(\bar{\mathcal{L}})/\!/\mathrm{GL}_r(\mathbb{C})$ is a point. It follows that $(I, J, K) \in I(r, n)$ if and only if $\mathrm{mod}(X, \mathrm{GL}_r) = 0$. □

4 A parametrization of Schubert varieties

In this section, we introduce a parametrization of the Schubert varieties in G/P, and give some examples.

4.1 The general case

4.1.1 — Let G be a complex reductive group. Let $T \subset B$ be a maximal torus and a Borel subgroup of G.

Let Φ (resp. Φ^+) denote the set of roots (resp. positive roots) of G. Set $\Phi^- = -\Phi^+$. Let Δ denote the set of simple roots. Let us consider the set $X(T)^+$ of dominant characters of T. Let W denote its Weyl group.

4.1.2 — Let P be a standard (ie which contains B) parabolic subgroup of G and L denote its Levi subgroup containing T. Let W_L denote the Weyl group of L and Φ_L denote the set of roots of L. We consider the homogeneous space G/P. Its point base is denoted by P.

For $w \in W/W_L$, we consider the associated Schubert variety $\Omega(w)$ which is the closure of BwP/P.

If G/P is a Grassmannian, the Schubert varieties are classically parametrized by partitions (see Paragraphs 2.1.1 and 2.1.3). We are going to generalize this parametrization. The set of weights of T acting on the tangent space $T_P G/P$ is $-(\Phi^+ \backslash \Phi_L)$. Set

$$\Lambda(G/P) = -(\Phi^+ \backslash \Phi_L).$$

Let W^P denote the set of minimal length representatives of elements in W/W_L. Let $w \in W^P$. Consider $w^{-1}\Omega(w)$: it is a closed T-stable subvariety of G/P containing P and smooth at P. The tangent space $T_P w^{-1}\Omega(w)$ is called *the centered tangent space of* $\Omega(w)$. We set:

$$\Lambda_w = \{\alpha \in \Lambda(G/P) \,:\, \alpha \text{ is not a weight of } T \text{ in } T_P w^{-1}\Omega(w)\}.$$

Let $\mathcal{P}(\Lambda(G/P))$ denote the set of parts of $\Lambda(G/P)$. We have the following easy lemma

Lemma 2 *We have* $\Lambda_w = \{\alpha \in \Lambda(G/P) \,:\, -w\alpha \in \Phi^+\}$, *and the map* $W^P \longrightarrow \mathcal{P}(\Lambda(G/P))$, $w \mapsto \Lambda_w$ *is injective. Moreover, the codimension of* $\Omega(w)$ *is the cardinality of* Λ_w.

Proof. Since $(w^{-1}Bw).P/P$ is open $\Omega(w)$, the weights of T in $T_P w^{-1}\Omega(w)$ are $w^{-1}.\Phi^+ \cap \Lambda(G/P)$. So,

$$\Lambda_w = \{\alpha \in \Lambda(G/P) \,:\, w\alpha \in \Phi - \Phi^+ = \Phi^-\}.$$

In particular, $w^{-1}\Phi^+ \cap \Lambda(G/P) = \Lambda(G/P)\backslash\Lambda_w$. Since w has minimal length in its class in W/W_L, $w^{-1}\Phi^+ \cap \Phi_L = \Phi_L^+$. Since, $\Phi = (-\Lambda(G/P)) \cup \Phi_L \cup \Lambda(G/P)$, this implies that $w^{-1}\Phi^+$ is determined by Λ_w. This implies the injectivity.

The last assertion is obvious, since T acts on $T_P G/P$ without multiplicity. □

4.1.3 — We write $\alpha \prec \beta$ if $\beta - \alpha$ is a non negative combination of positive roots.

If λ is a one parameter subgroup of G, we denote by $P(\lambda)$ the set of $g \in G$ such that $\lim_{t\to 0} \lambda(t)g\lambda(t^{-1})$ exists in G. Then, $P(\lambda)$ is a parabolic subgroup of G and any parabolic subgroup of G can be obtained in such a way. Let us fix a one parameter subgroup λ of T such that $P = P(\lambda)$. Let $\langle \cdot\,, \cdot \rangle$ denote the natural paring between one parameter subgroups and characters of T.

Lemma 3 *Let $\alpha \in \Lambda_w$ and $\beta \in \Lambda(G/P)$. We assume that $\langle \lambda, \alpha \rangle = \langle \lambda, \beta \rangle$ and $\beta \prec \alpha$.*

Then, $\beta \in \Lambda_w$.

Proof. We have to prove that $w\beta \in \Phi^-$. But $w\beta = w\alpha + w(\beta - \alpha)$. Since $\langle \lambda, \beta - \alpha \rangle = 0$, $\beta - \alpha$ belongs to the root lattice of L. But, $\beta \prec \alpha$; so, $\beta - \alpha$ is a non negative combination of negative roots of L. Since $w \in W^P$, $w\Phi_L^- \subset \Phi^-$. Finally, $w(\beta - \alpha)$ is a non negative combination of negative roots. If follows that $w\beta \prec w\alpha$ and $w\beta \in \Phi^-$. □

Lemma 3 means that Λ_w is an order ideal on each strata given by λ.

4.2 The case SL_n

4.2.1 — Let V be a n-dimensional vector space and set $G = \mathrm{SL}(V)$. Let $\mathcal{B} = (e_1, \cdots, e_n)$ be a basis of V. Let T be the maximal torus of G consisting of diagonal matrices in $\mathcal{B}$ and B the Borel subgroup of G consisting of upper triangular matrices. Let ε_i denote the character of T which maps $\mathrm{diag}(t_1, \cdots, t_n)$ to t_i; we have $X(T) = \oplus_i \mathbb{Z}\varepsilon_i / \mathbb{Z}\sum_i \varepsilon_i$. Here, we have:

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j \,:\, i < j\}$$
$$\Delta = \{\alpha_r = \varepsilon_r - \varepsilon_{r+1} \,:\, r = 1, \cdots, n-1\}.$$

The Weyl group W of G is the symmetric group S_n acting on n letters. We will denote by $F(r)$ the span of $e_1, \cdots, e_r$.

4.2.2 — Let α_r be a simple root, P_r be the corresponding maximal standard parabolic subgroup of G and L_r be its Levi subgroup containing T. The homogeneous space G/P_r with base point P_r is the Grassmannian $\mathbb{G}(r, n)$ of r-dimensional subspaces of V with base point $F(r)$. The tangent space $T_{F(r)}\mathbb{G}(r, n)$ identifies with $\mathrm{Hom}(F(r), V/F(r))$. The natural action of L_r which is isomorphic to $\mathrm{S}(\mathrm{GL}(F(r)) \times \mathrm{GL}(V/F(r)))$ makes this identification equivariant.

Consider $\Lambda(\mathbb{G}(r, n)) = \Phi^- \backslash \Phi_{L_r}$ as in Paragraph 4.1.2:

$$\Lambda(\mathbb{G}(r, n)) = \{\varepsilon_i - \varepsilon_j \,:\, 1 \leq j \leq r < i \leq n\}.$$

We now represent $\Lambda(\mathbb{G}(r, n))$ by a rectangle with $r \times (n - r)$ boxes: the box at line i and the row j represents the root $\varepsilon_{r+i} - \varepsilon_j$.

Note that Lemma 3 asserts in this case that the Λ_w's are Young diagrams (oriented as Figure 2 shown).

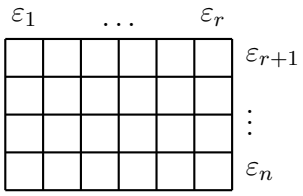


Figure 1: $\Lambda(\mathbb{G}(r,n))$

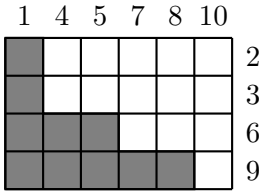


Figure 2: An example of Λ_I

4.2.3 — If $I \in \mathcal{P}(r,n)$, we set $F(I) = \text{Span}(e_i \,:\, i \in I)$. Let $I = \{i_1 < \cdots < i_r\}$ and $\Omega(I)$ the corresponding Schubert variety, that is the closure of $B.F(I)$. Set $\{i_{r+1} < \cdots < i_n\} = I^c$. Set $w_I = (i_1, \cdots, i_n) \in S_n = W$; then, $w_I \in W^{P_r}$ and represents $\Omega(I)$. Set $\Lambda_I = \Lambda_{w_I}$; we have:

$$\Lambda_I = \{\varepsilon_i - \varepsilon_j \,:\, w_I(j) < w_I(i) \text{ and } j \leq r < i\}.$$

To obtain Λ_I on Figure 1, one can proceeds as follows. Index the columns (resp. lines) of Figure 1 by I (resp. I^c). Now, a given box belongs to Λ_I if and only if the index of its column is less that those of its line. For example, if $I = \{1, 4, 5, 7, 8, 10\} \in \mathcal{P}(6,10)$, Λ_I is the set of black boxes on Figure 2.

Note that Λ_I is the **complementary of the transpose of** λ_I as defined in Paragraph 2.1.3.

4.2.4 — We now consider the case of a two step flag manifold $\mathcal{F}l_n(r_1,r_2)$. Here, $\Lambda(\mathcal{F}l_n(r_1,r_2))$ is the union of three rectangles of size $r_1 \times (n-r_2)$, $(r_2-r_1)\times(n-r_2)$ and $r_1 \times (r_2-r_1)$ (see Figure 3).

The Schubert varieties are naturally parametrized by the set $\mathcal{S}(\mathcal{F}l_n(r_1,r_2))$ of the pairs $(I_1,\, I_2) \in \mathcal{P}(r_1,n)\times\mathcal{P}(r_2,n)$ such that $I_1 \subset I_2$. Let $p = (I_1, I_2) \in \mathcal{S}(\mathcal{F}l_n(r_1,r_2))$. To obtain Λ_p on Figure 3, one can proceed as follows. Index the r_1 first columns (resp. r_2-r_1 first lines) of Figure 3 by I_1 (resp. I_2-I_1). Index the following r_2-r_1 columns (resp. $n-r_2$ lines) of Figure 3 by I_2-I_1 (resp. $[1,n]-I_2$). Now, a given box belongs to Λ_p if and only if the index

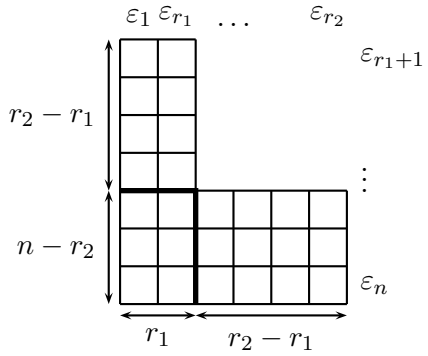


Figure 3: $\Lambda(\mathcal{F}l_n(r_1, r_2))$

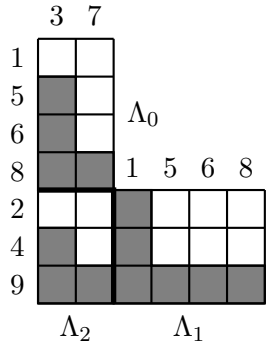


Figure 4: An example of Λ_p for $p \in \mathcal{S}(\mathcal{F}l_n(r_1, r_2))$

of its column is less that those of its line. For example, if $n = 9$, $I_1 = \{3, 7\}$ and $I_2 = I_1 \cup \{1, 5, 6, 8\}$, one obtains Λ_p on Figure 4.

4.2.5 — We now consider the following characteristic function:

$$\begin{array}{rccl} \chi_p : & [1;n] & \longrightarrow & \{0,1,2\} \\ & & & \left\{ \begin{array}{lll} 1 & \text{if} & i \in I_1, \\ 2 & \text{if} & i \in I_2 - I_1, \\ 0 & \text{if} & i \notin I_2. \end{array} \right. \\ & i & \longmapsto & \end{array}$$

We think about χ_p like a word of length n with letters in $\{0, 1, 2\}$. If one cancels the letters 2 of this word, one obtains the characteristic function of a part p_2 of $[1; n-(r_2-r_1)]$ with r_1 elements. If one cancels the letters 1 of this word and then replaces 2 by 1, one obtains the characteristic function of a part p_1 of $[1; n-r_1]$ with r_2-r_1 elements. If one cancels the letters 0 of this word and then replaces 2 by 0, one obtains the characteristic function of a part p_0 of $[1; r_2]$ with r_1 elements. We just defined a map:

$$\begin{array}{ccc} \mathcal{S}(\mathcal{F}l_n(r_1,r_2)) & \longrightarrow & \mathcal{P}(r_1, n+r_1-r_2) \times \mathcal{P}(r_2-r_1, n-r_1) \times \mathcal{P}(r_1,r_2) \\ p & \longmapsto & (p_2, p_1, p_0). \end{array}$$

Proposition 3 *With above notation, Λ_i is the partition associated to the part p_i, for $i = 1, 2$ and 0.*

Proof. The proof is direct with the description of Λ_p made in Paragraph 4.2.4. □

Remark. Lemma 3 means that Λ_p is the union of three Young diagrams like on Figure 4. It should be interesting to have a description of the triples of such Young diagrams which appear.

4.2.6 — We now consider the particular case when $n - r_2 = r_1$. So consider $\mathcal{F}l_n(r, n-r)$. In this case $\Lambda(G/P)$ is symmetric under the diagonal dashed line on Figure 5 below. Let τ denote this symmetry.

For $i \in [1; n]$, we set $\overline{i} = n + 1 - i$. The symmetry τ corresponds to the involution $\overline{\square}$. More precisely, we have:

Lemma 4 *Let $p = (I_1, I_2) \in \mathcal{S}(\mathcal{F}l_n(r, n-r))$. Set $J_1 = I_1$, $J_2 = I_2 - I_1$ and $J_3 = J_2^c$.*

Consider $(J_1', J_2', J_3') = (\overline{J_3}, \overline{J_2}, \overline{J_1})$; and $p' = (J_1', J_1' \cup J_2') \in \mathcal{S}(\mathcal{F}l_n(r, n-r))$.

Then, $\tau(\Lambda_p) = \Lambda_{p'}$.

Proof. The proof is direct with the description of Λ_p made in Paragraph 4.2.4. □

4.3 The case Sp_{2n}

4.3.1 — Root system.

Let V be a $2n$-dimensional vector space and $\mathcal{B} = (e_1, \cdots, e_{2n})$ be a basis of V. Let us consider the bilinear symplectic form with matrix

$$\omega = \begin{pmatrix} & & & & & 1 \\ & & 0 & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & 0 & \\ -1 & & & & & \end{pmatrix}.$$

Let G be the associated symplectic group. Set $T = \{\mathrm{diag}(t_1, \cdots, t_n, t_n^{-1}, \cdots, t_1^{-1}) : t_i \in \mathbb{C}^*\}$. Let B be the Borel subgroup of G consisting of upper triangular matrices of G. For $i \in [1, n]$, let ε_i denote the character of T which maps $\mathrm{diag}(t_1, \cdots, t_n, t_n^{-1}, \cdots, t_1^{-1})$ to t_i; we have $X(T) = \oplus_i \mathbb{Z}\varepsilon_i$. Here, we have:

$$\Phi^+ = \{\varepsilon_i \pm \varepsilon_j \,:\, 1 \leq i < j \leq n\} \cup \{2\varepsilon_i \,:\, 1 \leq i \leq n\},$$
$$\Delta = \{\alpha_1 = \varepsilon_1 - \varepsilon_2,\, \alpha_2 = \varepsilon_2 - \varepsilon_3, \cdots,\, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n,\, \alpha_n = 2\varepsilon_n\}.$$

If $i \in [1; 2n]$, we set $\overline{i} = 2n + 1 - i$. The Weyl group W of G is a subgroup of the Weyl group S_{2n} of $\mathrm{SL}(V)$:

$$W = \{w \in S_{2n} \,:\, w(\overline{i}) = \overline{w(i)} \;\; \forall i \in [1; 2n]\}.$$

We will denote by $F(r)$ (resp. $\overline{F}(r)$) the span of $e_1, \cdots, e_r$ (resp. $e_{\overline{1}}, \cdots, e_{\overline{r}}$). We will denote by $V(r)$ the span of $e_{r+1}, \cdots, e_{\overline{r+1}}$.

4.3.2 — Tangent space of isotropic Grassmanians. Let α_r be a simple root, P_r be the corresponding maximal parabolic subgroup of G and L_r be its Levi subgroup containing T. The homogeneous space G/P_r with base point P_r is the isotropic Grassmannian $\mathbb{G}_\omega(r, 2n)$ of r-dimensional subspaces M of V such that $\omega(M, M) = 0$ with base point $F(r)$.

Note that $V = F(r) \oplus V(r) \oplus \overline{F}(r)$. Moreover, $F(r)^{\perp_\omega} = F(r) \oplus V(r)$, and ω identifies $\overline{F}(r)$ with the dual of $F(r)$. The tangent space $T_{F(r)}\mathbb{G}_\omega(r, 2n)$ identifies with $\mathrm{Hom}(F(r), V(r)) \oplus S^2F(r)^*$. The natural action of L_r which is isomorphic to $\mathrm{GL}(F(r)) \times \mathrm{Sp}(V(r))$ makes this identification equivariant.

For convenience we set for $i = 1, \cdots, n$, $\varepsilon_{\overline{i}} := -\varepsilon_i$. Then,

$$\Phi^- = \{\varepsilon_i - \varepsilon_j \,:\, 1 \leq j \quad < i \leq \overline{j} \leq 2n\}, \text{ and}$$
$$\Lambda(\mathbb{G}_\omega(r, 2n)) = \{\varepsilon_i - \varepsilon_j \,:\, 1 \leq j \leq r < i \leq \overline{j} \leq 2n\}.$$

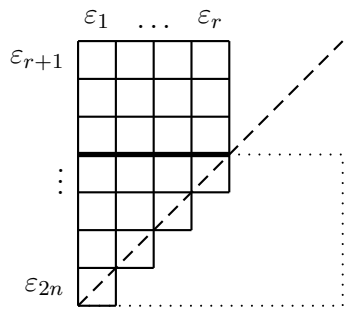


Figure 5: Roots of $T_{F(r)}\mathbb{G}_\omega(r,2n)$

We now represent each element of $\Lambda(\mathbb{G}_\omega(r,2n))$ by a box on Figure 5. The box at line i and column j corresponds to $\varepsilon_{r+i}-\varepsilon_j$.

The boxes corresponding to roots of $S^2F(r)^*$ (resp. $\mathrm{Hom}(F(r),V(r))$) are in the triangular (resp. rectangular) part of Figure 5.

4.3.3 — Schubert varieties of isotropic Grassmanians. If $I \in \mathcal{P}(r,2n)$ then we set $\overline{I} = \{\overline{i} \,:\, i \in I\}$ and

$$\mathcal{S}(\mathbb{G}_\omega(r,2n)) := \{I \in \mathcal{P}(r,2n) \,:\, I \cap \overline{I} = \emptyset\}.$$

The subspace $F(I)$ belongs to $\mathbb{G}_\omega(r,2n)$ if and only if $I \in \mathcal{S}(\mathbb{G}_\omega(r,2n))$; so, the Schubert varieties $\Psi(I)$ of $\mathbb{G}_\omega(r,2n)$ are indexed by $I \in \mathcal{S}(\mathbb{G}_\omega(r,2n))$. If $I = \{i_1 < \cdots < i_r\} \in \mathcal{S}(r,2n)$, we set $i_{\overline{k}} = \overline{i_k}$ and write $(I \cup \overline{I})^c = \{i_{r+1} < \cdots < i_{\overline{r+1}}\}$. Then, the element of W^{P_r} which corresponds to $\Psi(I)$ is $w_I = (i_1, \cdots, i_{2n})$.

4.3.4 — We now want to describe $\Lambda_I = \Lambda_{w_I}$. Consider $p = (I \subset \overline{I}^c) \in \mathcal{S}(\mathcal{F}l_{2n}(r,2n-r))$. We draw Λ_p on Figure 5 including the dotted part.

Proposition 4 *(i) The part Λ_p is symmetric relatively to the dashed line.*

(ii) The part Λ_I is the intersection of $\Lambda(\mathbb{G}_\omega(r,2n))$ and Λ_p.

Proof. The first assertion is a direct consequence of Lemma 4. Consider W as a subgroup of S_{2n} like in Paragraph 4.3.1. Then, w_I is the element of S_{2n} corresponding to the Schubert class p in $\mathcal{S}(\mathcal{F}l_{2n}(r,2n-r))$ like in Paragraph 4.1.2. The second assertion follows. □

4.4 The case SO_{2n+1}

4.4.1 — Root system.

Let V be a $2n+1$-dimensional vector space and $\mathcal{B} = (e_1, \cdots, e_{2n+1})$ be a basis of V. We denote by $(x_1, \cdots, x_{2n+1})$ the dual basis. If $i \in [1; 2n+1]$, we set $\overline{i} = 2n+2-i$. Let G be the special orthogonal group associated to the quadratic form

$$Q = x_{n+1}^2 + \sum_{i=1}^{n} x_i x_{\overline{i}}.$$

Set $T = \{\mathrm{diag}(t_1, \cdots, t_n, 1, t_n^{-1}, \cdots, t_1^{-1}) \,:\, t_i \in \mathbb{C}^*\}$. Let B the Borel subgroup of G consisting of upper triangular matrices of G. Let ε_i denote the character of T which maps $\mathrm{diag}(t_1, \cdots, t_n, 1, t_n^{-1}, \cdots, t_1^{-1})$ to t_i; we have $X(T) = \oplus_{i=1}^n \mathbb{Z}\varepsilon_i$. Here, we have:

$$\Phi^+ = \{\varepsilon_i \pm \varepsilon_j \,:\, 1 \leq i < j \leq n\} \cup \{\varepsilon_i \,:\, 1 \leq i \leq n\},$$
$$\Delta = \{\alpha_1 = \varepsilon_1 - \varepsilon_2,\, \alpha_2 = \varepsilon_2 - \varepsilon_3, \cdots,\, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n,\, \alpha_n = \varepsilon_n\}.$$

The Weyl group W of G is a subgroup of the Weyl group S_{2n+1} of $\mathrm{SL}(V)$:

$$W = \{w \in S_{2n+1} \,:\, w(\overline{i}) = \overline{w(i)} \;\; \forall i \in [1; 2n+1]\}.$$

We will denote by $F(r)$ (resp. $\overline{F}(r)$) the span of $e_1, \cdots, e_r$ (resp. $e_{\overline{1}}, \cdots, e_{\overline{r}}$). We will denote by $V(r)$ the span of $e_{r+1}, \cdots, e_{\overline{r+1}}$.

4.4.2 — Tangent space of orthogonal Grassmanians. Let α_r be a simple root, P_r be the corresponding maximal parabolic subgroup of G and L_r be its Levi subgroup containing T. For $r \leq n$, we denote by $\mathbb{G}_Q(r, 2n+1)$ the orthogonal Grassmannian of r-dimensional subspaces M of V such that $Q_{|M} = 0$. The homogeneous space G/P_r with base point P_r is $\mathbb{G}_Q(r, 2n+1)$ with base point $F(r)$.

Note that $V = F(r) \oplus V(r) \oplus \overline{F}(r)$. Moreover, $F(r)^{\perp_Q} = F(r) \oplus V(r)$, and Q identifies $\overline{F}(r)$ with the dual of $F(r)$. The tangent space $T_{F(r)}\mathbb{G}_Q(r, 2n+1)$ identifies with $\mathrm{Hom}(F(r), V(r)) \oplus \bigwedge^2 F(r)^*$. The natural action of L_r which is isomorphic to $S(\mathrm{GL}(F(r)) \times \mathrm{O}(V(r)))$ makes this identification equivariant.

We set for $i \in [1, n]$, $\varepsilon_{\overline{i}} := -\varepsilon_i$, and $\varepsilon_{n+1} = 0$. Then, we have:

$$\Phi^- = \{\varepsilon_i - \varepsilon_j \,:\, j < i < \overline{j}\}, \text{ and}$$
$$\Lambda(\mathbb{G}_Q(r, 2n+1)) = \{\varepsilon_i - \varepsilon_j \,:\, j \leq r < i < \overline{j}\}.$$

We now represent each element of $\Lambda(\mathbb{G}_Q(r, 2n+1))$ by a box on Figure 6.

The boxes corresponding to roots of $\bigwedge^2 F(r)^*$ (resp. $\mathrm{Hom}(F(r), V(r))$) are in the triangular (resp. rectangular) part of Figure 6.

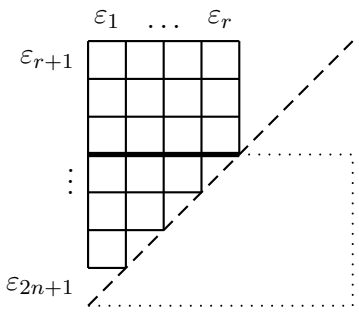


Figure 6: Roots of $T_{F(r)}\mathbb{G}_Q(r, 2n+1)$

4.4.3 — **Schubert varieties of orthogonal Grassmanians.** If $I \in \mathcal{P}(r, 2n+1)$ then we set $\overline{I} = \{\overline{i} \,:\, i \in I\}$ and

$$\mathcal{S}(\mathbb{G}_Q(r, 2n+1)) := \{I \in \mathcal{P}(r, 2n+1) \,:\, I \cap \overline{I} = \emptyset\}.$$

The subspace $F(I)$ belongs to $\mathbb{G}_Q(r, 2n+1)$ if and only if $I \in \mathcal{S}(\mathbb{G}_Q(r, 2n+1))$; so, the Schubert varieties $\Psi(I)$ of $\mathbb{G}_Q(r, 2n+1)$ are indexed by $I \in \mathcal{S}(\mathbb{G}_Q(r, 2n+1))$. If $I = \{i_1 < \cdots < i_r\} \in \mathcal{S}(r, 2n+1)$, we set $i_{\overline{k}} = \overline{i_k}$ and write $(I \cup \overline{I})^c = \{i_{r+1} < \cdots < i_{\overline{r+1}}\}$. Then, the element of W^{P_r} which corresponds to $\Psi(I)$ is $w_I = (i_1, \cdots, i_{2n+1})$.

4.4.4 — We now want to describe $\Lambda_I = \Lambda_{w_I}$. Consider $p = (I \subset \overline{I}^c) \in \mathcal{S}(\mathcal{F}l_{2n+1}(r, 2n+1-r))$. We draw Λ_p on Figure 6 including the dotted part. Then, we obtain easily:

Proposition 5 *(i) The part Λ_p is symmetric relatively to the dashed line.*

(ii) The part Λ_I is the intersection of $\Lambda(\mathbb{G}_Q(r, 2n+1))$ and Λ_p.

5 Levi-movability

In this section, we introduce the Belkale-Kumar notion of Levi-movability (see [BK06]).

5.1 Cohomology of G/P

5.1.1 — Let σ_w denote the Poincaré dual of the homology class of $\Omega(w)$. We have:

$$\mathrm{H}^*(G/P, \mathbb{Z}) = \oplus_{w \in W^P} \mathbb{Z}\sigma_w.$$

The dual of the class σ_w is denoted by $\sigma_w^\vee$. Note that σ_e is the class of the point. Let σ_1, σ_2, σ_3 be three Schubert classes. If there exists an integer d such that $\sigma_1.\sigma_2.\sigma_3 = d\sigma_e$, we set $c_{123} = d$ and we set $c_{123} = 0$ otherwise. These coefficients are the (symmetrized) structure coefficients of the product of $\mathrm{H}^*(G/P, \mathbb{Z})$ in the Schubert basis in the following sense:

$$\sigma_{w_1}.\sigma_{w_2} = \sum_{w \in W^P} c_{w_1 w_2 w} \sigma_w^\vee;$$

and $c_{123} = c_{213} = c_{132}$.

5.1.2 — Let w_1, w_2 and w_3 in W^P. Let us consider the three tangent spaces T_1, T_2 and T_3 of the $w_i^{-1}Bw_iP/P$'s at the point P. Using the transversality theorem of Kleiman, Belkale-Kumar showed in [BK06, Proposition 2] the following important lemma:

Lemma 5 *The coefficient $c_{w_1 w_2 w_3}$ is non zero if and only if there exist $p_1, p_2, p_3 \in P$ such that the natural map*

$$T_P(G/P) \longrightarrow \frac{T_P(G/P)}{p_1 T_1} \oplus \frac{T_P(G/P)}{p_2 T_2} \oplus \frac{T_P(G/P)}{p_3 T_3},$$

is an isomorphism.

Then, Belkale-Kumar defined Levi-movability:

Definition. The triple $(\sigma_{w_1}, \sigma_{w_2}, \sigma_{w_3})$ is said to be *Levi-movable* if there exist $l_1, l_2, l_3 \in L$ such that the natural map

$$T_P(G/P) \longrightarrow \frac{T_P(G/P)}{l_1 T_1} \oplus \frac{T_P(G/P)}{l_2 T_2} \oplus \frac{T_P(G/P)}{l_3 T_3},$$

is an isomorphism.

We set:

$$c_{w_1 w_2 w_3}^{\odot_0} = \begin{cases} c_{w_1 w_2 w_3} & \text{if } (\sigma_{w_1}, \sigma_{w_2}, \sigma_{w_3}) \text{ is Levi} - \text{movable;} \\ 0 & \text{otherwise.} \end{cases}$$

Note that in [RR09], an equivalent characterization of Levi-movability is given. We define on the group $\mathrm{H}^*(G/P, \mathbb{Z})$ a bilinear product $\odot_0$ by the formula:

$$\sigma_{w_1} \odot_0 \sigma_{w_2} = \sum_{w \in W^P} c_{w_1 w_2 w}^{\odot_0} \sigma_w^\vee.$$

By [BK06, Definition 18], we have:

Theorem 9 *The product $\odot_0$ is commutative, associative and satisfies Poincaré duality.*

6 Cones associated to groups

6.1 The tensor product cone

In this section, we will define the generalization of the Horn cone for any semisimple group G. We will also recall some results about these cones.

6.1.1 — The Borel-Weil theorem. Let P be a parabolic subgroup of G. Let ν be a character of B. Let $\mathbb{C}_\nu$ denote the field $\mathbb{C}$ endowed with the action of B defined by $b.\tau = \nu(b^{-1})\tau$ for all $\tau \in \mathbb{C}_\nu$ and $b \in B$. The fiber product $G \times_B \mathbb{C}_\nu$ is a G-linearized line bundle on G/B, denoted by $\mathcal{L}_\nu$. In fact, the map $X(B) = X(T) \longrightarrow \mathrm{Pic^G(G/B)}$, $\nu \longmapsto \mathcal{L}_\nu$ is an isomorphism. Moreover, $\mathcal{L}_\nu$ is generated by its sections if and only if it has non zero sections if and only if ν is dominant; and, $\mathrm{H}^0(G/B, \mathcal{L}_\nu)$ is isomorphic to the dual V_ν^* of the irreducible G-module V_ν of highest weight ν.

6.1.2 — We set: $X(T)_\mathbb{Q} = X(T) \otimes \mathbb{Q}$. The set of triples $(\nu_1, \nu_2, \nu_3) \in (X(T)^+)^3$ such that $V_{\nu_1} \otimes V_{\nu_2} \otimes V_{\nu_3}$ contains non zero G-invariant vectors is a finitely generated semigroup. We will denote by $\mathcal{L}R(G)$ the convex hull in $X(T)^3_\mathbb{Q}$ of this semigroup: it is a closed convex rational polyhedral cone.

Set $X = (G/B)^3$. Identifying $X(T^3)$ with $X(T)^3$, for any $(\nu_1, \nu_2, \nu_3) \in X(T)^3$, we obtain a G^3-linearized line bundle $\mathcal{L}_{\nu_1,\nu_2,\nu_3}$ on X. Applying the Borel-Weil theorem, we obtain

$$\mathcal{L}R(G) = \{(\nu_1\ , \nu_2, \nu_3) \in X(T)^3 \otimes \mathbb{Q} \ : \ \exists n > 0 \ \ \mathrm{H}^0(X, \mathcal{L}^{\otimes n}_{\nu_1,\nu_2,\nu_3})^G \neq \{0\}\}.$$

Since G is assumed to be semisimple, we have isomorphisms $X(T^3)_\mathbb{Q} \simeq \mathrm{Pic^{G^3}(X)_\mathbb{Q}} \simeq \mathrm{Pic^G(X)_\mathbb{Q}}$. With these identifications, $\mathcal{L}R(G)$ is the closure of $\mathcal{AC}^G((G/B)^3)$ (see for example [Res07, Proposition]).

6.1.3 — Let α be a simple root of G, P_α denote the associated maximal standard parabolic subgroup and L_α denote its Levi subgroup containing T. Set $W_\alpha = W_{L_\alpha}$. Consider the one parameter subgroup $\omega_{\alpha^\vee}$ (with usual notation) of the center of L_α. We now state the main result of [BK06]:

Theorem 10 *Here G is assumed to be semisimple. Let $(\nu_1, \nu_2, \nu_3) \in X(T)^3_{\mathbb{Q}}$ dominant. Then, $(\nu_1, \nu_2, \nu_3) \in \mathcal{L}R(G)$ if and only if*

$$\langle w_1\omega_{\alpha^\vee}, \nu_1\rangle + \langle w_2\omega_{\alpha^\vee}, \nu_2\rangle + \langle w_3\omega_{\alpha^\vee}, \nu_3\rangle \leq 0, \tag{9}$$

for all simple root α and all triple $(w_1, w_2, w_3) \in W/W_\alpha$ with $c^{\odot_0}_{w_1w_2w_3} = 1$.

Let α and $(w_1, w_2, w_3) \in W/W_\alpha$ be as in the theorem. The set of $(\nu_1, \nu_2, \nu_3) \in \mathcal{L}R(G)$ for which Inequality (9) becomes an equality is a face of $\mathcal{L}R(G)$ denoted by $\mathcal{F}(\alpha, w_1, w_2, w_3)$. The following statement is proved in [Res07]:

Theorem 11 *Let α and $(w_1, w_2, w_3) \in W/W_\alpha$ be as in Theorem 10. Then, $\mathcal{F}(\alpha, w_1, w_2, w_3)$ is a codimension one face of $\mathcal{L}R(G)$ intersecting the strictly dominant chamber.*

6.1.4 — We now want to understand better the faces $\mathcal{F}(\alpha, w_1, w_2, w_3)$. Consider the fixed point set $X^{\omega_{\alpha^\vee}}$ of $\omega_{\alpha^\vee}$ acting on X. Then,

$$C(w_1, w_2, w_3) = L_\alpha w_1^{-1}B \times L_\alpha w_2^{-1}B \times L_\alpha w_3^{-1}B$$

is an irreducible component of $X^{\omega_{\alpha^\vee}}$. Note that $B_L = B \cap L_\alpha$ is a Borel subgroup of L_α. If each w_i belongs to W^P, we fix the following isomorphism between $(L_\alpha/B_L)^3$ and $C(w_1, w_2, w_3)$ by

$$(l_1B_L, l_2B_L, l_3B_L) \longmapsto l_1w_1^{-1}B_L \times l_2w_2^{-1}B_L \times l_3w_3^{-1}B_L.$$

In particular, the group $\mathrm{Pic}^{\mathrm{L}^3_\alpha}(\mathrm{C}(\mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3))$ is isomorphic to $\mathrm{Pic}^{\mathrm{L}^3_\alpha}((\mathrm{L}_\alpha/\mathrm{B}_\mathrm{L})^3)$; that is, to $X(T)^3$. With these identifications the restriction morphism $\mathrm{Pic}^{\mathrm{G}^3}(\mathrm{X}) \longrightarrow \mathrm{Pic}^{\mathrm{L}^3_\alpha}(\mathrm{C}(\mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3))$ is

$$\begin{array}{rccc} \rho_{w_1w_2w_3} : & X(T)^3 & \longrightarrow & X(T)^3 \\ & (\nu_1, \nu_2, \nu_3) & \longmapsto & (w_1^{-1}\nu_1, w_2^{-1}\nu_2, w_3^{-1}\nu_3). \end{array}$$

The following statement is [Res09, Lemma 1]:

Theorem 12 *Let α and $(w_1, w_2, w_3) \in W/W_\alpha$ be such that $\sigma_{w_1}.\sigma_{w_2}.\sigma_{w_3} \neq 0$. Then, for any $(\nu_1, \nu_2, \nu_3) \in \mathcal{L}R(G)$,*

$$\langle w_1\omega_{\alpha^\vee}, \nu_1\rangle + \langle w_2\omega_{\alpha^\vee}, \nu_2\rangle + \langle w_3\omega_{\alpha^\vee}, \nu_3\rangle \leq 0,$$

holds. Let $\mathcal{F}(\alpha, w_1, w_2, w_3)$ denote the corresponding face. If $(\nu_1, \nu_2, \nu_3) \in X(T)^3 \otimes \mathbb{Q}$ is dominant then $(\nu_1, \nu_2, \nu_3) \in \mathcal{F}(\alpha, w_1, w_2, w_3)$ if and only if $\rho_{w_1w_2w_3}(\nu_1, \nu_2, \nu_3)$ belongs to $\mathcal{L}R(L_\alpha)$.

Corollary 1 *Let α and $(w_1, w_2, w_3) \in W/W_\alpha$ be as in Theorem 12. Then, $c^{\odot 0}_{w_1 w_2 w_3} = 1$ if and only if $\mathcal{F}(\alpha, w_1, w_2, w_3)$ intersects the interior of the dominant chamber of $X(T^3)_{\mathbb{Q}}$.*

Proof. The direct implication is a consequence of Theorem 11. Conversely, the cone $\mathcal{L}R(L_\alpha)$ (see for example [Res07,]) has codimension one (the rank of the center of L_α) in $X(T)^3_{\mathbb{Q}}$. So, since $\mathcal{F}(\alpha, w_1, w_2, w_3)$ intersects the interior of the dominant chamber of $X(T^3)_{\mathbb{Q}}$, Theorem 12 implies that $\mathcal{F}(\alpha, w_1, w_2, w_3)$ has codimension one. So, the corresponding inequality has to appear in Theorem 10. This implies that $c^{\odot 0}_{w_1 w_2 w_3} = 1$. □

6.2 The eigencone

Let us fix a maximal compact subgroup K of G in such a way that $T \cap K$ is a Cartan subgroup of K. Let $\mathfrak{k}$ and $\mathfrak{t}$ denote the Lie algebras of K and T. Let $\mathfrak{t}^+$ be the Weyl chamber of $\mathfrak{t}$ corresponding to B. Let $\sqrt{-1}$ denote the usual complex number. It is well known that $\sqrt{-1}\mathfrak{t}^+$ is contained in $\mathfrak{k}$ and that the map:

$$\begin{array}{ccc} \mathfrak{t}^+ & \longrightarrow & \mathfrak{k}/K \\ \xi & \longmapsto & K.(\sqrt{-1}\xi) \end{array}$$

is an homeomorphism. Consider the set

$$\Gamma(K) := \{(\xi_1, \xi_2, \xi_3) \in (\mathfrak{h}^+)^3 \,:\, K.(\sqrt{-1}\xi_1) + K.(\sqrt{-1}\xi_2) + K.(\sqrt{-1}\xi_3) \ni 0\}.$$

Let $\mathfrak{k}^*$ (resp. $\mathfrak{t}^*$) denote the dual (resp. complex dual) of $\mathfrak{k}$ (resp. $\mathfrak{t}$). Let $\mathfrak{t}^{*+}$ denote the dominant chamber of $\mathfrak{t}^*$ corresponding to B. By taking the tangent map at the identity, one can embed $X(T)^+$ in $\mathfrak{t}^{*+}$. Note that, this embedding induces a rational structure on the complex vector space $\mathfrak{t}^*$. In particular, we can embed $\mathcal{L}R(G)$ in $(\mathfrak{t}^{*+})^3$: let $\tilde{\mathcal{L}R}(G)$ denote the so obtained part of $\mathfrak{t}^{*+}$.

Now, using the Cartan-Killing form, we identify $\mathfrak{t}^+$ and $\mathfrak{t}^{*+}$. In particular, we can embed $\Gamma(K)$ in $(\mathfrak{t}^{*+})^3$; the so obtained cone is denoted by $\tilde{\Gamma}(K)$.

Theorem 13 *The set $\Gamma(K)$ is a closed convex polyhedral cone. Moreover, $\tilde{\mathcal{L}R}(G)$ is the set of the rational points of $\tilde{\Gamma}(K)$.*

7 About the cohomology of $\mathbb{G}_\omega(r, 2n)$

This section is concerned by coefficient structures of the cohomology of ordinary and isotropic Grassmannians. To avoid any confusion, those concerning

ordinary and isotropic Grassmanians will be denoted with c and d respectively. Note that, since ordinary Grassmannian is cominuscule, $c^{\odot_0} = c$.

7.1 — The following result is due to Belkale-Kumar:

Proposition 6 *Let I, I', $I'' \in \mathcal{S}(\mathbb{G}_\omega(r,2n))$ such that $|\Lambda_I| + |\Lambda_{I'}| + |\Lambda_{I''}| = \dim \mathbb{G}_\omega(r,2n)$. Let p, p' and $p'' \in \mathcal{S}(\mathcal{F}l_{2n}(r,2n-r))$ associated respectively to I, I' and I'' as in Paragraph 4.3.4. The following are equivalent:*

(i) $d^{\odot_0}_{II'I''} \neq 0$;

(ii) $|\Lambda_{p_0}| + |\Lambda_{p'_0}| + |\Lambda_{p''_0}| = 2r(n-r)$ *and* $d_{II'I''} \neq 0$;

(iii) $d_{p_2p'_2p''_2} \neq 0$ *and* $c_{p_0\,p'_0\,p''_0} \neq 0$.

Proof. This is essentially [BK07, Theorem 30]. We include a brief discussion for completeness.

The equivalence between the two first assertions is [RR09,]. We use notation of Paragraph 4.3.1 for Sp_{2n}. Let P be the standard parabolic subgroup of Sp_{2n} such that $\mathbb{G}_\omega(r,2n) = \mathrm{Sp}_{2n}/P$ and L be the usual Levi subgroup of P. Consider the decomposition of $T_P\mathbb{G}_\omega(r,2n)$ as sum of irreducible L-modules. The centered tangent space of $\Omega_I(\mathbb{G}_\omega(r,2n))$ decomposes as the sum of those of $\Omega_{p_0}(\mathbb{G}(r,2n-r))$ and those of $\Omega_{p_2}(\mathbb{G}_\omega(r,2r))$. Since (I,I',I'') is Levi-movable, one immediately deduces that (p_2,p'_2,p''_2) and (p_0,p'_0,p''_0) are. In particular, Lemma 5 implies that $d_{p_2p'_2p''_2} \neq 0$ and $c_{p_0\,p'_0\,p''_0} \neq 0$.

The fact that the last assertion implies the second one is the difficult part of [BK07, Theorem 30]. □

7.2 — Here comes our main result about cohomology of $\mathbb{G}_\omega(r,2n)$; it allows to characterize the condition $d^{\odot_0}_{II'I''} = 1$ in terms of the Littlewood-Richardson coefficients.

Theorem 14 *Let I, I', $I'' \in \mathcal{S}(\mathbb{G}_\omega(r,2n))$ such that $|\Lambda_I| + |\Lambda_{I'}| + |\Lambda_{I''}| = \dim \mathbb{G}_\omega(r,2n)$. Let p, p' and $p'' \in \mathcal{S}(\mathcal{F}l_{2n}(r,2n-r))$ associated respectively to I, I' and I'' as in Paragraph 4.3.4. The following are equivalent:*

(i) $d^{\odot_0}_{II'I''} = 1$;

(ii) $d_{p_2p'_2p''_2} = 1$ *and* $c_{p_0\,p'_0\,p''_0} = 1$;

(iii) $c_{p_2p'_2p''_2} = 1$ *and* $c_{p_0\,p'_0\,p''_0} = 1$.

Proof. We first prove that Assertion (i) implies Assertion (ii). Proposition 6 implies that $d_{p_2p_2'p_2''} \neq 0$ and $c_{p_0\,p_0'\,p_0''} \neq 0$. Now, by Corollary 1, it is sufficient to prove that the two faces $\mathcal{F}_2$ and $\mathcal{F}_0$ of $\mathcal{L}R(\mathrm{Sp}_{2r})$ and $\mathcal{L}R(\mathrm{Gl}_{2n-r})$ corresponding to these coefficients intersect the strictly dominant chambers.

We first make more explicit the description of the face $\mathcal{F}(r, I, I', I'')$ of $\mathcal{L}R(\mathrm{Sp}_{2n})$ associated to $d^{\odot_0}_{II'I''} = 1$ as in Theorem 11. Let us use notation of Section 4.3 for the data associated to the group Sp_{2n}. The elements of L_r have the following form:

$$A = \left(\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & A_2 & 0 \\ \hline 0 & 0 & {}^tA_1^{-1} \end{array}\right), \tag{10}$$

where $A_1 \in \mathrm{GL}_r$ and $A_2 \in \mathrm{Sp}_{2(n-r)}$. Moreover, the central one parameter subgroup $\omega_{\omega_r^\vee}$ of L_r is obtain for $A_1 = t.\mathrm{I}_r$ and $A_2 = \mathrm{I}_{2n-r}$. Let $\nu = \sum \nu_i\varepsilon_i \in X(T)$. Let us recall that $\overline{i} = 2n + 1 - i$ and set $\nu_{\overline{i}} = -\nu_i$ for $i \in [1, n]$. A direct computation shows that

$$\langle w_I\omega_{\alpha_r^\vee}, \nu\rangle = \sum_{i\in I} \nu_i. \tag{11}$$

Let $A \in L_r \cap U_{2n}$ like in 10. By juxtaposition of the spectrums of $\sqrt{-1}A_1$, $\sqrt{-1}A_2$ and $\sqrt{-1}A_3$ (each one in non increasing order), we obtain a point $\xi(A)$ in $\mathbb{R}^{2n}$. We now assume that $\nu = w_I^{-1}\xi(A)$ and ν is dominant. This means that when one applies w_I to $\xi(A)$, one obtains an ordered point in $\mathbb{R}^{2n}$. This implies that the eigenvalues of $\sqrt{-1}A_1$ and $\sqrt{-1}A_2$ are respectively the ν_i with $i \in I$ and $i \in (I \cup \overline{I})^c$. In particular, we have

$$\sum_{i\in I} \nu_i = \sqrt{-1}\,\mathrm{tr}(A_1). \tag{12}$$

Let us consider the isomorphism $\rho_{II'I''}$ of $X(T)^3$. Let (λ, μ, ν) be a regular point in $\mathcal{F}(r, I, I', I'')$. By Theorem 12, $\rho_{II'I''}(\lambda, \mu, \nu)$ belongs to $\mathcal{L}R(L)$. By Theorem 13, there exist six matrices $A_1, B_1, C_1 \in u_r(\mathbb{C})$ and $A_2, B_2, C_2 \in u_{2(n-r)}(\mathbb{C}) \cap \mathrm{Lie}(\mathrm{Sp}_{2(n-r)})$ such that if

$$A = \left(\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & A_2 & 0 \\ \hline 0 & 0 & -{}^tA_1 \end{array}\right),\ B = \left(\begin{array}{c|c|c} B_1 & 0 & 0 \\ \hline 0 & B_2 & 0 \\ \hline 0 & 0 & -{}^tB_1 \end{array}\right),\ C = \left(\begin{array}{c|c|c} C_1 & 0 & 0 \\ \hline 0 & C_2 & 0 \\ \hline 0 & 0 & -{}^tC_1 \end{array}\right),$$

we have

$$A + B + C = 0, \text{ and}$$
$$(\xi(A), \xi(B), \xi(C)) = \rho_{II'I''}(\lambda, \mu, \nu).$$

Now, consider the three following matrices of $\mathrm{Sp}_{2r} \cap U_{2r}$:

$$\bar{A} = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & -{}^tA_1 \end{array}\right), \bar{B} = \left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & -{}^tB_1 \end{array}\right), \text{ and } \bar{C} = \left(\begin{array}{c|c} C_1 & 0 \\ \hline 0 & -{}^tC_1 \end{array}\right).$$

Let α, β and γ be the spectrums of $\sqrt{-1}\bar{A}$, $\sqrt{-1}\bar{B}$ and $\sqrt{-1}\bar{C}$. Since the eigenvalues of $\sqrt{-1}A_2$ are the ν_i's with $i \in (I \cup \overline{I})^c$, we have: $\sum_{i \in p_0} \alpha_i = \sum_{i \in I} \nu_i$. We deduce that

$$0 = \mathrm{tr}(A_1) + \mathrm{tr}(B_1) + \mathrm{tr}(C_1) = \sum_{p_0} \alpha_i + \sum_{p_0'} \beta_i + \sum_{p_0''} \gamma_i.$$

We deduce that (α, β, γ) is a regular point in the face $\mathcal{F}_2$.

In a similar way,

$$\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right) + \left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right) + \left(\begin{array}{c|c} C_1 & 0 \\ \hline 0 & C_2 \end{array}\right) = 0,$$

provides a regular point in $\mathcal{F}_0$.

We now prove that Assertion (ii) implies Assertion (iii). This implication is only concerned about $\mathbb{G}(r, 2r)$ and $\mathbb{G}_\omega(r, 2r)$: we may assume that $r = n$. Let us assume that $d_{II'I''} = d^{\odot_0}_{II'I''} = 1$. By [BK07, Corollary 11], the following product in $\mathrm{H}^*(\mathbb{G}(n, 2n))$ is non zero:

$$\sigma_I(\mathbb{G}(n, 2n)).\sigma_{I'}(\mathbb{G}(n, 2n)).\sigma_{I''}(\mathbb{G}(n, 2n)) \neq 0.$$

Now, by Corollary 1 it is sufficient to prove that the face $\mathcal{F}^A$ of $\mathcal{L}R(\mathrm{SL}_{2n})$ corresponding to (I, I', I'') contains regular points. Let $\mathcal{F}^C$ be the face of $\mathcal{L}R(\mathrm{Sp}_{2n})$ corresponding to $d_{II'I''} = 1$. By Theorems 11 and 13, there exist $A, B, C \in u_n(\mathbb{C})$ such that

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & -{}^tA \end{array}\right) + \left(\begin{array}{c|c} B & 0 \\ \hline 0 & -{}^tB \end{array}\right) + \left(\begin{array}{c|c} C & 0 \\ \hline 0 & -{}^tC \end{array}\right) = 0,$$

and the spectrum (α, β, γ) of these three matrices give a regular point in $\mathcal{F}^C$. Since

$$\mathrm{tr}(A) + \mathrm{tr}(B) + \mathrm{tr}(C) = \sum_I \alpha_i + \sum_{I'} \beta_i + \sum_{I''} \gamma_i,$$

we just obtained a regular point in $\mathcal{F}^A$.

Still assuming that $r = n$, we now want to prove that Assertion (iii) implies Assertion (ii). Consider the inclusion of $\mathbb{G}_\omega(n, 2n)$ in $\mathbb{G}(n, 2n)$. Let $\Omega_I(\mathbb{G}(n, 2n))$, $\Omega_{I'}(\mathbb{G}(n, 2n))$ and $\Omega_{I''}(\mathbb{G}(n, 2n))$ be the three Schubert varieties of $\mathbb{G}(n, 2n)$ corresponding to I, I' and I'' and the standard flag in the basis of Paragraph 4.3.1. Since $c_{II'I''} = 1$, [Sot10, Theorem 2] implies that for general g, g' and g'' in Sp_{2n} the intersection $g\Omega_I(\mathbb{G}(n, 2n)) \cap g'\Omega_{I'}(\mathbb{G}(n, 2n)) \cap g''\Omega_{I''}(\mathbb{G}(n, 2n))$ is transverse and reduced to one point F. Let us consider the orthogonal $F^{\perp_\omega}$ of F for ω. Since $g \in \mathrm{Sp}_{2n}$, $F^{\perp_\omega}$ belongs to $g\Omega_I(\mathbb{G}(n, 2n))$; and finally to the intersection. We deduce that $F = F^{\perp_\omega}$ belongs to $\mathbb{G}_\omega(n, 2n)$. So, the intersection $g\Omega_I(\mathbb{G}_\omega(n, 2n)) \cap g'\Omega_{I'}(\mathbb{G}_\omega(n, 2n)) \cap g''\Omega_{I''}(\mathbb{G}_\omega(n, 2n))$ is reduced to one point F for general g, g' and g'' in Sp_{2n}. We deduce that $d_{II'I''} = 1$.

It remains to prove that Assertion (iii) implies Assertion (i). By the preceding argue, Assertion (ii) holds. Since $\mathbb{G}_\omega(r, 2r)$ is cominuscule, we may assume that $r < n$. Now, Proposition 6 implies that $d_{II'I''} \neq 0$. It remains to prove that the corresponding face $\mathcal{F}(r, I, I', I'')$ of $\mathcal{L}R(\mathrm{Sp}_{2n})$ contains regular points. Let us consider the three Schubert classes $\sigma_p(\mathcal{F}l_{2n}(r, 2n - r))$, $\sigma_{p'}(\mathcal{F}l_{2n}(r, 2n - r))$ and $\sigma_{p''}(\mathcal{F}l_{2n}(r, 2n - r))$ of $H^*(\mathcal{F}l_{2n}(r, 2n - r))$ corresponding to p, p' and p''. Since $c_{p_2 p'_2 p''_2} \neq 0$ and $c_{p_0\, p'_0\, p''_0} \neq 0$, (p, p', p'') is Levi-movable. Let d be the positive integer such that

$$\sigma_p(\mathcal{F}l_{2n}(r, 2n - r)) \odot_0 \sigma_{p'}(\mathcal{F}l_{2n}(r, 2n - r)) \odot_0 \sigma_{p''}(\mathcal{F}l_{2n}(r, 2n - r)) = d[\mathrm{pt}].$$

By [Ric09] (see also [Ric08] or [Res08b]), d is the product of $c_{p_0\, p'_0\, p''_0}$ and another Littlewood-Richardson coefficient c. The fact that $c_{p_0\, p'_0\, p''_0} = 1$ allows to apply Theorem 7 to c: $c = c_{p_2 p'_2 p''_2} . c_{p_0\, p'_0\, p''_0}$. We deduce that $d = 1$.

By [Res08a], by saturating the two inequalities $\varphi_{II'I''}$ and $\varphi_{\overline{I}^c\, \overline{I'}^c\, \overline{I''}^c}$, one obtains a face $\mathcal{F}$ of $\mathcal{L}R(\mathrm{SL}_{2n})$ intersecting the strictly dominant chamber and of codimension two.

Let T^A be the diagonal maximal torus of SL_{2n}. Let θ be the $\mathbb{Z}$-linear involution of $X(T^A)$ mapping ε_i on $-\varepsilon_{2n+1-i}$, with notation of Paragraph 4.2.1. Since θ corresponds to duality for representations, $\mathcal{L}R(\mathrm{SL}_{2n})$ is stable by the automorphism (θ, θ, θ) of $X(T^A)^3 \otimes \mathbb{Q}$. Note that the character group of the maximal torus of Sp_{2n} defined in Paragraph 4.3.1 identifies by restriction with the set of θ-fixed points in $X(T^A)$. Moreover, by [BK07, Theorem 1], $\mathcal{L}R(\mathrm{Sp}_{2n})$ is precisely the set of points in $\mathcal{L}R(\mathrm{Sl}_{2n})$ fixed by (θ, θ, θ).

Since $\varphi_{II'I''} \circ \theta = \varphi_{\overline{I}^c\, \overline{I'}^c\, \overline{I''}^c}$, $\mathcal{F}$ is stable by (θ, θ, θ). By convexity $\mathcal{F}$ contains regular θ-fixed points. We deduce using [BK07, Theorem 1], that $\mathcal{F}(r, I, I', I'')$ contains regular points. □

7.1 Examples

We now give some examples performed with the Anders Buch's quantum calculator [Buc].

7.1.1 — Whereas multiplicative formulas exist for structure coefficients of the Belkale-Kumar product (see [Ric08, Ric09, Res08b]), no such formula seems to explain Theorem 14:

Set $r = 3$ and $n = 5$. If $I = I' = I'' = \{3, 7, 10\}$ then $d_{II'I''}^{\odot_0} = 2$, $d_{p_2p_2'p_2''} = 2$ and $c_{p_0\,p_0'\,p_0''} = 2$.

7.1.2 — We now consider $\mathbb{G}_\omega(n, 2n)$ and observe relations between d_{IJK} and c_{IJK} for $I, J, K \in \mathcal{S}(\mathbb{G}_\omega(n, 2n)) \subset \mathcal{P}(n, 2n) = \mathcal{S}(\mathbb{G}(n, 2n))$. Since $\mathbb{G}_\omega(n, 2n)$ and $\mathbb{G}(n, 2n)$ are cominuscule, the Belkale-Kumar product and the ordinary one coincide here. Let δ_I denote the number of diagonal elements in $\Lambda_I(\mathbb{G}_\omega(n, 2n))$. Theorem 14 shows that

$$d_{IJK} = 1 \iff c_{IJK} = 1.$$

Assume that $d_{IJK} = 1$. The fact that c_{IJK} is non zero implies that the sum of the codimensions of the three corresponding Schubert varieties of $\mathbb{G}(n, 2n)$ equals the dimension of $\mathbb{G}(n, 2n)$. One can easily check that this means that $\delta_I + \delta_J + \delta_K = n$. The following example shows that this is not true if d_{IJK} is only assumed to be non zero:

Set $n = 4$, $I = \{1, 2, 4, 6\}$ and $J = K = \{4, 6, 7, 8\}$. Then $d_{IJK} = 2$ and $\delta_I + \delta_J + \delta_K = 3 + 1 + 1 = 5$. In particular, $c_{IJK} = 0$.

7.1.3 — For I, J, K in $\mathcal{S}(\mathbb{G}_\omega(n, 2n))$ such that $c_{IJK} = 1$, we obviously have $\delta_I + \delta_J + \delta_K = n$. The following example shows that this is not true if c_{IJK} is only assumed to be non zero.

Set $n = 4$, $I = J = \{2, 4, 6, 8\}$ and $K = \{3, 4, 7, 8\}$. Then $c_{IJK} = 2$ and $\delta_I + \delta_J + \delta_K = 6$. In particular, $d_{IJK} = 0$.

7.1.4 — We now assume that $\delta_I + \delta_J + \delta_K = n$ and $|\Lambda_I(\mathbb{G}_\omega(n, 2n))| + |\Lambda_J(\mathbb{G}_\omega(n, 2n))| + |\Lambda_K(\mathbb{G}_\omega(n, 2n))| = \frac{n(n+1)}{2}$. The Belkale-Kumar-Sottile theorem (see [Sot10, Theorem 2]) implies that

$$\begin{aligned} &c_{IJK} \geq d_{IJK} \text{ and} \\ &c_{IJK} - d_{IJK} \text{ is even.} \end{aligned}$$

We already noticed that c_{IJK} and d_{IJK} can be different for dimension reasons. The following example shows that they can be different for other reasons.

Set $n = 5$, $I = J = \{2, 4, 6, 8, 10\}$ and $K = \{3, 6, 7, 9, 10\}$. Then

$$d_{IJK} = 4 \quad \text{and} \quad c_{IJK} = 6.$$

8 About the cohomology of $\mathbb{G}_Q(r, 2n+1)$

This section is concerned by coefficient structures of the cohomology of ordinary and orthogonal Grassmanians. To avoid any confusion, those concerning ordinary and isotropic Grassmanians will be denoted with c and e respectively.

8.1 — The following is [BK07, Theorem 41] :

Proposition 7 *Let I, I', $I'' \in \mathcal{S}(\mathbb{G}_Q(r, 2n+1))$ such that $|\Lambda_I| + |\Lambda_{I'}| + |\Lambda_{I''}| = \dim \mathbb{G}_Q(r, 2n+1)$. Let p, p' and $p'' \in \mathcal{S}(\mathcal{F}l_{2n+1}(r, 2n+1-r))$ associated respectively to I, I' and I'' as in Paragraph 4.4.4. The following are equivalent:*

(i) $e^{\odot_0}_{II'I''} \neq 0$;

(ii) $|\Lambda_{p_0}| + |\Lambda_{p'_0}| + |\Lambda_{p''_0}| = r(2n+1-2r)$ *and* $e_{II'I''} \neq 0$;

(iii) $e_{p_2 p'_2 p''_2} \neq 0$ *and* $c_{p_0\, p'_0\, p''_0} \neq 0$.

8.2 — Here comes our main result about cohomology of $\mathbb{G}_Q(r, 2n+1)$; it allows to characterize the condition $e^{\odot_0}_{II'I''} = 1$ in terms of the Littlewood-Richardson coefficients.

Theorem 15 *Let I, I', $I'' \in \mathcal{S}(\mathbb{G}_Q(r, 2n+1))$ such that $|\Lambda_I| + |\Lambda_{I'}| + |\Lambda_{I''}| = \dim \mathbb{G}_Q(r, 2n+1)$. Let p, p' and $p'' \in \mathcal{S}(\mathcal{F}l_{2n+1}(r, 2n+1-r))$ associated respectively to I, I' and I'' as in Paragraph 4.4.4. The following are equivalent:*

(i) $e^{\odot_0}_{II'I''} = 1$;

(ii) $e_{p_2 p'_2 p''_2} = 1$ *and* $c_{p_0\, p'_0\, p''_0} = 1$;

(iii) $c_{p_2 p'_2 p''_2} = 1$ *and* $c_{p_0\, p'_0\, p''_0} = 1$.

Proof. The proof which is similar to those of Theorem 14 is left to the reader. □

9 Quivers

9.1 Definitions

Let Q be a quiver (that is, a finite oriented graph) with vertexes Q_0 and arrows Q_1. We assume that Q has no oriented cycle. An arrow $a \in Q_1$ has initial vertex ia and terminal one ta. A representation R of Q is a family $(V(s))_{s\in Q_0}$ of finite dimensional vector spaces and a family of linear maps $u(a) \in \mathrm{Hom}(V(ia), V(ta))$ indexed by $a \in Q_1$. The dimension vector of R is the family $(\dim(V(s)))_{s\in Q_0} \in \mathbb{N}^{Q_0}$.

Let us fix $\alpha \in \mathbb{N}^{Q_0}$ and a vector space $V(s)$ of dimension $\alpha(s)$ for each $s \in Q_0$. Set

$$\mathrm{Rep}(Q, \alpha) = \bigoplus_{a\in Q_1} \mathrm{Hom}(V(ia), V(ta)).$$

The group $\mathrm{GL}(\alpha) = \prod_{s\in Q_0} \mathrm{GL}(V(s))$ acts naturally on $\mathrm{Rep}(Q, \alpha)$.

For $\alpha, \beta \in \mathbb{N}^{Q_0}$ two vector dimensions, the Ringle form is defined by:

$$\langle \alpha, \beta \rangle = \sum_{s\in Q_0} \alpha(s)\beta(s) - \sum_{a\in Q_1} \alpha(ia)\beta(ta).$$

If there exists $R \in \mathrm{Rep}(Q, \alpha)$ whose the stabilizer in $\mathrm{GL}(\alpha)$ has dimension one, α is said to be a *Schur root*. If α is a Schur root then $\langle \alpha, \alpha \rangle \leq 1$; α is said to be *real* if $\langle \alpha, \alpha \rangle = 1$.

We call $\alpha = \alpha_1 + \cdots + \alpha_s$ the *canonical decomposition of* α if a general representation of dimension α decomposes into indecomposable representations of dimensions $\alpha_1,\ \alpha_2, \cdots, \alpha_s$. A vector dimension α is said to be *quasihomogeneous* if $\mathrm{Rep}(Q, \alpha)$ contains a dense $\mathrm{GL}(\alpha)$-orbit.

9.2 A Kac theorem

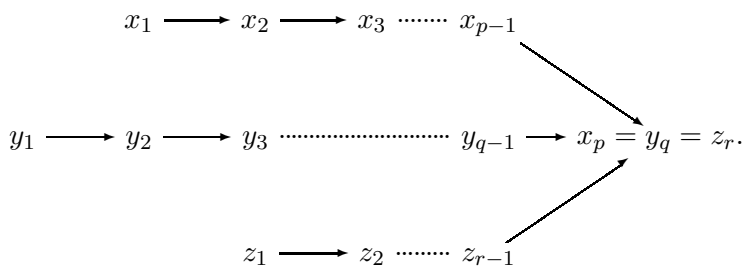
We have the following characterization of quasihomogeneous vector dimension:

Theorem 16 *(see [Kac82, Proposition 4]) Let $\alpha = \alpha_1 + \cdots + \alpha_s$ be the canonical decomposition of α. Then α is quasihomogeneous if and only if $\alpha_1, \cdots, \alpha_s$ are real Schur roots.*

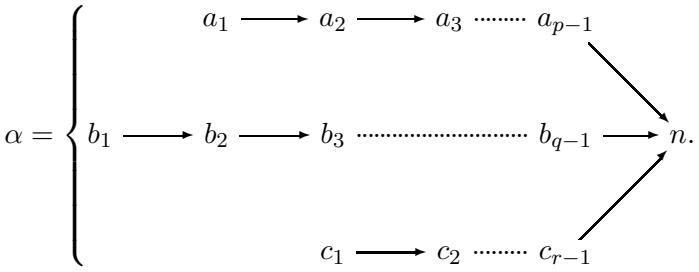
In [DW02], Derksen-Weyman describe an efficient algorithm to compute the canonical decomposition of a vector dimension. With Theorem 16, this gives **an algorithm to decide if a given vector dimension is quasihomogeneous.**

9.3 A particular quiver

Consider the following quiver T_{pqr} with $p+q+r-2$ vertexes and $p+q+r-3$ arrows:



Consider a vector dimension α of T_{pqr}:



We have the following well known

Lemma 6 *We assume the α is increasing on each harm. Then, the following are equivalent:*

(i) α is quasihomogeneous;

(ii) $\mathcal{F}l_n(a_1,\cdots,a_{p-1})\times\mathcal{F}l_n(b_1,\cdots,b_{q-1})\times\mathcal{F}l_n(c_1,\cdots,c_{r-1})$ is quasihomogeneous under GL_n.

Proof. Let R be a general representation of T_{pqr} of dimension α. If s is a vertex of T_{pqr}, $V(s)$ denotes the vector space of R at s and $u(s)$ the linear map (if there exists) associated to the arrow a in T_{pqr} such that $ia = s$. Since α is increasing on each harm, for all $a \in Q_1$, the linear map $u(a)$ is injective. In particular, the flag:

$$\xi_x = V(x_p) \supset u(x_{p-1})(V(x_{p-1})) \supset (u(x_{p-1}) \circ u(x_{p-2}))(V(x_{p-2})) \supset \cdots$$

has dimension $n > a_{p-1} > a_{p-2} \cdots$. So, we obtain a point (ξ_x, ξ_y, ξ_z) in $\mathcal{F}l_n(a_1,\cdots,a_{p-1}) \times \mathcal{F}l_n(b_1,\cdots,b_{q-1}) \times \mathcal{F}l_n(c_1,\cdots,c_{r-1})$. It is easy to see

that $\mathrm{GL}(\alpha).R$ is dense in $\mathrm{Rep}(Q,\alpha)$ if and only if $\mathrm{GL}_n.(\xi_x,\xi_y,\xi_z)$ is dense in $\mathcal{F}l_n(a_1,\cdots,a_{p-1})\times\mathcal{F}l_n(b_1,\cdots,b_{q-1})\times\mathcal{F}l_n(c_1,\cdots,c_{r-1})$. □

With the paragraph following Theorem 16, Lemma 6 implies the

Proposition 8 *The Derksen-Weyman algorithm allows to decide if the* GL_n*-variety*

$$\mathcal{F}l_n(a_1,\cdots,a_{p-1})\times\mathcal{F}l_n(b_1,\cdots,b_{q-1})\times\mathcal{F}l_n(c_1,\cdots,c_{r-1})$$

is quasihomogeneous.

Remark. It would be interesting to have a classification of the triples of parabolic subgroups (P,Q,R) of $G=\mathrm{GL}_n$ such that $G/P\times G/Q\times G/R$ is quasihomogeneous; instead an algorithm to decide if it is. In [MWZ99], Magyar-Weyman-Zelevinsky gives a classification of such triples such that $G/P\times G/Q\times G/R$ contains finitely many orbits. If one of P, Q, R is a Borel subgroup these two conditions are actually equivalent. Indeed, if $G/B\times G/Q\times G/R$ is quasihomogeneous, $G/Q\times G/R$ is a spherical G-variety and contains by [Bri86] finitely many B-orbits. The case when $P=Q=R$ is maximal was obtained in [Pop07].

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