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# A DIRECT LEBEAU-ROBBIANO STRATEGY FOR THE OBSERVABILITY OF HEAT-LIKE SEMIGROUPS

LUC MILLER

*Dedicated to David L. Russell on the occasion of his 70th birthday*

**ABSTRACT.** This paper generalizes and simplifies abstract results of Miller and Seidman on the cost of fast control/observation. It deduces final-observability of an evolution semigroup from some stationary observability property on some spaces associated to the generator, e.g. spectral subspaces when the semigroup has an integral representation via spectral measures. Contrary to the original Lebeau-Robbiano strategy, it does not have recourse to null-controllability and it yields the optimal bound of the cost when applied to the heat equation, i.e.  $c_0 \exp(c/T)$ , or to the heat diffusion in potential wells observed from cones, i.e.  $c_0 \exp(c/T^\beta)$  with optimal  $\beta$ . It also yields simple upper-bounds for  $c$ .

This paper also gives a geometric lower bound on the rate in the estimate on sums of eigenfunctions of the Laplacian on which the original Lebeau-Robbiano strategy was based. Jerison and Lebeau had proved that this rate is positive.

## 1. INTRODUCTION

This paper concerns the so-called “Lebeau-Robbiano strategy” for the null-controllability of linear evolutions systems like the heat equation. The Lebeau-Robbiano strategy was originally devised for the heat flux on a bounded domain of  $\mathbb{R}^d$  observed from some open subset of this domain. It originally starts from the interior observability estimate for sums of eigenfunctions of the Dirichlet Laplacian proved by some Carleman estimates at the end of the nineties in joint papers of Lebeau with Jerison, Robbiano and Zuazua, cf. § 3.8.

In the last decade, many people have contributed applications, e.g. to nodal sets of sums of Laplacian eigenfunctions in [JL99], to coupled wave and heat equations in the same domain in [LZ98], to the heat equation in unbounded domains in [Mil05], to anomalous diffusions in [Mil06c], cf. § 4.1, to structural damping, e.g. the plate equation with square root damping, in [Mil06a, AC07], cf. § 4.2, to thermoelasticity without rotatory inertia in [BN02, Mil07, Cok07, Sei08], to the heat transmission problem in [LRR07], to diffusions in a potential well of  $\mathbb{R}^d$  in [Mil08], cf. § 4.3, to the heat equation discretized in time or space in [Zhe08, BHLR09]. We also refer to the survey [LRL09].

The Lebeau-Robbiano strategy was already stated in abstract settings with bounds on the cost of fast control of the form  $c_0 \exp(c/T^\beta)$  in [Mil06c, Sei08]. Our goal is to retain the most general features of both papers while simplifying the proof to improve the estimate of the cost.

The paper [Mil06c] concerns semigroups generated by negative self-adjoint operators, introduces some notion of observability on spectral subspaces, cf. § 3.6.

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*Date:* August 30, 2009. On november 11, 2009, added the “estimate without  $\limsup$ ” in thm.2.2, lem.2.3, lem.3.1, the first paragraph in § 3.2, § 5 and the second paragraph in § 3.8, a remark after thm.4.3, § acknowledgment, [FI96, HO07], and corrected some typos.

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It links the exponent  $\beta$  in the fast control cost estimate to some exponent in this notion, but falls just short of the optimal exponent. It combines final-observability and null-controllability as in the original setting, but does not use Weyl's eigenvalues asymptotics, not even the discreteness of the spectrum of  $A$ . The assumptions brought out in [Mil07] and introduced as an abstract framework in [Sei08] allow generators which are not self-adjoint, but do not apply to the semigroups considered in [Mil06c, Mil08]. Thus the notion of relative observability on growth spaces adopted in § 2.2 is a little more general. The paper [Sei08] achieves the breakthrough of reaching the exponent  $\beta = 1$  which is optimal for the heat equation, but it adds approximate null-controllability as another layer to the strategy.

Here, the strategy goes directly from relative observability on growth spaces to the estimate of fast final-observability cost, and reaches the optimal exponents  $\beta$  for the observation from cones of heat diffusion in potential wells  $V(x) = |x|^{2k}$ ,  $k \in \mathbb{N}^*$ , cf. § 4.3. Its sheer simplicity yields straightforward upper-bounds of the rate  $c$  in the fast control cost estimate. Since it leaves null-controllability out, it can be seen as a shortcut to the original Lebeau-Robbiano strategy.

Section 2 introduces the abstract setting, states and proves the direct Lebeau-Robbiano strategy. The abstract setting is connected to the original setting in § 3.8. Section 3 gives further background, four lemmas which may be of independent interest and some open problems. Section 4 describes the application of the main result to the P.D.E. problems considered in [Mil06c, Mil06a, Mil08]. Section 5 gives a lower bound on the rate in the estimate for sums of eigenfunctions in the original setting.

## 2. SETTING AND MAIN RESULT

**2.1. Observability cost.** We consider the abstract differential equation

$$(1) \quad \dot{\phi}(t) = A\phi(t), \quad \phi(0) = x \in \mathcal{E}, \quad t \geq 0,$$

where  $A : \mathcal{D}(A) \subset \mathcal{E} \rightarrow \mathcal{E}$  is the generator of a strongly continuous semigroup  $(e^{tA})_{t \geq 0}$  on a Hilbert space  $\mathcal{E}$ . The solution is  $\phi(t) = e^{tA}x$ . Although we may think of  $A$  as a nonpositive self-adjoint operator with an orthonormal basis of eigenfunctions for example, cf. § 3.6, our setting has applications where  $A$  has no eigenvalues (e.g. in § 4.1 when  $M = \mathbb{R}^d$ ) or  $A$  is not a self-adjoint operator bounded from above (e.g.  $A$  does not even generate an analytic semigroup in § 4.2 for  $\gamma < 1/2$ , cf. [CT90, HO07]).

We also consider an observation operator  $C \in \mathcal{L}(\mathcal{D}(A), \mathcal{F})$  admissible for this semigroup, cf. § 3.1, i.e.  $C$  is a continuous operator from  $\mathcal{D}(A)$  with the graph norm to another Hilbert space  $\mathcal{F}$  and satisfies (norms in  $\mathcal{E}$  and  $\mathcal{F}$  are both denoted  $\|\cdot\|$ )

$$(2) \quad \int_0^T \|Ce^{tA}x\|^2 dt \leq \text{Adm}_T \|x\|^2, \quad x \in \mathcal{D}(A), \quad T > 0.$$

N.b. the admissibility constant  $T \mapsto \text{Adm}_T > 0$  is nondecreasing. We may think of  $C$  as a bounded operator from  $\mathcal{E}$  to itself for example.

We say that  $(A, C, T)$  is *observable at cost*  $\kappa_T > 0$  if

$$(3) \quad \|e^{TA}x\|^2 \leq \kappa_T \int_0^T \|Ce^{tA}x\|^2 dt, \quad x \in \mathcal{D}(A).$$

N.b.  $\sqrt{\kappa_T}$  does not grow more than the semigroup as  $T \rightarrow +\infty$ , e.g.  $T \mapsto \kappa_T$  is nonincreasing when  $A$  is nonpositive self-adjoint, cf. § 3.2. This final-observability of (1) through  $C$  in time  $T > 0$  is equivalent to a controllability property for which  $\kappa_T$  is the ratio of the size of the input annihilating the disturbance to the size of

this disturbance, cf. § 3.2. We are interested in the asymptotic growth of  $\kappa_T$  as  $T \rightarrow 0$  and think of  $\kappa_T$  as the cost of fast control.

The crucial lemma to bound this cost here is (cf. a continuous version in § 3.3)

**Lemma 2.1.** *If the approximate observability estimate (§ 3.5 justifies this name)*

$$(4) \quad f(t)\|e^{tA}x\|^2 - f(qt)\|x\|^2 \leq \int_0^t \|Ce^{\tau A}x\|^2 d\tau, \quad x \in \mathcal{D}(A), \quad t \in (0, T'],$$

holds with  $f(t) \rightarrow 0$  as  $t \rightarrow 0^+$ ,  $q \in (0, 1)$  and  $T' > 0$ , then  $\kappa_T \leq 1/f((1-q)T)$  for  $T \in (0, T']$ , i.e. the fast control cost does not grow more than the inverse of  $f$ .

*Proof.* Let  $T \leq T'$ . Let  $T_0 = T$ ,  $T_{k+1} = T_k - \tau_k$ ,  $\tau_k = q^k(1-q)T$ ,  $k \in \mathbb{N}$ . The series  $\sum \tau_k = T$  defines a disjoint partition  $\cup (T_{k+1}, T_k] = (0, T]$ . Applying (4) to  $x = e^{T_{k+1}A}y$  and  $t = \tau_k$  yields

$$f(\tau_k)\|e^{T_k A}y\|^2 - f(\tau_{k+1})\|e^{T_{k+1}A}y\|^2 \leq \int_{T_{k+1}}^{T_k} \|Ce^{tA}y\|^2 dt, \quad y \in \mathcal{D}(A), \quad k \in \mathbb{N}.$$

Adding these inequalities yields

$$f(\tau_0)\|e^{T A}y\|^2 - f(\tau_k)\|e^{T_k A}y\|^2 \leq \int_{T_k}^T \|Ce^{tA}y\|^2 dt, \quad y \in \mathcal{D}(A), \quad k \in \mathbb{N}.$$

Taking the limit  $k \rightarrow \infty$  completes the proof since  $f(\tau_k) \rightarrow 0$  and the continuous function  $t \mapsto \|e^{tA}y\|^2$  is bounded on the compact set  $[0, T]$ .  $\square$

**2.2. Relative observability on growth subspaces.** We assume that there is a nondecreasing family of semigroup invariant spaces  $\mathcal{E}_\lambda \subset \mathcal{E}$ ,  $\lambda > 0$  (i.e.  $e^{tA}\mathcal{E}_\lambda \subset \mathcal{E}_\lambda \subset \mathcal{E}_{\lambda'}$ ,  $t > 0$ ,  $\lambda' > \lambda$ ) satisfying the growth condition (namely some decay)

$$(5) \quad \|e^{tA}x\| \leq m_0 e^{-\lambda t} \|x\|, \quad x \perp \mathcal{E}_\lambda, \quad t \in (0, T_0), \quad \lambda > 0.$$

We call them *growth spaces*. We think of them as spectral subspaces of  $A$ , i.e.  $\sigma(A|_{\mathcal{E}_\lambda^\perp}) \subset \{z \in \sigma(A) \mid \operatorname{Re} z \leq -\lambda\}$ , and we think of (5) as a spectrally determined growth property, cf. § 3.6.

We also assume that there is an admissible observation operator  $C_0$  satisfying the bound relative to  $C$  on growth spaces with exponent  $\alpha \in (0, 1)$  and rate  $a > 0$

$$(6) \quad \|C_0 x\|^2 \leq a_0 e^{2a\lambda^\alpha} \|C x\|^2, \quad x \in \mathcal{E}_\lambda, \quad \lambda > 0.$$

We call  $C_0$  a *reference operator* and the property (6) of  $C$ : *observability on growth subspaces relatively to  $C_0$* . We think of  $C_0$  as a simple operator with a good estimate of fast control like the identity operator, cf. § 3.7.

**2.3. Main result.** When the reference operator  $C_0$  satisfies the observability cost estimate with exponent  $\beta > 0$  and rate  $b > 0$

$$(7) \quad \|e^{T A}x\|^2 \leq b_0 e^{\frac{2b}{T^\beta}} \int_0^T \|C_0 e^{tA}x\|^2 dt, \quad x \in \mathcal{D}(A), \quad T \in (0, T_0),$$

we claim that  $C$  satisfies an observability cost estimate with exponent<sup>1</sup>  $\max\left\{\beta, \frac{\alpha}{1-\alpha}\right\}$ :

**Theorem 2.2.** *Under the assumptions (5), (6) and (7) with  $\beta = \frac{\alpha}{1-\alpha}$ , the system  $(A, C, T)$  is observable at a cost  $\kappa_T$  such that  $2c = \limsup_{T \rightarrow 0} T^\beta \ln \kappa_T < \infty$ .*

*More precisely, this rate  $c$  is bounded in terms of an implicitly defined  $s > 0$ :*

$$(8) \quad c \leq c_* := \left(\frac{b}{a}(\beta+1)\right)^{\frac{\beta+1}{\beta}} \frac{\beta^\beta}{s^{\frac{(\beta+1)^2}{\beta}}}, \quad \text{with } s(s+\beta+1)^\beta = (\beta+1)\beta^{\frac{\beta^2}{\beta+1}} \frac{b^{\frac{1}{\beta+1}}}{a}.$$

<sup>1</sup>N.b.  $\alpha$  or  $\beta$  may be increased so that the match  $\beta = \frac{\alpha}{1-\alpha}$  is achieved as in the statement of theorem 2.2.

Moreover, if the admissibility constant in (2) satisfies  $\text{Adm}_T \rightarrow 0$  as  $T \rightarrow 0$ , then there exists  $T' > 0$  such that  $\kappa_T \leq 4a_0b_0 \exp(\frac{2c_*}{T^\beta})$  for  $T \in (0, T']$ .

Since  $c > 0$  for some “meaningful” example, cf. § 4.3, there are no lower  $\beta$  such that  $\limsup_{T \rightarrow 0} T^\beta \ln \kappa_T < \infty$  under these assumptions.

N.b. the condition  $\text{Adm}_T \rightarrow 0$  as  $T \rightarrow 0$  for the better bound in theorem 2.2 holds for example when  $C$  is bounded from  $\mathcal{E}$  to  $\mathcal{F}$ , cf. § 3.1.

**Corollary 1.** *Under the same assumptions as theorem 2.2, the cost rate  $c$  is bounded more explicitly in the following cases:*

- (i) If (6) holds with  $\alpha = \frac{1}{2}$  (i.e.  $\beta = 1$ ) then  $c_* = 4b^2 \left( \sqrt{a + 2\sqrt{b}} - \sqrt{a} \right)^{-4}$ .
- (ii) If (7) holds for any  $b$  then  $c \leq a^{\beta+1}(\beta+1)^{\beta(\beta+1)}\beta^{-\beta^2}$ .
- (iii) If (6) holds for any  $a$  then  $c \leq b$ .
- (iv) If  $b > a^{\beta+1}(\beta+1)^{\beta(\beta+1)}\beta^{-\beta^2}$  then  $c_* \leq \left(\frac{b}{a}\right)^{\frac{1}{\alpha}} \left( \frac{b^{(1-\alpha)^2}}{a^{1-\alpha}} - \frac{(\beta+1)^\alpha}{\beta\alpha^2} \right)^{-1}$ .

N.b. (ii) applies to the identity operator as reference operator  $C_0$ , cf. § 3.7.

Theorem 2.2 for  $\alpha = \frac{1}{2}$  (i.e.  $\beta = 1$ ) is due to Seidman with some less precise and less simple cost rate bound than (8); e.g. [Sei08, theorem 2.4] proves<sup>2</sup>  $c \leq 8a^2$  in case (ii) with  $\beta = 1$  instead of  $c \leq 4a^2$ , and does not state (i).

With the exponential bound  $b_0e^{bT^{-\beta}}$  in (7) replaced by a polynomial bound  $\frac{b_0}{T^b}$  (so that (ii) applies), the papers [Mil06c, Mil07] only prove  $\limsup_{T \rightarrow 0} T^\beta \ln \kappa_T < \infty$  for  $\beta > \frac{\alpha}{1-\alpha}$ , hence fall short of the optimal exponent.

**2.4. Proof of the main result.** We shall use lemma 2.1 in the following form.

**Lemma 2.3.** *If the approximate observability estimate*

$$(9) \quad f(T) \|e^{TA}x\|^2 - g(T) \|x\|^2 \leq \int_0^T \|Ce^{tA}x\|^2 dt, \quad x \in \mathcal{D}(A), \quad T \in (0, T_0],$$

holds with  $f(T) = f_0 \exp(-2/(d_2T)^\beta)$  and  $g(T) = g_0 \exp(-2/(d_1T)^\beta)$ , where  $f_0, g_0, d_1 < d_2$  are positive, then for all  $d \in (0, d_2 - d_1)$  there exists  $T' \in (0, T_0]$  such that  $\kappa_T \leq f_0^{-1} \exp(2/(dT)^\beta)$  for  $T \in (0, T']$ .

Moreover, if  $g_0 \leq f_0$  then we may take  $d = d_2 - d_1$  and  $T' = T_0$ .

*Proof.* To apply lemma 2.1, we compute the least  $q$  such that  $g(T) \leq f(qT)$  for all  $T \in (0, T']$ . We find  $q = \frac{d_1}{d_2} h(T')$  with  $h(T') = (1 + \inf_{t \in (0, T')} t^\beta d_1^\beta \ln \frac{f_0}{g_0})^{-\frac{1}{\beta}}$  where the parenthesis is 1 when  $g_0 \leq f_0$  and positive when  $T'$  is small enough. Now  $\kappa_T \leq \frac{1}{f((1-q)T)} = \frac{1}{f_0} \exp(\frac{1}{(d_3T)^\beta})$  with  $d_3 = d_2 - d_1 h(T') \rightarrow d_2 - d_1$  as  $T' \rightarrow 0$ .  $\square$

We proceed with the proof of theorem 2.2. Plugging (6) in (7) yields

$$(10) \quad \|e^{\tau A}\phi\|^2 \leq a_0b_0 e^{2a\lambda^\alpha + \frac{2b}{\tau^\beta}} \int_0^\tau \|Ce^{tA}\phi\|^2 dt, \quad \phi \in \mathcal{E}_\lambda, \quad \tau \in (0, T_0).$$

Given  $x \in \mathcal{D}(A)$  and  $T \in (0, T_0)$ , we introduce an observation time  $\tau = \varepsilon T$  with  $\varepsilon \in (0, 1)$ , a spectral threshold  $\lambda$  defined by  $(r\lambda)^\alpha = \frac{1}{r^\beta}$  with  $r > 0$ , the orthogonal projection of  $x$  on  $\mathcal{E}_\lambda$  denoted  $x_\lambda$ , and  $x_\lambda^\perp = x - x_\lambda$ .

Since  $\mathcal{E}_\lambda$  is semigroup invariant, we may apply (10) to  $\phi = e^{(1-\varepsilon)TA}x_\lambda$  and obtain:

$$(11) \quad \|e^{TA}x_\lambda\|^2 \leq \frac{1}{4f(T)} \int_{(1-\varepsilon)T}^T \|Ce^{tA}x_\lambda\|^2 dt, \quad f(T) = \frac{1}{4a_0b_0} \exp\left(-\frac{2}{T^\beta} \frac{a + br^\alpha}{r^\alpha \varepsilon^\beta}\right).$$

<sup>2</sup>Indeed [Sei08, theorem 2.4] states  $c \leq 2a^2$ , but correcting the factor 2 into  $\frac{1}{2}$  in the definition of  $\underline{d}(s)$  in [Sei08, theorem 2.1] only proves  $c \leq 8a^2$ .

We put the factor 4 in the definition of  $f$  because we shall use twice the inequality:

$$(12) \quad \|y + z\|^2 \leq 2(\|y\|^2 + \|z\|^2), \quad y \in \mathcal{E}, \quad z \in \mathcal{E}.$$

Using (12) then (2) yields

$$(13) \quad \int_{(1-\varepsilon)T}^T \|Ce^{tA}x_\lambda\|^2 dt \leq 2 \int_{(1-\varepsilon)T}^T \|Ce^{tA}x\|^2 dt + 2 \text{Adm}_{\varepsilon T} \|e^{(1-\varepsilon)TA}x_\lambda^\perp\|^2.$$

Using (12) again, then (11) and finally (13) yields

$$f(T) \|e^{TA}x\|^2 \leq \int_{(1-\varepsilon)T}^T \|Ce^{tA}x\|^2 dt + \text{Adm}_{\varepsilon T} \|e^{(1-\varepsilon)TA}x_\lambda^\perp\|^2 + 2f(T) \|e^{TA}x_\lambda^\perp\|^2.$$

Applying (5) to  $x_\lambda^\perp$  yields

$$f(T) \|e^{TA}x\|^2 - m_0^2 \left( \text{Adm}_{\varepsilon T} e^{-2(1-\varepsilon)T\lambda} + 2f(T) e^{-2T\lambda} \right) \|x_\lambda^\perp\|^2 \leq \int_0^T \|Ce^{tA}x\|^2 dt.$$

Since  $\|x_\lambda^\perp\| \leq \|x\|$ ,  $\text{Adm}_{\varepsilon T} \leq \text{Adm}_{T_0}$  and  $f(T) \leq f(T_0)$ , we deduce the approximate observability estimate

$$(14) \quad f(T) \|e^{TA}x\|^2 - m_0^2 (\text{Adm}_{T_0} + 2f(T_0)) e^{-2(1-\varepsilon)T\lambda} \|x\|^2 \leq \int_0^T \|Ce^{tA}x\|^2 dt.$$

Recalling that here  $\beta = \frac{\alpha}{1-\alpha}$  so that  $T\lambda = 1/(r\varepsilon^{\beta/\alpha}T^\beta)$ , this proves (9) with

$$f_0 = \frac{1}{4a_0b_0}, g_0 = m_0^2 (\text{Adm}_{T_0} + 2f(T_0)), d_2 = \frac{\varepsilon}{(ar^{-\alpha} + b)^{\frac{1}{\beta}}} \text{ and } d_1 = \varepsilon^{\frac{1}{\alpha}} \left( \frac{r}{1-\varepsilon} \right)^{\frac{1}{\beta}}.$$

As  $T_0 \rightarrow 0$ , if  $\text{Adm}_{T_0} \rightarrow 0$  then  $g_0 \rightarrow 0$ , hence (9) still holds with  $g_0 = f_0$  with a smaller  $T_0$ . Therefore lemma 2.3 proves the theorem for  $c_* = (d_2 - d_1)^{-\beta}$ , for any  $\varepsilon \in (0, 1)$  and  $r > 0$ .

Now, introducing for convenience  $\gamma = \frac{1}{\beta}$  and  $s = \frac{\varepsilon}{1-\varepsilon}$ , we are left with maximizing with respect to  $r > 0$  and  $s > 0$ :

$$d_{a,b}(r, s) = d_2 - d_1 = \frac{\varepsilon}{(ar^{-\alpha} + b)^\gamma} - \left( \frac{r}{1-\varepsilon} \right)^\gamma \varepsilon^{\frac{1}{\alpha}} = \frac{s}{s+1} r^\gamma (h^{-\gamma}(r) - s^\gamma),$$

where  $h(r) = ar^{\frac{\gamma}{\gamma+1}} + br$ , since  $\frac{1}{\alpha} = 1 + \gamma$ ,  $\varepsilon = \frac{s}{s+1}$  and  $1 - \alpha = \frac{1}{\beta+1} = \frac{\gamma}{\gamma+1}$ . N.b.  $d_{a,b}(r, s) > 0$  for  $r$  small enough already proves  $c < \infty$ .

The optimality condition  $\nabla d_{a,b} = 0$  writes successively, abbreviating  $h = h(r)$ ,

$$\begin{cases} \gamma r^{\gamma-1} (h^{-\gamma} - s^\gamma) = \frac{r^\gamma \gamma h'}{h^{\gamma+1}}, & \begin{cases} h^{\gamma+1} (h^{-\gamma} - s^\gamma) = r h' = \frac{\gamma}{\gamma+1} h + \left(1 - \frac{\gamma}{\gamma+1}\right) br, \\ h = (\gamma s^\gamma (s+1) + s^\gamma)^{-\frac{1}{\gamma}} = \frac{1}{s} (\gamma s + \gamma + 1)^{-\frac{1}{\gamma}}. \end{cases} \end{cases}$$

Plugging the last equation ( $h$  in terms of  $s$ ) in the former yields  $r$  in terms of  $s$ :

$$\frac{br}{\gamma+1} = h^{\gamma+1} (h^{-\gamma} - s^\gamma) - \frac{\gamma}{\gamma+1} h = h^{\gamma+1} \left( \frac{h^{-\gamma}}{\gamma+1} - s^\gamma \right) = \frac{\gamma}{\gamma+1} (sh)^{\gamma+1},$$

hence  $r = \gamma b^{-1} (\gamma s + \gamma + 1)^{-\frac{\gamma+1}{\gamma}}$ . Plugging this once in  $h(r)$  in terms of  $s$  yields

$$\gamma s + \gamma + 1 = s (\gamma s + \gamma + 1)^{\frac{\gamma+1}{\gamma}} h = s \frac{\gamma}{br} h = s \gamma \left( \frac{a}{b} r^{-\frac{1}{\gamma+1}} + 1 \right).$$

Simplifying  $\gamma s$  and plugging  $r$  in terms of  $s$  again yields the equation for  $s$  in (8):

$$s^\gamma (\gamma s + \gamma + 1) = \left( \frac{\gamma+1}{a} \right)^\gamma \left( \frac{b}{\gamma} \right)^{\frac{\gamma^2}{\gamma+1}},$$

which has a unique solution since the L.H.S. increases from 0 to  $+\infty$  as  $s$  does. We still denote  $s$  this solution. The corresponding  $r = \gamma b^{-1} (\gamma s + \gamma + 1)^{-\frac{\gamma+1}{\gamma}}$  satisfies

$r^{\frac{1}{\gamma+1}} = \left(\frac{\gamma}{b}\right)^{\frac{1}{\gamma+1}} (\gamma s + \gamma + 1)^{-\frac{1}{\gamma}} = s \frac{a}{b} \frac{\gamma}{\gamma+1}$ . The second equation of the first system traducing the optimality condition  $\nabla d_{a,b} = 0$  yields:

$$d_{a,b}(r, s) = \frac{s}{s+1} r^\gamma (h^{-\gamma}(r) - s^\gamma) = \gamma s^{\gamma+1} r^\gamma = \gamma s^{(\gamma+1)^2} \left(\frac{a}{b} \frac{\gamma}{\gamma+1}\right)^{\gamma(\gamma+1)}.$$

Now  $c_* = d_{a,b}^{-\beta}(r, s)$  is (8) since  $1 + \gamma = \frac{\beta+1}{\beta}$ ,  $\frac{\gamma+1}{\gamma} = \beta + 1$  and  $\frac{(\gamma+1)^2}{\gamma} = \frac{(\beta+1)^2}{\beta}$ . Corollary 1 is deduced by the following arguments

- (i) The positive solution of the quadratic equation in (8) is  $s = \sqrt{1 + \frac{2\sqrt{b}}{a}} - 1$ .
- (ii) Eliminating  $b$  from (8) yields  $c_* = (a/(\beta+1))^{\beta+1} \beta^{-\beta^2} (s + \beta + 1)^{(\beta+1)^2}$ , and the implicit equation yields  $s \rightarrow 0$  as  $b \rightarrow 0$ .
- (iii) Eliminating  $a$  from (8) yields  $c_* = b(s + \beta + 1)^{\beta+1} / s^{\beta+1}$ , and the implicit equation yields  $s \rightarrow \infty$  as  $a \rightarrow 0$ .
- (iv) The easiest lower bound for  $s$  is  $s + \beta + 1 \geq (\beta + 1)^{\frac{1}{\beta}} \beta^{\frac{\beta}{\beta+1}} \frac{b^{\frac{1}{(\beta+1)^2}}}{a^{\frac{1}{\beta+1}}}$ , obtained by plugging  $s + \beta + 1 \geq s$  in its implicit equation.

### 3. COMMENTS

**3.1. Admissibility.** Any  $C \in \mathcal{L}(\mathcal{E}, \mathcal{F})$  satisfies the admissibility condition (2) with  $\text{Adm}_T = T\|C\|^2$ . The more general setting in § 2.1 is canonical (cf. [Wei89]) and required in many P.D.E. problems, e.g. when the heat flux is observed on the boundary rather than an open subset of the domain. Although it should be sufficient for any P.D.E. problems, it might be useful to circumvent the admissibility assumption:

**Lemma 3.1.** *The conclusion  $2c = \limsup_{T \rightarrow 0} T^\beta \ln \kappa_T \leq 2c_*$  of theorem 2.2 is still valid if we replace the assumption that  $C \in \mathcal{L}(\mathcal{D}(A), \mathcal{F})$  satisfies the admissibility condition (2) by the following time smoothing effect assumption:*

$$(15) \quad \forall x \in \mathcal{E}, \forall t > 0, e^{tA}x \in \mathcal{D}(A), \quad \text{and} \quad \limsup_{t \rightarrow 0} t^\beta \ln \|Ae^{tA}\| = 0.$$

*Proof.* In the proof of theorem 2.2, the admissibility condition (2) is only used once, for  $x = x_\lambda^\perp$  in the growth space  $\mathcal{E}_\lambda$ , in this manner:

$$(16) \quad \int_{(1-\varepsilon)T}^T \|Ce^{tA}x\|^2 dt \leq \text{Adm}_{T_0} e^{-2(1-\varepsilon)T\lambda} \|x\|^2, \quad x \in \mathcal{E}_\lambda, \quad T \in (0, T_0),$$

and this only affects the definition of the function  $g(T) = g_0 \exp(-2/(d_1 T)^\beta)$  used in (9). Recall that  $\varepsilon \in (0, 1)$  and  $r > 0$  have been fixed (in order to maximize  $d_{a,b}$ ). We shall prove, for any  $\delta \in (0, 1)$  small enough, any  $g_1 > 0$  and some smaller  $T_0$ ,

$$(17) \quad \int_{(1-\varepsilon)T}^T \|Ce^{\tau A}x\|^2 d\tau \leq g_1 e^{-2(1-(1+\delta)\varepsilon)T\lambda} \|x\|^2, \quad x \in \mathcal{E}_\lambda, \quad T \in (0, T_0).$$

Indeed replacing (16) by (17), (9) still holds with  $g_0$  and  $d_1$  replaced by  $g_1$  and  $d_{1,\delta} = \varepsilon^{\frac{1}{\alpha}} \left(\frac{r}{1-(1+\delta)\varepsilon}\right)^{\frac{1}{\beta}}$ . Since  $d_{1,\delta} \rightarrow d_1$  as  $\delta \rightarrow 0$  this will not affect the range of  $d$  obtained by applying lemma 2.3, nor the conclusion of theorem 2.2.

With the graph norm on  $\mathcal{D}(A)$ ,  $C \in \mathcal{L}(\mathcal{D}(A), \mathcal{F})$  means  $\|x\| \leq \|C\|(\|x\| + \|Ax\|)$ . We only need to prove (17) with  $C$  replaced by  $A$  since the proof of (17) with  $C$  replaced by the identity is the same, only shorter. We use the small parameter  $\delta \in (0, 1)$  to decompose the lower integration time in (17) in this geometric way:

$(1 - \varepsilon)T = (1 - (1 + \delta)\varepsilon)T + (1 - \delta)\delta\varepsilon T + \delta^2\varepsilon T = \tau_1 + \tau_2 + \tau_3$ . According to (15),  $\tau^\beta \ln \|Ae^{\tau A}\| \leq \delta^{2\beta+2}$ ,  $\tau \in (0, T_0)$ , for a smaller  $T_0$ . This with  $\tau = \tau_3$  and (5) yield

$$\|Ae^{tA}x\| \leq e^{\frac{\delta^{2\beta+2}}{\tau_3^\beta}} \|e^{(t-\tau_3)A}x\| \leq e^{\frac{\delta^2}{(\varepsilon T)^\beta}} m e^{-(t-\tau_3)\lambda} \|x\| \leq m e^{\frac{\delta^2}{(\varepsilon T)^\beta}} e^{-(\tau_1+\tau_2)\lambda} \|x\|,$$

for all  $t \in ((1 - \varepsilon)T, T_0)$ . Recalling  $\tau_2\lambda = \frac{(1-\delta)\delta\varepsilon}{r\varepsilon^{\beta/\alpha}T^\beta}$  and bounding the length of the integration interval by  $T_0$ , the proof of (17) with  $C$  replaced by  $A$  now reduces to

$$T_0 m^2 e^{\frac{2\delta^2}{(\varepsilon T)^\beta}} e^{-2\tau_2\lambda} = T_0 m^2 e^{\frac{-2\delta c_\delta}{T^\beta}} \leq g_1, \quad T \in (0, T_0),$$

where  $c_\delta \rightarrow \varepsilon/(r\varepsilon^{\beta/\alpha}) > 0$  as  $\delta \rightarrow 0$ . This does hold for  $T_0 \leq g_1/m^2$  and any  $\delta$  small enough for  $c_\delta$  to be positive.  $\square$

The idea of dispensing with the admissibility assumption is due to Marius Tucsnak and Gerald Tenenbaum in the case where  $A$  is a nonpositive self-adjoint operator with an orthonormal basis of eigenfunctions. Indeed, that  $A$  is nonpositive self-adjoint implies that  $A$  generates a bounded analytic semigroup, which is equivalent to the usual time smoothing effect,  $\sup_{t>0} \|tAe^{tA}\| < \infty$ , which implies the weaker effect (15) assumed in lemma 3.1. N.b. although  $A$  in § 4.2 for  $\gamma < 1/2$  does not generate an analytic semigroup, it is proved in [HO07, theorem 4.2] that  $\sup_{t>0} \|t^{\frac{1}{2\gamma}} Ae^{tA}\| < \infty$ , which also implies (15).

**3.2. Controllability cost.** From the definition of  $\kappa_T$  in (3), we have, for  $T' < T$ ,  $\kappa_T \leq \|e^{(T-T')A}\|^2 \kappa_{T'}$ . This justifies our claim in § 2.1 that  $\sqrt{\kappa_T}$  does not grow more than the semigroup as  $T \rightarrow +\infty$ . Moreover, if  $\kappa_t \leq g(t)$ ,  $t \in (0, T']$ ,  $g$  nonincreasing, then  $\kappa_t \leq Mg(t)$ ,  $t \in (0, T]$ , with  $M = \sup_{s \in (0, T-T')} \|e^{sA}\|^2 < \infty$ . This justifies that we restrict to some bounded intervals  $(0, T']$  in the statements of our results.

The dual problem to the final-observability of (1) is the null-controllability of

$$(18) \quad \dot{f}(t) = A^*f(t) + Bu(t), \quad f(0) = f_0 \in \mathcal{E}, \quad t \geq 0,$$

with input  $u \in L^2([0, T], \mathcal{F})$  and control operator  $B = C^* \in \mathcal{L}(\mathcal{F}, \mathcal{D}(A^*))$  ( $A^*$  denotes the adjoint of  $A$  and  $\mathcal{D}(A^*)'$  denotes the dual space of  $\mathcal{D}(A^*)$  in  $\mathcal{E}$ ). Since  $C$  satisfies the admissibility condition (2),  $B$  satisfies  $\|\int_0^T e^{tA^*} Bu(t) dt\|^2 \leq K_T \int_0^T \|u(t)\|^2 dt$ , and the solution of (18) is  $f(T) = e^{TA^*} f_0 + \int_0^T e^{(T-t)A^*} Bu(t) dt$ . More precisely, if  $(A, C, T)$  is observable at cost  $\kappa_T$  then, for all  $f_0$ , there is a  $u$  such that  $f(T) = 0$  and  $\int_0^T \|u(t)\|^2 dt \leq \kappa_T \|f_0\|^2$  (cf. [DR77]).

The study of the cost of fast controls was initiated by Seidman in [Sei84] with a result on the heat equation obtained by Russell's method in [Rus73]. We refer to the surveys [Sei05, Mil06b] and the more recent paper [TT07]. An application to reachability is given in § 3.4.

**3.3. Integrated observability estimate.** Lemma 2.1 can be seen as the discrete version of the following lemma which has been used with  $f(t) = \exp(-c/t)$  when proving observability by some parabolic global Carleman estimates (cf. e.g. [FI96, FCZ00]).

**Lemma 3.2.** *In the setting of § 2.1, if the integrated observability estimate*

$$(19) \quad \int_0^T f(t) \|e^{tA}x\|^2 dt \leq \int_0^T \|Ce^{tA}x\|^2 dt, \quad x \in \mathcal{D}(A), \quad T \in (0, T_0),$$

holds with  $T_0 > 0$  and  $f$  an increasing function such that  $f(t) \rightarrow 0$  as  $t \rightarrow 0^+$ , then  $\kappa_T \leq M_T/f(T)$  for  $T \in (0, T_0)$  with  $M_T = \sup_{t \in [0, T]} \|e^{tA}\|^2 \leq M_{T_0} < \infty$ , i.e. the fast control cost does not grow more than the inverse of  $f$ . Conversely, if (3) holds for  $T \in (0, T_0)$  then (19) holds for  $T \in (0, T_0)$  with  $f(T) = 1/(T_0\kappa_T)$ .



*Proof.* The implication results from  $\|e^{TA}x\|^2 = \|e^{(T-t)A}e^{tA}x\|^2 \leq M_T \|e^{tA}x\|^2$  and  $f(t) \leq f(T)$  for  $t \in (0, T)$ . The converse results from integrating (3) on  $(0, T)$  and  $\int_0^T \int_0^\tau \|Ce^{tA}x\|^2 dt d\tau \leq \int_0^T \int_0^T \|Ce^{tA}x\|^2 dt d\tau = T \int_0^T \|Ce^{tA}x\|^2 dt$ .  $\square$

**3.4. Reachability.** As the input  $u$  varies, the final state  $f(T)$  of (18) spans the set of states which are reachable from  $f_0$  in time  $T$ , denoted  $\mathcal{R}(T, f_0)$ . Assuming  $(A, C, T)$  is observable for all  $T > 0$ , the usual duality in § 3.2 implies that this reachability set  $\mathcal{R} = \mathcal{R}(T, f_0)$  does not depend on  $T$  and  $f_0$  (by an argument due to Seidman in [Sei79], cf. [Mil06b, footnote 7]) and satisfies  $e^{tA}(\mathcal{E}) \subset \mathcal{R}$ ,  $t > 0$ .

The following lemma provides further information on the reachability set when a cost estimate as in theorem 2.2 is available.

**Lemma 3.3.** *In the setting of § 2.1, assume  $A$  is self-adjoint and  $\sigma(A) \subset (-\infty, \lambda_1]$ , and consider the fractional powers  $A_\beta = -(-A + \lambda_1)^\beta$ ,  $\beta > 0$ .*

*For all exponents  $\beta > 0$ ,  $\alpha = \frac{\beta}{\beta+1}$ , and rates  $b > 0$ ,  $c > b(\beta+1)$ ,  $a > (b\beta)^{\frac{1}{\beta+1}}/\alpha$ , for all  $T_0 > 0$ , there exists  $c_0 > 0$  such that*

$$(20) \quad \|e^{A_\alpha}x\|^2 \leq c_0 e^{\frac{2c}{T^\beta}} \int_0^T e^{-\frac{2b}{t^\beta}} \|e^{tA}x\|^2 dt, \quad x \in \mathcal{D}(A), \quad T \in (0, T_0).$$

*If  $(A, C, T)$  is observable at a cost  $\kappa_T$  such that  $2b_0 = \limsup_{T \rightarrow 0} T^\beta \ln \kappa_T < \infty$ , then the reachability set satisfies  $e^{A_\alpha}(\mathcal{E}) \subset \mathcal{R}$  for  $\alpha = \frac{\beta}{\beta+1}$  and  $a > (b_0\beta)^{\frac{1}{\beta+1}}/\alpha$ .*

*Proof.* We first deduce the reachability statement from the previous one. For any  $b > b_0$ , (3) holds with  $\kappa_T = \exp(2b/T^\beta)$ ,  $T \in (0, T_0)$ , for  $T_0$  small enough. The converse in lemma 3.2 proves that the integral in (20) is bounded by some multiple of the integral in (3). Plugging this in (20) yields a  $c_1 > 0$  such that

$$\|e^{A_\alpha}x\|^2 \leq c_1 e^{\frac{2c}{T^\beta}} \int_0^T \|Ce^{tA}x\|^2 dt, \quad x \in \mathcal{D}(A), \quad T \in (0, T_0).$$

The same duality argument (cf. [DR77, (3.22)]) deduces  $e^{A_\alpha}(\mathcal{E}) \subset \mathcal{R}(T, 0) = \mathcal{R}$ .

Given  $x \in \mathcal{D}(A)$  and  $T \in (0, T_0)$ , using the spectral measure  $dE_x(\lambda)$  of  $A$  for  $x$ :

$$\|e^{A_\alpha}x\|^2 = \int_{\sigma(A)} e^{-2a(\lambda_1 - \lambda)^\alpha} dE_x, \quad \int_0^T f(t) \|e^{tA}x\|^2 dt = \int_0^T \int_{\sigma(A)} f(t) e^{2t\lambda} dE_x dt.$$

Hence (20) reduces to  $\int_0^T e^{-2j_\lambda(t)} dt \geq \frac{1}{c_0} e^{-\frac{2c}{T^\beta}} e^{-2a(\lambda_1 + \lambda)^\alpha}$  for  $\lambda \geq -\lambda_1$ , and further to (by changing  $\lambda$  into  $\lambda - \lambda_1$ , with  $c_1 = c_0 \min\{1, e^{2T_0\lambda_1}\}$ ):

$$(21) \quad \int_0^T e^{-2j_\lambda(t)} dt \geq \frac{1}{c_1} e^{-\frac{2c}{T^\beta}} e^{-2a\lambda^\alpha}, \quad T \in (0, T_0), \quad \lambda \geq 0,$$

where  $j_\lambda(t) = \frac{b}{t^\beta} + t\lambda$  satisfies  $j_\lambda(t) \geq j_\lambda(t_\lambda) = \frac{t_\lambda\lambda}{\alpha}$ ,  $t_\lambda = \left(\frac{b\beta}{\lambda}\right)^{\frac{1}{\beta+1}}$ ,  $\lambda > 0$ .

On the one hand, if  $t_\lambda < T$ , then

$$\int_0^T e^{-2j_\lambda(t)} dt \geq \int_{\delta t_\lambda}^{t_\lambda} e^{-2j_\lambda(t)} dt \geq (1 - \delta) t_\lambda e^{-2j_\lambda(\delta t_\lambda)}, \quad \delta \in (0, 1),$$

with  $j_\lambda(\delta t_\lambda) = \left(\frac{1}{\beta\delta^\beta} + \delta\right) t_\lambda\lambda = a_\delta \lambda^\alpha$ ,  $a_\delta = (b\beta)^{\frac{1}{\beta+1}} \left(\frac{1}{\beta\delta^\beta} + \delta\right) \xrightarrow{\delta \rightarrow 1} \frac{(b\beta)^{\frac{1}{\beta+1}}}{\alpha}$ ,

hence (21) holds for  $c = 0$  and  $a > \frac{(b\beta)^{\frac{1}{\beta+1}}}{\alpha}$  by choosing  $\delta$  such that  $a > a_\delta$ .

On the other hand, if  $\lambda \leq \frac{b\beta}{T^{\beta+1}}$  then

$$\int_0^T e^{-2j_\lambda(t)} dt \geq \int_{\delta T}^T e^{-2j_\lambda(t)} dt \geq (1-\delta)T e^{-2j_\lambda(\delta T)}, \quad \delta \in (0, 1),$$

$$\text{with } j_\lambda(\delta T) \leq \frac{b}{(\delta T)^\beta} + (\delta T) \frac{b\beta}{T^{\beta+1}} = \frac{c_\delta}{T^\beta}, \quad c_\delta = b \left( \frac{1}{\delta^\beta} + \delta\beta \right) \xrightarrow{\delta \rightarrow 1} b(\beta+1),$$

hence (21) holds for  $a = 0$  and  $c > b(\beta+1)$  by choosing  $\delta$  such that  $c > c_\delta$ .  $\square$

Concerning the heat semigroup in § 3.8, as a corollary to the cost upper bound in § 3.2 under the geometrical optics condition, this lemma with  $\beta = 1$  proves that  $e^{-a\sqrt{-\Delta}}\phi_0$  is reachable for  $a > \sqrt{3}L\Omega$ ,  $\phi_0 \in L^2(M)$ , cf. [Mil06b, corollary 10]. In dimension one a better result is due to Fattorini and Russell, cf. [FR71, (3.19)]: if  $M$  is a segment of length  $L$  controlled from one endpoint then  $e^{-a\sqrt{-\Delta}}\phi_0$  is reachable for all  $a > L$ ,  $\phi_0 \in L^2(M)$  (this cannot be proved by the same method for  $a < L$ , cf. [FR71, (3.20)]). Whether “the optimal” rate  $a$  such that  $e^{-a\sqrt{-\Delta}}(L^2(M)) \subset \mathcal{R}$  can be expressed geometrically in the general setting of § 3.8 is an open question, e.g. is it  $\sup_{y \in M} \text{dist}(y, \bar{\Omega})$ ?

**3.5. Approximate observability.** The following lemma clarifies the connection of (4) in lemma 2.1 to approximate controllability, and therefore to [Sei08].

**Lemma 3.4.** *If  $(A, C, T)$  satisfy the approximate observability estimate*

$$(22) \quad \|e^{TA}x\|^2 \leq \kappa \int_0^T \|Ce^{tA}x\|^2 dt + \varepsilon \|x\|^2, \quad x \in \mathcal{D}(A),$$

for some positive  $\kappa$  and  $\varepsilon$ , then approximate null-controllability of (18) holds, i.e. for all  $f_0$  there exists  $u$  such that  $\|f(T)\|^2 \leq \varepsilon \|f_0\|^2$  and  $\int_0^T \|u(t)\|^2 dt \leq \kappa \|f_0\|^2$ .

*Proof.* Consider the strictly convex  $\mathcal{C}^1$  functional  $J$  defined on  $\mathcal{E}$  by density as

$$J(x) = \frac{\kappa}{2} \int_0^T \|Ce^{tA}x\|^2 dt + \frac{\varepsilon}{2} \|x\|^2 + \langle e^{TA}x, f_0 \rangle, \quad x \in \mathcal{D}(A).$$

Assumption (22) implies  $J(x) \geq \frac{1}{2} \|e^{TA}x\|^2 + \langle e^{TA}x, f_0 \rangle$ , hence  $J$  is coercive. Therefore  $J$  has a unique minimizer  $\psi_0 \in \mathcal{E}$ , i.e.  $J(\psi_0) = \inf_{x \in \mathcal{E}} J(x)$ , and

$$0 = \nabla J(\psi_0) = \kappa \int_0^T e^{tA*} BCe^{tA}\psi_0 dt + \varepsilon \psi_0 + e^{TA*} f_0.$$

This equation also says that the input  $u(t) = \kappa Ce^{tA}\psi_0$  in (18) yields the final state  $f(T) = -\varepsilon \psi_0$ . In terms of these  $u$  and  $f(T)$ ,  $\langle \nabla J(\psi_0), \psi_0 \rangle = 0$  writes

$$(23) \quad \frac{1}{\kappa} \int_0^T \|u(t)\|^2 dt + \frac{1}{\varepsilon} \|f(T)\|^2 = \kappa \int_0^T \|Ce^{tA}\psi_0\|^2 dt + \varepsilon \|\psi_0\|^2 = -\langle e^{TA}\psi_0, f_0 \rangle.$$

Plugging this in (22) yields  $\|e^{TA}\psi_0\|^2 \leq -\langle e^{TA}\psi_0, f_0 \rangle \leq \|e^{TA}\psi_0\| \|f_0\|$ . Hence  $\|e^{TA}\psi_0\| \leq \|f_0\|$ . This allows to bound (23) as

$$\frac{1}{\kappa} \int_0^T \|u(t)\|^2 dt + \frac{1}{\varepsilon} \|f(T)\|^2 \leq \|e^{TA}\psi_0\| \|f_0\| \leq \|f_0\|^2.$$

Which completes the proof of  $\|f(T)\|^2 \leq \varepsilon \|f_0\|^2$  and  $\int_0^T \|u(t)\|^2 dt \leq \kappa \|f_0\|^2$ .  $\square$

**3.6. Normal semigroups and spectral spaces.** For a normal semigroup (i.e.  $A$  is normal, e.g.  $A$  is self-adjoint), the natural growth spaces are their spectral spaces. It has a spectral decomposition  $E$  (a.k.a. projection-valued measure) which commutes with any operator which commutes with  $A$  and defines spectral projections  $E_\lambda = E(\{z \in \sigma(A) \mid \operatorname{Re} z > -\lambda\})$  and spectral spaces  $\mathcal{E}_\lambda = E_\lambda(\mathcal{E})$ . Then (5) holds if the semigroup satisfies the spectral bound equal growth bound condition (e.g. it is eventually norm-continuous, e.g.  $A$  is self-adjoint and bounded from above). E.g. in [Mil06c]  $A$  is negative self-adjoint. N.b. (5) never holds for unitary groups (i.e.  $A$  is skew-adjoint, e.g. Schrödinger or wave equations).

If there is an orthonormal basis  $\{e_n\}$  of  $\mathcal{E}$  such that  $-Ae_n = \lambda_n e_n$ , then the spectral spaces are just spanned by linear combinations of normalized eigenfunctions  $\mathcal{E}_\lambda = \operatorname{Span}\{e_n\}_{\lambda_n < \lambda}$  and (6) is an estimate on sums of eigenfunctions of  $A$ .

For  $A = -\Delta$  on  $\mathcal{E} = L^2(\mathbb{R}^d)$ , the spectral decomposition is the Fourier transform:  $f(\widehat{-\Delta})\phi(\xi) = f(|\xi|^2)\hat{\phi}(\xi)$ ,  $\phi \in L^2(\mathbb{R}^d)$ , thus  $\phi \in \mathcal{E}_\lambda$  just means  $\hat{\phi}(\xi) = 0$  for  $|\xi|^2 > \lambda$ , i.e.  $\phi$  is the restriction to the real axis of an entire function  $\tilde{\phi}$  such that  $|\tilde{\phi}(z)| \leq ce^{\sqrt{\lambda}|\operatorname{Im} z|}$  by the Paley-Wiener theorem. When  $C_0$  is the identity operator,  $C$  is the multiplication by the characteristic function of the exterior of a ball and  $\mathcal{F} = \mathcal{E}$ , [Mil05] proves (6) with exponent  $\alpha = \frac{1}{2}$  by Carleman estimates as in § 3.8. It is an open problem to obtain an explicit bound on the rate  $a$  in (6), e.g. by complex analysis.

**3.7. Reference operator.** Any  $A$  satisfies the fast control cost estimate

$$(24) \quad \|e^{TA}x\|^2 \leq \frac{M_T}{T} \int_0^T \|e^{tA}x\|^2 dt, \quad x \in \mathcal{D}(A), \quad T \in (0, T_0),$$

with  $M_T = \sup_{t \in [0, T]} \|e^{tA}\|^2 \leq M_{T_0} < \infty$ . Thus the cost estimate (7) holds for any exponent  $\beta > 0$  and rate  $b > 0$  when  $C_0$  is the identity operator.

For a system of coupled P.D.E.,  $C_0$  can be the observation of a single component as in § 4.2, e.g. the operator  $\mathcal{C}_M$  in [Mil07]: for this reference operator, (5), (6), (7) are stated in this form in [Mil07, Propositions 4, 3, 2] respectively. The assumptions (5) and (6) are called  $[H]$  in the abstract framework of [Sei08].

**3.8. Original example.** For  $A = \Delta$  the Dirichlet Laplacian on a compact smooth connected Riemannian manifold  $M$ ,  $\mathcal{E} = \mathcal{F} = L^2(M)$ ,  $\mathcal{E}_\lambda$  the spectral spaces of  $A$  (cf. § 3.6),  $C_0$  the identity operator, and  $C$  the multiplication by the characteristic function of an open subset  $\Omega \neq \emptyset$  of  $\overline{M}$ , [LZ98, theorem 3] and [JL99, theorem 14.6] prove the estimate on sums of eigenfunctions (6) with exponent  $\alpha = \frac{1}{2}$  using the semiclassical local elliptic Carleman estimates of [LR95].

The exponent  $\alpha = \frac{1}{2}$  is always sharp in this setting as proved in [JL99, proposition 14.9] (cf. also [LRL09, proposition 5.5]). The cost estimate  $\liminf_{T \rightarrow 0} T \ln \kappa_T \geq \sup_{y \in M} \operatorname{dist}(y, \overline{\Omega})^2/2$  proved in [Mil04, theorem 2.1] combined with theorem 2.2 provides another proof of this sharpness. Combined with corollary 1(ii) with  $\beta = 1$ , this cost estimate yields the following lower bound on the rate  $a$  in the estimate on sums of eigenfunctions (6):  $a \geq \sup_{y \in M} \operatorname{dist}(y, \overline{\Omega})/4$ . Theorem 5.1 in the § 5 improves this lower bound by a factor 2 using a direct proof in the spirit of [JL99, proposition 14.9].

In this general setting, the cost upper bound  $\limsup_{T \rightarrow 0} T \ln \kappa_T < \infty$  is due to Seidman (it is deduced in [Sei08] from the estimate on sums of eigenfunctions cited above, and the first such upper bound was proved in [Sei84]) and the cost lower bound  $\liminf_{T \rightarrow 0} T \ln \kappa_T \geq \sup_{y \in M} \operatorname{dist}(y, \overline{\Omega})^2/2$  is due to [Mil04] (the first lower bound was proved in dimension one in [Güi85]). In the Euclidean case, this upper bound was proved in [FCZ00] by global Carleman estimates with singular weights of the Èmanuilov type (with a less precise lower bound). Under the geometrical

optics condition on  $\Omega$ , a more precise upper bound is deduced in [Mil04] by the control transmutation method from the observability of the wave group in [BLR92]:  $\limsup_{T \rightarrow 0} T \ln \kappa_T \leq c L_\Omega^2$ , where  $L_\Omega$  is the length of the longest generalized geodesic in  $\bar{M}$  which does not intersect  $\Omega$ , and  $c$  is determined by a one-dimensional observability estimate for which  $c \leq (2\frac{36}{37})^2$ , improved into  $c \leq \frac{3}{2}$  in [TT07].

**3.9. “Converse” to the main result.** The following lemma is a very partial converse to theorem 2.2: only for sequences of eigenfunctions of  $A$  and  $C_0 = \text{id}$ .

**Lemma 3.5.** *Assume that  $(A, C)$  satisfies the observability cost estimate with exponent  $\beta > 0$  and rate  $b > 0$*

$$(25) \quad \|e^{TA}x\|^2 \leq b_0 e^{\frac{2b}{T^\beta}} \int_0^T \|Ce^{tA}x\|^2 dt, \quad x \in \mathcal{D}(A), \quad T \in (0, T_0).$$

Any sequence  $(e_n)$  in  $\mathcal{D}(A)$  such that  $-Ae_n = \lambda_n e_n$  and  $\lim \lambda_n = +\infty$ , must satisfy

$$(26) \quad \|e_n\|^2 \leq \frac{b_0}{2\lambda_n} e^{2a\lambda_n^\alpha} \|Ce_n\|^2, \quad \alpha = \frac{\beta}{\beta+1}, \quad a = \frac{\beta+1}{\beta^\alpha} b^{\frac{1}{\beta+1}}, \quad \lambda_n \text{ large enough.}$$

In particular, if the sequence satisfies for some exponent  $\alpha > 0$  and rate  $a > 0$ :

$$(27) \quad \|e_n\|^2 \geq a_0 e^{2a\lambda_n^\alpha} \|Ce_n\|^2, \quad \lambda_n \text{ large enough,}$$

then the observability cost in (3) satisfies  $\limsup_{T \rightarrow 0} T^\beta \ln \kappa_T > 0$  with  $\beta = \frac{\alpha}{1-\alpha}$ .

*Proof.* Applying (25) to  $x = e_n$  yields  $e^{-2T\lambda_n} \|e_n\|^2 \leq b_0 e^{\frac{2b}{T^\beta}} \int_0^T \|Ce_n\|^2 e^{-2t\lambda_n} dt$ ,

hence  $\|e_n\|^2 \leq \frac{b_0}{2\lambda_n} e^{2h(T)} \|Ce_n\|^2$ , with  $h(T) = \frac{b}{T^\beta} + T\lambda_n$ . Minimizing  $h$  yields

$$h(T_n) = \frac{\beta+1}{\beta^\alpha} b^{\frac{1}{\beta+1}} \lambda_n^\alpha \text{ at } T_n = \left(\frac{\beta b}{\lambda_n}\right)^{\frac{1}{\beta+1}} \text{ with } T_n < T_0 \text{ for } \lambda_n \text{ large enough.}$$

We prove the last statement of lemma 3.5 by contradiction. If the observability cost in (3) satisfies  $\limsup_{T \rightarrow 0} T^\beta \ln \kappa_T = 0$  with  $\beta = \frac{\alpha}{1-\alpha}$ , then (25) holds for any  $b > 0$  with  $T_0$  small enough, hence (26) holds for any  $a > 0$ , which refutes (27).  $\square$

## 4. APPLICATIONS

**4.1. Anomalous diffusions.** Let  $M$  be a smooth connected complete  $d$ -dimensional Riemannian manifold with metric  $g$  and boundary  $\partial M$ . When  $\partial M \neq \emptyset$ ,  $M$  denotes the interior and  $\bar{M} = M \cup \partial M$ . Let  $\Delta$  denote the Laplace-Beltrami operator on  $L^2(M)$  with domain  $\mathcal{D}(\Delta) = H_0^1(M) \cap H^2(M)$  defined by  $g$ . N.b. the results are already interesting when  $(M, g)$  is a smooth connected domain of the Euclidean space  $\mathbb{R}^d$ , so that  $\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_d^2}$ .

In this application, the state and input spaces are  $\mathcal{E} = \mathcal{F} = L^2(M)$ , the growth spaces are the spectral spaces of § 3.6, the reference operator  $C_0$  is the identity operator and the observation operator  $C$  is the multiplication by the characteristic function  $\chi_\Omega$  of an open subset  $\Omega \neq \emptyset$  of  $\bar{M}$ , i.e. it truncates the input function outside the control region  $\Omega$ . If  $\bar{M}$  is not compact, assume that  $\Omega$  is the exterior of a compact set  $K$  such that  $K \cap \bar{\Omega} \cap \partial M = \emptyset$ .

For  $A = \Delta$ , (6) holds with exponent  $\alpha = \frac{1}{2}$ , cf. § 3.8 for compact  $M$ , and [Mil05] otherwise. Hence for  $A = -(-\Delta)^\gamma$ , (6) holds with exponent  $\alpha = \frac{1}{2\gamma}$ . Applying theorem 2.2 improves on [Mil06c, theorem 2]:

**Theorem 4.1.** *For all  $\gamma > 1/2$ , the anomalous diffusion:*

$$\partial_t \phi + (-\Delta)^\gamma \phi = \chi_\Omega u, \quad \phi(0) = \phi_0 \in L^2(M), \quad u \in L^2([0, T] \times M),$$

is null-controllable in any time  $T > 0$ . Moreover the cost  $\kappa_T$  (cf. § 3.2) satisfies  $\limsup_{T \rightarrow 0} T^\beta \ln \kappa_T < \infty$  with  $\beta = \frac{1}{2\gamma-1}$ .

When the manifold  $M$  is the whole Euclidean space  $\mathbb{R}^d$ , the fractional Laplacian  $-(-\Delta)^\gamma$  with  $\gamma \in (0, 1]$  generates the rotationally invariant  $2\gamma$ -stable Lévy process. For  $\gamma = 1$  this process is the Brownian motion  $B_t$  on  $\mathbb{R}^d$ , and for  $\gamma < 1$  it is subordinated to  $B_t$  by a strictly  $\gamma$ -stable subordinator  $T_t$ , so that it writes  $B_{T_t}$ . The convolution kernels of the corresponding semigroups are the rotationally invariant Lévy stable probability distributions, in particular the Gaussian distribution for  $\gamma = 1$  and the Cauchy distribution for  $\gamma = 1/2$ . For  $\gamma < 1$  these distributions have “heavy tails”, i.e. far away they decrease like a power as opposed to the exponential decrease found in the Gaussian, which accounts for the “superdiffusive” behavior of the semigroup. The more restrictive range  $\gamma \in (1/2, 1)$  is the most widely used to model anomalously fast diffusions, and it turns out that the controllability result theorem 4.1 applies to this range of fractional superdiffusions only.

When the manifold  $M$  is a domain of the Euclidean space  $\mathbb{R}^d$ , the Markov process generated by the fractional Dirichlet Laplacian  $-(-\Delta)^\gamma$  with  $\gamma \in (0, 1]$  can be obtained by killing the Brownian motion on  $\mathbb{R}^d$  upon exiting the domain then subordinating the killed Brownian motion by the subordinator  $T_t$  introduced above.

**4.2. Structural damping.** Let  $\mathcal{A}$  be a positive self-adjoint and boundedly invertible operator on another Hilbert space  $\mathcal{H}$  (with norm still denoted  $\|\cdot\|$ ). Let  $\mathcal{D}(\mathcal{A})$  denote its domain with the norm  $\zeta \mapsto \|\mathcal{A}\zeta\|$ . Let  $\mathcal{H}_\lambda$ ,  $\lambda > 0$ , denote the spectral spaces of  $\mathcal{A}$  corresponding to  $\{z \in \sigma(\mathcal{A}) \mid \operatorname{Re} z < \lambda\}$  (cf. § 3.6). We consider two observation operators  $\mathcal{C}_0$  and  $\mathcal{C}$  in  $\mathcal{L}(\mathcal{D}(\mathcal{A}), \mathcal{F})$  satisfying relative observability:

$$(28) \quad \|\mathcal{C}_0 z\|^2 \leq d_0 e^{2d\lambda^\delta} \|\mathcal{C}z\|^2, \quad z \in \mathcal{H}_\lambda, \quad \lambda > 0,$$

and a control operator  $\mathcal{B} = \mathcal{C}^* \in \mathcal{L}(\mathcal{F}, \mathcal{D}(\mathcal{A})')$  ( $\mathcal{D}(\mathcal{A})'$  denotes the dual space of  $\mathcal{D}(\mathcal{A})$  in  $\mathcal{H}$ ).

To give a precise meaning to the solution of the structurally damped system

$$(29) \quad \begin{aligned} \ddot{\zeta}(t) + \rho \mathcal{A}^{2\gamma} \dot{\zeta}(t) + \mathcal{A}^2 \zeta(t) &= \mathcal{B}u(t), \\ \zeta(0) = \zeta_0 \in \mathcal{D}(\mathcal{A}), \quad \dot{\zeta}(0) = \zeta_1 \in \mathcal{H}, \quad u &\in L^2([0, T], \mathcal{F}), \end{aligned}$$

with structural dissipation power  $\gamma \in (0, 1)$ , we write it as a first order system.

The state space is  $\mathcal{E} = \mathcal{D}(\mathcal{A}) \times \mathcal{H}$ . The semigroup generator  $A$  is

$$A = \begin{pmatrix} 0 & I \\ -\mathcal{A}^2 & -\rho \mathcal{A}^{2\gamma} \end{pmatrix}, \quad \mathcal{D}(A) = \{(z_0, z_1) \in \mathcal{E} \mid \mathcal{A}z_0 + \rho \mathcal{A}^{2\gamma-1} z_1 \in \mathcal{D}(\mathcal{A})\}.$$

It inherits from  $-\mathcal{A}$  the necessary and sufficient properties of Lumer-Phillips for generating a contraction semigroup.

The observation and control operators are the projection  $C_0 : \mathcal{E} \rightarrow \mathcal{H}$  defined by  $C_0(z_0, z_1) = z_1$ ,  $C = \mathcal{C}C_0$ , and  $B$  defined in § 3.2. We assume that  $C$  is admissible for the semigroup generated by  $A$ , i.e. (2). It results from [AL03] that (7) holds for any  $\beta > 0$  and  $b > 0$ .

We extend the action of the orthogonal projection  $H_\lambda$  on  $\mathcal{H}_\lambda$  from  $\mathcal{H}$  to  $\mathcal{E}$  according to  $H_\lambda(z_0, z_1) = (H_\lambda z_0, H_\lambda z_1)$ . The growth condition (5) is proved in [Mil06a, Proposition 1] for the growth spaces  $\mathcal{E}_\lambda = H_\mu(\mathcal{E})$  with  $\lambda = \min\left\{\frac{\rho}{2}, \frac{1}{\rho}\right\} \mu^{2 \min\{\gamma, 1-\gamma\}}$ , using [CT90] (or [HO07]). For this choice of growth spaces, (28) implies the relative observability (6) with exponent  $\alpha = \frac{\delta}{2 \min\{\gamma, 1-\gamma\}}$  and rate  $a = \frac{d}{\min\left\{\frac{\rho}{2}, \frac{1}{\rho}\right\}^\alpha}$ .

Applying theorem 2.2 and corollary 1(ii) improves on [Mil06a, theorem 1]:

**Theorem 4.2.** *Recall that  $\delta$  and  $d$  are the exponent and rate in the main assumption (28). For all  $\rho > 0$  and  $\gamma \in (\delta/2, 1 - \delta/2)$ , for all  $\zeta_0$  and  $\zeta_1$ , there is an input*

$u$  such that the solution  $\zeta$  of (29) satisfies  $\zeta(T) = \dot{\zeta}(T) = 0$  and the cost estimate :

$$\int_0^T \|u(t)\|^2 dt \leq b_0 \exp\left(\frac{2b}{T^\beta}\right) (\|\mathcal{A}\zeta_0\|^2 + \|\zeta_1\|^2), \quad \zeta_0 \in \mathcal{D}(\mathcal{A}), \quad \zeta_1 \in \mathcal{H}, \quad T \text{ small,}$$

$$\text{with } \beta = \left(\frac{2}{\delta} \min\{\gamma, 1 - \gamma\} - 1\right)^{-1}, \quad \text{and any } b > \frac{d^{\beta+1}}{\min\left\{\frac{\rho}{2}, \frac{1}{\rho}\right\}^\beta} \frac{(\beta+1)^{\beta(\beta+1)}}{\beta^{\beta^2}}.$$

We refer to [CR82, LT98] for the motivation of the abstract model (29). The main application is to the plate equation with square root damping and interior control in  $\Omega$  with hinged boundary conditions on a manifold  $M$ , in the framework of § 4.1:

$$(30) \quad \begin{aligned} \ddot{\zeta} - \rho \Delta \dot{\zeta} + \Delta^2 \zeta &= \chi_\Omega u \quad \text{on } [0, T] \times M, \quad \zeta = \Delta \zeta = 0 \quad \text{on } [0, T] \times \partial M, \\ \zeta(0) &= \zeta_0 \in H^2(M) \cap H_0^1(M), \quad \dot{\zeta}(0) = \zeta_1 \in L^2(M), \quad u \in L^2([0, T] \times M). \end{aligned}$$

Applying theorem 4.2 instead of [Mil06a, theorem 1] to  $\mathcal{A} = -\Delta$  with  $\delta = \gamma = \frac{1}{2}$  improves on the value of  $\beta$  in the first part of [Mil06a, theorem 2] (cf. also [AC07]). Under the geometrical optics condition in [BLR92] that the length  $L_\Omega$  of the longest generalized geodesic in  $\overline{M}$  which does not intersect  $\Omega$  is not  $\infty$ , the second part of [Mil06a, theorem 2] estimates the cost rate: for all  $\rho \in (0, 2)$ , the control cost of (30) satisfies the estimate in theorem 4.2 with  $\beta = 1$  and any  $b > b_1 L_\Omega^2$  for some  $b_0$  and  $b_1$  which do not depend on  $\Omega$  and  $\rho$  (cf. [Mil06a, note added in proof]), hence e.g. (cf. [LZZ00], [ET06, Appendix]) the minimal null-control input  $u$  converges to the minimal null-control input for the undamped plate equation as  $\rho \rightarrow 0$ .

**4.3. Diffusion in a potential well.** We consider a power  $k \in \mathbb{N}^*$  and the potential well  $V(x) = |x|^{2k}$ ,  $x \in \mathbb{R}^d$ . The Schrödinger operator  $A = \Delta - V$  with domain  $\mathcal{D}(A) = \{\phi \in H^2(\mathbb{R}^d) \mid \int |V\phi|^2 < \infty\}$  is negative self-adjoint and has compact resolvent. Let  $\chi_\Gamma$  denote the multiplication by the characteristic function of any non empty open cone  $\Gamma = \{x \in \mathbb{R}^d \mid |x| > r_0, x/|x| \in \Omega_0\}$ , where  $r_0 \geq 0$  and  $\Omega_0$  is an open subset of the unit sphere.

In this application, the state and input spaces are  $\mathcal{E} = \mathcal{F} = L^2(\mathbb{R}^d)$ , the growth spaces are the spectral spaces of § 3.6, the reference operator  $C_0$  is the identity operator and the observation operator  $C$  is the multiplication by  $\chi_\Gamma$  as in § 4.1, i.e. it truncates the input function outside the control region  $\Gamma$ .

In [Mil08], (6) with exponent  $\alpha = \frac{1}{2}(1 + \frac{1}{k})$  is proved and some radial eigenfunctions concentrating at some “equator” such that (27) holds are exhibited (cf. [Mil08, § 4.2.2]) allowing to deduce from theorem 2.2 and lemma 3.5:

**Theorem 4.3.** *For all  $k > 1$ , the diffusion in the potential well  $V(x) = |x|^{2k}$ :*

$$\partial_t \phi - \Delta \phi + V \phi = \chi_\Gamma u, \quad \phi(0) = \phi_0 \in L^2(\mathbb{R}^d), \quad u \in L^2([0, T] \times \mathbb{R}^d),$$

*is null-controllable in any time  $T > 0$ . Moreover the cost  $\kappa_T$  (cf. § 3.2) satisfies:  $\kappa = \limsup_{T \rightarrow 0} T^\beta \ln \kappa_T < \infty$  with  $\beta = 1 + \frac{2}{k-1}$ .*

*If there is a vector space of dimension 2 in  $\mathbb{R}^d$  which does not intersect the closure  $\overline{\Omega_0}$  of the subset  $\Omega_0$  of the unit sphere defining the cone  $\Gamma$  then  $\kappa \neq 0$ .*

When  $\Gamma$  is a bounded set instead of a cone, some radial eigenfunctions such that (26) fails are exhibited in [Mil08, § 4.2.3] allowing to deduce from lemma 3.5 that  $\kappa = \limsup_{T \rightarrow 0} T^\beta \ln \kappa_T = +\infty$  with  $\beta = 1 + \frac{2}{k-1}$ . Whether null-controllability from bounded sets  $\Gamma$  holds for  $k > 1$  remains open.

As in § 4.1, the semigroup considered here is a well known model of diffusion. It can be interpreted as a Brownian diffusion on  $\mathbb{R}^d$  killed at the rate  $V$ .

## 5. ON THE RATE IN THE ESTIMATE ON SUMS OF EIGENFUNCTIONS.

In the context of § 3.8, this section proves the following independent result.

**Theorem 5.1.** *Consider the Laplace-Beltrami operator  $\Delta$  with Dirichlet boundary condition on a compact smooth connected Riemannian manifold  $M$  and an open subset  $\Omega \neq \emptyset$  of  $\overline{M}$ .*

*If the observability estimate*

$$\int_M |\phi(x)|^2 dx \leq c_0 e^{c\sqrt{\lambda}} \int_\Omega |\phi(x)|^2 dx,$$

*holds for any  $\lambda > 0$  and any sum of eigenfunctions*

$$\phi = \sum_{\mu \in \sigma(-\Delta) \cap (0, \lambda)} \varphi_\mu, \quad -\Delta \varphi_\mu = \mu \varphi_\mu,$$

*then  $c \geq \sup_{y \in M} \text{dist}(y, \overline{\Omega})$ .*

*Proof.* The proof relies on an estimate of the heat kernel in small time. We could use Varadhan's formula as in [Mil04, theorem 2.1] or many other stronger (usually less general) Gaussian estimates. We choose to use the following  $L^2$  estimate proved e.g. in [Tay96, chapter 6, (2.22)] as a direct consequence of the finite propagation speed for the wave equation and Kannai's formula

$$e^{t\Delta} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{4t}} \cos(s\sqrt{-\Delta}) ds,$$

using the original idea of [CGT82]:

$$(31) \quad \|C e^{t\Delta} C'\|_{\mathcal{L}(L^2)} \leq \frac{1}{\sqrt{4\pi t}} \int_{|s| \geq \rho} e^{-\frac{s^2}{4t}} ds \leq e^{-\frac{\rho^2}{4t}},$$

where  $C$  (resp.  $C'$ ) denotes the multiplication by the characteristic function of  $\Omega$  (resp. an open subset  $\Omega'$ ), and  $\rho = \text{dist}(\Omega, \Omega') = \inf \{\text{dist}(x, y) \mid x \in \Omega, y \in \Omega'\}$ .

Let  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$  denote the norm and scalar product in  $L^2(M)$ . Consistently with the functional calculus for self-adjoint operators, let  $\mathbf{1}_{-\Delta < \lambda^2}$  denote the projection on the space spanned by the eigenfunctions of  $-\Delta$  with eigenvalues less than  $\lambda^2$ . From now on we fix an eigenvalue  $\mu^2$  of  $-\Delta$  and  $\varphi_\mu$  a corresponding normalized eigenfunction. The unique continuation property for elliptic operators implies that  $Y = \{y \in M \setminus \overline{\Omega} \mid \varphi_\mu(y) \neq 0\}$  is an open dense set in  $M \setminus \overline{\Omega}$ , so that the supremum in theorem 5.1 can be taken over  $y \in Y$  instead of  $y \in M$ . Since eigenfunctions are continuous, for any  $y \in Y$  we may find a small ball  $\Omega'$  with center  $y$  such that  $|\int_{\Omega'} \varphi_\mu(x) dx| > 0$  and  $\text{dist}(\Omega, \Omega')$  is arbitrarily close to  $\text{dist}(y, \overline{\Omega})$ .

With these notations, equivalently to theorem 5.1, we assume

$$(32) \quad \|\phi\| \leq a_0 e^{a\lambda} \|C\phi\|, \quad \phi \in \mathbf{1}_{-\Delta < \lambda^2} L^2(M), \quad \lambda > 0,$$

and prove  $2a \geq \rho = \text{dist}(\Omega, \Omega')$  for any open subset  $\Omega' \neq \emptyset$  with characteristic function  $\chi$  satisfying

$$(33) \quad |\langle \chi, \varphi_\mu \rangle| > 0.$$

For any  $\lambda > 0$ , we consider  $\phi_\lambda = e^{\frac{\rho}{2\lambda}\Delta} \chi$  and  $\phi = \mathbf{1}_{-\Delta < \lambda^2} \phi_\lambda$ . In (31)  $C'$  is the multiplication by  $\chi$ , so that  $C'\chi = \chi^2 = \chi$  and (31) yields  $\|C\phi_\lambda\| \leq e^{-\frac{\rho^2}{4t}} \|\chi\|$ . Since  $\phi_\lambda - \phi = \mathbf{1}_{-\Delta \geq \lambda^2} e^{\frac{\rho}{2\lambda}\Delta} \chi$ , the spectral theorem yields  $\|C(\phi_\lambda - \phi)\| \leq \|\phi_\lambda - \phi\| \leq e^{-\frac{\rho^2}{4t}} \|\chi\|$ . Adding these inequalities yields  $\|C\phi\| \leq 2e^{-\frac{\rho^2}{4t}} \|\chi\|$ . Since  $\|\varphi_\mu\| = 1$  and  $-\Delta \varphi_\mu = \mu^2 \varphi_\mu$ , we also have  $\|\phi\| \geq |\langle \phi, \varphi_\mu \rangle| = |\langle \chi, e^{\frac{\rho}{2\lambda}\Delta} \varphi_\mu \rangle| = e^{-\frac{\rho\mu^2}{2\lambda}} |\langle \chi, \varphi_\mu \rangle|$ . Plugging these inequalities in (32) and using (33) yields

$$0 < e^{-\frac{\rho\mu}{2}} |\langle \chi, \varphi_\mu \rangle| \leq 2a_0 e^{(a-\rho/2)\lambda} \|\chi\|, \quad \lambda \geq \mu.$$

Taking the limit as  $\lambda \rightarrow \infty$  in this inequality forbids  $a < \rho/2$ .  $\square$

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