A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups

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A DIRECT LEBEAU-ROBBIANO STRATEGY
FOR THE OBSERVABILITY OF HEAT-LIKE SEMIGROUPS

Dedicated to David L. Russell on the occasion of his 70th birthday

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Abstract. This paper generalizes and simplifies abstract results of Miller and Seidman on the cost of fast control/observation. It deduces final-observability of an evolution semigroup from a spectral inequality, i.e. some stationary observability property on some spaces associated to the generator, e.g. spectral subspaces when the semigroup has an integral representation via spectral measures. Contrary to the original Lebeau-Robbiano strategy, it does not have recourse to null-controllability and it yields the optimal bound of the cost when applied to the heat equation, i.e. $c_0 \exp(c/T)$, or to the heat diffusion in potential wells observed from cones, i.e. $c_0 \exp(c/T^\beta)$ with optimal β. It also yields simple upper bounds for the cost rate c in terms of the spectral rate.

This paper also gives geometric lower bounds on the spectral and cost rates for heat, diffusion and Ginzburg-Landau semigroups, including on non-compact Riemannian manifolds, based on L^2 Gaussian estimates.

1. **Introduction.** This paper concerns the so-called "Lebeau-Robbiano strategy" for the null-controllability of linear evolutions systems like the heat equation. The Lebeau-Robbiano strategy was originally devised for the heat flux on a bounded domain of $\mathbb{R}^d$ observed from some open subset of this domain. It originally starts from the interior observability estimate for sums of eigenfunctions of the Dirichlet Laplacian proved by some Carleman estimates at the end of the nineties in joint papers of Lebeau with Jerison, Robbiano and Zuazua, cf. § 2.4.

In the last decade, many people have contributed applications, e.g. to nodal sets of sums of Laplacian eigenfunctions in [24], to coupled wave and heat equations in the same domain in [29], to the heat equation in unbounded domains in [34], to anomalous diffusions in [37], cf. § 4.1, to structural damping, e.g. the plate equation with square root damping, in [36, 3], cf. § 4.2, to thermoelastic plates without rotatory inertia in [6, 39, 12, 46], to the heat transmission problem in [31], to diffusions in a potential well of $\mathbb{R}^d$ in [40], cf. § 4.3, to the heat equation discretized in time or space in [51, 7], to semigroups generated by non-selfadjoint elliptic operators in [27]. We also refer to the survey [30].

The Lebeau-Robbiano strategy was already stated in abstract settings with bounds on the cost of fast control of the form $c_0 \exp(c/T^\beta)$ in [37, 46]. Our goal is to retain the most general features of both papers while simplifying the proof to improve the estimate of the cost.

The paper [37] concerns semigroups generated by negative self-adjoint operators, introduces some notion of observability on spectral subspaces, cf. § 3.6. It links the exponent β in the fast control cost estimate to some exponent in this notion, but falls just short of the optimal exponent. It combines final-observability and null-controllability as in the original setting, but does not use Weyl's eigenvalues asymptotics, not even the discreteness of the spectrum of A. The assumptions brought out in [39] and introduced as an abstract framework in [46] allow generators which are not self-adjoint, but do not apply to the semigroups considered in [37, 40]. Thus the notion of relative observability on growth spaces adopted in § 2.2 is a little more general. The paper [46] achieves the breakthrough of reaching the exponent $\beta = 1$ which is optimal for the heat equation, but it adds approximate null-controllability as another layer to the strategy.

Here, the strategy goes directly from relative observability on growth spaces to the estimate of fast final-observability cost, and reaches the optimal exponents β for the observation from cones of heat diffusion in potential wells $V(x) = |x|^{2k}$, $k \in \mathbb{N}^*$, cf. § 4.3. Its sheer simplicity yields straightforward upper bounds of the rate c in the fast control cost estimate. Since it leaves null-controllability out, it can be seen as a shortcut to the original Lebeau-Robbiano strategy.

Section 2 introduces the abstract setting, states and proves the direct Lebeau-Robbiano strategy. The abstract result is connected to the original Lebeau-Robbiano setting in § 2.4. Section 3 gives further background, four lemmas which may be of independent interest and some open problems. Section 4 describes the application of the main result to the P.D.E. problems considered in [37, 36, 40]. Section 5 gives lower bounds on the rates in the cost of fast control and in the observability on spectral subspaces (e.g. the estimate for sums of eigenfunctions in § 2.4).

2. Setting and main result.

2.1. Observability cost.
We consider the abstract differential equation

$$\dot{\phi}(t) = A\phi(t), \quad \phi(0) = x \in \mathcal{E}, \quad t \geq 0, \tag{1}$$

where $A : \mathcal{D}(A) \subset \mathcal{E} \to \mathcal{E}$ is the generator of a strongly continuous semigroup $(e^{tA})_{t\geq 0}$ on a Hilbert space $\mathcal{E}$. The solution is $\phi(t) = e^{tA}x$. Although we may think of A as a nonpositive self-adjoint operator with an orthonormal basis of eigenfunctions for example, cf. § 3.6, our setting has applications where A has no eigenvalues (e.g. in § 4.1 when $M = \mathbb{R}^d$) or A is not a self-adjoint operator bounded from above (e.g. A does not even generate an analytic semigroup in § 4.2 for $\gamma < 1/2$, cf. [11, 23]).

We also consider an observation operator $C \in \mathcal{L}(\mathcal{D}(A), \mathcal{F})$ admissible for this semigroup, cf. § 3.1, i.e. C is a continuous operator from $\mathcal{D}(A)$ with the graph norm to another Hilbert space $\mathcal{F}$ and satisfies (norms in $\mathcal{E}$ and $\mathcal{F}$ are both denoted $\|\cdot\|$)

$$\int_0^T \|Ce^{tA}x\|^2 dt \leq \mathrm{Adm}_T \|x\|^2, \quad x \in \mathcal{D}(A), \quad T > 0. \tag{2}$$

N.b. the admissibility constant $T \mapsto \mathrm{Adm}_T > 0$ is nondecreasing. We may think of C as a bounded operator from $\mathcal{E}$ to itself for example.

We say that (A, C, T) *is observable at cost* $\kappa_T > 0$ if

$$\|e^{TA}x\|^2 \le \kappa_T \int_0^T \|Ce^{tA}x\|^2 dt, \quad x \in \mathcal{D}(A). \tag{3}$$

N.b. as $T \to +\infty$, $\sqrt{\kappa_T}$ does not grow more than the semigroup and, e.g., when A is nonpositive self-adjoint $T \mapsto \kappa_T$ is nonincreasing and decays at least like $1/T$, cf. § 3.2. This final-observability of (1) through C in time $T > 0$ is equivalent to a controllability property for which κ_T is the ratio of the size of the input annihilating the disturbance to the size of this disturbance, cf. § 3.2. We are interested in the asymptotic growth of κ_T as $T \to 0$ and think of κ_T as the cost of fast control.

The crucial lemma to bound this cost here is (cf. a continuous version in § 3.3)

Lemma 2.1. *If the approximate observability estimate (§ 3.5 justifies this name)*

$$f(t)\|e^{tA}x\|^2 - f(qt)\|x\|^2 \le \int_0^t \|Ce^{\tau A}x\|^2 d\tau, \quad x \in \mathcal{D}(A), \quad t \in (0, T'], \tag{4}$$

holds with $f(t) \to 0$ *as* $t \to 0^+$, $q \in (0,1)$ *and* $T' > 0$, *then* $\kappa_T \le 1/f((1-q)T)$ *for* $T \in (0, T']$, *i.e. the fast control cost does not grow more than the inverse of* f.

Proof. Let $T \le T'$. Let $T_0 = T$, $T_{k+1} = T_k - \tau_k$, $\tau_k = q^k(1-q)T$, $k \in \mathbb{N}$. The series $\sum \tau_k = T$ defines a disjoint partition $\cup (T_{k+1}, T_k] = (0, T]$. Applying (4) to $x = e^{T_{k+1}A}y$ and $t = \tau_k$ yields

$$f(\tau_k)\|e^{T_k A}y\|^2 - f(\tau_{k+1})\|e^{T_{k+1}A}y\|^2 \le \int_{T_{k+1}}^{T_k} \|Ce^{tA}y\|^2 dt, \quad y \in \mathcal{D}(A), \quad k \in \mathbb{N}.$$

Adding these inequalities yields, since the left hand side is a telescoping series,

$$f(\tau_0)\|e^{TA}y\|^2 - f(\tau_k)\|e^{T_k A}y\|^2 \le \int_{T_k}^T \|Ce^{tA}y\|^2 dt, \quad y \in \mathcal{D}(A), \quad k \in \mathbb{N}.$$

Taking the limit $k \to \infty$ completes the proof since $f(\tau_k) \to 0$ and the continuous function $t \mapsto \|e^{tA}y\|$ is bounded on the compact set $[0, T]$. $\qquad\square$

2.2. Relative observability on growth subspaces.
We assume that there is a nondecreasing family of semigroup invariant spaces $\mathcal{E}_\lambda \subset \mathcal{E}$, $\lambda > 0$ (i.e. $e^{tA}\mathcal{E}_\lambda \subset \mathcal{E}_\lambda \subset \mathcal{E}_{\lambda'}$, $t > 0$, $\lambda' > \lambda$) satisfying the semigroup growth property (namely some time-decay) with exponent $\nu \in (0,1)$ and rate $m \ge 0$

$$\|e^{tA}x\| \le m_0 e^{m\lambda^\nu} e^{-\lambda t}\|x\|, \quad x \perp \mathcal{E}_\lambda, \quad t \in (0, T_0), \quad \lambda > 0. \tag{5}$$

We call them *growth spaces*. We think of them as spectral subspaces of A, i.e. $\sigma(A_{\restriction \mathcal{E}_\lambda^\perp}) \subset \{z \in \sigma(A) \mid \operatorname{Re} z \le -\lambda\}$, and we think of (5) as a spectrally determined growth property, cf. § 3.6.

We also assume that there is an observation operator $C_0 \in \mathcal{L}(\mathcal{D}(A), \mathcal{F})$ satisfying the bound relative to C on growth spaces with exponent $\alpha \in (0,1)$ and rate $a > 0$

$$\|C_0 x\|^2 \le a_0 e^{2a\lambda^\alpha}\|Cx\|^2, \quad x \in \mathcal{E}_\lambda, \quad \lambda > 0. \tag{6}$$

We call C_0 a *reference operator* and the property (6) of C: *observability on growth subspaces relatively to* C_0. We think of C_0 as a simple operator with a good estimate of fast control like the identity operator, cf. § 3.7.

2.3. **Main result.** When the reference operator C_0 satisfies the observability cost estimate with exponent $\beta > 0$ and rate $b > 0$

$$\|e^{TA}x\|^2 \le b_0 e^{\frac{2b}{T^\beta}} \int_0^T \|C_0 e^{tA}x\|^2 dt, \quad x \in \mathcal{D}(A), \quad T \in (0, T_0), \tag{7}$$

we claim that C satisfies a similar estimate with exponent[1] $\max\left\{\beta, \frac{\alpha}{1-\alpha}, \frac{\nu}{1-\nu}\right\}$:

Theorem 2.2. *Under the assumptions* (5), (6) *and* (7) *with* $\beta = \frac{\alpha}{1-\alpha} = \frac{\nu}{1-\nu}$, *the system* (A, C, T) *is observable at a cost* κ_T *such that* $2c = \limsup_{T \to 0} T^\beta \ln \kappa_T < \infty$.
More precisely, this rate c *is bounded in terms of an implicitly defined* $s > 0$:

$$c \le c_* := \left(\frac{(\beta+1)b}{a+m}\right)^{\frac{\beta+1}{\beta}} \frac{\beta^\beta}{s^{\frac{(\beta+1)^2}{\beta}}}, \quad \text{with } s(s+\beta+1)^\beta = (\beta+1)\beta^{\frac{\beta^2}{\beta+1}} \frac{b^{\frac{1}{\beta+1}}}{a+m}. \tag{8}$$

Moreover, if the admissibility constant in (2) *satisfies* $\mathrm{Adm}_T \to 0$ *as* $T \to 0$, *then there exists* $T' > 0$ *such that* $\kappa_T \le 4a_0 b_0 \exp(\frac{2c_*}{T^\beta})$ *for* $T \in (0, T']$.
Since $c > 0$ *for some "meaningful" example, cf. § 4.3, there are no lower* β *such that* $\limsup_{T \to 0} T^\beta \ln \kappa_T < \infty$ *under these assumptions.*

N.b. the condition $\mathrm{Adm}_T \to 0$ as $T \to 0$ for the better bound in theorem 2.2 holds for example when C is bounded from $\mathcal{E}$ to $\mathcal{F}$, cf. § 3.1.

Corollary 1. *Under the same assumptions as theorem* 2.2, *the cost rate* c *is bounded more explicitly in the following cases, with the abbreviation* $a_m = a + m$:

i. *If* (6) *holds with* $\alpha = \frac{1}{2}$ *(i.e.* $\beta = 1$*) then* $c_* = 4b^2 \left(\sqrt{a_m + 2\sqrt{b}} - \sqrt{a_m}\right)^{-4}$.

ii. *If* (7) *holds for any* b *then* $c \le a_m^{\beta+1}(\beta+1)^{\beta(\beta+1)}\beta^{-\beta^2}$.

iii. *If* (6) *holds for any* a *then* $c \le b$.

iv. *If* $b > a_m^{\beta+1}(\beta+1)^{\beta(\beta+1)}\beta^{-\beta^2}$ *then* $c_* \le \left(\frac{b}{a_m}\right)^{\frac{1}{\alpha}} \left(\frac{b^{(1-\alpha)^2}}{a_m^{1-\alpha}} - \frac{(\beta+1)^\alpha}{\beta^{\alpha^2}}\right)^{-1}$.

N.b. (ii) applies to the identity operator as reference operator C_0, cf. § 3.7.
N.b. if (5) holds with $m = 0$ or for any $m > 0$ then a_m can be replaced by a.
Theorem 2.2 for $\alpha = \frac{1}{2}$ (i.e. $\beta = 1$) and $m = 0$ is due to Seidman with some less precise and less simple cost rate bound than (8); e.g., in the case (ii) with $\beta = 1$ and $m = 0$ which applies to the original setting in § 2.4, [46, theorem 2.4] proves[2] $c \le 8a^2$ instead of $c \le 4a^2$, and does not state (i).

With the exponential bound $b_0 e^{bT^{-\beta}}$ in (7) replaced by a polynomial bound $\frac{b_0}{T^b}$ (so that (ii) applies), the papers [37, 39] only prove $\limsup_{T \to 0} T^\beta \ln \kappa_T < \infty$ for $\beta > \frac{\alpha}{1-\alpha}$, hence fall short of the optimal exponent.

2.4. **Original example.** For $A = \Delta$ the Laplace-Beltrami operator with Dirichlet boundary condition on a compact smooth connected Riemannian manifold M, $\mathcal{E} = \mathcal{F} = L^2(M)$, $\mathcal{E}_\lambda =$ the spectral spaces of A (cf. § 3.6), C_0 the identity operator, and C the multiplication by the characteristic function of an open subset $\Omega \ne \emptyset$ of M,

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[1]N.b. ν, α or β may be increased so that the match $\beta = \frac{\alpha}{1-\alpha} = \frac{\nu}{1-\nu}$ is achieved as in the statement of theorem 2.2.

[2]Indeed [46, theorem 2.4] states $c \le 2a^2$, but correcting the factor 2 into $\frac{1}{2}$ in the definition of $\underline{d}(s)$ in [46, theorem 2.1] only proves $c \le 8a^2$.

[29, theorem 3] and [24, theorem 14.6] prove (6) with exponent $\alpha = \frac{1}{2}$ using the semiclassical local elliptic Carleman estimates of [28]. In this case (6) writes as

$$\int_M |v(x)|^2 dx \le a_0 e^{2a\sqrt{\lambda}} \int_\Omega |v(x)|^2 dx, \quad \lambda > 0,$$

for any sum of eigenfunctions $v = \sum_{\mu \le \lambda} \varphi_\mu$, $-\Delta\varphi_\mu = \mu\varphi_\mu$. Since (5) holds with $m = 0$ (cf. § 3.6) and (7) holds with any $b > 0$ (cf. § 3.7), the corollary 1(ii) of theorem 2.2 proves that this estimate on sums of eigenfunctions implies the bound on fast control for any $c > 4a^2$:

$$\int_M |\phi(T,x)|^2 dx \le \kappa_T \int_0^T \int_\Omega |\phi(t,x)|^2 dx\,dt, \quad \kappa_T = c_0 e^{\frac{2c}{T}}, \quad T \in [0, T_0],$$

for any solution of the Cauchy problem $\partial_t\phi - \Delta\phi = 0$, $\phi(0,\cdot) \in L^2(M)$.

The exponent $\alpha = \frac{1}{2}$ is always sharp in this setting as proved in [24, proposition 14.9] (cf. also [30, proposition 5.5]), i.e. the above estimate on sums of eigenfunctions implies $a > 0$. Theorem 5.3 improves this into $a \ge \sup_{y \in M} \mathrm{dist}(y, \overline{\Omega})/2$. N.b. the cost estimate in [33, theorem 2.1] (cf. also theorem 5.1) combined with $c > 4a^2$ given by theorem 2.2 only proves a lower bound on a which is worse by a factor 2. This could mean that "something is lost" in the proof of theorem 2.2.

In this general setting, the cost upper bound $\limsup_{T \to 0} T \ln \kappa_T < \infty$ is due to Seidman (it is deduced in [46] from the above estimate on sums of eigenfunctions, and the first such upper bound was proved in [44]) and the cost lower bound $\liminf_{T \to 0} T \ln \kappa_T \ge \sup_{y \in M} \mathrm{dist}(y, \overline{\Omega})^2/2$ is due to [33] (the first lower bound was proved in dimension one in [22]). In the Euclidean case, this upper bound was proved in [19] by global Carleman estimates with singular weights of the Èmanuilov type (with a less precise lower bound). Under the geometrical optics condition on Ω, a more precise upper bound is deduced in [33] by the control transmutation method from the observability of the wave group in [5]: $\limsup_{T \to 0} T \ln \kappa_T \le c_* L_\Omega^2$, where L_Ω is the length of the longest generalized geodesic in M which does not intersect Ω, and c_* is determined by a one-dimensional observability estimate for which $c_* \le (2\frac{36}{37})^2$, improved into $c_* \le \frac{3}{2}$ in [49].

2.5. **Proof of the main result.** We shall use lemma 2.1 in the following form.

Lemma 2.3. *If the approximate observability estimate*

$$f(T)\|e^{TA}x\|^2 - g(T)\|x\|^2 \le \int_0^T \|Ce^{tA}x\|^2 dt, \quad x \in \mathcal{D}(A), \quad T \in (0, T_0], \qquad (9)$$

holds with $f(T) = f_0\exp(-2/(d_2 T)^\beta)$ and $g(T) = g_0\exp(-2/(d_1 T)^\beta)$, where f_0, g_0, $d_1 < d_2$ are positive, then for all $d \in (0, d_2 - d_1)$ there exists $T' \in (0, T_0]$ such that $\kappa_T \le f_0^{-1}\exp(2/(dT)^\beta)$ for $T \in (0, T')$.

Moreover, if $g_0 \le f_0$ then we may take $d = d_2 - d_1$ and $T' = T_0$.

Proof. To apply lemma 2.1, we compute the least q such that $g(T) \le f(qT)$ for all $T \in (0, T')$. We find $q = \frac{d_1}{d_2}h(T')$ with $h(T') = (1 + \inf_{t \in (0,T')} t^\beta d_1^\beta \ln\frac{f_0}{g_0})^{-\frac{1}{\beta}}$ where the parenthesis is 1 when $g_0 \le f_0$ and positive when T' is small enough. Now $\kappa_T \le \frac{1}{f((1-q)T)} = \frac{1}{f_0}\exp(\frac{1}{(d_3 T)^\beta})$ with $d_3 = d_2 - d_1 h(T') \to d_2 - d_1$ as $T' \to 0$. $\square$

We proceed with the proof of theorem 2.2. For ease of exposition, we start with the case $m = 0$ in (5) and complete the general case at the very end of § 2.5.

Plugging (6) in (7) yields

$$\|e^{\tau A}\phi\|^2 \le a_0 b_0 e^{2a\lambda^\alpha + \frac{2b}{\tau^\beta}} \int_0^\tau \|Ce^{tA}\phi\|^2 dt, \quad \phi \in \mathcal{E}_\lambda, \quad \tau \in (0, T_0). \qquad (10)$$

Given $x \in \mathcal{D}(A)$ and $T \in (0, T_0)$, we introduce an observation time $\tau = \varepsilon T$ with $\varepsilon \in (0,1)$, a spectral threshold λ defined by $(r\lambda)^\alpha = \frac{1}{\tau^\beta}$ with $r > 0$, the orthogonal projection of x on $\mathcal{E}_\lambda$ denoted x_λ, and $x_\lambda^\perp = x - x_\lambda$.

Since $\mathcal{E}_\lambda$ is semigroup invariant, we may apply (10) to $\phi = e^{(1-\varepsilon)TA}x_\lambda$ and obtain:

$$\|e^{TA}x_\lambda\|^2 \le \frac{1}{4f(T)} \int_{(1-\varepsilon)T}^T \|Ce^{tA}x_\lambda\|^2 dt, f(T) = \frac{1}{4a_0 b_0} \exp\left(-\frac{2}{T^\beta} \frac{a + br^\alpha}{r^\alpha \varepsilon^\beta}\right). \quad (11)$$

We put the factor 4 in the definition of f because we shall use twice the inequality:

$$\|y + z\|^2 \le 2(\|y\|^2 + \|z\|^2), \quad y \in \mathcal{E}, \quad z \in \mathcal{E}. \qquad (12)$$

Using (12) then (2) yields

$$\int_{(1-\varepsilon)T}^T \|Ce^{tA}x_\lambda\|^2 dt \le 2\int_{(1-\varepsilon)T}^T \|Ce^{tA}x\|^2 dt + 2\operatorname{Adm}_{\varepsilon T}\|e^{(1-\varepsilon)TA}x_\lambda^\perp\|^2. \qquad (13)$$

Using (12) again, then (11) and finally (13) yields

$$f(T)\|e^{TA}x\|^2 \le \int_{(1-\varepsilon)T}^T \|Ce^{tA}x\|^2 dt + \operatorname{Adm}_{\varepsilon T}\|e^{(1-\varepsilon)TA}x_\lambda^\perp\|^2 + 2f(T)\|e^{TA}x_\lambda^\perp\|^2.$$

Applying (5) with $m = 0$ to $x_\lambda^\perp$ yields

$$f(T)\|e^{TA}x\|^2 - m_0^2\left(\operatorname{Adm}_{\varepsilon T} e^{-2(1-\varepsilon)T\lambda} + 2f(T)e^{-2T\lambda}\right)\|x_\lambda^\perp\|^2 \le \int_0^T \|Ce^{tA}x\|^2 dt.$$

Since $\|x_\lambda^\perp\| \le \|x\|$, $\operatorname{Adm}_{\varepsilon T} \le \operatorname{Adm}_{T_0}$ and $f(T) \le f(T_0)$, we deduce the approximate observability estimate

$$f(T)\|e^{TA}x\|^2 - m_0^2\left(\operatorname{Adm}_{T_0} + 2f(T_0)\right)e^{-2(1-\varepsilon)T\lambda}\|x\|^2 \le \int_0^T \|Ce^{tA}x\|^2 dt. \qquad (14)$$

Recalling that here $\beta = \frac{\alpha}{1-\alpha}$ so that $T\lambda = 1/(r\varepsilon^{\beta/\alpha}T^\beta)$, this proves (9) with

$$f_0 = \frac{1}{4a_0 b_0}, g_0 = m_0^2\left(\operatorname{Adm}_{T_0} + 2f(T_0)\right), d_2 = \frac{\varepsilon}{(ar^{-\alpha} + b)^{\frac{1}{\beta}}} \text{ and } d_1 = \varepsilon^{\frac{1}{\alpha}}\left(\frac{r}{1-\varepsilon}\right)^{\frac{1}{\beta}}.$$

As $T_0 \to 0$, if $\operatorname{Adm}_{T_0} \to 0$ then $g_0 \to 0$, hence (9) still holds with $g_0 = f_0$ with a smaller T_0. Therefore lemma 2.3 proves the theorem for $c_* = (d_2 - d_1)^{-\beta}$, for any $\varepsilon \in (0,1)$ and $r > 0$.

Now, introducing for convenience $\gamma = \frac{1}{\beta}$ and $s = \frac{\varepsilon}{1-\varepsilon}$, we are left with maximizing with respect to $r > 0$ and $s > 0$:

$$d_{a,b}(r,s) = d_2 - d_1 = \frac{\varepsilon}{(ar^{-\alpha} + b)^\gamma} - \left(\frac{r}{1-\varepsilon}\right)^\gamma \varepsilon^{\frac{1}{\alpha}} = \frac{s}{s+1}r^\gamma\left(h^{-\gamma}(r) - s^\gamma\right),$$

where $h(r) = ar^{\frac{\gamma}{\gamma+1}} + br$, since $\frac{1}{\alpha} = 1 + \gamma$, $\varepsilon = \frac{s}{s+1}$ and $1 - \alpha = \frac{1}{\beta+1} = \frac{\gamma}{\gamma+1}$. N.b. $d_{a,b}(r,s) > 0$ for r small enough already proves $c < \infty$.

The optimality condition $\nabla d_{a,b} = 0$ writes successively, abbreviating $h = h(r)$,

$$\begin{cases} \gamma r^{\gamma-1}\left(h^{-\gamma} - s^\gamma\right) = \frac{r^\gamma \gamma h'}{h^{\gamma+1}}, \\ \frac{1}{(s+1)^2}\left(h^{-\gamma} - s^\gamma\right) = \frac{\gamma s^\gamma}{s+1}, \end{cases} \begin{cases} h^{\gamma+1}\left(h^{-\gamma} - s^\gamma\right) = rh' = \frac{\gamma}{\gamma+1}h + \left(1 - \frac{\gamma}{\gamma+1}\right)br, \\ h = \left(\gamma s^\gamma(s+1) + s^\gamma\right)^{-\frac{1}{\gamma}} = \frac{1}{s}\left(\gamma s + \gamma + 1\right)^{-\frac{1}{\gamma}}. \end{cases}$$

Plugging the last equation (h in terms of s) in the former yields r in terms of s:

$$\frac{br}{\gamma+1} = h^{\gamma+1}\left(h^{-\gamma} - s^\gamma\right) - \frac{\gamma}{\gamma+1}h = h^{\gamma+1}\left(\frac{h^{-\gamma}}{\gamma+1} - s^\gamma\right) = \frac{\gamma}{\gamma+1}(sh)^{\gamma+1},$$

hence $r = \gamma b^{-1}(\gamma s + \gamma + 1)^{-\frac{\gamma+1}{\gamma}}$. Plugging this once in $h(r)$ in terms of s yields

$$\gamma s + \gamma + 1 = s(\gamma s + \gamma + 1)^{\frac{\gamma+1}{\gamma}} h = s\frac{\gamma}{br}h = s\gamma\left(\frac{a}{b}r^{-\frac{1}{\gamma+1}} + 1\right).$$

Simplifying γs and plugging r in terms of s again yields the equation for s in (8):

$$s^\gamma(\gamma s + \gamma + 1) = \left(\frac{\gamma+1}{a}\right)^\gamma \left(\frac{b}{\gamma}\right)^{\frac{\gamma^2}{\gamma+1}},$$

which has a unique solution since the L.H.S. increases from 0 to $+\infty$ as s does. We still denote s this solution. The corresponding $r = \gamma b^{-1}(\gamma s + \gamma + 1)^{-\frac{\gamma+1}{\gamma}}$ satisfies $r^{\frac{1}{\gamma+1}} = \left(\frac{\gamma}{b}\right)^{\frac{1}{\gamma+1}}(\gamma s + \gamma + 1)^{-\frac{1}{\gamma}} = s\frac{a}{b}\frac{\gamma}{\gamma+1}$. The second equation of the first system traducing the optimality condition $\nabla d_{a,b} = 0$ yields:

$$d_{a,b}(r,s) = \frac{s}{s+1}r^\gamma\left(h^{-\gamma}(r) - s^\gamma\right) = \gamma s^{\gamma+1}r^\gamma = \gamma s^{(\gamma+1)^2}\left(\frac{a}{b}\frac{\gamma}{\gamma+1}\right)^{\gamma(\gamma+1)}.$$

Now $c_* = d_{a,b}^{-\beta}(r,s)$ is (8) with $m = 0$ since $1 + \gamma = \frac{\beta+1}{\beta}$, $\frac{\gamma+1}{\gamma} = \beta + 1$ and $\frac{(\gamma+1)^2}{\gamma} = \frac{(\beta+1)^2}{\beta}$.

Corollary 1 in the case $a_m = a$ is deduced by the following arguments.

i. The positive solution of the quadratic equation in (8) is $s = \sqrt{1 + \frac{2\sqrt{b}}{a}} - 1$.

ii. Eliminating b from (8) yields $c_* = (a/(\beta+1))^{\beta+1}\beta^{-\beta^2}(s+\beta+1)^{(\beta+1)^2}$, and the implicit equation yields $s \to 0$ as $b \to 0$.

iii. Eliminating a from (8) yields $c_* = b(s+\beta+1)^{\beta+1}/s^{\beta+1}$, and the implicit equation yields $s \to \infty$ as $a \to 0$.

iv. The easiest lower bound for s is $s + \beta + 1 \geq (\beta+1)^{\frac{1}{\beta}}\beta^{(\frac{\beta}{\beta+1})^2}\dfrac{b^{\frac{1}{(\beta+1)^2}}}{a^{\frac{1}{\beta+1}}}$, obtained by plugging $s + \beta + 1 \geq s$ in its implicit equation.

We now complete the general case $m \neq 0$ in (5). The proof uses (5) only once: in the equation before (14). We may divide this equation by $e^{m\lambda^\alpha}$ and keep the same right hand side since $e^{-m\lambda^\alpha} \leq 1$. This yields (14) with $f(T)$ replaced by $f(T)e^{-m\lambda^\alpha}$. This amounts to replacing a by $a + m$ in the definition of d_2 after (14) and therefore in the conclusion (8).

3. Comments.

3.1. **Admissibility.** Any $C \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ satisfies the admissibility condition (2) with $\text{Adm}_T = T\|C\|^2$. The more general setting in § 2.1 is canonical (cf. [50]) and required in many P.D.E. problems, e.g. when the heat flux is observed on the boundary rather than an open subset of the domain. Although it should be sufficient for any P.D.E. problems, it might be useful to circumvent the admissibility assumption:

Lemma 3.1. *The conclusion $2c = \limsup_{T\to 0} T^\beta \ln \kappa_T \leq 2c_*$ of theorem 2.2 is still valid if we replace the assumption that $C \in \mathcal{L}(\mathcal{D}(A), \mathcal{F})$ satisfies the admissibility condition (2) by the following* time smoothing effect *assumption:*

$$\forall x \in \mathcal{E}, \ \forall t > 0, \ e^{tA}x \in \mathcal{D}(A), \quad and \quad \limsup_{t\to 0} t^\beta \ln\|Ae^{tA}\| = 0. \tag{15}$$

Proof. In the proof of theorem 2.2, the admissibility condition (2) is only used once, for $x = x_\lambda^\perp$ orthogonal to the growth space $\mathcal{E}_\lambda$, in this manner:

$$\int_{(1-\varepsilon)T}^T \|Ce^{tA}x\|^2 dt \le \mathrm{Adm}_{T_0}\, m_0^2 e^{2m\lambda^\nu} e^{-2(1-\varepsilon)T\lambda}\|x\|^2,\ x \perp \mathcal{E}_\lambda,\ T \in (0, T_0),\quad (16)$$

and this only affects the definition of the function $g(T) = g_0 \exp(-2/(d_1 T)^\beta)$ used in (9). Recall that $\varepsilon \in (0,1)$ and $r > 0$ have been fixed (in order to maximize $d_{a,b}$). We shall prove, for any $\delta \in (0,1)$ small enough, any $g_1 > 0$ and some smaller T_0,

$$\int_{(1-\varepsilon)T}^T \|Ce^{\tau A}x\|^2 d\tau \le g_1 e^{2m\lambda^\nu} e^{-2(1-(1+\delta)\varepsilon)T\lambda}\|x\|^2,\ x \perp \mathcal{E}_\lambda,\ T \in (0, T_0).\quad (17)$$

Indeed replacing (16) by (17), (9) still holds with g_0 and d_1 replaced by g_1 and $d_{1,\delta} = \varepsilon^{\frac{1}{\alpha}} \left(\frac{r}{1-(1+\delta)\varepsilon}\right)^{\frac{1}{\beta}}$. Since $d_{1,\delta} \to d_1$ as $\delta \to 0$ this will not affect the range of d obtained by applying lemma 2.3, nor the conclusion of theorem 2.2.

With the graph norm on $\mathcal{D}(A)$, $C \in \mathcal{L}(\mathcal{D}(A), \mathcal{F})$ means $\|Cx\| \le \|C\|(\|x\| + \|Ax\|)$. We only need to prove (17) with C replaced by A since the proof of (17) with C replaced by the identity is the same, only shorter. We use the small parameter $\delta \in (0,1)$ to decompose the lower integration bound in (17) in this geometric way: $(1-\varepsilon)T = (1-(1+\delta)\varepsilon)T + (1-\delta)\delta\varepsilon T + \delta^2 \varepsilon T = \tau_1 + \tau_2 + \tau_3$. According to (15), $\tau^\beta \ln\|Ae^{\tau A}\| \le \delta^{2\beta+2}$, $\tau \in (0, T_0)$, for a smaller T_0. This with $\tau = \tau_3$ and (5) yield

$$\|Ae^{tA}x\| \le e^{\frac{\delta^{2\beta+2}}{\tau_3^\beta}} \|e^{(t-\tau_3)A}x\| \le e^{\frac{\delta^2}{(\varepsilon T)^\beta}} m_\lambda e^{-(t-\tau_3)\lambda}\|x\| \le m_\lambda e^{\frac{\delta^2}{(\varepsilon T)^\beta}} e^{-(\tau_1+\tau_2)\lambda}\|x\|,$$

for all $t \in ((1-\varepsilon)T, T_0)$, where $m_\lambda = m_0 e^{m\lambda^\nu}$. Recalling $\tau_2\lambda = \frac{(1-\delta)\delta\varepsilon}{r\varepsilon^{\beta/\alpha}T^\beta}$ and bounding the length of the integration interval by T_0, the proof of (17) with C replaced by A now reduces to

$$T_0 m_0^2 e^{\frac{2\delta^2}{(\varepsilon T)^\beta}} e^{-2\tau_2\lambda} = T_0 m_0^2 e^{\frac{-2\delta c_\delta}{T^\beta}} \le g_1,\quad T \in (0, T_0),$$

where $c_\delta \to \varepsilon/(r\varepsilon^{\beta/\alpha}) > 0$ as $\delta \to 0$. This does hold for $T_0 \le g_1/m^2$ and any δ small enough for c_δ to be positive. $\qquad\square$

The idea of dispensing with the admissibility assumption is due to Marius Tucsnak and Gerald Tenenbaum in the case where A is a nonpositive self-adjoint operator with an orthonormal basis of eigenfunctions. Indeed, that A is nonpositive self-adjoint implies that A generates a bounded analytic semigroup, which is equivalent to the usual time smoothing effect, $\sup_{t>0}\|tAe^{tA}\| < \infty$, which implies the weaker effect (15) assumed in lemma 3.1. N.b. although A in § 4.2 for $\gamma < 1/2$ does not generate an analytic semigroup, it is proved in [23, theorem 4.2] that $\sup_{t>0}\|t^{\frac{1}{2\gamma}}Ae^{tA}\| < \infty$, which also implies (15).

3.2. **Controllability cost.** From the definition of κ_T in (3), we have, for $T' < T$, $\kappa_T \le \|e^{(T-T')A}\|^2 \kappa_{T'}$. This justifies our claim in § 2.1 that $\sqrt{\kappa_T}$ does not grow more than the semigroup as $T \to +\infty$ and does not increase when the semigroup is contractive. Moreover, if $\kappa_t \le g(t)$, $t \in (0, T']$, g nonincreasing, then $\kappa_t \le M_0^2 g(t)$, $t \in (0, T]$, with $M_0 = \sup_{s\in(0,T-T')}\|e^{sA}\| < \infty$. This justifies that we restrict to some bounded intervals $(0, T']$ in the statements of our results. When the semigroup is bounded by $M = \sup_{t\ge0}\|e^{tA}\|$, the cost bound $\sup_{T>T'}\kappa_T \le M^2\kappa_{T'}$ improves

into the decay: $\sup_{T>T'} T\kappa_T \le 2M^2 T'\kappa_{T'}$. Indeed, let $\tau_k = kT'$, $k \in \mathbb{N}$, and $n = \lfloor T/T' \rfloor$, so that $\tau_n \le T < \tau_{n+1}$. Since $\|e^{TA}x\| \le M\|e^{\tau_k A}x\|$ for $k \le n$,

$$n\|e^{TA}x\|^2 \le M^2 \kappa_{T'} \sum_{k=1}^{n} \int_{\tau_{k-1}}^{\tau_k} \|Ce^{tA}x\|^2 dt = M^2 \kappa_{T'} \int_0^{\tau_n} \|Ce^{tA}x\|^2 dt.$$

Since $\tau_n \le T \le \tau_{2n}$, the proof of $\kappa_T \le M^2 \kappa_{T'}/n \le M^2 \kappa_{T'}(2T'/T)$ is completed.

The dual problem to the final-observability of (1) is the null-controllability of

$$\dot{f}(t) = A^* f(t) + Bu(t), \quad f(0) = f_0 \in \mathcal{E}, \quad t \ge 0, \tag{18}$$

with input $u \in L^2([0,T], \mathcal{F})$ and control operator $B = C^* \in \mathcal{L}(\mathcal{F}, \mathcal{D}(A^*)')$ (A^* denotes the adjoint of A and $\mathcal{D}(A^*)'$ denotes the dual space of $\mathcal{D}(A^*)$ in $\mathcal{E}$). Since C satisfies the admissibility condition (2), B satisfies $\|\int_0^T e^{tA^*} Bu(t)dt\|^2 \le K_T \int_0^T \|u(t)\|^2 dt$, and the solution of (18) is $f(T) = e^{TA^*} f_0 + \int_0^T e^{(T-t)A^*} Bu(t)dt$. More precisely, if (A, C, T) is observable at cost κ_T then, for all f_0, there is a u such that $f(T) = 0$ and $\int_0^T \|u(t)\|^2 dt \le \kappa_T \|f_0\|^2$ (cf. [15]).

The study of the cost of fast controls was initiated by Seidman in [44] with a result on the heat equation obtained by Russell's method in [42]. We refer to the surveys [45, 38] and the more recent paper [49]. An application to reachability is given in § 3.4.

3.3. Integrated observability estimate.
Lemma 2.1 can be seen as the discrete version of the following lemma which has been used with $f(t) = \exp(-c/t)$ when proving observability by some parabolic global Carleman estimates (cf. e.g. [20, 19]).

Lemma 3.2. *In the setting of* § 2.1*, if the integrated observability estimate*

$$\int_0^T f(t)\|e^{tA}x\|^2 dt \le \int_0^T \|Ce^{tA}x\|^2 dt, \quad x \in \mathcal{D}(A), \quad T \in (0, T_0), \tag{19}$$

holds with $T_0 > 0$ and f an increasing function such that $f(t) \to 0$ as $t \to 0^+$, then, for any $\varepsilon \in (0,1)$, $\kappa_T \le M_{\varepsilon T}/(\varepsilon T f((1-\varepsilon)T))$, $T \in (0, T_0)$, with $M_{\varepsilon T} \le M_T := \sup_{t \in [0,T]} \|e^{tA}\|^2 \le M_{T_0} < \infty$, i.e. the growth of the fast control cost is almost bounded by the inverse of f.

Conversely, if (3) holds for $T \in (0, T_0)$ then (19) holds with $f(t) = 1/(T_0 \kappa_t)$.

Proof. The implication results from $\|e^{TA}x\|^2 = \|e^{(T-t)A} e^{tA}x\|^2 \le M_T \|e^{tA}x\|^2$ and $f(t) \le f(T)$ for $t \in (0, T)$: for $\varepsilon \in (0, 1)$,

$$\varepsilon T f((1-\varepsilon)T)\|e^{TA}x\|^2 \le M_{\varepsilon T} \int_{T(1-\varepsilon)}^T f(t)\|e^{tA}\|^2 dt \le M_{\varepsilon T} \int_0^T \|Ce^{tA}\|^2 dt.$$

Writting (3) as $\kappa_\tau^{-1}\|e^{\tau A}x\|^2 \le \int_0^\tau \|Ce^{tA}x\|^2 dt$, the converse results from integrating:

$$\int_0^T \int_0^\tau \|Ce^{tA}x\|^2 dt d\tau \le \int_0^T \int_0^T \|Ce^{tA}x\|^2 dt d\tau = T \int_0^T \|Ce^{tA}x\|^2 dt.$$

$\square$

3.4. Reachability.
As the input u varies, the final state $f(T)$ of (18) spans the set of states which are reachable from f_0 in time T, denoted $\mathcal{R}(T, f_0)$. Assuming (A, C, T) is observable for all $T > 0$, the usual duality in § 3.2 implies that this reachability set $\mathcal{R} = \mathcal{R}(T, f_0)$ does not depend on T and f_0 (by an argument due to Seidman in [43], cf. [38, footnote 7]) and satisfies $e^{tA}(\mathcal{E}) \subset \mathcal{R}$, $t > 0$.

The following lemma provides further information on the reachability set when a cost estimate as in theorem 2.2 is available.

Lemma 3.3. *In the setting of* § *2.1, assume A is self-adjoint and $\sigma(A) \subset (-\infty, \lambda_1]$, and consider the fractional powers $A_\beta = -(-A + \lambda_1)^\beta$, $\beta > 0$.*

For all exponents $\beta > 0$, $\alpha = \frac{\beta}{\beta+1}$, and rates $b > 0$, $c > b(\beta+1)$, $a > (b\beta)^{\frac{1}{\beta+1}}/\alpha$, for all $T_0 > 0$, there exists $c_0 > 0$ such that

$$\|e^{aA_\alpha}x\|^2 \le c_0 e^{\frac{2c}{T^\beta}} \int_0^T e^{-\frac{2b}{t^\beta}} \|e^{tA}x\|^2 dt, \quad x \in \mathcal{D}(A), \quad T \in (0, T_0). \tag{20}$$

If (A, C, T) is observable at a cost κ_T such that $2b_0 = \limsup_{T \to 0} T^\beta \ln \kappa_T < \infty$, then the reachability set satisfies $e^{aA_\alpha}(\mathcal{E}) \subset \mathcal{R}$ for $\alpha = \frac{\beta}{\beta+1}$ and $a > (b_0\beta)^{\frac{1}{\beta+1}}/\alpha$.

Proof. We first deduce the reachability statement from the previous one. For any $b > b_0$, (3) holds with $\kappa_T = \exp(2b/T^\beta)$, $T \in (0, T_0)$, for T_0 small enough. The converse in lemma 3.2 proves that the integral in (20) is bounded by some multiple of the integral in (3). Plugging this in (20) yields a $c_1 > 0$ such that

$$\|e^{aA_\alpha}x\|^2 \le c_1 e^{\frac{2c}{T^\beta}} \int_0^T \|Ce^{tA}x\|^2 dt, \quad x \in \mathcal{D}(A), \quad T \in (0, T_0).$$

The same duality argument (cf. [15, (3.22)]) deduces $e^{aA_\alpha}(\mathcal{E}) \subset \mathcal{R}(T, 0) = \mathcal{R}$.

Given $x \in \mathcal{D}(A)$ and $T \in (0, T_0)$, using the spectral measure $dE_x(\lambda)$ of A for x:

$$\|e^{aA_\alpha}x\|^2 = \int_{\sigma(A)} e^{-2a(\lambda_1 - \lambda)^\alpha} dE_x, \quad \int_0^T f(t)\|e^{tA}x\|^2 dt = \int_0^T \int_{\sigma(A)} f(t)e^{2t\lambda} dE_x dt.$$

Hence (20) reduces to $\int_0^T e^{-2j_\lambda(t)} dt \ge \frac{1}{c_0} e^{-\frac{2c}{T^\beta}} e^{-2a(\lambda_1 + \lambda)^\alpha}$ for $\lambda \ge -\lambda_1$, and further to (by changing λ into $\lambda - \lambda_1$, with $c_1 = c_0 \min\{1, e^{2T_0\lambda_1}\}$):

$$\int_0^T e^{-2j_\lambda(t)} dt \ge \frac{1}{c_1} e^{-\frac{2c}{T^\beta}} e^{-2a\lambda^\alpha}, \quad T \in (0, T_0), \quad \lambda \ge 0, \tag{21}$$

where $\quad j_\lambda(t) = \frac{b}{t^\beta} + t\lambda \quad$ satisfies $\quad j_\lambda(t) \ge j_\lambda(t_\lambda) = \frac{t_\lambda \lambda}{\alpha}, t_\lambda = \left(\frac{b\beta}{\lambda}\right)^{\frac{1}{\beta+1}}, \lambda > 0.$

On the one hand, if $t_\lambda < T$, then

$$\int_0^T e^{-2j_\lambda(t)} dt \ge \int_{\delta t_\lambda}^{t_\lambda} e^{-2j_\lambda(t)} dt \ge (1-\delta)t_\lambda e^{-2j_\lambda(\delta t_\lambda)}, \quad \delta \in (0, 1),$$

with $\quad j_\lambda(\delta t_\lambda) = \left(\frac{1}{\beta\delta^\beta} + \delta\right) t_\lambda \lambda = a_\delta \lambda^\alpha, \quad a_\delta = (b\beta)^{\frac{1}{\beta+1}} \left(\frac{1}{\beta\delta^\beta} + \delta\right) \xrightarrow{\delta \to 1} \frac{(b\beta)^{\frac{1}{\beta+1}}}{\alpha},$

hence (21) holds for $c = 0$ and $a > \frac{(b\beta)^{\frac{1}{\beta+1}}}{\alpha}$ by choosing δ such that $a > a_\delta$.

On the other hand, if $\lambda \le \frac{b\beta}{T^{\beta+1}}$ then

$$\int_0^T e^{-2j_\lambda(t)} dt \ge \int_{\delta T}^T e^{-2j_\lambda(t)} dt \ge (1-\delta)T e^{-2j_\lambda(\delta T)}, \quad \delta \in (0, 1),$$

with $\quad j_\lambda(\delta T) \le \frac{b}{(\delta T)^\beta} + (\delta T)\frac{b\beta}{T^{\beta+1}} = \frac{c_\delta}{T^\beta}, \quad c_\delta = b\left(\frac{1}{\delta^\beta} + \delta\beta\right) \xrightarrow{\delta \to 1} b(\beta+1),$

hence (21) holds for $a = 0$ and $c > b(\beta+1)$ by choosing δ such that $c > c_\delta$. $\qquad \square$

Concerning the heat semigroup in § 2.4, as a corollary to the cost upper bound in § 3.2 under the geometrical optics condition, this lemma with $\beta = 1$ proves that $e^{-a\sqrt{-\Delta}}\phi_0$ is reachable for $a > \sqrt{3}L_\Omega$, $\phi_0 \in L^2(M)$, cf. [38, corollary 10]. In dimension one a better result is due to Fattorini and Russell, cf. [18, (3.19)]: if M is a segment of length L controlled from one endpoint then $e^{-a\sqrt{-\Delta}}\phi_0$ is reachable for all $a > L$, $\phi_0 \in L^2(M)$ (this cannot be proved by the same method for $a < L$, cf. [18, (3.20)]). Whether "the optimal" rate a such that $e^{-a\sqrt{-\Delta}}(L^2(M)) \subset \mathcal{R}$ can be expressed geometrically in the general setting of § 2.4 is an open question, e.g. is it $\sup_{y \in M} \text{dist}(y, \overline{\Omega})$?

3.5. **Approximate observability.** The following lemma clarifies the connection of (4) in lemma 2.1 to approximate controllability, and therefore to [46].

Lemma 3.4. *Given the time $T > 0$, the cost $\kappa > 0$ and the approximation rate $\varepsilon > 0$, the following two properties are equivalent.*

i. Approximate observability of (A, C, T):

$$\|e^{TA}x\|^2 \le \kappa \int_0^T \|Ce^{tA}x\|^2 dt + \varepsilon\|x\|^2, \quad x \in \mathcal{D}(A).$$

ii. Approximate null-controllability of (18):

$$\forall f_0 \in \mathcal{E}, \quad \exists u \in L^2([0,T], \mathcal{F}), \quad \frac{1}{\kappa} \int_0^T \|u(t)\|^2 dt + \frac{1}{\varepsilon}\|f(T)\|^2 \le \|f_0\|^2.$$

Proof. Consider the strictly convex $\mathcal{C}^1$ functional J defined on $\mathcal{E}$ by density as

$$J(x) = \frac{\kappa}{2} \int_0^T \|Ce^{tA}x\|^2 dt + \frac{\varepsilon}{2}\|x\|^2 + \langle e^{TA}x, f_0 \rangle, \quad x \in \mathcal{D}(A).$$

Property (i) implies $J(x) \ge \frac{1}{2}\|e^{TA}x\|^2 + \langle e^{TA}x, f_0 \rangle$, hence J is coercive. Therefore J has a unique minimizer $\psi_0 \in \mathcal{E}$, i.e. $J(\psi_0) = \inf_{x \in \mathcal{E}} J(x)$, and

$$0 = \nabla J(\psi_0) = \kappa \int_0^T e^{tA^*}BCe^{tA}\psi_0 dt + \varepsilon\psi_0 + e^{TA^*}f_0.$$

This equation also says that the input $u(t) = \kappa Ce^{tA}\psi_0$ in (18) yields the final state $f(T) = -\varepsilon\psi_0$. In terms of these u and $f(T)$, $\langle \nabla J(\psi_0), \psi_0 \rangle = 0$ writes

$$\frac{1}{\kappa} \int_0^T \|u(t)\|^2 dt + \frac{1}{\varepsilon}\|f(T)\|^2 = \kappa \int_0^T \|Ce^{tA}\psi_0\|^2 dt + \varepsilon\|\psi_0\|^2 = -\langle e^{TA}\psi_0, f_0 \rangle. \quad (22)$$

Plugging this in property (i) yields $\|e^{TA}\psi_0\|^2 \le -\langle e^{TA}\psi_0, f_0 \rangle \le \|e^{TA}\psi_0\|\|f_0\|$. Hence $\|e^{TA}\psi_0\| \le \|f_0\|$. This allows to bound (22) as in property (ii).

Conversely, taking the duality product of $x \in \mathcal{D}(A)$ with a final state of (18) $f(T) = \int_0^T e^{tA^*}Bu(t)dt + e^{TA^*}f_0$ yields $\langle f_0, e^{TA}x \rangle = -\int_0^T \langle u, Ce^{TA}x \rangle dt + \langle f(T), x \rangle$. Using the Cauchy-Schwarz inequality in $\mathcal{E}$, $L^2(0, T)$ and $\mathbb{R}^2$ yields

$$|\langle f_0, e^{TA}x \rangle|^2 \le \left(\frac{1}{\kappa} \int_0^T \|u(t)\|^2 dt + \frac{1}{\varepsilon}\|f(T)\|^2 \right) \left(\kappa \int_0^T \|Ce^{tA}x\|^2 dt + \varepsilon\|x\|^2 \right).$$

Choosing $f_0 = e^{TA}x$ completes the proof that property (ii) implies property (i). $\square$

3.6. **Growth condition, normal semigroups and spectral spaces.** If the growth spaces are closed and satisfy $\sup \operatorname{Re} \sigma(A_{\rceil \mathcal{E}_\lambda^\perp}) = -\lambda$, then the growth condition (5) for a given $\lambda > 0$ as $t \to \infty$ says that the restriction of the semigroup to $\mathcal{E}_\lambda^\perp$ satisfies the spectral bound equal growth bound condition (this condition is satisfied by any eventually norm-continuous semigroup, e.g. differentiable semigroup, e.g. A is self-adjoint and bounded from above). Yet the growth bound of this restricted semigroup for small t may get worse as $\lambda \to \infty$. This justifies allowing $m \neq 0$ in the growth condition (5). E.g. the growth condition for some non-selfadjoint elliptic operators A stated in [27, Proposition 4.12], which comes naturally from the Laplace representation of the semigroup and resolvent estimates, is precisely of the form (5) for some $m > 0$. When A is only mildly non-normal as in § 4.2, the loss is only polynomial in λ, hence (5) holds for any $m > 0$.

On the contrary, for a normal semigroup (i.e. A is normal and the real part of its spectrum is bounded from above, e.g. A is negative self-adjoint as in [37]) the natural growth spaces are its spectral spaces and (5) always holds with $m_0 = 1$ and $m = 0$. Indeed, it has a spectral decomposition E (a.k.a. projection-valued measure) which commutes with any operator which commutes with A, defines spectral projections $E_\lambda = E(\{z \in \sigma(A) \mid \operatorname{Re} z > -\lambda\})$ and spectral spaces $\mathcal{E}_\lambda = E_\lambda(\mathcal{E})$, and provides the integral representation $e^{tA} = \int_{\sigma(A)} e^{tz} dE(z)$ hence this growth condition (5). N.b. for unitary groups (i.e. A is skew-adjoint, e.g. Schrödinger or wave equations) $\mathcal{E}_\lambda = E_\lambda(\mathcal{E}) = \mathcal{E}$, $\lambda > 0$, so that (5) is trivial but (6) is never satisfied in applications.

If there is an orthonormal basis $\{e_n\}$ of $\mathcal{E}$ such that $-Ae_n = \lambda_n e_n$, then the spectral spaces are just spanned by linear combinations of normalized eigenfunctions $\mathcal{E}_\lambda = \operatorname{Span}\{e_n\}_{\lambda_n < \lambda}$ and (6) is an estimate on sums of eigenfunctions of A.

For $A = \Delta$ on $\mathcal{E} = L^2(\mathbb{R}^d)$, the spectral decomposition is the Fourier transform: $\widehat{f(-\Delta)}\phi(\xi) = f(|\xi|^2)\hat{\phi}(\xi)$, $\phi \in L^2(\mathbb{R}^d)$, thus $\phi \in \mathcal{E}_\lambda$ just means $\hat{\phi}(\xi) = 0$ for $|\xi|^2 > \lambda$, i.e. ϕ is the restriction to the real axis of an entire function $\tilde{\phi}$ such that $|\tilde{\phi}(z)| \leq ce^{\sqrt{\lambda}|\operatorname{Im} z|}$ by the Paley-Wiener theorem. When C_0 is the identity operator, C is the multiplication by the characteristic function of the exterior of a ball and $\mathcal{F} = \mathcal{E}$, [34] proves (6) with exponent $\alpha = \frac{1}{2}$ by Carleman estimates as in § 2.4. It is an open problem to obtain an explicit bound on the rate a in (6), e.g. by complex analysis.

3.7. **Reference operator.** Any A satisfies the fast control cost estimate

$$\|e^{TA}x\|^2 \leq \frac{M_T}{T} \int_0^T \|e^{tA}x\|^2 dt, \quad x \in \mathcal{D}(A), \quad T \in (0, T_0), \tag{23}$$

with $M_T = \sup_{t \in [0,T]} \|e^{tA}\|^2 \leq M_{T_0} < \infty$. Thus the cost estimate (7) holds for any exponent $\beta > 0$ and rate $b > 0$ when C_0 is the identity operator.

For a system of coupled P.D.E., C_0 can be the observation of a single component as in § 4.2, e.g. the operator $\mathcal{C}_M$ in [39]: for this reference operator, (5) with $m = 0$, (6), and (7) with any $b > 0$, are stated in this form in [39, Propositions 4, 3, 2] respectively. The assumptions (5) with $m = 0$ and (6) are called $[H]$ in the abstract framework of [46].

3.8. **"Converse" to the main result.** The following lemma is a very partial converse to theorem 2.2: only for sequences of eigenfunctions of A and $C_0 = \operatorname{id}$.

Lemma 3.5. *Assume that (A, C) satisfies the observability cost estimate with exponent $\beta > 0$ and rate $b > 0$*

$$\|e^{TA}x\|^2 \le b_0 e^{\frac{2b}{T^\beta}} \int_0^T \|Ce^{tA}x\|^2 dt, \quad x \in \mathcal{D}(A), \quad T \in (0, T_0). \qquad (24)$$

Any sequence (e_n) in $\mathcal{D}(A)$ such that $-Ae_n = \lambda_n e_n$ and $\lim \lambda_n = +\infty$, must satisfy

$$\|e_n\|^2 \le \frac{b_0}{2\lambda_n} e^{2a\lambda_n^\alpha} \|Ce_n\|^2, \, \alpha = \frac{\beta}{\beta + 1}, \, a = \frac{\beta + 1}{\beta^\alpha} b^{\frac{1}{\beta+1}}, \, \lambda_n \text{ large enough}. \qquad (25)$$

In particular, if the sequence satisfies for some exponent $\alpha > 0$ and rate $a > 0$:

$$\|e_n\|^2 \ge a_0 e^{2a\lambda_n^\alpha} \|Ce_n\|^2, \quad \lambda_n \text{ large enough}, \qquad (26)$$

then the observability cost in (3) satisfies $\limsup_{T \to 0} T^\beta \ln \kappa_T > 0$ with $\beta = \frac{\alpha}{1-\alpha}$.

Proof. Applying (24) to $x = e_n$ yields $e^{-2T\lambda_n} \|e_n\|^2 \le b_0 e^{\frac{2b}{T^\beta}} \int_0^T \|Ce_n\|^2 e^{-2t\lambda_n} dt$,

hence $\|e_n\|^2 \le \frac{b_0}{2\lambda_n} e^{2h(T)} \|Ce_n\|^2$, with $h(T) = \frac{b}{T^\beta} + T\lambda_n$. Minimizing h yields

$h(T_n) = \frac{\beta+1}{\beta^\alpha} b^{\frac{1}{\beta+1}} \lambda_n^\alpha$ at $T_n = \left(\frac{\beta b}{\lambda_n}\right)^{\frac{1}{\beta+1}}$ with $T_n < T_0$ for λ_n large enough.

We prove the last statement of lemma 3.5 by contradiction. If the observability cost in (3) satisfies $\limsup_{T \to 0} T^\beta \ln \kappa_T = 0$ with $\beta = \frac{\alpha}{1-\alpha}$, then (24) holds for any $b > 0$ with T_0 small enough, hence (25) holds for any $a > 0$, which refutes (26). $\square$

4. Applications.

4.1. Anomalous diffusions.
Let M be a smooth connected complete d-dimensional Riemannian manifold with metric g and boundary ∂M. When $\partial M \ne \emptyset$, M denotes the interior and $\overline{M} = M \cup \partial M$. Let Δ denote the Laplace-Beltrami operator on $L^2(M)$ with domain $\mathcal{D}(\Delta) = H_0^1(M) \cap H^2(M)$ defined by g. N.b. the results are already interesting when (M, g) is a smooth connected domain of the Euclidean space $\mathbb{R}^d$, so that $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$.

In this application, the state and input spaces are $\mathcal{E} = \mathcal{F} = L^2(M)$, the growth spaces are the spectral spaces of § 3.6, the reference operator C_0 is the identity operator and the observation operator C is the multiplication by the characteristic function χ_Ω of an open subset $\Omega \ne \emptyset$ of $\overline{M}$, i.e. it truncates the input function outside the control region Ω. If M is not compact, assume that Ω is the exterior of a compact set K such that $K \cap \overline{\Omega} \cap \partial M = \emptyset$.

For $A = \Delta$, (6) holds with exponent $\alpha = \frac{1}{2}$, cf. § 2.4 for compact M, and [34] otherwise. Hence for $A = -(-\Delta)^\gamma$, (6) holds with exponent $\alpha = \frac{1}{2\gamma}$. Applying theorem 2.2 improves on [37, theorem 2]:

Theorem 4.1. *For all $\gamma > 1/2$, the anomalous diffusion:*

$$\partial_t \phi + (-\Delta)^\gamma \phi = \chi_\Omega u, \ \phi(0) = \phi_0 \in L^2(M), \ u \in L^2([0, T] \times M),$$

is null-controllable in any time $T > 0$. Moreover the cost κ_T (cf. § 3.2) satisfies $\limsup_{T \to 0} T^\beta \ln \kappa_T < \infty$ with $\beta = \frac{1}{2\gamma - 1}$.

When the manifold M is the whole Euclidean space $\mathbb{R}^d$, the fractional Laplacian $-(-\Delta)^\gamma$ with $\gamma \in (0, 1]$ generates the rotationally invariant 2γ-stable Lévy process. For $\gamma = 1$ this process is the Brownian motion B_t on $\mathbb{R}^d$, and for $\gamma < 1$ it is subordinated to B_t by a strictly γ-stable subordinator T_t, so that it writes B_{T_t}. The

convolution kernels of the corresponding semigroups are the rotationally invariant Lévy stable probability distributions, in particular the Gaussian distribution for $\gamma = 1$ and the Cauchy distribution for $\gamma = 1/2$. For $\gamma < 1$ these distributions have "heavy tails", i.e. far away they decrease like a power as opposed to the exponential decrease found in the Gaussian, which accounts for the "superdiffusive" behavior of the semigroup. The more restrictive range $\gamma \in (1/2, 1)$ is the most widely used to model anomalously fast diffusions, and it turns out that the controllability result theorem 4.1 applies to this range of fractional superdiffusions only.

When the manifold M is a domain of the Euclidean space $\mathbb{R}^d$, the Markov process generated by the fractional Dirichlet Laplacian $-(-\Delta)^\gamma$ with $\gamma \in (0, 1]$ can be obtained by killing the Brownian motion on $\mathbb{R}^d$ upon exiting the domain then subordinating the killed Brownian motion by the subordinator T_t introduced above.

4.2. **Structural damping.** Let $\mathcal{A}$ be a positive self-adjoint and boundedly invertible operator on another Hilbert space $\mathcal{H}$ (with norm still denoted $\|\cdot\|$). Let $\mathcal{D}(\mathcal{A})$ denote its domain with the norm $\zeta \mapsto \|\mathcal{A}\zeta\|$. Since $-\mathcal{A}$ is normal, we may consider its spectral decomposition H, its spectral projections $H_\mu = H(\{z \in \sigma(\mathcal{A}) \mid \operatorname{Re} z < \mu\})$ and spectral spaces $\mathcal{H}_\mu = H_\mu(\mathcal{H})$. (cf. § 3.6). We consider an observation operator $\mathcal{C}$ in $\mathcal{L}(\mathcal{D}(\mathcal{A}), \mathcal{F})$ satisfying observability on $\mathcal{H}_\mu$ relative to the identity operator:

$$\|z\|^2 \leq d_0 e^{2d\mu^\delta} \|\mathcal{C}z\|^2, \quad z \in \mathcal{H}_\mu, \quad \mu > 0, \tag{27}$$

and the corresponding control operator $\mathcal{B} = \mathcal{C}^* \in \mathcal{L}(\mathcal{F}, \mathcal{D}(\mathcal{A})')$ ($\mathcal{D}(\mathcal{A})'$ denotes the dual space of $\mathcal{D}(\mathcal{A})$ in $\mathcal{H}$).

To give a precise meaning to the solution of the structurally damped system

$$\begin{aligned}
&\ddot{\zeta}(t) + \rho \mathcal{A}^{2\gamma} \dot{\zeta}(t) + \mathcal{A}^2 \zeta(t) = \mathcal{B}u(t), \\
&\zeta(0) = \zeta_0 \in \mathcal{D}(\mathcal{A}), \quad \dot{\zeta}(0) = \zeta_1 \in \mathcal{H}, \quad u \in L^2([0, T], \mathcal{F}),
\end{aligned} \tag{28}$$

with structural dissipation power $\gamma \in (0, 1)$, we write it as a first order system.

The state space is $\mathcal{E} = \mathcal{D}(\mathcal{A}) \times \mathcal{H}$. The semigroup generator A is

$$A = \begin{pmatrix} 0 & I \\ -\mathcal{A}^2 & -\rho \mathcal{A}^{2\gamma} \end{pmatrix}, \mathcal{D}(A) = \left\{ (z_0, z_1) \in \mathcal{E} \mid \mathcal{A}z_0 + \rho \mathcal{A}^{2\gamma-1} z_1 \in \mathcal{D}(\mathcal{A}) \right\}.$$

It inherits from $-\mathcal{A}$ the necessary and sufficient properties of Lumer-Phillips for generating a contraction semigroup.

The observation and control operators are the projection $C_0 : \mathcal{E} \to \mathcal{H}$ defined by $C_0(z_0, z_1) = z_1$, $C = \mathcal{C}C_0$, and B defined in § 3.2. We assume that C is admissible for the semigroup generated by A, i.e. (2). Since the cost estimate for C_0 given in [4] is polynomial in $1/T$, (7) holds for any $\beta > 0$ and $b > 0$.

For $\mu > 0$ and $z = (z_0, z_1) \in \mathcal{H} \times \mathcal{H}$, we denote $H_{z_0,z_1}(\mu) = \langle H(\mu)z_0, z_1 \rangle$ where H is the spectral decomposition of $\mathcal{A}$. We define the matrix valued function M and the positive Hermitian matrix valued measure $E_{z,z}$ by

$$M(\mu) = \begin{pmatrix} 0 & -1 \\ \mu^2 & \rho\mu^{2\gamma} \end{pmatrix}, \quad E_{z,z} = \begin{pmatrix} H_{z_0,z_0} & H_{z_0,z_1} \\ H_{z_1,z_0} & H_{z_1,z_1} \end{pmatrix}.$$

As proved in [36, Lemma 3], the roots $\lambda_\pm = r \pm s$ of $P_\mu(\lambda) = \det(M(\mu) - \lambda I)$ satisfy $\min\{\operatorname{Re}\lambda_+, \operatorname{Re}\lambda_-\} \geq \min\left\{\frac{\rho}{2}, \frac{1}{\rho}\right\} \mu^{2\min\{\gamma, 1-\gamma\}}$ for $\mu \geq 1$. Therefore we define the growth spaces as $\mathcal{E}_\lambda = \mathcal{H}_\mu \times \mathcal{H}_\mu$ with $\lambda = \min\left\{\frac{\rho}{2}, \frac{1}{\rho}\right\} \mu^{2\min\{\gamma, 1-\gamma\}}$. Plugging in the

spectral representation $\langle e^{tA}z, z\rangle = \int_{\sigma(\mathcal{A})} e^{-tM(\mu)} dE_{z,z}(\mu)$ the explicit formula

$$e^{-tM(\mu)} = e^{-tr} \left(t\operatorname{shc}(st)M(\mu) + (\cosh(st) + rt\operatorname{shc}(st))I \right), \quad t > 0,$$

(where the cardinal hyperbolic sine function is the continuous and even function defined by $\operatorname{shc}(0) = 1$ and $\operatorname{shc}(t) = \sinh(t)/t$ for $t \neq 0$) proves the growth condition (5) with any $m > 0$ (indeed the loss μ^2 is only polynomial in λ instead of exponential).

For this choice of growth spaces, (27) implies the relative observability (6) with exponent $\alpha = \frac{\delta}{2\min\{\gamma, 1-\gamma\}}$ and rate $a = \frac{d}{\min\left\{\frac{\rho}{2}, \frac{1}{\rho}\right\}^\alpha}$.

Applying theorem 2.2 and corollary 1(ii) improves on [36, theorem 1]:

Theorem 4.2. *Recall that δ and d are the exponent and rate in the main assumption* (27). *For all $\rho > 0$ and $\gamma \in (\delta/2, 1 - \delta/2)$, for all ζ_0 and ζ_1, there is an input u such that the solution ζ of* (28) *satisfies $\zeta(T) = \dot{\zeta}(T) = 0$ and the cost estimate:*

$$\int_0^T \|u(t)\|^2 dt \leq b_0 \exp\left(\frac{2b}{T^\beta}\right) \left(\|\mathcal{A}\zeta_0\|^2 + \|\zeta_1\|^2\right), \quad \zeta_0 \in \mathcal{D}(\mathcal{A}), \ \zeta_1 \in \mathcal{H}, \ T \text{ small},$$

with $\beta = \left(\frac{2}{\delta}\min\{\gamma, 1-\gamma\} - 1\right)^{-1}$, and any $b > \dfrac{d^{\beta+1}}{\min\left\{\frac{\rho}{2}, \frac{1}{\rho}\right\}^\beta} \dfrac{(\beta+1)^{\beta(\beta+1)}}{\beta^{\beta^2}}.$

We refer to [10, 26] for the motivation of the abstract model (28). The main application is to the plate equation with square root damping and interior control in Ω with hinged boundary conditions on a manifold M, in the framework of § 4.1:

$$\ddot{\zeta} - \rho\Delta\dot{\zeta} + \Delta^2\zeta = \chi_\Omega u \quad \text{on} \ \ [0, T] \times M, \quad \zeta = \Delta\zeta = 0 \ \ \text{on} \ \ [0, T] \times \partial M,$$
$$\zeta(0) = \zeta_0 \in H^2(M) \cap H_0^1(M), \quad \dot{\zeta}(0) = \zeta_1 \in L^2(M), \quad u \in L^2([0, T] \times M). \tag{29}$$

Applying theorem 4.2 instead of [36, theorem 1] to $\mathcal{A} = -\Delta$ with $\delta = \gamma = \frac{1}{2}$, hence $\beta = 1$, improves on the value of β in the first part of [36, theorem 2] (cf. also [3]). Under the geometrical optics condition in [5] that the length L_Ω of the longest generalized geodesic in $\overline{M}$ which does not intersect Ω is not ∞, the second part of [36, theorem 2] estimates the cost rate: for all $\rho \in (0, 2)$, the control cost of (29) satisfies the estimate in theorem 4.2 with $\beta = 1$ and any $b > b_1 L_\Omega^2$ for some b_0 and b_1 which do not depend on Ω and ρ (cf. [36, note added in proof]), hence e.g. (cf. [32], [16, Appendix]) the minimal null-control input u converges to the minimal null-control input for the undamped plate equation as $\rho \to 0$.

4.3. **Diffusion in a potential well.** We consider a power $k \in \mathbb{N}^*$ and the potential well $V(x) = |x|^{2k}$, $x \in \mathbb{R}^d$. The Schrödinger operator $A = \Delta - V$ with domain $\mathcal{D}(A) = \left\{\phi \in H^2(\mathbb{R}^d) \mid \int |V\phi|^2 < \infty\right\}$ is negative self-adjoint and has compact resolvent. Let χ_Γ denote the multiplication by the characteristic function of any non empty open cone $\Gamma = \left\{x \in \mathbb{R}^d \mid |x| > r_0, x/|x| \in \Omega_0\right\}$, where $r_0 \geq 0$ and Ω_0 is an open subset of the unit sphere.

In this application, the state and input spaces are $\mathcal{E} = \mathcal{F} = L^2(\mathbb{R}^d)$, the growth spaces are the spectral spaces of § 3.6, the reference operator C_0 is the identity operator and the observation operator C is the multiplication by χ_Γ as in § 4.1, i.e. it truncates the input function outside the control region Γ.

In [40], (6) with exponent $\alpha = \frac{1}{2}(1 + \frac{1}{k})$ is proved and some radial eigenfunctions concentrating at some "equator" such that (26) holds are exhibited (cf. [40, § 4.2.2]) allowing to deduce from theorem 2.2 and lemma 3.5:

Theorem 4.3. *For all $k > 1$, the diffusion in the potential well $V(x) = |x|^{2k}$:*

$$\partial_t \phi - \Delta\phi + V\phi = \chi_\Gamma u, \ \ \phi(0) = \phi_0 \in L^2(\mathbb{R}^d), \ u \in L^2([0,T] \times \mathbb{R}^d),$$

is null-controllable in any time $T > 0$. Moreover the cost κ_T (cf. § 3.2) satisfies:
$\kappa = \limsup_{T\to 0} T^\beta \ln \kappa_T < \infty$ with $\beta = 1 + \frac{2}{k-1}$.
   If there is a vector space of dimension 2 in $\mathbb{R}^d$ which does not intersect the closure $\overline{\Omega}_0$ of the subset Ω_0 of the unit sphere defining the cone Γ then $\kappa \neq 0$.

When Γ is a bounded set instead of a cone, some radial eigenfunctions such that (25) fails are exhibited in [40, § 4.2.3] allowing to deduce from lemma 3.5 that $\kappa = \limsup_{T\to 0} T^\beta \ln \kappa_T = +\infty$ with $\beta = 1 + \frac{2}{k-1}$. Whether null-controllability from bounded sets Γ holds for $k > 1$ remains open.

As in § 4.1, the semigroup considered here is a well known model of diffusion. It can be interpreted as a Brownian diffusion on $\mathbb{R}^d$ killed at the rate V.

5. **Lower bounds for the cost and spectral rates.** The setting of this independent section is slightly more general than in § 2.4. As in § 4.1, M is a smooth complete Riemannian manifold and Δ is the Laplace-Beltrami operator with Dirichlet boundary condition on $\mathcal{H} = L^2(M)$ which is both the state space $\mathcal{E}$ and the input space $\mathcal{F}$. *In this section we denote in the same way an open subset of M, its characteristic function and the multiplication by this function which is a bounded operator on $\mathcal{H}$.* With this notation, the observation operator is $C = \Omega$ where $\Omega \neq \emptyset$ is an open subset of M. In this section $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ still denotes the generator of a $\mathcal{C}_0$-semigroup $(e^{tA})_{t\geq 0}$ on $\mathcal{H}$.

The main assumption in this section is the following L^2 *Gaussian estimate*: for all open subset $\omega \subset M$ and $d < \text{dist}(\Omega, \omega) := \inf_{(x,y)\in\Omega\times\omega} \text{dist}(x,y)$,

$$\|\Omega e^{tA}\omega\| \leq d_0 e^{-\frac{d^2}{4t}}\|\omega\|, \quad t \in [0,T_0]. \tag{30}$$

The lower bounds in this section are given in terms of the following distance:

$$d_\Omega = \sup_{y\in M} \text{dist}(\overline{\Omega}, y), \tag{31}$$

i.e. the furthest from Ω a point of M can be. A simple example to keep in mind was considered at the end of § 3.6: $M = \mathbb{R}^d$, $A = \Delta$, Ω is the exterior of a ball, hence d_Ω in (31) is the radius of this ball.

In the particular case of the heat semigroup on a compact manifold considered in § 2.4, Gaussian estimates were already the main tool in the geometric lower bound for the cost rate in [33, theorem 2.1] and the proof that the spectral rate is positive in [24, proposition 14.9] and [30, proposition 5.5]. But these proofs used pointwise Gaussian estimates and Weyl's asymptotics for eigenvalues.

The L^2 Gaussian estimate (30) provides simpler proofs where A need not even have eigenvalues. As shown in § 5.1, it does not only apply to $A = \Delta$ but also e.g. to the linear Ginzburg-Landau equation on M complete or compact with Dirichlet boundary condition, real smooth potential V bounded from below and real ρ,

$$(1 + \rho i)\partial_t \phi + (-\Delta + V)\phi = 0, \quad t \geq 0. \tag{32}$$

N.b. [35] gives an upper bound of the cost rate for this equation in terms of the length of the longest generalized geodesic in M which does not intersect Ω.

5.1. **Semigroups satisfying L^2 Gaussian estimates.** When A is a nonpositive self-adjoint operator, the semigroup satisfies this stronger L^2 Gaussian estimate

$$\|\Omega e^{zA}\omega\|_{\mathcal{L}(\mathcal{H})} \leq e^{-\frac{d^2}{4}\operatorname{Re}\frac{1}{z}}, \quad \operatorname{Re} z > 0, \quad \Omega \subset M, \quad \omega \subset M, \tag{33}$$

where $d = \operatorname{dist}(\Omega, \omega)$, ω and Ω are open subsets. Following the theme of [9], this is an easy consequence of the propagation of the support with speed less than one for the (even) solution of the corresponding wave equation:

$$\Omega \cos(t\sqrt{-A})\omega = 0, \quad t \in (0, d), \quad \Omega \subset M, \quad \omega \subset M. \tag{34}$$

The key idea is to represent the semigroup in terms of the wave group

$$e^{zA} = \frac{1}{\sqrt{4\pi z}} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{4z}} \cos(s\sqrt{-A}) ds, \quad \operatorname{Re} z > 0. \tag{35}$$

Indeed, (33) results from plugging (34) in (35) and taking $1/t = \operatorname{Re} 1/z$ in

$$\frac{1}{\sqrt{4\pi t}} \int_{|s| \geq d} e^{-\frac{s^2}{4t}} ds \leq e^{-\frac{d^2}{4t}}, \quad t > 0.$$

This proof can be found e.g. in [48, chapter 6, (2.22)] for $A = \Delta$, and in [13, theorem 3.4]. N.b. the converse holds using the Paley-Wiener theorem, i.e. (33) implies (34). Cf. [13, § 3] for a deeper study of these estimates and their relationship.

The transmutation formula (35) results directly from the integral representation of functions of A via spectral measures and the Fourier transform. In this context of short time asymptotics of diffusion semigroups, it was first used by Kannai in [25]. The control transmutation method in [35] is based on analogous formulas for both the controlled solution and the input.

The L^2 Gaussian estimates (33) for real z are known as *Davies-Gaffney estimates*. Indeed, Gaffney's argument in [21] used to prove such estimates without (34) in [14] needs very little smoothness, cf. [41, § 2] and [13, theorem 3.3].

If A satisfies (33) then $(1 + i\rho)^{-1}(A + \lambda_0 I)$, with $\rho \in \mathbb{R}$ and $\lambda_0 \geq 0$ satisfies (30) with $d_0 = e^{T_0 \lambda_0}$ (with $d_0 = 1$ if $\lambda_0 \leq 0$). In particular, for a potential $V \in \mathcal{C}^\infty(M)$ such that $V(x) \geq -\lambda_0$, for all $x \in M$, $A = \Delta - V - \lambda_0$ (defined by Friedrichs extension from $\mathcal{C}_c^\infty(M)$) satisfies (34), hence (33), therefore the generator $(1 + i\rho)^{-1}(\Delta - V)$ satisfies (30) (n.b. [13, theorem 3.3] proves that A still satisfies (33) for $V \in L^1_{loc}(M)$ on a complete M). Hence theorems 5.1, 5.2 and 5.3 apply to the linear Ginzburg-Landau equation (32).

When it is not assumed that A is self-adjoint, but only that it is the generator of a cosine operator function Cos, then the transmutation formula (35) holds with $s \mapsto \operatorname{Cos}(s)$ replacing $s \mapsto \cos(s\sqrt{-A})$, cf. e.g. [2, Weierstrass formula (3.102)], [47, 8, 17]. Since a cosine operator function satisfies a growth bound $\|\operatorname{Cos}(s)\|_{\mathcal{L}(\mathcal{H})} \leq M_0 e^{Ms}$, $s \geq 0$, the finite propagation speed (34) for Cos implies the weaker L^2 Gaussian estimate (30) where t is bounded and the limit value $d = \operatorname{dist}(\Omega, \omega)$ is excluded.

E.g. theorem 5.1 still applies to the diffusion semigroup with generator,

$$A\phi = \sum_{j,k=1}^{d} \partial_{x_j}(g_{jk}\partial_{x_k}\phi) + \sum_{j=1}^{d} b_j \partial_{x_j}\phi + V\phi, \ \mathcal{D}(A) = \left\{ \phi \in H_0^1(M) \mid A\phi \in L^2(M) \right\},$$

where M is a C^2 connected bounded domain in $\mathbb{R}^d$, b_j and V are complex valued and bounded on M, $g_{jk} \in \mathcal{C}^1(M)$, the matrix $G = (g_{ij})$ is real symmetric and $0 < G \leq I$ uniformly on M. Indeed these assumptions ensure that A is a generator of a cosine operator function, cf. [2, theorem 7.2.3], and that the support propagates

with speed less than one, cf. [1, 48]. N.b. if $b_j = 0$ and V is real then A is self-adjoint, theorems 5.2 and 5.3 also hold and [7] proves that (39) does hold.

5.2. **Lower bound for the cost rate.**

Theorem 5.1. *If A satisfies the Gaussian estimate* (30) *and the cost bound*

$$\|e^{TA}v\|^2 \le c_0 e^{\frac{2c}{T}} \int_0^T \|\Omega e^{tA}v\|^2 dt, \quad v \in \mathcal{D}(A), \quad T \in (0, T_0), \tag{36}$$

then $c \ge d_\Omega^2/4$ where d_Ω is the distance defined in (31).

Proof. Given $d < d_\Omega$, by the definition (31), there is an open ball $\omega \subset M$ such that $\mathrm{dist}(\Omega, \omega) > d$. Taking $v = \omega$ in (36), applying (30) and taking the limit $T \to 0$ yields a contradiction for $c \le d^2/4$:

$$0 \ne \|\omega\|^2 \leftarrow \|e^{TA}v\|^2 \le c_0 e^{\frac{2c}{T}} \int_0^T \|\Omega e^{tA}v\|^2 dt \le T c_0 d_0^2 \|\omega\|^2 e^{\frac{2(c-d^2/4)}{T}} \to 0.$$

Hence $c > d^2/4$. Taking the limit $d \to d_\Omega$ completes the proof. $\qquad\square$

In the remaining part of § 5, we need spectral subspaces to state our results. Therefore we assume that A is the generator of a normal semigroup (cf. § 3.6) and

$$\mathcal{H}_\lambda \text{ is the spectral subspace of } \mathcal{H} \text{ relative to } \left\{ z \in \sigma(-A) \,|\, \mathrm{Re}\, z > \lambda^2 \right\}. \tag{37}$$

N.b. λ was an "eigenvalue" in $\mathcal{E}_\lambda$, now it is a "square-root of an eigenvalue" in $\mathcal{H}_\lambda$.

The next theorem makes a weaker assumption than the previous one but draws the same conclusion when taking the limit $\varepsilon \to 0$.

Theorem 5.2. *If A is the generator of a normal semigroup, satisfies the Gaussian estimate* (30) *and the cost bound for some $\varepsilon \in (0, 4/d_\Omega^2]$:*

$$\|e^{TA}v\|^2 \le c_0 e^{\frac{2c}{T}} \int_0^T \|\Omega e^{tA}v\|^2 dt, \quad v \in \mathcal{H}_{\frac{1}{\varepsilon T}}, \quad T \in (0, T_0), \tag{38}$$

then $(1+\varepsilon)c \ge d_\Omega^2/4$ where d_Ω is the distance defined in (31).

Proof. Let d and ω be as in the proof of theorem 5.1. We consider $\phi = e^{\varepsilon TA}\omega$ and its projection v on $\mathcal{H}_{\frac{1}{\varepsilon T}}$, i.e. $v = \mathbf{1}_{-A<(\varepsilon T)^{-2}}\phi$ and $\phi - v = \mathbf{1}_{A \le -(\varepsilon T)^{-2}} e^{\varepsilon TA}\omega$. The spectral representation of functions of A and $\varepsilon \le 4/d_\Omega^2$ yield

$$\|e^{tA}(\phi - v)\| \le e^{-\frac{t+\varepsilon T}{(\varepsilon T)^2}}\|\omega\| \le e^{-\frac{1}{(\varepsilon T)}}\|\omega\| \le e^{-d_\Omega^2/(4(1+\varepsilon)T)}\|\omega\|.$$

Plugging this and (30) for ϕ in (38) yields $(1+\varepsilon)c \ge d^2/4$ as in the proof of theorem 5.1. Taking the limit $d \to d_\Omega$ completes the proof. $\qquad\square$

Both theorems 5.1 and 5.2 were proved in [33, theorem 2.1] in the setting of § 2.4.

5.3. **Lower bound for the spectral rate.**

Theorem 5.3. *If A is the generator of a normal semigroup, satisfies the Gaussian estimate* (30) *and the spectral observability estimate on $\mathcal{H}_\lambda$ defined in* (37)

$$\|v\| \le a_0 e^{a\lambda}\|\Omega v\|, \quad \lambda > 0, \quad v \in \mathcal{H}_\lambda, \tag{39}$$

then $a \ge d_\Omega/2$ where d_Ω is the distance defined in (31).

Proof. Let d and ω be as in the proof of theorem 5.1. For any $\lambda > 0$ and $t \leq T_0$, we consider $\phi = e^{tA}\omega$ and its projection v on $\mathcal{H}_\lambda$, i.e. $v = \mathbf{1}_{-A < \lambda^2}\,\phi$ and $\phi - v = \mathbf{1}_{A \leq -\lambda^2}\,e^{tA}\omega$. The spectral representation of functions of A and (30) yield

$$\|\Omega(\phi - v)\| \leq \|\phi - v\| \leq e^{-t\lambda^2}\|\omega\| \quad \text{and} \quad \|\Omega\phi\| \leq e^{-\frac{d^2}{4t}}\|\omega\|.$$

We choose $t = d/(2\lambda)$ to make the right-hand sides of these inequalities equal. Plugging them in (39) and taking the limit $\lambda \to \infty$ yield a contradiction for $a < d/2$:

$$0 \neq \|\omega\| \leftarrow \|v\| \leq a_0 e^{a\lambda}\|\Omega v\| \leq a_0 e^{a\lambda}(\|\Omega\phi\| + \|\Omega(\phi - v)\|) \leq 2a_0\|\omega\|e^{(a - d/2)\lambda} \to 0.$$

Hence $a \geq d/2$. Taking the limit $d \to d_\Omega$ completes the proof. $\qquad\square$

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