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# Technical appendix to “Adaptive estimation of stationary Gaussian fields”

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## Abstract

This is a technical appendix to “Adaptive estimation of stationary Gaussian fields” [6]. We present several proofs that have been skipped in the main paper. These proofs are organised as in Section 8 of [6].

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## 1. Proof of Proposition 8.1

*Proof of Proposition 8.1.* First, we recall the notations introduced in [3]. Let  $N$  be a positive integer. Then,  $\mathcal{I}_N$  stands for the family of subsets of  $\{1, \dots, N\}$  of size less than 2. Let  $\mathcal{T}$  be a set of vectors indexed by  $\mathcal{I}_N$ . In the sequel,  $\mathcal{T}$  is assumed to be a compact subset of  $\mathbb{R}^{(N(N+1)/2)+1}$ . The following lemma states a slightly modified version of the upper bound in remark 7 in [3].

**Lemma 1.1.** *Let  $T$  be a supremum of Rademacher chaos indexed by  $\mathcal{I}_N$  of the form*

$$T := \sup_{t \in \mathcal{T}} \left| \sum_{\{i,j\}} U_i U_j t_{\{i,j\}} + \sum_{i=1}^N t_{\{i\}} + t_{\emptyset} \right|,$$

where  $U_1, \dots, U_N$  are independent Rademacher random variables. Then for any  $x > 0$ ,

$$\mathbb{P}\{T \geq \mathbb{E}[T] + x\} \leq 4 \exp\left(-\frac{x^2}{L_1 \mathbb{E}[D]^2} \wedge \frac{x}{L_2 E}\right), \quad (1)$$

where  $D$  and  $E$  are defined by:

$$D := \sup_{t \in \mathcal{T}} \sup_{\alpha: \|\alpha\|_2 \leq 1} \left| \sum_{i=1}^N U_i \sum_{j \neq i} \alpha_j t_{\{i,j\}} \right|,$$

$$E := \sup_{t \in \mathcal{T}} \sup_{\alpha^{(1)}, \alpha^{(2)}, \|\alpha^{(1)}\|_2 \leq 1, \|\alpha^{(2)}\|_2 \leq 1} \left| \sum_{i=1}^N \sum_{j \neq i} t_{\{i,j\}} \alpha_i^{(1)} \alpha_j^{(2)} \right|.$$

Contrary to the original result of [3], the chaos are not assumed to be homogeneous. Besides, the  $t_{\{i\}}$  are redundant with  $t_{\emptyset}$ . In fact, we introduced this family in order to emphasize the connection with Gaussian chaos in the next result.

A suitable application of the central limit theorem enables to obtain a corresponding bound for Gaussian chaos of order 2.

**Lemma 1.2.** *Let  $T$  be a supremum of Gaussian chaos of order 2.*

$$T := \sup_{t \in \mathcal{T}} \left| \sum_{\{i,j\}} t_{\{i,j\}} Y_i Y_j + \sum_i t_i Y_i^2 + t_{\emptyset} \right|, \quad (2)$$

where  $Y_1, \dots, Y_N$  are independent standard Gaussian random variable. Then, for any  $x > 0$ ,

$$\mathbb{P}\{T \geq \mathbb{E}[T] + x\} \leq \exp\left(-\frac{x^2}{\mathbb{E}[D]^2 L_1} \wedge \frac{x}{EL_2}\right), \quad (3)$$

where

$$D := \sup_{t \in \mathcal{T}} \sup_{\alpha \in \mathbb{R}^N, \|\alpha\|_2 \leq 1} \sum_{i,j} Y_i (1 + \delta_{i,j}) \alpha_j t_{\{i,j\}},$$

$$E := \sup_{t \in \mathcal{T}} \sup_{\alpha_1, \|\alpha_1\|_2 \leq 1} \sup_{\alpha_2, \|\alpha_2\|_2 \leq 1} \sum_{i,j} \alpha_{1,i} \alpha_{2,j} t_{\{i,j\}} (1 + \delta_{i,j}).$$

The proof of this Lemma is postponed to the end of this section. To conclude, we derive the result of Proposition 8.1 from this last lemma. For any matrix  $R \in F$ , we define the vector  $t^R \in \mathbb{R}^{nr(nr+1)/2+1}$  indexed by  $\mathcal{I}_{nr}$  as follows

$$t_{\{(i,k),(j,l)\}}^R := \delta_{k,l}(2 - \delta_{i,j}) \frac{R[i,j]}{n}, \quad t_{\{(i,k)\}}^R := \frac{R[i,i]}{n}, \quad \text{and } t_{\emptyset}^R := -\text{tr}(R),$$

where  $\delta_{i,j}$  is the indicator function of  $i = j$ . In order to apply Lemma 1.2 with  $N = nr$  and  $\mathcal{T} = \{t^R | R \in F\}$ , we have to work out the quantities  $D$  and  $E$ .

$$\begin{aligned} D &= \sup_{t^R \in \mathcal{T}} \sup_{\alpha \in \mathbb{R}^{nr}, \|\alpha\|_2 \leq 1} \left\{ \sum_{i=1}^r \sum_{k=1}^n Y_{[i,k]} \sum_{j=1}^r \sum_{l=1}^n t_{ij}^{R,k,l} (1 + \delta_{i,j} \delta_{k,l}) \alpha_j^l \right\} \\ &= \sup_{R \in F} \sup_{\alpha \in \mathbb{R}^{nr}, \|\alpha\|_2 \leq 1} 2 \left\{ \sum_{i=1}^r \sum_{k=1}^n Y_{[i,k]} \sum_{j=1}^r \frac{R[i,j] \alpha_j^k}{n} \right\} \\ &= \sup_{R \in F} \sup_{\alpha \in \mathbb{R}^{nr}, \|\alpha\|_2 \leq 1} \frac{2}{n} \left\{ \sum_{k=1}^n \sum_{j=1}^r \alpha_j^k \left( \sum_{i=1}^r Y_{[i,k]} R[i,j] \right) \right\}. \end{aligned}$$

Applying Cauchy-Schwarz identity yields

$$\begin{aligned} D^2 &= \frac{4}{n^2} \sup_{R \in F} \left\{ \sum_{k=1}^n \sum_{j=1}^r \left( \sum_{i=1}^r Y_{[i,k]} R_{[i,j]} \right)^2 \right\} \\ &= \frac{4}{n} \sup_{R \in F} \text{tr}(R \overline{Y Y^*} R^*) . \end{aligned} \quad (4)$$

Let us now turn the constant  $E$

$$\begin{aligned} E &= \sup_{t^R \in \mathcal{T}} \sup_{\substack{\alpha_1, \alpha_2 \in \mathbb{R}^{nr} \\ \|\alpha_1\|_2 \leq 1, \|\alpha_2\|_2 \leq 1}} \sum_{1 \leq i, j \leq r} \sum_{1 \leq k, l \leq n} (1 + \delta_{ij} \delta_{kl}) t_{i,j}^{R,kl} \alpha_{1,i}^k \alpha_{2,j}^l \\ &= \sup_{R \in F} \sup_{\substack{\alpha_1, \alpha_2 \in \mathbb{R}^{nr} \\ \|\alpha_1\|_2 \leq 1, \|\alpha_2\|_2 \leq 1}} \frac{2}{n} \sum_{1 \leq i, j \leq r} \sum_{1 \leq k \leq n} R_{[i,j]} \alpha_{1,i}^k \alpha_{2,j}^k . \end{aligned}$$

From this last expression, it follows that  $E$  is a supremum of  $L_2$  operator norms

$$E = \frac{2}{n} \sup_{R \in F} \varphi_{\max} \left( \text{Diag}^{(n)}(R) \right) ,$$

where  $\text{Diag}^{(n)}(R)$  is the  $(nr \times nr)$  block diagonal matrix such that each diagonal block is made of the matrix  $R$ . Since the largest eigenvalue of  $\text{Diag}^{(n)}(R)$  is exactly the largest eigenvalue of  $R$ , we get

$$E = \frac{2}{n} \sup_{R \in F} \varphi_{\max}(R) . \quad (5)$$

Applying Proposition 1.2 and gathering identities (4) and (5) yields

$$\mathbb{P}(Z \geq \mathbb{E}(Z) + t) \leq \exp \left[ - \left( \frac{t^2}{L_1 \mathbb{E}(V)} \wedge \frac{t}{L_2 B} \right) \right] ,$$

where  $B = E$  and  $V = D^2$ .  $\square$

*Proof of Lemma 1.1.* This result is an extension of Corollary 4 in [3]. We shall closely follow the sketch of their proof adapting a few arguments. First, we upper bound the moments of  $(T - \mathbb{E}(T))_+$ . Then, we derive the deviation inequality from it. Here,  $x_+ = \max(x, 0)$ .

**Lemma 1.3.** *For all real numbers  $q \geq 2$ ,*

$$\|(T - \mathbb{E}(T))_+\|_q \leq \sqrt{Lq} \mathbb{E}(D) + LqE , \quad (6)$$

where  $\|T\|_q^q$  stands for the  $q$ -th moment of the random variable  $T$ . The quantities  $D$  and  $E$  are defined in Lemma 1.1.

By Lemma 1.3, for any  $t \geq 0$  and any  $q \geq 2$ ,

$$\begin{aligned} \mathbb{P}(T \geq \mathbb{E}(T) + t) &\leq \frac{\mathbb{E}[(T - \mathbb{E}(T))_+^q]}{t^q} \\ &\leq \left( \frac{\sqrt{Lq}\mathbb{E}(D) + LqE}{t} \right)^q. \end{aligned}$$

The right-hand side is at most  $2^{-q}$  if  $\sqrt{Lq}\mathbb{E}(D) \leq t/4$  and  $LqE \leq t/4$ . Let us set

$$q_0 := \frac{t^2}{16L\mathbb{E}(D)^2} \wedge \frac{t}{4LE}.$$

If  $q_0 \geq 2$ , then  $\mathbb{P}(T \geq \mathbb{E}(T) + t) \leq 2^{-q_0}$ . On the other hand if  $q_0 < 2$ , then  $4 \times 2^{-q_0} \geq 1$ . It follows that

$$\mathbb{P}(T \geq \mathbb{E}(T) + t) \leq 4 \exp\left(-\frac{\log(2)}{4L} \left[ \frac{t^2}{4\mathbb{E}(D)^2} \wedge \frac{t}{E} \right]\right).$$

□

*Proof of Lemma 1.3.* This result is based on the entropy method developed in [3]. Let  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  be a measurable function such that  $T = f(U_1, \dots, U_N)$ . In the sequel,  $U'_1, \dots, U'_N$  denote independent copies of  $U_1, \dots, U_N$ . The random variable  $T'_i$  and  $V^+$  are defined by

$$\begin{aligned} T'_i &:= f(U_1, \dots, U_{i-1}, U'_i, U_{i+1}, \dots, U_N), \\ V^+ &:= \mathbb{E} \left[ \sum_{i=1}^N (T - T'_i)_+^2 \middle| U_1^N \right], \end{aligned}$$

where  $U_1^N$  refers to the set  $\{U_1, \dots, U_N\}$ . Theorem 2 in [3] states that for any real  $q \geq 2$ ,

$$\|(T - \mathbb{E}(T))_+\|_q \leq \sqrt{Lq} \|\sqrt{V^+}\|_q. \quad (7)$$

To conclude, we only have bound the moments of  $\sqrt{V^+}$ . By definition,

$$T = \sup_{t \in \mathcal{T}} \left| \sum_{\{i,j\}} U_i U_j t_{\{i,j\}} + \sum_{i=1}^N t_{\{i\}} + t_\emptyset \right|.$$

Since the set  $\mathcal{T}$  is compact, this supremum is achieved almost surely at an element  $t^0$  of  $\mathcal{T}$ . For any  $1 \leq i \leq N$ ,

$$(T - T'_i)_+^2 \leq \left( (U_i - U'_i) \left| \sum_{j \neq i} U_j t^0_{\{i,j\}} \right| \right)^2.$$

Gathering this bound for any  $i$  between 1 and  $N$ , we get

$$\begin{aligned}
V^+ &\leq \sum_{i=1}^N \mathbb{E} \left[ \left( (U_i - U'_i) \left| \sum_{j \neq i} U_j t^0 \{i, j\} \right| \right)^2 \middle| U_1^N \right] \\
&\leq 2 \sum_{i=1}^N \left[ \sum_{j \neq i} U_j t^0 \{i, j\} \right]^2 \\
&\leq 2 \sup_{\alpha \in \mathbb{R}^N, \|\alpha\|_2 \leq 1} \left[ \sum_{i=1}^N \alpha_i \left( \sum_{j \neq i} t_{\{i, j\}}^0 U_j \right) \right]^2 \\
&\leq 2 \sup_{t \in \mathcal{T}} \sup_{\alpha \in \mathbb{R}^N, \|\alpha\|_2 \leq 1} \sum_{i=1}^N \left[ U_i \sum_{j \neq i} \alpha_j t_{\{i, j\}} \right]^2 = 2D^2 .
\end{aligned}$$

Combining this last bound with (7) yields

$$\begin{aligned}
\|(T - \mathbb{E}(T))_+\|_q &\leq \sqrt{Lq} \sqrt{2} \|D\|_q \\
&\leq \sqrt{Lq} \left[ \mathbb{E}(D) + |(D - \mathbb{E}(D))_+\|_q \right] . \tag{8}
\end{aligned}$$

Since the random variable  $D$  defined in Lemma 1.1 is a measurable function  $f_2$  of the variables  $U_1, \dots, U_N$ , we apply again Theorem 2 in [3].

$$\|(D - \mathbb{E}(D))_+\|_q \leq \sqrt{Lq} \left\| \sqrt{V_2^+} \right\|_q ,$$

where  $V_2^+$  is defined by

$$V_2^+ := \mathbb{E} \left[ \sum_{i=1}^N (D - D'_i)_+^2 \middle| U_1^N \right] ,$$

and  $D'_i := f_2(U_1, \dots, U_{i-1}, U'_i, U_{i+1}, \dots, U_N)$ . As previously, the supremum in  $D$  is achieved at some random parameter  $(t^0, \alpha^0)$ . We therefore upper bound  $V_2^+$  as previously.

$$\begin{aligned}
V_2^+ &\leq \sum_{i=1}^N \mathbb{E} \left[ \left( (U_i - U'_i) \left( \sum_{j \neq i} \alpha_j^0 t_{\{i, j\}}^0 \right) \right)^2 \middle| U_1^N \right] \\
&\leq 2 \sum_{i=1}^N \left( \sum_{j \neq i} \alpha_j^0 t_{\{i, j\}}^0 \right)^2 \\
&\leq 2 \sup_{\alpha^{(2)} \in \mathbb{R}^N, \|\alpha\|_2 \leq 1} \left( \sum_{i=1}^N \alpha_j^{(2)} \sum_{j \neq i} \alpha_i^0 t_{\{i, j\}} \right)^2 = 2E^2 .
\end{aligned}$$

Gathering this upper bound with (8) yields

$$\|(T - \mathbb{E}(T))_+\|_q \leq \sqrt{Lq} \mathbb{E}(D) + LqE .$$

□

*Proof of Lemma 1.2.* We shall apply the central limit theorem in order to transfer results for Rademacher chaos to Gaussian chaos. Let  $f$  be the unique function satisfying  $T = f(y_1, \dots, y_N)$  for any  $(y_1, \dots, y_N) \in \mathbb{R}^N$ . As the set  $\mathcal{T}$  is compact, the function  $f$  is known to be continuous. Let  $(U_i^{(j)})_{1 \leq i \leq N, j \geq 0}$  an i.i.d. family of Rademacher variables. For any integer  $n > 0$ , the random variables  $Y^{(n)}$  and  $T^{(n)}$  are defined by

$$\begin{aligned} Y^{(n)} &:= \left( \sum_{j=1}^n \frac{U_1^{(j)}}{\sqrt{n}}, \dots, \sum_{j=1}^n \frac{U_N^{(j)}}{\sqrt{n}} \right), \\ T^{(n)} &:= f\left(Y^{(n)}\right). \end{aligned}$$

Clearly,  $T^{(n)}$  is a supremum of Rademacher chaos of order 2 with  $nN$  variables and a constant term. By the central limit theorem,  $T^{(n)}$  converges in distribution towards  $T$  as  $n$  tends to infinity. Consequently, deviation inequalities for the variables  $T^{(n)}$  transfer to  $T$  as long as the quantities  $\mathbb{E}[D^{(n)}]$ ,  $E^{(n)}$ , and  $\mathbb{E}[T^{(n)}]$  converge.

We first prove that the sequence  $T^{(n)}$  converges in expectation towards  $T$ . As  $T^{(n)}$  converges in distribution, it is sufficient to show that the sequence  $T^{(n)}$  is asymptotically uniformly integrable. The set  $\mathcal{T}$  is compact, thus there exists a positive number  $t_\infty$  such that

$$\begin{aligned} T^{(n)} &\leq t_\infty \left[ \sum_{i,j} |Y_i^{(n)} Y_j^{(n)}| + 1 \right] \\ &\leq t_\infty \left[ 1 + (N+1)/2 \sum_{i=1}^N \left(Y_i^{(n)}\right)^2 \right]. \end{aligned}$$

It follows that

$$\left(T^{(n)}\right)^2 \leq t_\infty^2 \left(\frac{N+1}{2}\right)^2 \frac{N+2}{2} \left[ 1 + \sum_{i=1}^N \left(Y_i^{(n)}\right)^4 \right]. \quad (9)$$

The sequence  $Y_i^{(n)}$  does not only converge in distribution to a standard normal distribution but also in moments (see for instance [1] p.391). It follows that  $\overline{\lim} \mathbb{E} \left[ \left(T^{(n)}\right)^2 \right] \leq \infty$  and the sequence  $f\left(Y^{(n)}\right)$  is asymptotically uniformly integrable. As a consequence,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ T^{(n)} \right] = \mathbb{E}[T].$$

Let us turn to the limit of  $\mathbb{E}[D^{(n)}]$ . As the variable  $T^{(n)}$  equals

$$T^{(n)} = \sup_{t \in \mathcal{T}} \left| \sum_{\{i,j\}} t_{\{i,j\}} \sum_{1 \leq k,l \leq n} \frac{U_i^{(k)} U_j^{(l)}}{n} + \sum_i t_i \sum_{1 \leq k \leq n} \frac{U_i^{(k)}}{\sqrt{n}} \sum_{l \neq k} \frac{U_i^{(l)}}{\sqrt{n}} + t_\emptyset + \sum_i t_i \right|,$$

it follows that

$$\begin{aligned} D^{(n)} &= \sup_{t \in \mathcal{T}} \sup_{\alpha \in \mathbb{R}^{nN}, \|\alpha\|_2 \leq 1} \left| \sum_{1 \leq i \leq N} \sum_{1 \leq k \leq n} U_i^{(k)} \left\{ \sum_{j \neq i} \frac{t_{\{i,j\}}}{n} \sum_{1 \leq l \leq n} \alpha_j^{(l)} + 2 \sum_{l \neq k} \frac{t_{\{i\}}}{n} \alpha_i^{(l)} \right\} \right| \\ &\leq \sup_{t \in \mathcal{T}} \sup_{\alpha \in \mathbb{R}^{nN}, \|\alpha\|_2 \leq 1} \left\{ \sum_i \frac{U_i^{(k)}}{\sqrt{n}} \sum_j (1 + \delta_{i,j}) t_{\{i,j\}} \frac{\sum_{1 \leq l \leq n} \alpha_j^{(l)}}{\sqrt{n}} \right\} + A^{(n)}, \quad (10) \end{aligned}$$

where the random variable  $A^{(n)}$  is defined by

$$A^{(n)} := \sup_{t \in \mathcal{T}} \sup_{\alpha \in \mathbb{R}^{nN}, \|\alpha\|_2 \leq 1} \sum_{i=1}^N \sum_{j=1}^n t_{\{i\}} \frac{U_i^{(j)}}{n} \alpha_i^j.$$

Straightforwardly, one upper bounds  $A^{(n)}$  by  $t_\infty/n \sqrt{\sum_{i=1}^N \sum_{j=1}^n (U_i^{(j)})^2}$  and its expectation satisfies

$$\mathbb{E} \left( |A^{(n)}| \right) \leq t_\infty \sqrt{\frac{N}{n}},$$

which goes to 0 when  $n$  goes to infinity. Thus, we only have to upper bound the expectation of the first term in (10). Clearly, the supremum is achieved only when for all  $1 \leq j \leq N$ , the sequence  $(\alpha_j^{(l)})_{1 \leq l \leq n}$  is constant. In such a case, the sequence  $(\alpha_j^{(1)})_{1 \leq j \leq N}$  satisfies  $\|\alpha^{(1)}\|_2 \leq 1/\sqrt{n}$ . it follows that

$$\mathbb{E} [D^{(n)}] = \mathbb{E} \left\{ \sup_{t \in \mathcal{T}} \sup_{\alpha \in \mathbb{R}^N, \|\alpha\|_2 \leq 1} \mathbb{E} \left[ \sum_i Y_i^{(n)} \sum_j (1 + \delta_{i,j}) \alpha_j \right] \right\} + \mathcal{O} \left( \frac{1}{\sqrt{n}} \right).$$

Let  $g$  be the function defined by

$$g(y_1, \dots, y_N) = \sup_{t \in \mathcal{T}} \sup_{\alpha \in \mathbb{R}^N, \|\alpha\|_2 \leq 1} \left[ \sum_i y_i \sum_j (1 + \delta_{i,j}) \alpha_j \right],$$

for any  $(y_1, \dots, y_N) \in \mathbb{R}^N$ . The function  $g(\cdot)$  is measurable and continuous as the supremum is taken over a compact set. As a consequence,  $g(Y^{(n)})$  converges in distribution towards  $g(Y)$ . As previously, the sequence is asymptotically uniformly integrable since its moment of order 2 is uniformly upper bounded. It follows that  $\lim \mathbb{E} [D^{(n)}] = \mathbb{E} [D]$ .

Third, we compute the limit of  $E^{(n)}$ . By definition,

$$\begin{aligned} E^{(n)} &= \sup_{t \in \mathcal{T}} \sup_{\alpha_1, \alpha_2 \in \mathbb{R}^{nN}, \|\alpha_1\|_2 \leq 1, \|\alpha_2\|_2 \leq 1} \sum_{i=1}^N \sum_{k=1}^n \alpha_{1,i}^k \left[ \sum_{j \neq i} \sum_{l=1}^n \alpha_{2,j}^{(l)} \frac{t_{\{i,j\}}}{n} + 2 \sum_{l \neq k} \alpha_{2,i}^{(l)} \frac{t_{\{i\}}}{n} \right] \\ &= \sup_{t \in \mathcal{T}} \sup_{\alpha_1, \alpha_2, \|\alpha_1\|_2 \leq 1, \|\alpha_2\|_2 \leq 1} \sum_{i=1}^N \sum_{j=1}^n (1 + \delta_{i,j}) \frac{t_{\{i,j\}}}{n} \left[ \sum_{k=1}^n \sum_{l=1}^n \alpha_{1,i}^{(k)} \alpha_{2,j}^{(l)} \right] + \mathcal{O} \left( \frac{1}{n} \right). \end{aligned}$$



As for the computation of  $D^{(n)}$ , the supremum is achieved when the sequences  $(\alpha_{1,i}^k)_{1 \leq k \leq n}$  and  $(\alpha_{2,j}^l)_{1 \leq l \leq n}$  are constant for any  $i \in \{1, \dots, N\}$ . Thus, we only have to consider the supremum over the vectors  $\alpha_1$  and  $\alpha_2$  in  $\mathbb{R}^N$ .

$$E^{(n)} = \sup_{t \in \mathcal{T}} \sup_{\alpha_1, \alpha_2 \in \mathbb{R}^N \|\alpha_i\|_2 \leq 1} \sum_{i=1}^N \sum_{j=1}^N (1 + \delta_{ij}) t_{i,j} \alpha_{1,i} \alpha_{2,j} + \mathcal{O}\left(\frac{1}{n}\right).$$

It follows that  $E^{(n)}$  converges towards  $E$  when  $n$  tends to infinity.

The random variable  $T^{(n)} - \mathbb{E}(T^{(n)})$  converges in distribution towards  $T - \mathbb{E}(T)$ . By Lemma 1.1 ,

$$\mathbb{P}(T - \mathbb{E}(T) \geq x) \leq \underline{\lim} \exp\left(-\frac{x^2}{\mathbb{E}[D^{(n)}]^2 L_1} \wedge \frac{x}{E^{(n)} L_2}\right),$$

for any  $x > 0$ . Combining this upper bound with the convergence of the sequences  $D^{(n)}$  and  $E^{(n)}$  allows to conclude.  $\square$

## 2. Proof of Theorem 3.1

*Proof of Lemma 8.3.* We only consider here the anisotropic case, since the isotropic case is analogous. This result is based on the deviation inequality for suprema of Gaussian chaos of order 2 stated in Proposition 8.1. For any model  $m'$  belonging to  $\mathcal{M}$ , we shall upper bound the quantities  $\mathbb{E}(Z_{m'})$ ,  $B_{m'}$ , and  $\mathbb{E}(W_{m'})$  defined in (42) in [6].

1. Let us first consider the expectation of  $Z_{m'}$ . Let  $U'_{m,m'}$  be the new vector space defined by

$$U'_{m,m'} := U_{m,m'} \frac{\sqrt{D_\Sigma}}{p},$$

where  $U_{m,m'}$  is introduced in the proof of Lemma 8.2 in [6]. This new space allows to handle the computation with the canonical inner product in the space of matrices. Let  $\mathcal{B}_{m^2, m'^2}^{(2)}$  be the unit ball of  $U'_{m,m'}$  with respect to the canonical inner product. If  $R$  belongs to  $U_{m,m'}$ , then  $\|R\|_{\mathcal{H}'} = \|R\sqrt{D_\Sigma}/p\|_F$ , where  $\|\cdot\|_F$  stands for the Frobenius norm.

$$\begin{aligned} Z_{m'} &= \sup_{R \in \mathcal{B}_{m^2, m'^2}^{\mathcal{H}'}} \frac{1}{p^2} \text{tr} [R D_\Sigma (\overline{\mathbf{Y}\mathbf{Y}^*} - I_{p^2})] \\ &= \sup_{R \in \mathcal{B}_{m^2, m'^2}^{(2)}} \text{tr} \left[ R \frac{\sqrt{D_\Sigma}}{p} (\overline{\mathbf{Y}\mathbf{Y}^*} - I_{p^2}) \right] \\ &= \left\| \Pi_{U'_{m,m'}} \frac{\sqrt{D_\Sigma}}{p} (\overline{\mathbf{Y}\mathbf{Y}^*} - I_{p^2}) \right\|_F, \end{aligned} \tag{11}$$

where  $\Pi_{U'_{m,m'}}$  refers to the orthogonal projection with respect to the canonical inner product onto the space  $U'_{m,m'}$ . Let  $F_1, \dots, F_{d_{m^2,m'^2}}$  denote an orthonormal basis of  $U'_{m,m'}$ .

$$\begin{aligned}
\mathbb{E}(Z_{m'}^2) &= \sum_{i=1}^{d_{m^2,m'^2}} \mathbb{E} \left[ \text{tr}^2 \left( F_i \sqrt{\frac{D_\Sigma}{p^2}} (\overline{\mathbf{Y}\mathbf{Y}^*} - I_{p^2}) \right) \right] \\
&= \sum_{i=1}^{d_{m^2,m'^2}} \mathbb{E} \left[ \sum_{j=1}^{p^2} F_{i[j,j]} \frac{\sqrt{D_{\Sigma[j,j]}}}{p} (\overline{\mathbf{Y}\mathbf{Y}^*}_{[j,j]} - 1) \right]^2 \\
&= \sum_{i=1}^{d_{m^2,m'^2}} \frac{2}{np^2} \text{tr}(F_i D_\Sigma F_i) \\
&\leq \sum_{i=1}^{d_{m^2,m'^2}} \frac{2\varphi_{\max}(D_\Sigma)}{np^2} = \frac{2d_{m^2,m'^2}\varphi_{\max}(\Sigma)}{np^2}.
\end{aligned}$$

Applying Cauchy-Schwarz inequality, it follows that

$$\mathbb{E}(Z_{m'}) \leq \sqrt{\frac{2d_{m^2,m'^2}\varphi_{\max}(\Sigma)}{np^2}}. \quad (12)$$

2. Using the identity (11), the quantity  $B_{m'}$  equals

$$B_{m'} = \frac{2}{n} \sup_{R \in \mathcal{B}_{m^2,m'^2}^{(2)}} \varphi_{\max} \left( R \frac{\sqrt{D_\Sigma}}{p} \right).$$

As the operator norm is under-multiplicative and as it dominates the Frobenius norm, we get the following bound

$$B_{m'} \leq \frac{2\sqrt{\varphi_{\max}(\Sigma)}}{np}. \quad (13)$$

3. Let us turn to bounding the quantity  $\mathbb{E}(W_{m'})$ . Again, by introducing the ball  $\mathcal{B}_{m^2,m'^2}^{(2)}$ , we get

$$\begin{aligned}
W_{m'} &= \frac{4}{n} \sup_{R \in \mathcal{B}_{m^2,m'^2}^{(2)}} \frac{1}{p^2} \text{tr} [R \overline{\mathbf{Y}\mathbf{Y}^*} D_\Sigma R] \\
&\leq \frac{4\varphi_{\max}(\Sigma)}{np^2} \sup_{R \in \mathcal{B}_{m^2,m'^2}^{(2)}} \text{tr} [R \overline{\mathbf{Y}\mathbf{Y}^*} R] \\
&\leq \frac{4\varphi_{\max}(\Sigma)}{np^2} \left( 1 + \sup_{R \in \mathcal{B}_{m^2,m'^2}^{(2)}} \text{tr} [R (\overline{\mathbf{Y}\mathbf{Y}^*} - I_{p^2}) R] \right).
\end{aligned}$$

Let  $F_1, \dots, F_{d_{m^2, m'^2}}$  an orthonormal basis of  $U'_{m, m'}$  and let  $\lambda$  be a vector in  $\mathbb{R}^{d_{m^2, m'^2}}$ . We write  $\|\lambda\|_2$  for its  $L_2$  norm.

$$\begin{aligned} & \mathbb{E} \left( \sup_{R \in \mathcal{B}_{m^2, m'^2}^{(2)}} \text{tr} [R (\overline{\mathbf{Y}\mathbf{Y}^*} - I_{p^2}) R]^2 \right) \\ &= \mathbb{E} \left( \sup_{\|\lambda\|_2 \leq 1} \sum_{i, j=1}^{d_{m^2, m'^2}} \lambda_i \lambda_j \text{tr} [F_i F_j (\overline{\mathbf{Y}\mathbf{Y}^*} / n - I_{p^2})] \right)^2 \\ &\leq \sum_{i, j=1}^{d_{m^2, m'^2}} \mathbb{E} \left( \text{tr} [F_i F_j (\mathbf{Y}\mathbf{Y}^* / n - I_{p^2})]^2 \right). \end{aligned}$$

The second inequality is a consequence of Cauchy-Schwarz inequality in  $\mathbb{R}^{(d_{m^2, m'^2})^2}$  since the  $l_2$  norm of the vector  $(\lambda_i \lambda_j)_{1 \leq i, j \leq d_{m^2, m'^2}} \in \mathbb{R}^{d_{m^2, m'^2}^2}$  is bounded by 1. Since the matrices  $F_i$  are diagonal, we get

$$\mathbb{E} \left( \sup_{R \in \mathcal{B}_{m^2, m'^2}^{(2)}} \text{tr} [R (\mathbf{Y}\mathbf{Y}^* / n - I) R]^2 \right) \leq \frac{2}{n} \sum_{i, j=1}^{d_{m^2, m'^2}} \|F_i F_j\|_2^2.$$

It remains to bound the norm of the products  $F_i F_j$  for any  $i, j$  between 1 and  $d_{m^2, m'^2}$ .

$$\sum_{i, j=1}^{d_{m^2, m'^2}} \|F_i F_j\|_2^2 = \sum_{i, j=1}^{d_{m^2, m'^2}} \sum_{k=1}^{p^2} F_i[k, k]^2 F_j[k, k]^2 = \sum_{k=1}^{p^2} \left( \sum_{i=1}^{d_{m^2, m'^2}} F_i[k, k]^2 \right)^2.$$

For any  $k \in \{1, \dots, p^2\}$ ,  $\sum_{i=1}^{d_{m^2, m'^2}} F_i[k, k]^2 \leq 1$  since  $(F_1, \dots, F_{d_{m^2, m'^2}})$  form an orthonormal family. Hence, we get

$$\sum_{i, j=1}^{d_{m^2, m'^2}} \|F_i F_j\|_2^2 \leq \sum_{k=1}^{p^2} \sum_{i=1}^{d_{m^2, m'^2}} F_i[k, k]^2 = d_{m^2, m'^2}.$$

All in all, we have proved that

$$\mathbb{E}(W_{m'}) \leq \frac{4\varphi_{\max}(\Sigma)}{np^2} \left[ 1 + \sqrt{\frac{2d_{m^2, m'^2}}{n}} \right]. \quad (14)$$

Gathering these three bounds and applying Proposition 8.1 allows to obtain the following deviation inequality:

$$\begin{aligned} & \mathbb{P} \left( Z_{m'} \geq \sqrt{\frac{2\varphi_{\max}(\Sigma)}{n}} \left\{ \sqrt{1 + \alpha/2} \sqrt{d_{m^2, m'^2}} + \xi \right\} \right) \\ &\leq \exp \left\{ - \left[ \frac{[(\sqrt{1 + \alpha/2} - 1) \sqrt{d_{m^2, m'^2}} + \xi]^2}{2L_1(1 + \sqrt{2d_{m^2, m'^2}/n})} \wedge \frac{\sqrt{n}[(\sqrt{1 + \alpha/2} - 1) \sqrt{d_{m^2, m'^2}} + \xi]}{\sqrt{2}L_2} \right] \right\} \\ &\leq \exp \left\{ - \left[ \frac{\omega_{m, m'}^2}{2L_1(1 + \sqrt{2d_{m^2, m'^2}/n})} \wedge \frac{\sqrt{n}\omega_{m, m'}}{\sqrt{2}L_2} \right] - \left[ \frac{\xi\omega_{m, m'}}{L_1[1 + \sqrt{2d_{m^2, m'^2}/n}]} \wedge \frac{\sqrt{n}\xi}{\sqrt{2}L_2} \right] \right\}, \end{aligned}$$

where  $\omega_{m,m'} = \left(\sqrt{1+\alpha/2}-1\right)\sqrt{d_{m^2,m'^2}}$ . As  $n$  and  $d_{m^2,m'^2}$  are larger than one, there exists a universal constant  $L'_2$  such that

$$\left[ \frac{(\sqrt{1+\alpha/2}-1)^2 d_{m^2,m'^2}}{2L_1(1+\sqrt{2d_{m^2,m'^2}/n})} \wedge \frac{\sqrt{n}(\sqrt{1+\alpha/2}-1)\sqrt{d_{m^2,m'^2}}}{\sqrt{2}L_2} \right] \geq 4L'_2\sqrt{d_{m^2,m'^2}} \left[ (\sqrt{1+\alpha/2}-1)^2 \wedge (\sqrt{1+\alpha/2}-1) \right].$$

Since the vector space  $U_{m,m'}$  contains all the matrices  $D(\theta')$  with  $\theta'$  belonging to  $m'$ ,  $d_{m^2,m'^2}$  is larger than  $d_{m'}$ . Besides, by concavity of the square root function, it holds that  $\sqrt{1+\alpha/2}-1 \geq \alpha[4\sqrt{1+\alpha/2}]^{-1}$ . Setting  $L'_1 := [4L_1(1+\sqrt{2})]^{-1} \wedge [\sqrt{2}L_2]^{-1}$  and arguing as previously leads to

$$\frac{\xi(\sqrt{1+\alpha/2}-1)\sqrt{d_{m^2,m'^2}}}{L_1(1+\sqrt{2d_{m^2,m'^2}/n})} \wedge \frac{\sqrt{n}\xi}{\sqrt{2}L_2} \geq L'_1\xi \left[ \frac{\alpha}{\sqrt{1+\alpha/2}} \wedge \sqrt{n} \right].$$

Gathering these two inequalities allows us to conclude that

$$\begin{aligned} & \mathbb{P} \left( Z_{m'} \geq \sqrt{\frac{2\varphi_{\max}(\Sigma)}{n}} \left\{ \sqrt{(1+\alpha/2)d_{m^2,m'^2}} + \xi \right\} \right) \\ & \leq \exp \left\{ -L'_2\sqrt{d_{m'}} \left( \frac{\alpha}{\sqrt{1+\alpha/2}} \wedge \frac{\alpha^2}{1+\alpha/2} \right) - L'_1\xi \left[ \frac{\alpha}{\sqrt{1+\alpha/2}} \wedge \sqrt{n} \right] \right\}. \end{aligned}$$

□

*Proof of Lemma 8.4 in [6].* The approach falls in two parts. First, we relate the dimensions  $d_m$  and  $d_{m^2}$  to the number of nodes of the torus  $\Lambda$  that are closer than  $r_m$  or  $2r_m$  to the origin  $(0,0)$ . We recall that the quantity  $r_m$  is introduced in Definition 2.1 of [6]. Second, we compute a nonasymptotic upper bound of the number of points in  $\mathbb{Z}^2$  that lie in the disc of radius  $r$ . This second step is quite tedious and will only give the main arguments.

Let  $m$  be a model of the collection  $\mathcal{M}_1$ . By definition,  $m$  is the set of points lying in the disc of radius  $r_m$  centered on  $(0,0)$ . Hence,

$$\Theta_m = \text{vect} \{ \Psi_{i,j}, (i,j) \in m \},$$

where the matrices  $\Psi_{i,j}$  are defined by Eq. (14) in [6]. As  $\Psi_{i,j} = \Psi_{-i,-j}$ , the dimension  $d_m$  of  $\Theta_m$  is exactly the number of orbits of  $m$  under the action of the central symmetry  $s$ .

As  $d_{m^2}$  is defined as the dimension of the space  $U_m$ , it also corresponds to the dimension of the space

$$\text{vect} \{ C(\theta), \theta \in \Theta_m \} + \text{vect} \{ C(\theta)^2, \theta \in \Theta_m \}, \quad (15)$$

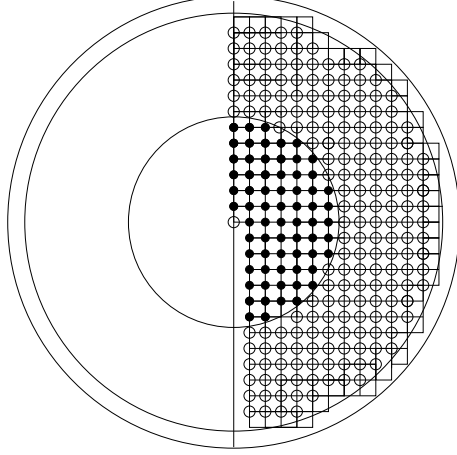


FIGURE 1. The black dots represent the orbit space of  $m$  and the white dots represent the remaining points of the orbit space of  $\mathcal{N}(m)$ .

which is clearly in one to one correspondence with  $U_m$ . Straightforward computations lead to the following identity:

$$\begin{aligned} C(\Psi_{i_1, j_1})C(\Psi_{i_2, j_2}) &= C(\Psi_{i_1+i_2, j_1+j_2}) [1 + s_{i_1+i_2, j_1+j_2}] \\ &\quad + C(\Psi_{i_1-i_2, j_1-j_2}) [1 + s_{i_1-i_2, j_1-j_2}] , \end{aligned}$$

where  $s_{x,y}$  is the indicator function of  $x = -x$  and  $y = -y$  in the torus  $\Lambda$ . Combining this property with the definition of  $\Theta_m$ , we embed the space (15) in the space

$$\text{vect} \{ C(\Psi_{i_1+i_2, j_1+j_2}), (i_1, j_1), (i_2, j_2) \in m \cup \{(0, 0)\} \} ,$$

and this last space is in one to one correspondence with

$$\text{vect} \{ \Psi_{i_1+i_2, j_1+j_2}, (i_1, j_1), (i_2, j_2) \in m \cup \{(0, 0)\} \} . \quad (16)$$

In the sequel,  $\mathcal{N}(m)$  stands for the set

$$\{(i_1 + i_2, j_1 + j_2), (i_1, j_1), (i_2, j_2) \in m \cup \{(0, 0)\} \} .$$

Thus, the dimension  $d_{m^2}$  is smaller or equal to the number of orbits of  $\mathcal{N}(m)$  under the action of the symmetry  $s$ .

To conclude, we have to compare the number of orbits in  $m$  and the number of orbits in  $\mathcal{N}(m)$ . We distinguish two cases depending whether  $2r_m + 1 \leq p$  or  $2r_m + 1 > p$ . First, we assume that  $2r_m + 1 \leq p$ . For such values the disc of radius  $r_m$  centered on the points  $(0, 0)$  is not overlapping itself on the torus except on a set of null Lebesgue measure. In the sequel,  $[x]$  refers to the largest integer smaller than  $x$ . We represent the orbit space of  $m$  as in Figure 1. To any of these points, we associate a square of size 1. If we add  $2 + 2[r_m]$  squares to

the  $d_m$  first squares, we remark that the half disc centered on  $(0,0)$  and with length  $r_m$  is contained in the reunion of these squares. Then, we get

$$d_m + 2 + 2\lfloor r_m \rfloor \geq \frac{\pi r_m^2}{2}. \quad (17)$$

The points in  $\mathcal{N}(m)$  are closer than  $2r_m$  from the origin. Consequently, all the squares associated to representants of  $\mathcal{N}(m)$  are included in the disc of radius  $2r_m + \sqrt{2}$ .

$$d_{m^2} + 2 + 2\lfloor 2r_m \rfloor \leq \frac{\pi}{2} \left\{ 2r_m + \sqrt{2} \right\}^2.$$

Combining these two inequalities, we are able to upper bound  $d_{m^2}$

$$\begin{aligned} 2 + 2\lfloor 2r_m \rfloor + d_{m^2} &\leq 4 \left\{ 1 + \frac{\sqrt{2}}{2r_m} \right\}^2 (d_m + 1 + 2\lfloor r_m \rfloor), \\ d_{m^2} &\leq 4 \left\{ 1 + \frac{\sqrt{2}}{2r_m} \right\}^2 d_m + 4 \left\{ 1 + \frac{\sqrt{2}}{2r_m} \right\}^2 (1 + 2\lfloor r_m \rfloor). \end{aligned}$$

Applying again inequality (17), we upper bound  $r_m$ :

$$r_m \leq \frac{2}{\pi} \left[ 1 + \sqrt{1 + \frac{\pi}{2}(1 + d_m)} \right].$$

Gathering these two last bounds yields

$$d_{m^2} \leq 4 \left\{ 1 + \frac{\sqrt{2}}{2r_m} \right\}^2 \left[ 1 + \frac{1}{d_m} \left( 1 + \frac{4}{\pi} \left[ 1 + \sqrt{1 + \frac{\pi}{2}(1 + d_m)} \right] \right) \right] d_m.$$

This upper bound is equivalent to  $4d_m$ , when  $d_m$  goes to infinity. Computing the ratio  $d_{m^2}/d_m$  for every model  $m$  of small dimension allows to conclude.

Let us turn to the case  $2r_m + 1 > p$ . Suppose that  $p$  is larger or equal to 9. The lower bound (17) does not necessarily hold anymore. Indeed, the disc is overlapping with itself because of toroidal effects. Nevertheless, we obtain a similar lower bound by replacing  $r_m$  by  $(p-1)/2$ :

$$d_m + 2 + 2\lfloor \frac{p-1}{2} \rfloor \geq \frac{\pi(p-1)^2}{8}.$$

The number of orbits of  $\Lambda$  under the action of the symmetry  $s$  is  $(p^2+1)/2$  if  $p$  is odd and  $[(p+1)^2-1]/2$  if  $p$  is even. It follows that  $d_{m^2} \leq [(p+1)^2-1]/2$ . Gathering these two bounds, we get

$$\frac{d_{m^2}}{d_m} \leq \frac{(p+1)^2}{\pi(p-1)^2/4 - 2(p+1)}.$$

This last quantity is smaller than 4 for any  $p \geq 9$ . An exhaustive computation of the ratios when  $p < 9$  allows to conclude.

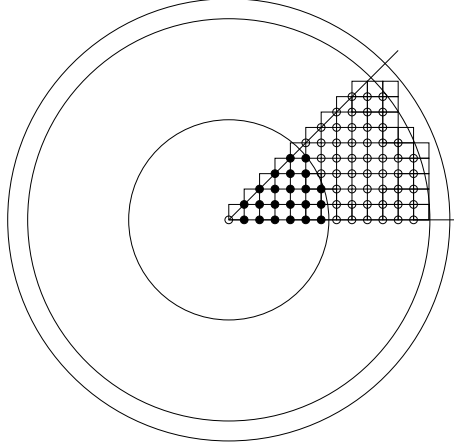


FIGURE 2. The black dots represent the orbit space of  $m$  under the action of  $G$  and the white dots represent the remaining points of the orbit space of  $\mathcal{N}^{\text{iso}}(m)$ .

Let us turn to the isotropic case. Arguing as previously, we observe that the dimension  $d_m^{\text{iso}}$  is the number of orbits of the set  $m$  under the action of the group  $G$  introduced in in [6] Sect.1.1 whereas  $d_{m^2}$  is smaller or equal to the number of orbits of  $\mathcal{N}^{\text{iso}}(m)$  under the action of  $G$ . As for anisotropic models, we choose represent these orbits on the torus and associate squares of size 1 (see Figure 2). Assuming that  $r_m < (p-1)/2$ , we bound  $d_m$  and  $d_{m^2}$ .

$$d_m + 1 \geq \frac{1}{8}\pi r_m^2 + \frac{1}{2} \lfloor \frac{\sqrt{2}r_m}{2} \rfloor,$$

$$d_{m^2} \leq 4 \left\{ 1 + \frac{\sqrt{2}}{2r_m} \right\}^2 \frac{1}{8}\pi r_m^2 + \frac{1}{2} \lfloor \sqrt{2}r_m \rfloor.$$

Gathering these two inequalities, we get

$$d_{m^2} \leq 4 \left\{ 1 + \frac{\sqrt{2}}{2r_m} \right\}^2 d_m.$$

As a consequence,  $d_{m^2}$  is smaller than  $4d_m$  when  $d_m$  goes to infinity. As previously, computing the ratio  $d_{m^2}/d_m$  for models  $m$  of small dimension allows to conclude. The case  $r_m > (p-1)/2$  is handled as for the anisotropic case.  $\square$

### 3. Proofs of the minimax bounds

*Proof of Lemma 8.5 in [6].* This lower bound is based on an application of Fano's approach. See [7] for a review of this method and comparisons with Le Cam's and

Assouad's Lemma. The proof follows three main steps: First, we upper bound the Kullback-Leibler entropy between distributions corresponding to  $\theta_1$  and  $\theta_2$  in the hypercube. Second, we find a set of points in the hypercube well separated with respect to the Hamming distance. Finally, we conclude by applying Birgé's version of Fano's lemma.

**Lemma 3.1.** *The Kullback-Leibler entropy between two mean zero-Gaussian vectors of size  $p^2$  with precision matrices  $(I_{p^2} - C(\theta_1)) / \sigma^2$  and  $(I_{p^2} - C(\theta_2)) / \sigma^2$  equals*

$$\mathcal{K}(\theta_1, \theta_2) = 1/2 \left[ \log \left( \frac{|I_{p^2} - C(\theta_1)|}{|I_{p^2} - C(\theta_2)|} \right) + \text{tr} \left( [I_{p^2} - C(\theta_2)] [I_{p^2} - C(\theta_1)]^{-1} \right) - p^2 \right],$$

where for any square matrix  $A$ ,  $|A|$  refers to the determinant of  $A$ .

This statement is classical and its proof is omitted. The matrices  $(I_{p^2} - C(\theta_1))$  and  $(I_{p^2} - C(\theta_2))$  are diagonalizable in the same basis since they are symmetric block circulant (Lemma A.1 in [6]). Transforming vectors of size  $p^2$  into  $p \times p$  matrices, we respectively define  $\lambda_1$  and  $\lambda_2$  as the  $p \times p$  matrices of eigenvalues of  $(I_{p^2} - C(\theta_1))$  and  $(I_{p^2} - C(\theta_2))$ . It follows that

$$\mathcal{K}(\theta_1, \theta_2) = 1/2 \sum_{1 \leq i, j \leq p} \left( \frac{\lambda_{2[i,j]}}{\lambda_{1[i,j]}} - \log \left( \frac{\lambda_{2[i,j]}}{\lambda_{1[i,j]}} \right) - 1 \right).$$

For any  $x > 0$ , the following inequality holds

$$x - 1 - \log(x) \leq \frac{9}{64} \left( x - \frac{1}{x} \right)^2.$$

It is easy to establish by studying the derivative of corresponding functions. As a consequence,

$$\begin{aligned} \frac{\lambda_{2[i,j]}}{\lambda_{1[i,j]}} - \log \left( \frac{\lambda_{2[i,j]}}{\lambda_{1[i,j]}} \right) - 1 &\leq \frac{9}{64} \left( \frac{\lambda_{2[i,j]}}{\lambda_{1[i,j]}} - \frac{\lambda_{1[i,j]}}{\lambda_{2[i,j]}} \right)^2 \\ &\leq \frac{9}{64} \left( \frac{1}{\lambda_{1[i,j]}} + \frac{1}{\lambda_{2[i,j]}} \right)^2 (\lambda_{1[i,j]} - \lambda_{2[i,j]})^2 \end{aligned} \quad (18)$$

Let us first consider the anisotropic case. Let  $m$  be a model in  $\mathcal{M}_1$  and let  $\theta'$  belong  $\Theta_m \cap \mathcal{B}_1(0_p, 1)$ . We also consider a positive radius  $r$  such that  $(1 - \|\theta'\|_1 - 2rd_m)$  is positive. For any  $\theta_1, \theta_2$  in  $\mathcal{C}_m(\theta', r)$  the matrices  $(I_{p^2} - C(\theta_1))$  and  $(I_{p^2} - C(\theta_2))$  are diagonally dominant and their eigenvalues  $\lambda_{1[i,j]}$  and  $\lambda_{2[i,j]}$  are larger than  $1 - \|\theta'\|_1 - 2rd_m$ .

$$\begin{aligned} \mathcal{K}(\theta_1, \theta_2) &\leq \frac{9}{16(1 - \|\theta'\|_1 - 2rd_m)^2} \sum_{1 \leq i, j \leq p} (\lambda_{1[i,j]} - \lambda_{2[i,j]})^2 \\ &\leq \frac{9}{16(1 - \|\theta'\|_1 - 2rd_m)^2} \|C(\theta_1) - C(\theta_2)\|_F^2 \\ &\leq \frac{9d_m r^2 p^2}{8(1 - \|\theta'\|_1 - 2rd_m)^2}. \end{aligned} \quad (19)$$



We recall that  $\|\cdot\|_F$  refers to the Frobenius norm in the space of matrices.

Let us state Birgé's version of Fano's lemma [2] and a combinatorial argument known under the name of Varshamov-Gilbert's lemma. These two lemma are taken from [4] and respectively correspond to Corollary 2.18 and Lemma 4.7.

**Lemma 3.2. (Birgé's lemma)** *Let  $(S, d)$  be some pseudo-metric space and  $\{\mathbb{P}_s, s \in S\}$  be some statistical model. Let  $\kappa$  denote some absolute constant smaller than one. Then for any estimator  $\hat{s}$  and any finite subset  $T$  of  $S$ , setting  $\delta = \min_{s, t \in T, s \neq t} d(s, t)$ , provided that  $\max_{s, t \in T} \mathcal{K}(\mathbb{P}_s, \mathbb{P}_t) \leq \kappa \log |T|$ , the following lower bound holds for every  $p \geq 1$ ,*

$$\sup_{s \in S} \mathbb{E}_s [d^p(s, \hat{s})] \geq 2^{-p} \delta^p (1 - \kappa) .$$

**Lemma 3.3. (Varshamov-Gilbert's lemma)** *Let  $\{0, 1\}^d$  be equipped with Hamming distance  $d_H$ . There exists some subset  $\Phi$  of  $\{0, 1\}^d$  with the following properties*

$$d_H(\phi, \phi') > d/4 \text{ for every } (\phi, \phi') \in \Phi^2 \text{ with } \phi \neq \phi' \text{ and } \log |\Phi| \geq \frac{d}{8} .$$

Applying Lemma 3.2 with Hamming distance  $d_H$  and the set  $\Phi$  introduced in Lemma 3.3 yields

$$\sup_{\theta \in \mathcal{C}_m(\theta', r)} \mathbb{E}_\theta \left[ d_H(\hat{\theta}, \theta) \right] \geq \frac{d_m}{8} (1 - \kappa) , \quad (20)$$

provided that

$$\frac{9d_m r^2 p^2 n}{8(1 - \|\theta'\|_1 - 2rd_m)^2} \leq \frac{\kappa d_m}{8} . \quad (21)$$

Let us express (20) in terms of the Frobenius  $\|\cdot\|_F$  norm.

$$\sup_{\theta \in \mathcal{C}_m(\theta', r)} \mathbb{E}_\theta \left[ \|C(\hat{\theta}) - C(\theta)\|_F^2 \right] \geq \frac{d_m r^2 p^2}{4} (1 - \kappa) .$$

Since for every  $\theta$  in the hypercube,  $\sigma^{-2}(I_{p^2} - C(\theta))$  is diagonally dominant, its largest eigenvalue is smaller than  $2\sigma^{-2}$ . The loss function  $l(\hat{\theta}, \theta)$  equals  $\sigma^2/p^2 \text{tr}\{[C(\hat{\theta}) - C(\theta)](I - C(\theta))^{-1}[C(\hat{\theta}) - C(\theta)]\}$ . It follows that

$$\sup_{\theta \in \mathcal{C}_m(\theta', r)} \mathbb{E}_\theta \left[ l(\hat{\theta}, \theta) \right] \geq \sigma^2 \frac{d_m r^2}{8} (1 - \kappa) . \quad (22)$$

Condition (21) is equivalent to  $r^2(1 - \|\theta'\|_1 - 2rd_m)^{-2} \leq \kappa/(9p^2n)$ . If we assume that

$$r^2 \leq \frac{\kappa(1 - \|\theta'\|_1)^2}{18p^2n} , \quad (23)$$

then  $1 - \|\theta'\|_1 - 2rd_m \geq (1 - \|\theta'\|_1) \left(1 - 2d_m \sqrt{\kappa/(18np^2)}\right)$ . This last quantity is larger than  $(1 - \|\theta'\|_1)/\sqrt{2}$  if  $d_m$  is smaller than  $1.5(\sqrt{2}-1)\sqrt{np^2/\kappa}$ . Gathering inequality (22) and condition (23), we get the lower bound

$$\begin{aligned} \inf_{\widehat{\theta}} \sup_{\theta \in \text{Co}[\mathcal{C}_m(\theta', r)]} \mathbb{E}_\theta \left[ l(\widehat{\theta}, \theta) \right] &\geq \inf_{\widehat{\theta}} \sup_{\theta \in \mathcal{C}_m \left[ \theta', r \wedge (1 - \|\theta'\|_1) \sqrt{\frac{\kappa}{18p^2 n}} \right]} \mathbb{E}_\theta \left[ l(\widehat{\theta}, \theta) \right] \\ &\geq L \left( r^2 \wedge \frac{(1 - \|\theta'\|_1)^2}{np^2} \right) d_m \sigma^2 . \end{aligned}$$

One handles models of dimension  $d_m$  between  $1.5(\sqrt{2}-1)\sqrt{np^2/\kappa}$  and  $\sqrt{np}$  by changing the constant  $L$  in the last lower bound.

Let us turn to sets of isotropic GMRFs. The proof is similar to the non-isotropic case, except for a few arguments. Let  $m$  belongs to the collection  $\mathcal{M}_1$  and let  $\theta'$  be an element of  $\Theta_m^{\text{iso}} \cap \mathcal{B}_1(0_p, 1)$ . Let  $r$  be such that  $1 - \|\theta'\|_1 - 8d_m^{\text{iso}}$  is positive. If  $\theta_1$  and  $\theta_2$  belong to the hypercube  $\mathcal{C}_m^{\text{iso}}(\theta', r)$ , then

$$\mathcal{K}(\theta_1, \theta_2) \leq \frac{9d_m r^2 p^2}{2(1 - \|\theta'\|_1 - 8rd_m^{\text{iso}})^2} .$$

Applying Lemma 3.2 and 3.3, it follows that

$$\inf_{\widehat{\theta}} \sup_{\theta \in \mathcal{C}_m^{\text{iso}}(\theta', r)} \mathbb{E}_\theta \left[ d_H(\widehat{\theta}, \theta) \right] \geq \frac{d_m^{\text{iso}}}{8}(1 - \kappa) ,$$

provided that  $4.5d_m r^2 p^2 n(1 - \|\theta'\|_1 - 8rd_m^{\text{iso}})^{-2} \leq \kappa d_m^{\text{iso}}/8$ . As a consequence,

$$\inf_{\widehat{\theta}} \sup_{\theta \in \mathcal{C}_m^{\text{iso}}(\theta', r)} \mathbb{E}_\theta \left[ l(\widehat{\theta}, \theta) \right] \geq \frac{d_m^{\text{iso}} r^2}{8}(1 - \kappa) ,$$

if  $r^2 (1 - \|\theta'\|_1 - 8rd_m^{\text{iso}})^{-2} \leq \kappa(36p^2 n)^{-1}$ . We conclude by arguing as in the isotropic case.  $\square$

*Proof of lemma 8.6 in [6].* Let  $m$  be a model in  $\mathcal{M}_1$ ,  $r$  be a positive number smaller than  $1/(4d_m)$ , and  $\theta$  be an element of the convex hull of  $\mathcal{C}_m(0_p, r)$ . The covariance matrix of the vector  $X^v$  is  $\Sigma = \sigma^2 [I - C(\theta)]^{-1}$ . Since the field  $X$  is stationary,  $\text{Var}_\theta(X_{[0,0]})$  equals any diagonal element of  $\Sigma$ . In particular,  $\text{Var}_\theta(X_{[0,0]})$  corresponds to the mean of the eigenvalues of  $\Sigma$ . The matrix  $(I - C(\theta))$  is block circulant. As in the proof of Lemma 20, we note  $\lambda$  the  $p \times p$  matrix of the eigenvalues of  $(I_{p^2} - C(\theta))$ . By Lemma A.1 in [6],

$$\lambda_{[i,j]} = 1 + \sum_{(k,l) \in \Lambda} \theta_{[k,l]} \cos \left[ 2\pi \left( \frac{ik}{p} + \frac{jl}{p} \right) \right] ,$$

for any  $1 \leq i, j \leq p$ . Since  $\theta$  belongs to the convex hull of  $\mathcal{C}_m(0_p, r)$ ,  $\theta_{[k,l]}$  is zero if  $(k, l) \notin m$  and  $|\theta_{[k,l]}| \leq r$  if  $(k, l) \in m$ . Thus  $\sum_{(k,l) \in \Lambda} |\theta_{[k,l]}|$  is smaller than  $1/2$ . Applying Taylor-Lagrange inequality, we get

$$\frac{1}{1+x} \leq 1 - x + \frac{x^2}{(1-|x|)^3},$$

for any  $x$  between  $-1$  and  $1$ . It follows that

$$\lambda_{[i,j]}^{-1} \leq 1 - \sum_{k,l \in \Lambda} \theta_{[k,l]} \cos \left[ 2\pi \left( \frac{ik}{p} + \frac{jl}{p} \right) \right] + 8 \left\{ \sum_{k,l \in \Lambda} \theta_{[k,l]} \cos \left[ 2\pi \left( \frac{ik}{p} + \frac{jl}{p} \right) \right] \right\}^2.$$

Summing this inequality for all  $(i, j) \in \{1, \dots, p\}^2$ , the first order term turns out to be  $\text{tr}[C(\theta)]/p^2$  which is zero whereas the second term equals  $8\text{tr}[C(\theta)^2]/p^2$ . Since there are less than  $2d_m$  non-zero terms on each line of the matrix  $C(\theta)$ , its Frobenius norm is smaller than  $2d_m p^2 r^2$ . Consequently, we obtain

$$\text{Var}_\theta(X_{[0,0]}) \leq \sigma^2 (1 + 16d_m r^2).$$

□

*Proof of Lemma 8.7 in [6].* This property seems straightforward but the proof is a bit tedious. Let  $i$  be a positive integer smaller than  $\text{Card}(\mathcal{M}_1)$ . By definition of the radius  $r_m$  in Equation (10) in [6], the model  $m_{i+1}$  is the set of nodes in  $\Lambda \setminus \{(0, 0)\}$  at a distance smaller or equal to  $r_{m_{i+1}}$  from  $(0, 0)$ , whereas the model  $m_i$  only contains the points in  $\Lambda \setminus \{(0, 0)\}$  at a distance strictly smaller than  $r_{m_{i+1}}$  from the origin.

Let us first assume that  $2r_{m_{i+1}} \leq p$ . In such a case, the disc centered on  $(0, 0)$  with radius  $r_{m_{i+1}}$  does not overlap with itself on the torus  $\Lambda$ . To any node in the neighborhood  $m_{i+1}$  and to the node  $(0, 0)$ , we associate the square of size 1 centered on it. All these squares do not overlap and are included in the disc of radius  $r_{m_{i+1}} + \sqrt{2}/2$ . Hence, we get the upper bound  $2d_{m_{i+1}} + 1 \leq \pi(r_{m_{i+1}} + \sqrt{2}/2)^2$ . Similarly, the disc of radius  $r_{m_{i+1}} - \sqrt{2}/2$  is included in the union of the squares associated to the nodes  $m_i \cup \{0, 0\}$ . It follows that  $2d_{m_i} + 1$  is larger or equal to  $\pi(r_{m_{i+1}} - \sqrt{2}/2)^2$ . Gathering these two inequalities, we obtain

$$\frac{d_{m_{i+1}}}{d_{m_i}} \leq \frac{(r_{m_{i+1}} + \sqrt{2}/2)^2 - 1}{(r_{m_{i+1}} - \sqrt{2}/2)^2 - 1},$$

if  $r_{m_{i+1}}$  is larger than  $1 + \sqrt{2}/2$ . If  $r_{m_{i+1}}$  larger than 5, this upper bound is smaller than two. An exhaustive computation for models of small dimension allows to conclude.

If  $2r_{m_{i+1}} \geq p$  and  $2r_{m_i} < p$ , then the preceding lower bound of  $d_{m_i}$  and the preceding upper bound of  $d_{m_{i+1}}$  still hold. Finally, let us assume that  $2r_{m_i} \geq p$ . Arguing as previously, we conclude that  $2d_{m_i} + 1 \geq \pi(p/2 - \sqrt{2}/2)^2$ . The largest dimension of a model  $m \in \mathcal{M}_1$  is  $(p^2 - 1)/2$  if  $p$  is odd and  $((p + 1)^2 - 3)/2$  if  $p$  is even. Thus,  $d_{m_{i+1}} \leq [(p + 1)^2 - 3]/2$ . Gathering these two bounds yields

$$\frac{d_{m_{i+1}}}{d_{m_i}} \leq 4 \frac{(p + 1)^2 - 3}{(p - \sqrt{2})^2},$$

which is smaller than 2 if  $p$  is larger than 10. Exhaustive computations for small  $p$  allow to conclude.  $\square$

*Proof of Proposition 6.7 in [6].* This result derives from the upper bound of the risk of  $\tilde{\theta}_{\rho_1}$  stated in Theorem 3.1 and the minimax lower bound stated in Proposition 6.6 in [6].

Let  $\mathcal{E}(a)$  be a pseudo-ellipsoid that satisfies Assumption  $(\mathbb{H}_a)$  and such that  $a_1^2 \geq 1/(np^2)$ . For any  $\theta$  in  $\mathcal{E}(a) \cap \mathcal{B}_1(0_p, 1) \cap \mathcal{U}(\rho_2)$ , the penalty term satisfies  $\text{pen}(m) = K\sigma^2\rho_1^2\rho_2 d_m/np^2$  is larger than  $Kd_m\varphi_{\max}(\Sigma)/np^2$ . Applying Theorem 3.1, we upper bound the risk  $\theta_{\rho_1}$

$$\mathbb{E}_\theta \left[ l(\tilde{\theta}_{\rho_1}, \theta) \right] \leq L_1(K) \inf_{m \in \mathcal{M}_1} [l(\theta_{m, \rho_1}, \theta) + \text{pen}(m)] + L_2(K)\rho_2 \frac{\sigma^2}{np^2},$$

for any  $\theta \in \mathcal{E}(a) \cap \mathcal{B}_1(0_p, 1) \cap \mathcal{U}(\rho_2)$ . It follows that

$$\sup_{\theta \in \mathcal{E}(a) \cap \mathcal{B}_1(0_p, 1) \cap \mathcal{U}(\rho_2)} \mathbb{E}_\theta \left[ l(\tilde{\theta}_{\rho_1}, \theta) \right] \leq L(K) \inf_{m \in \mathcal{M}_1, d_m > 0} \left[ l(\theta_{m, \rho_1}, \theta) + \rho_1^2 \rho_2 \sigma^2 \frac{d_m}{np^2} \right].$$

Let  $i$  be a positive integer smaller or equal than  $\text{Card}(\mathcal{M}_1)$ . We know from Section 4.1 in [6] that the bias  $l(\theta_{m_i}, \theta)$  of the model  $m_i$  equals  $\text{Var}(X_{[0,0]}|X_{m_i}) - \sigma^2$ . Since  $\theta$  belongs to the set  $\mathcal{E}(a) \cap \mathcal{B}_1(0_p, 1)$ , the bias term is smaller or equal to  $a_{i+1}^2$  with the convention  $a_{\text{Card}(\mathcal{M}_1)+1}^2 = 0$ . Hence, the previous upper bound becomes

$$\begin{aligned} \mathbb{E}_\theta \left[ l(\tilde{\theta}_{\rho_1}, \theta) \right] &\leq L(K) \inf_{1 \leq i \leq \text{Card}(\mathcal{M}_1)} \left[ a_{i+1}^2 + \rho_1^2 \rho_2 \sigma^2 \frac{d_{m_i}}{np^2} \right] \\ &\leq L(K, \rho_1, \rho_2) \inf_{1 \leq i \leq \text{Card}(\mathcal{M}_1)} \left[ a_{i+1}^2 + \frac{\sigma^2 d_{m_i}}{np^2} \right]. \end{aligned} \quad (24)$$

Applying Proposition 6.6 in [6] to the set  $\mathcal{E}(a) \cap \mathcal{B}_1(0_p, 1) \cap \mathcal{U}(2)$ , we get

$$\begin{aligned} \inf_{\hat{\theta}} \sup_{\theta \in \mathcal{E}(a) \cap \mathcal{B}_1(0_p, 1) \cap \mathcal{U}(\rho_2)} \mathbb{E}_\theta \left[ l(\hat{\theta}, \theta) \right] &\geq \inf_{\hat{\theta}} \sup_{\theta \in \mathcal{E}(a) \cap \mathcal{B}_1(0_p, 1) \cap \mathcal{U}(2)} \mathbb{E}_\theta \left[ l(\hat{\theta}, \theta) \right] \\ &\geq L \sup_{1 \leq i \leq \text{Card}(\mathcal{M}_1)} \left( a_i^2 \wedge \sigma^2 \frac{d_{m_i}}{np^2} \right). \end{aligned}$$

Let us define  $i^*$  by

$$i^* := \sup \left\{ 1 \leq i \leq \text{Card}(\mathcal{M}_1), a_i^2 \geq \frac{\sigma^2 d_{m_i}}{np^2} \right\},$$

with the convention  $\sup \emptyset = 0$ . Since  $a_1^2 \geq \sigma^2/np^2$ ,  $i^*$  is larger or equal to one. It follows that

$$\inf_{\hat{\theta}} \sup_{\theta \in \mathcal{E}(a) \cap \mathcal{B}_1(0_p, \eta)} \mathbb{E}_\theta \left[ l(\hat{\theta}, \theta) \right] \geq L_2 \left( a_{i^*+1}^2 \vee \frac{\sigma^2 d_{m_{i^*}}}{np^2} \right).$$

Meanwhile, the upper bound (24) on the risk of  $\tilde{\theta}_{\rho_1}$  becomes

$$\mathbb{E}_\theta \left[ l(\tilde{\theta}_{\rho_1}, \theta) \right] \leq L(K, \rho_1, \rho_2) \left( a_{i^*+1}^2 + \frac{\sigma^2 d_{m_{i^*}}}{np^2} \right) \leq 2L(K, \rho_1, \rho_2) \left( a_{i^*+1}^2 \vee \frac{\sigma^2 d_{m_{i^*}}}{np^2} \right),$$

which allows to conclude.  $\square$

#### 4. Proof of the asymptotic risks bounds

*Proof of Corollary 4.6 in [6].* For the sake of simplicity, we assume that for any node  $(i, j) \in m$ , the nodes  $(i, j)$  and  $(-i, -j)$  are different in  $\Lambda$ . If this is not the case, we only have to slightly modify the proof in order to take account that  $\|\Psi_{i,j}\|_F^2$  may equal one. The matrix  $V$  is the covariance of the vector of size  $d_m$

$$(X_{i_1, j_1} + X_{-i_1, -j_1}, \dots, X_{i_{d_m}, j_{d_m}} + X_{-i_{d_m}, -j_{d_m}}). \quad (25)$$

Since the matrix  $\Sigma$  of  $X^v$  is positive,  $V$  is also positive. Moreover, its largest eigenvalue is larger than  $2\varphi_{\max}(\Sigma)$ .

Let us assume first the  $\theta$  belongs to  $\Theta_m^+$  and that Assumption  $(\mathbb{H}_1)$  is fulfilled. By the first result of Proposition 4.4 in [6],

$$\lim_{n \rightarrow +\infty} np^2 \mathbb{E} \left[ l(\hat{\theta}_{m, \rho_1}, \theta) \right] = 2\sigma^4 \text{tr} [IL_m V^{-1}] \geq \frac{\sigma^4}{\varphi_{\max}(\Sigma)} \text{tr} [IL_m] = 2\sigma^4 \frac{d_m}{\varphi_{\max}(\Sigma)},$$

which corresponds to the first lower bound (30) in [6].

Let us turn to the second result. We now assume that  $\theta$  satisfies Assumption  $(\mathbb{H}_2)$ . By the identity (28) of Proposition 4.4 in [6], we only have to lower bound the quantity  $\text{tr} [VW^{-1}]$ .

$$\text{tr} [V^{-1}W] \geq \varphi_{\max}(V)^{-1} \text{tr} [W] \geq \frac{1}{2\varphi_{\max}(\Sigma)} \text{tr} [W].$$

Since the matrix  $\Sigma^{-1} = \sigma^{-2} [I_{p^2} - C(\theta)]$  is diagonally dominant, its smallest eigenvalue is larger than  $\sigma^{-2}(1 - \|\theta\|_1)$ . The matrix  $(I_{p^2} - C(\theta_{m, \rho_1}))^2 (I_{p^2} - C(\theta))^{-2}$

is symmetric positive. It follows that  $W$  is also symmetric positive definite. Hence, we get

$$\begin{aligned} & \text{tr} [V^{-1}W] \\ & \geq \frac{\sigma^{-2}}{2} [1 - \|\theta\|_1] \sum_{k=1}^{d_m} \frac{\text{tr} \left[ C(\Psi_{i_k, j_k})^2 [I_{p^2} - C(\theta_{m, \rho_1})]^2 [I_{p^2} - C(\theta)]^{-2} \right]}{p^2}. \end{aligned} \quad (26)$$

The largest eigenvalue of  $(I_{p^2} - C(\theta))$  is smaller than 2 and the smallest eigenvalue of  $(I_{p^2} - C(\theta_{m, \rho_1}))$  is larger than  $1 - \|\theta_{m, \rho_1}\|_1$ . By Lemma A.1 in [6], these two matrices are jointly diagonalizable and the smallest eigenvalue of

$$(I_{p^2} - C(\theta_{m, \rho_1}))^2 (I_{p^2} - C(\theta))^{-2}$$

is therefore larger than  $(1 - \|\theta_{m, \rho_1}\|_1)^2/4$ . Gathering this lower bound with (26) yields

$$\text{tr} [V^{-1}W] \geq \frac{d_m \sigma^{-2}}{2} [1 - \|\theta\|_1] [1 - \|\theta_{m, \rho_1}\|_1]^2.$$

Lemma 4.1 in [6] states that  $\|\theta_{m, \rho_1}\|_1 \leq \|\theta\|_1$ . Combining these two lower bounds enables to conclude.  $\square$

*Proof of Example 4.8 in [6].*

**Lemma 4.1.** *For any  $\theta$  is the space  $\Theta_{m_1}^{+, \text{iso}}$ , the asymptotic variance term of  $\hat{\theta}_{m_1, \rho_1}^{\text{iso}}$  equals*

$$\lim_{n \rightarrow +\infty} np^2 \mathbb{E}_\theta \left[ l \left( \hat{\theta}_{m_1, \rho_1}^{\text{iso}}, \theta \right) \right] = 2\sigma^4 \frac{\text{tr} (H^2)}{\text{tr} (H^2 \Sigma)}.$$

*If  $\theta$  belongs to  $\Theta^{+, \text{iso}}$  and also satisfies  $(\mathbb{H}_2)$ , then*

$$\lim_{n \rightarrow +\infty} np^2 \mathbb{E}_\theta \left[ l \left( \hat{\theta}_{m_1, \rho_1}^{\text{iso}}, \theta_{m_1, \rho_1}^{\text{iso}} \right) \right] = 2 \frac{\text{tr} \left\{ \left[ (I - \theta_{m_1, \rho_1}^{\text{iso}} [1, 0] H) H \Sigma \right]^2 \right\}}{\text{tr} (H^2 \Sigma)}, \quad (27)$$

*where the  $p^2 \times p^2$  matrix  $H$  is defined as  $H := C(\Psi_{1,0}^{\text{iso}})$ .*

*Proof of Lemma 4.1.* Apply Proposition 4.4 in [6] noting that  $V = \text{tr}[H\Sigma H]/p^2$  and

$$W = \frac{\text{tr} \left\{ \left[ (I - \theta_{m_1, \rho_1}^{\text{iso}} [1, 0] H) H \Sigma \right]^2 \right\}}{\sigma^4 p^2}.$$

To prove the second result, we observe that  $\Theta_{m_1}^{+, \text{iso}}$  equals  $\Theta_{m_1, 2}^{+, \text{iso}}$ . It is stated for instance in Table 2 in [6].  $\square$

Since the matrix  $\theta$  belongs to  $\Theta_{m_1}^{+, \text{iso}}$ , we may apply the second result of Lemma 4.1. Straightforward computations lead to  $\text{tr}(H^2) = \|C(\Psi_{1,0}^{\text{iso}})\|_F^2 = 4p^2$  and

$$\text{tr}(H^2\Sigma) = 4p^2 [\text{Var}(X_{[0,0]}) + 2\text{cov}_\theta(X_{[0,0]}, X_{[1,1]}) + \text{cov}_\theta(X_{[0,0]}, X_{[2,0]})] .$$

Since the field  $X$  is an isotropic GMRF with four nearest neighbors,

$$X_{[0,0]} = \theta_{[1,0]} (X_{[1,0]} + X_{[-1,0]} + X_{[0,1]} + X_{[0,-1]}) + \epsilon_{[0,0]} ,$$

where  $\epsilon_{[0,0]}$  is independent from every variable  $X_{[i,j]}$  with  $(i,j) \neq 0$ . Multiplying this identity by  $X_{[1,0]}$  and taking the expectation yields

$$\text{cov}_\theta(X_{[0,0]}, X_{[1,0]}) = \theta_{[1,0]} [\text{Var}(X_{[0,0]}) + 2\text{cov}_\theta(X_{[0,0]}, X_{[1,1]}) + \text{cov}_\theta(X_{[0,0]}, X_{[2,0]})] .$$

Hence, we obtain  $\text{tr}(H^2\Sigma) = 4\text{cov}_\theta(X_{[0,0]}, X_{[1,0]})/\theta_{[1,0]}$  and

$$\frac{\text{tr}(H^2)}{\text{tr}(H^2\Sigma)} = \frac{\theta_{[1,0]}}{\text{cov}_\theta(X_{[0,0]}, X_{[1,0]})} ,$$

which concludes the first part of the proof.

This second part is based on the spectral representation of the field  $X$  and follows arguments which come back to Moran [5]. We shall compute the limit of  $\text{cov}_\theta(X_{[0,0]}, X_{[1,0]})$  when the size of  $\Lambda$  goes to infinity. As the field  $X$  is stationary on  $\Lambda$ , we may diagonalize its covariance matrix  $\Sigma$  applying Lemma A.1 in [6]. We note  $D_\Sigma$  the corresponding diagonal matrix defined by

$$D_{\Sigma[(i-1)p+j, (i-1)p+j]} = \sum_{k=1}^p \sum_{l=1}^p \text{cov}_\theta(X_{[0,0]}, X_{[k,l]}) \cos \left[ 2\pi \left( \frac{ki}{p} + \frac{lj}{p} \right) \right] ,$$

for any  $1 \leq i, j \leq p$ . Straightforwardly, we express  $\text{cov}_\theta(X_{[0,0]}, X_{[1,0]})$  as a linear combination of the eigenvalues

$$\text{cov}_\theta(X_{[0,0]}, X_{[1,0]}) = \frac{1}{p^2} \sum_{i=1}^p \sum_{j=1}^p \cos \left( 2\pi \frac{i}{p} \right) D_{\Sigma[(i-1)p+j, (i-1)p+j]} .$$

Applying Lemma A.1 in [6] to the matrix  $\Sigma^{-1}$  and noting that  $\theta \in \Theta^{\text{iso}, +}$  allows to get another expression of the eigenvalues of  $\Sigma$

$$D_{\Sigma[(i-1)p+j, (i-1)p+j]} = \frac{\sigma^2}{1 - 2\theta_{[1,0]} \left[ \cos \left( \frac{2\pi i}{p} \right) + \cos \left( \frac{2\pi j}{p} \right) \right]} .$$

We then combine these expression. By symmetry between  $i$  and  $j$  we get

$$\text{cov}_\theta(X_{[0,0]}, X_{[1,0]}) = \frac{\sigma^2}{2p^2} \sum_{i=1}^p \sum_{j=1}^p \frac{\cos \left( 2\pi \frac{i}{p} \right) + \cos \left( 2\pi \frac{j}{p} \right)}{1 - 2\theta_{[1,0]} \left[ \cos \left( 2\pi \frac{i}{p} \right) + \cos \left( 2\pi \frac{j}{p} \right) \right]} .$$

If we let  $p$  go to infinity, this sum converges to the following integral

$$\begin{aligned} & \lim_{p \rightarrow +\infty} \text{cov}_\theta (X_{[0,0]}, X_{[1,0]}) \\ &= \frac{\sigma^2}{2} \int_0^1 \int_0^1 \frac{\cos(2\pi x) + \cos(2\pi y)}{1 - 2\theta_{[1,0]} (\cos(2\pi x) + \cos(2\pi y))} dx dy \\ &= \frac{\sigma^2}{2\theta_{[1,0]}} \left[ -1 + \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{1}{1 - 2\theta_{[1,0]} [\cos(x) + \cos(y)]} dx dy \right]. \end{aligned}$$

This last elliptic integral is asymptotically equivalent to  $\log 16[4(1 - 4\theta_{[1,0]})]^{-1}$  when  $\theta_{[1,0]} \rightarrow 1/4$  as observed for instance by Moran [5]. We conclude by substituting this limit in expression (33) in [6].  $\square$

*Proof of Example 4.9 in [6].* First, we compute  $[\theta^{(p)}]_{m_1}^{\text{iso}}_{[1,0]}$ . By Lemma 4.1 in [6], it minimizes the function  $\gamma(\cdot)$  defined in (19) in [6] over the whole space  $\Theta_{m_1}^{\text{iso}}$ . We therefore obtain

$$[\theta^{(p)}]_{m_1}^{\text{iso}}_{[1,0]} = \frac{\text{tr} [\Sigma H]}{\text{tr} [\Sigma H^2]}.$$

Once again, we apply Lemma A.1 in [6] to simultaneously diagonalize the matrices  $H$  and  $\Sigma^{-1}$ . As previously, we note  $D_\Sigma$  the corresponding diagonal matrix of  $\Sigma$ .

$$\begin{aligned} D_{\Sigma[(i-1)p+j, (i-1)p+j]} &= \frac{\sigma^2}{1 - 2\alpha \left[ \cos \left( 2\pi \left( \frac{pi}{4p} + \frac{pj}{4p} \right) \right) + \cos \left( 2\pi \left( \frac{-pi}{4p} + \frac{pj}{4p} \right) \right) \right]} \\ &= \frac{\sigma^2}{1 - 4\alpha \cos \left( \pi \frac{i}{2} \right) \cos \left( \pi \frac{j}{2} \right)}. \end{aligned}$$

Analogously, we compute the diagonal matrix  $D(\Psi_{1,0}^{\text{iso}})$

$$D(\Psi_{1,0}^{\text{iso}})_{[(i-1)p+j, (i-1)p+j]} = 2 \left[ \cos \left( 2\pi \frac{i}{p} \right) + \cos \left( 2\pi \frac{j}{p} \right) \right].$$

Combining these two last expressions, we obtain

$$\text{tr}(H\Sigma) = \sum_{i=1}^p \sum_{j=1}^p \sigma^2 \frac{2 \left[ \cos \left( 2\pi \frac{i}{p} \right) + \cos \left( 2\pi \frac{j}{p} \right) \right]}{1 - 4\alpha \cos \left( \pi \frac{i}{2} \right) \cos \left( \pi \frac{j}{2} \right)}.$$

Let us split this sum in 16 parts depending on the congruence of  $i$  and  $j$  modulo 4. As each if of these 16 sums is shown to be zero, we conclude that  $\text{tr}(H\Sigma) = [\theta^{(p)}]_{m_1}^{\text{iso}}_{[1,0]} = 0$ . By Lemma 4.1, the asymptotic risk of  $\widehat{\theta^{(p)}}_{m_1}^{\text{iso}, \rho_1}$  therefore equals

$$\lim_{n \rightarrow +\infty} np^2 \mathbb{E}_{\theta^{(p)}} \left[ l \left( \widehat{\theta^{(p)}}_{m_1}^{\text{iso}, \rho_1}, [\theta^{(p)}]_{m_1}^{\text{iso}} \right) \right] = \frac{\text{tr}(H^4 \Sigma^2)}{\text{tr}(H^2 \Sigma)}.$$



First, we lower bound the numerator

$$\operatorname{tr}(H^4 \Sigma^2) = \sigma^4 \sum_{i=1}^p \sum_{j=1}^p \frac{\left\{ 2 \left[ \cos \left( 2\pi \frac{i}{p} \right) + \cos \left( 2\pi \frac{j}{p} \right) \right] \right\}^4}{\left\{ 1 - 4\alpha \cos \left( \pi \frac{i}{2} \right) \cos \left( \pi \frac{j}{2} \right) \right\}^2}.$$

As each term of this sum is non-negative, we may only consider the coefficients  $i$  and  $j$  which are congruent to 0 modulo 4.

$$\operatorname{tr}(H^4 \Sigma^2) \geq \sigma^4 \sum_{i=0}^{p/4-1} \sum_{j=0}^{p/4-1} \frac{16 \left[ \cos \left( 2\pi \frac{i}{p/4} \right) + \cos \left( 2\pi \frac{j}{p/4} \right) \right]^4}{(1 - 4\alpha)^2}.$$

If we let  $p$  go to infinity, we get the lower bound

$$\lim_{p \rightarrow +\infty} \frac{\operatorname{tr}(H^4 \Sigma^2)}{p^2} \geq \frac{\sigma^4}{(1 - 4\alpha)^2} \int_0^1 \int_0^1 [\cos(2\pi x) + \cos(2\pi y)]^4 dx dy.$$

Similarly, we upper bound  $\operatorname{tr}(H^2 \Sigma)$  and let  $p$  go to infinity

$$\lim_{p \rightarrow +\infty} \frac{\operatorname{tr}(H^2 \Sigma)}{p^2} \leq \frac{4\sigma^2}{1 - 4\alpha} \int_0^1 \int_0^1 [\cos(2\pi x) + \cos(2\pi y)]^2 dx dy.$$

Combining these two bounds allows to conclude

$$\lim_{p \rightarrow +\infty} \lim_{n \rightarrow +\infty} np^2 R_{\theta^{(p)}} \left( \widehat{\theta^{(p)}}_{m_1}^{\text{iso}, \rho_1}, [\theta^{(p)}]_{m_1}^{\text{iso}} \right) \geq \frac{L\sigma^2}{1 - 4\alpha}.$$

□

## 5. Miscellaneous

*Proof of Lemma 1.1 in [6].* Let  $\theta$  be a  $p \times p$  matrix that satisfies condition (3) in [6]. For any  $1 \leq i_1, i_2 \leq p$ , we define the  $p \times p$  submatrix  $C_{i_1, i_2}$  as

$$C_{i_1, i_2} [j_1, j_2] := C(\theta)_{[(i_1-1)p+j_1, (i_2-1)p+j_2]},$$

for any  $1 \leq j_1, j_2 \leq p$ . For the sake of simplicity, the subscripts  $(i_1, i_2)$  are taken modulo  $p$ . By definition of  $C(\theta)$ , it holds that  $C_{i_1, i_2} = C_{0, i_2 - i_1}$  for any  $1 \leq i_1, i_2 \leq p$ . Besides, the matrices  $C_{0, i}$  are circulant for any  $1 \leq i \leq p$ . In short, the matrix  $C(\theta)$  is of the form

$$C(\theta) = \begin{pmatrix} C_{0,1} & C_{0,2} & \cdots & C_{0,p} \\ \vdots & \vdots & \vdots & \vdots \\ C_{0,p} & C_{0,1} & \cdots & C_{0,p-1} \end{pmatrix},$$

where the matrices  $C_{0, i}$  are circulant. Let  $(i_1, i_2, j_1, j_2)$  be in  $\{1, \dots, p\}^4$ . By definition,

$$C(\theta)_{[(i_1-1)p+j_1, (i_2-1)p+j_2]} = \theta_{[i_2 - i_1, j_2 - j_1]}.$$

Since the matrix  $\theta$  satisfies condition (3) in [6],  $\theta_{[i_2-i_1, j_2-j_1]} = \theta_{[i_1-i_2, j_1-j_2]}$ . As a consequence,

$$C(\theta)_{[(i_1-1)p+j_1, (i_2-1)p+j_2]} = C(\theta)_{[(i_2-1)p+j_2, (i_1-1)p+j_1]} \text{ and } C(\theta) \text{ is symmetric.}$$

Conversely, let  $B$  be a  $p^2 \times p^2$  symmetric block circulant matrix. Let us define the matrix  $\theta$  of size  $p$  by

$$\theta_{[i,j]} := B_{[1, (i-1)p+j]} ,$$

for any  $1 \leq i, j \leq p$ . Since the matrix  $B$  is block circulant, it follows that  $C(\theta) = B$ . By definition,  $\theta_{[i,j]} = C(\theta)_{[1, (i-1)p+j]}$  and  $\theta_{[-i, -j]} = C(\theta)_{[(i-1)p+j, 1]}$  for any integers  $1 \leq i, j \leq p$ . Since the matrix  $B$  is symmetric, we conclude that  $\theta_{[i,j]} = \theta_{[-i, -j]}$ .  $\square$

*Proof of Lemma 2.2 in [6].* For any  $\theta' \in \Theta^+$ ,  $\gamma_{n,p}(\theta')$  is defined as

$$\gamma_{n,p}(\theta') = \frac{1}{p^2} \text{tr} \left[ (I_{p^2} - C(\theta')) \overline{\mathbf{X}^v \mathbf{X}^{v*}} (I_{p^2} - C(\theta')) \right] .$$

Applying Lemma A.1 in [6], there exists an orthogonal matrix  $P$  that simultaneously diagonalizes  $\Sigma$  and any matrix  $C(\theta')$ . Let us define  $\mathbf{Y}^i := \sqrt{\Sigma}^{-1} \mathbf{X}_i$  and  $D_\Sigma := P \Sigma P^*$ . Gathering these new notations yields

$$\gamma_{n,p}(\theta') = \frac{1}{p^2} \text{tr} \left[ (I_{p^2} - D(\theta')) D_\Sigma \overline{\mathbf{Y} \mathbf{Y}^*} (I_{p^2} - D(\theta')) \right] ,$$

where the vectors  $\mathbf{Y}^i$  are independent standard Gaussian random vectors. Except  $\overline{\mathbf{Y} \mathbf{Y}^*}$ , every matrix involved in this last expression is diagonal. Besides, the diagonal matrix  $D_\Sigma$  is positive since  $\Sigma$  is non-singular. Thus,

$$\text{tr} \left[ (I_{p^2} - D(\theta')) D_\Sigma \overline{\mathbf{Y} \mathbf{Y}^*} (I_{p^2} - D(\theta')) \right]$$

is almost surely a positive quadratic form on the vector space generated by  $I_{p^2}$  and  $D(\Theta^+)$ . Since the function  $D(\cdot)$  is injective and linear on  $\Theta^+$ , it follows that  $\gamma_{n,p}(\cdot)$  is almost surely strictly convex on  $\Theta^+$ .  $\square$

*Proof of Lemma 4.1 and Corollary 4.2 in [6].* The proof only uses the stationarity of the field  $X$  on  $\Lambda$  and the  $l_1$  norm of  $\theta$ . However, the computations are a bit cumbersome. Let  $\theta$  be an element of  $\Theta^+$ . By standard Gaussian properties, the expectation of  $X_{[0,0]}$  given the remaining covariates is

$$\mathbb{E}_\theta (X_{[0,0]} | X_{-\{0,0\}}) = \sum_{(i,j) \in \Lambda \setminus (0,0)} \theta_{[i,j]} X_{[i,j]} .$$

By assumption  $(\mathbb{H}_2)$ , the  $l_1$  norm of  $\theta$  is smaller than one. We shall prove by backward induction that for any subset  $A$  of  $\Lambda \setminus \{(0,0)\}$  the matrix  $\theta^A$  uniquely defined by

$$\mathbb{E}_\theta (X_{[0,0]}|X_A) = \sum_{(i,j) \in A} \theta^A_{[i,j]} X_{[i,j]} \text{ and } \theta^A_{[i,j]} = 0 \text{ for any } (i,j) \notin A$$

satisfies  $\|\theta^A\|_1 \leq \|\theta\|_1$ . The property is clearly true if  $A = \Lambda \setminus \{(0,0)\}$ . Suppose we have proved it for any set of cardinality  $q$  larger than one. Let  $A$  be a subset of  $\Lambda \setminus \{(0,0)\}$  of cardinality  $q-1$  and  $(i,j)$  be an element of  $\Lambda \setminus (A \cup \{(0,0)\})$ . Let us derive the expectation of  $X_{[0,0]}$  conditionally to  $X_A$  from the expectation of  $X_{[0,0]}$  conditionally to  $X_{A \cup \{(i,j)\}}$ .

$$\begin{aligned} \mathbb{E}_\theta (X_{[0,0]}|X_A) &= \mathbb{E}_\theta [\mathbb{E}(X_{[0,0]}|X_A)|X_{A \cup \{(i,j)\}}] \\ &= \sum_{(k,l) \in A} \theta^{A \cup \{(i,j)\}}_{[k,l]} X_{[k,l]} + \theta^{A \cup \{(i,j)\}}_{[i,j]} \mathbb{E}_\theta [X_{[i,j]}|X_A] \end{aligned} \quad (28)$$

Let us take the conditional expectation of  $X_{[i,j]}$  with respect to  $X_{A \cup \{(0,0)\}}$ . Since the field  $X$  is stationary on  $\Lambda$  and by the induction hypothesis, the unique matrix  $\theta^{A \cup \{(0,0)\}}_{(i,j)}$  defined by

$$\mathbb{E}_\theta (X_{[i,j]}|X_{A \cup \{(0,0)\}}) = \sum_{(k,l) \in A \cup \{(0,0)\}} \theta^{A \cup \{(0,0)\}}_{(i,j)}_{[k,l]} X_{[k,l]}$$

and  $\theta^{A \cup \{(0,0)\}}_{(i,j)}_{[k,l]} = 0$  for any  $(k,l) \notin A \cup \{(0,0)\}$  satisfies  $\|\theta^{A \cup \{(0,0)\}}_{(i,j)}\|_1 \leq \|\theta\|_1$ . Taking the expectation conditionally to  $X_A$  of this previous expression leads to

$$\mathbb{E}_\theta (X_{[i,j]}|X_A) = \sum_{(k,l) \in A} \theta^{A \cup \{(0,0)\}}_{(i,j)}_{[k,l]} X_{[k,l]} + \theta^{A \cup \{(0,0)\}}_{(i,j)}_{[0,0]} \mathbb{E} (X_{[0,0]}|X_A) . \quad (29)$$

Gathering identities (28) and (29) yields

$$\mathbb{E}_\theta (X_{[0,0]}|X_A) = \sum_{(k,l) \in A} \frac{\theta^{A \cup \{(i,j)\}}_{[k,l]} + \theta^{A \cup \{(i,j)\}}_{[i,j]} \theta^{A \cup \{(0,0)\}}_{(i,j)}_{[k,l]}}{1 - \theta^{A \cup \{(i,j)\}}_{[i,j]} \theta^{A \cup \{(0,0)\}}_{(i,j)}_{[0,0]}} X_{[k,l]} ,$$

since  $|\theta^{A \cup \{(i,j)\}}_{[i,j]} \theta^{A \cup \{(0,0)\}}_{(i,j)}_{[0,0]}| < 1$ . Then, we upper bound the  $l_1$  norm of  $\theta^A$

using that  $\|\theta^{A \cup \{(i,j)\}}\|_1$  and  $\|\theta_{(i,j)}^{A \cup \{(0,0)\}}\|_1$  are smaller or equal to  $\|\theta\|_1$ .

$$\begin{aligned}
& \|\theta^A\|_1 \\
& \leq \frac{\sum_{(k,l) \in A} |\theta^{A \cup \{j+1\}}_{[k,l]}| + \sum_{(k,l) \in A} |\theta^{A \cup \{(i,j)\}}_{[i,j]} \theta_{(i,j)}^{A \cup \{0,0\}}_{[k,l]}|}{1 - |\theta^{A \cup \{(i,j)\}}_{[i,j]} \theta_{(i,j)}^{A \cup \{(0,0)\}}_{[0,0]}|} \\
& \leq \frac{\|\theta\|_1 + |\theta^{A \cup \{(i,j)\}}_{[i,j]}| \left( \sum_{(k,l) \in A \cup \{(0,0)\}} |\theta_{(i,j)}^{A \cup \{(0,0)\}}_{[k,l]}| - 1 - |\theta_{(i,j)}^{A \cup \{(0,0)\}}_{[0,0]}| \right)}{1 - |\theta^{A \cup \{(i,j)\}}_{[i,j]} \theta_{i,j}^{A \cup \{(0,0)\}}_{[0,0]}|} \\
& \leq \frac{\|\theta\|_1 (1 + |\theta^{A \cup \{(i,j)\}}_{[i,j]}|) - |\theta^{A \cup \{(i,j)\}}_{[i,j]}| (1 + |\theta_{i,j}^{A \cup \{(0,0)\}}_{[0,0]}|)}{1 - |\theta^{A \cup \{(i,j)\}}_{[i,j]} \theta_{(i,j)}^{A \cup \{(0,0)\}}_{[0,0]}|} \\
& \leq \|\theta\|_1 + \frac{|\theta^{A \cup \{(i,j)\}}_{[i,j]}| (\|\theta\|_1 - 1) (1 + |\theta_{(i,j)}^{A \cup \{(0,0)\}}_{[0,0]}|)}{1 - |\theta^{A \cup \{(i,j)\}}_{[i,j]} \theta_{(i,j)}^{A \cup \{(0,0)\}}_{[0,0]}|}.
\end{aligned}$$

Since  $\|\theta\|_1$  is smaller than one, it follows that  $\|\theta^A\|_1 \leq \|\theta\|_1$ .

Let  $m$  be a model in the collection  $\mathcal{M}_1$ . Since  $m$  stands for a set of neighbors of  $(0,0)$ , we may define  $\theta^m$  as above. It follows that  $\|\theta^m\|_1 \leq \|\theta\|_1$ . Since the field  $X$  is stationary on the torus,  $X$  follows the same distribution as the field  $X^s$  defined by  $X^s_{[i,j]} = X_{[-i,-j]}$ . By uniqueness of  $\theta^m$ , we obtain that  $\theta^m_{[i,j]} = \theta^m_{[-i,-j]}$ . Thus,  $\theta^m$  belongs to the space  $\Theta_m$ . Moreover,  $\theta^m$  minimizes the function  $\gamma(\cdot)$  on  $\Theta_m$ . Since the  $l_1$  norm of  $\theta^m$  is smaller than one,  $\theta^m$  belongs to  $\Theta_{m,2}^+$ . The matrices  $\theta^m$  and  $\theta_{m,\rho_1}$  are therefore equal, which concludes the proof in the non-isotropic case.

Let us now turn to the isotropic case. Let  $\theta$  belong to  $\Theta^{\text{iso},+}$  and let  $m$  be a model in  $\mathcal{M}_1$ . As previously, the matrix  $\theta^m$  satisfies  $\|\theta^m\|_1 \leq \|\theta\|_1$ . Since the distribution of  $X$  is invariant under the action of the group  $G$ ,  $\theta^m$  belongs to  $\Theta_m^{\text{iso}}$ . Since  $\|\theta^m\|_1 \leq \|\theta\|_1$ ,  $\theta^m$  lies in  $\Theta_{m,2}^{+, \text{iso}}$ . It follows that  $\theta^m = \theta_{m,\rho_1}^{\text{iso}}$ .  $\square$

*Proof of Corollary 4.3 in [6].* Let  $\theta$  be a matrix in  $\Theta^+$  such that  $(\mathbb{H}_2)$  holds and let  $m$  be a model in  $\mathcal{M}_1$ . We decompose  $\gamma(\widehat{\theta}_{m,\rho_1})$  using the conditional

expectation of  $X_{[0,0]}$  given  $X_m$ .

$$\begin{aligned} \gamma(\widehat{\theta}_{m,\rho_1}) &= \mathbb{E}_\theta \left[ X_{[0,0]} - \sum_{(i,j) \in m} \widehat{\theta}_{m,\rho_1[i,j]} X_{[i,j]} \right]^2 \\ &= \mathbb{E}_\theta \left[ X_{[0,0]} - \mathbb{E}_\theta(X_{[0,0]} | X_m) \right]^2 \\ &+ \mathbb{E}_\theta \left[ \mathbb{E}_\theta(X_{[0,0]} | X_m) - \sum_{(i,j) \in m} \widehat{\theta}_{m,\rho_1[i,j]} X_{[i,j]} \right]^2. \end{aligned}$$

By Corollary (11) in [6], we know that

$$\mathbb{E}_\theta(X_{[0,0]} | X_m) = \sum_{(i,j) \in m} \theta_{m,\rho_1[i,j]} X_{[i,j]}.$$

Combining these two last identities yields

$$\gamma(\widehat{\theta}_{m,\rho_1}) = \gamma(\theta_{m,\rho_1}) + \mathbb{E}_\theta \left[ \sum_{(i,j) \in \Lambda \setminus \{(0,0)\}} (\theta_{m,\rho_1} - \widehat{\theta}_{m,\rho_1})_{[i,j]} X_{[i,j]} \right]^2.$$

Subtracting  $\gamma(\theta)$ , we obtain the first result. The proof is analogous in the isotropic case.  $\square$

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