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# Quantum Electrodynamics of Casimir Momentum

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We present a non-relativistic quantum theory for the linear momentum of electromagnetic zero-point fluctuations. We consider an harmonic oscillator subject to crossed, quasi-static magnetic and electric and coupled to the quantum vacuum. We derive a contribution of the quantum vacuum to the linear pseudo-momentum.

## INTRODUCTION

Casimir energy refers to the electromagnetic (EM) energy that shows up when dielectric or metallic objects interact with the quantum vacuum. It is undoubtedly one of the most fascinating phenomena in physics, with a rich history in the 20th century. Casimir forces become important on sub-micron scales and are thus believed to play an important role in nano-optics [1]. Casimir energy has been the subject of many speculations, such as its role in sonoluminescence [2] or in the cosmological constant problem [3].

The standard Casimir effect refers to the reduction of the EM zero-point energy when two ideal metallic plates approach [4]. Other well-known phenomena related to Casimir energy are Van Der Waals and Casimir-Polder forces between neutral atoms [5], the Lifshitz forces between dielectric media, and arguably the most famous among all, the Lamb shift of atomic levels. Shortly after its observation by Lamb in 1947 [6], Bethe explained the Lamb shift by the change in EM vacuum energy caused by the interaction of the atom with the quantum vacuum [7, 8]. The Lamb shifts in light atoms are now understood to be basically nonrelativistic QED phenomena, although full relativistic theory, including the contribution of several percents due to vacuum polarization, is necessary to come to the extraordinary agreement with experiment, unprecedented in physics. For the two-photon 1S-2S transition in atomic hydrogen, the shift is known up to several cycles [9].

Energy and momentum are naturally related by relativity. The search for "Casimir momentum" seems therefore obvious. In 2004 Feigel [10] proposed a quantum correction to the momentum of dielectric media exposed to static electric and magnetic fields. From classical electrodynamics it is easy to find the following expression for the linear momentum of a neutral, polarizable object with mass  $M$ ,

$$\mathbf{Q} = M\mathbf{v} - \alpha(0)\mathbf{E}_0 \times \mathbf{B}_0 \quad (1)$$

which is conserved in time, even if the external electric field  $\mathbf{E}_0$  is varied slowly in time. Here  $\alpha(0)$  is the static polarizability, with the dimension of a volume. The semi-

classical theory predicts a strongly diverging contribution of the quantum vacuum to Eq. (1), quite similar to the one encountered for Casimir energy. Fortunately, spatial gradients of Casimir energy - observable as forces - are often found not to diverge. Momentum however is an observable parameter and the divergence does pose a problem. It has been suggested that UV divergences are not physical and should disappear into the values attributed to physical observables, such as inertial mass, electric charge or cosmological constant [11]. If this is true it is not evident that the prediction of "Casimir momentum" found by Feigel will survive or be measurable. An obvious next question is what physical observable will then absorb the UV divergence of Casimir momentum. In this work we provide first answers to these questions. We use the method of mass renormalization first employed by Bethe and Kramers that results in a finite Casimir momentum of simple quantum objects.

It is now realized that Casimir momentum emerges quite generally in so-called bi-anisotropic media, in which also magnetic fields can induce an electric polarization. Except in media exposed to external EM fields, bi-anisotropy also occurs in moving dielectric media. This follows from the relativistic transformations of EM fields, and shall be discussed elsewhere [12]. In general, like spin, bi-anisotropic behavior and Casimir momentum can be viewed as a "remnants" of special relativity [13] in non-relativistic theory that often suffices to describe phenomena quantitatively. We will be obliged to go beyond the dipole approximation to treat the high wave numbers of zero-point fluctuation accurately.

## 3D HARMONIC OSCILLATOR IN EXTERNAL FIELDS

The quantum-mechanics of a 3D harmonic oscillator - composed of two particles with opposite charge  $q_1 = +e$  and  $q_2 = -e$  and masses  $m_i$  - exposed to crossed, homogeneous static EM fields  $\mathbf{E}_0, \mathbf{B}_0$  was discussed in detail and non-perturbationally by Dippel et al [14]. We will use  $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/M$  and  $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$  for the center of mass position and the interparticle distance, with *conjugate* momenta  $\mathbf{P}$  and  $\mathbf{p}$ , respectively. The external,

classical magnetic field can be described by the vector potential  $\mathbf{A}_0(\mathbf{r}) = \frac{1}{2}\mathbf{B}_0 \times \mathbf{r}$ . The total *kinetic* momentum is thus  $\mathbf{P}_{\text{kin}} = \mathbf{P} - e\mathbf{A}_0(\mathbf{r}_1) + e\mathbf{A}_0(\mathbf{r}_2) = \mathbf{P} - \frac{e}{2}\mathbf{B}_0 \times \mathbf{r}$ . Finally, the *pseudo*-momentum  $\mathbf{Q}$ , given by

$$\mathbf{Q} = \mathbf{P} + \frac{e}{2}\mathbf{B}_0 \times \mathbf{r} = \mathbf{P}_{\text{kin}} + e\mathbf{B}_0 \times \mathbf{r} \quad (2)$$

commutes with the atomic Hamiltonian and is the quantum-mechanical equivalent of Eq. (1). For the pur-

pose of this work it is convenient to choose eigenfunctions that simultaneously diagonalize the atomic Hamiltonian and the pseudo-momentum  $\mathbf{Q}$ , labeled by the eigenvalue  $\mathbf{Q}_0$ . It will be sufficient to ignore all contributions other than on those linear in either  $\mathbf{E}_0$  and  $\mathbf{B}_0$ , as indicated by the sign  $\asymp$ . In this approximation, the magneto-electric oscillator is unitary equivalent to an isotropic harmonic oscillator as expressed by,

$$|\mathbf{n}, \mathbf{Q}_0\rangle \asymp \exp\left(\frac{i}{\hbar}\mathbf{Q}_0 \cdot \mathbf{R}\right) \exp\left[-\frac{i}{2\hbar}(\mathbf{B}_0 \times \mathbf{r}) \cdot \mathbf{R}\right] \exp\left(-\frac{i}{\hbar}\mathbf{p}_0 \cdot \mathbf{r}\right) \exp\left(-\frac{i}{\hbar}\mathbf{p} \cdot \mathbf{r}_0\right) |\phi_{\mathbf{n}}\rangle$$

$\mathbf{n} = (n_x, n_y, n_z)$ ,  $n_i = 0, 1, \dots$  denotes the quantum levels of the oscillator. The first two exponentials on the right denote translational momentum of the center of mass, and governed by the conjugate momentum  $\mathbf{P}$ . The last pair of exponentials eliminate the static electric field from the picture, with the eigenfunctions of the oscillator shifted out of the center of mass over a distance  $\mathbf{r}_0 = e^{-1}\alpha(0)(\mathbf{E}_0 + \mathbf{Q}_0 \times \mathbf{B}_0/M)$ , and the reduced momentum shifted by  $\mathbf{p}_0 \asymp (2M)^{-1}(m_2 - m_1)\alpha(0)(\mathbf{E}_0 \times \mathbf{B}_0)$ . Here  $\alpha(0) = e^2/\mu\omega_0^2$  is the static polarizability of the oscillator. Due to the static magnetic field, the oscillator states  $|\phi_{\mathbf{n}}\rangle$  are in principle still anisotropic and even in  $B_0$ . The anisotropy is estimated by the small parameter  $eB_0a/(\hbar/a) \approx 10^{-5}$ , with  $a$  the atomic size. This anisotropy constitutes corrections nonlinear in the applied fields to the final result for the total momentum. We can therefore neglect it.

Upon taking the quantum-expectation value of Eq. (2) for the atomic ground state reveals that the eigenvalue  $\mathbf{Q}_0$  is just equal to the classical expression (1). The total energy of the oscillator in the ground state  $E_0 \asymp \frac{3}{2}\hbar\omega_0 + Q_0^2/2M$  is minimal when  $\mathbf{Q}_0 = 0$ , i.e. for a *finite* kinetic momentum.

## COUPLING TO EM QUANTUM VACUUM

In the following we couple the object above to the EM quantum vacuum and calculate the total momentum using second-order perturbation theory. The Hamiltonian

in the Schrödinger picture is given by

$$H = H_0 + H_F + W \quad (3)$$

$$H_0 = \sum_{i=1}^2 \left[ \frac{1}{2m_i}(\mathbf{p}_i - q_i\mathbf{A}_0(\mathbf{r}_i))^2 - q_i\mathbf{E} \cdot \mathbf{r}_i \right] + \frac{1}{2}\mu\omega_0^2\mathbf{r}^2$$

$$H_F = \sum_{\mathbf{k}\epsilon} \hbar\omega_{\mathbf{k}} \left[ a_{\mathbf{k}\epsilon}^\dagger a_{\mathbf{k}\epsilon} + \frac{1}{2} \right]$$

$$W = \sum_{i=1}^2 -\frac{q_i}{m_i}(\mathbf{p}_i - q_i\mathbf{A}_0(\mathbf{r}_i))\mathbf{A}(\mathbf{r}_i) + \frac{q_i^2}{2m_i}\mathbf{A}(\mathbf{r}_i)^2$$

The EM bath will be treated in the Coulomb gauge. A pseudo momentum  $\mathbf{K}$  exists that replaces  $\mathbf{Q}$  in that it commutes with  $H$  [15], even if the electric field is varied in time. It has contributions from both atom and radiation,

$$\mathbf{K} = \mathbf{Q} + e\Delta\mathbf{A} + \sum_{\mathbf{k}\epsilon} \hbar\mathbf{k} \left[ a_{\mathbf{k}\epsilon}^\dagger a_{\mathbf{k}\epsilon} + \frac{1}{2} \right] \quad (4)$$

The term  $e\Delta\mathbf{A} = e\mathbf{A}(\mathbf{r}_1) - e\mathbf{A}(\mathbf{r}_2)$  in  $\mathbf{K}$  guarantees the gauge-invariant contribution of the "longitudinal" vacuum field to the pseudo-momentum, in terms of the vector potential  $\mathbf{A}(\mathbf{r})$  quantized as usual inside a quantization volume  $V$ . The last term stems from the "transverse" electromagnetic field in the vicinity of the atom [15].

We wish to express  $\overline{\mathbf{K}} = \langle \Psi_0 | \mathbf{K} | \Psi_0 \rangle$ , of the perturbed total ground state  $|\Psi_0\rangle$  in terms of the two relevant vectors: the pseudo momentum  $\mathbf{Q}_0$  of the unperturbed oscillator and the magneto-electric vector  $\mathbf{E}_0 \times \mathbf{B}_0$ . In the absence of the interaction with the quantum vacuum, the eigenstates are just the direct products  $|\mathbf{n}, \mathbf{Q}_0, \mathbf{n}_{\mathbf{k}}\rangle = |\mathbf{n}, \mathbf{Q}_0\rangle \otimes |\mathbf{n}_{\mathbf{k}}\rangle$ , with unperturbed energies  $E_{\mathbf{n}\mathbf{Q}_0\mathbf{n}_{\mathbf{k}}} = E_{\mathbf{n}\mathbf{Q}_0} + \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}(n_{\mathbf{k}} + \frac{1}{2})$ . Here  $\mathbf{n}_{\mathbf{k}}$  is the occupation of the EM Fock states with photon momentum  $\hbar\mathbf{k}$ . The ground state follows from second-order perturbation in the coupling  $W$  to the quantum vacuum ( $\sum'$

avoids zeros in the denominator),

$$\begin{aligned}
|\Psi_0\rangle &= |\mathbf{0}, \mathbf{Q}_0, \mathbf{0}\rangle + \sum'_{\mathbf{lQn}} \frac{W_{\mathbf{lQn}, \mathbf{0Q}_0 \mathbf{0}}}{E_{\mathbf{0Q}_0 \mathbf{0}} - E_{\mathbf{lQn}}} |\mathbf{l}, \mathbf{Q}, \mathbf{n}\rangle \\
&+ \sum'_{\mathbf{lQn}} \sum'_{\mathbf{sQ'm}} \frac{W_{\mathbf{lQn}, \mathbf{sQ'm}} W_{\mathbf{sQ'm}, \mathbf{0Q}_0 \mathbf{0}}}{(E_{\mathbf{0Q}_0 \mathbf{0}} - E_{\mathbf{lQn}})(E_{\mathbf{0Q}_0 \mathbf{0}} - E_{\mathbf{sQ'm}})} |\mathbf{l}, \mathbf{Q}, \mathbf{n}\rangle \\
&\quad + \sum'_{\mathbf{lQn}} \frac{|W_{\mathbf{lQn}, \mathbf{0Q}_0 \mathbf{0}}|^2}{(E_{\mathbf{0Q}_0 \mathbf{0}} - E_{\mathbf{lQn}})^2} |\mathbf{l}, \mathbf{Q}, \mathbf{n}\rangle
\end{aligned}$$

Only the emission and subsequent re-absorption of one virtual photon contributes at this one-loop level of the theory, which generates an temporary recoil momentum  $\mathbf{Q}_0 - \hbar\mathbf{k}$  of the oscillator,

$$\begin{aligned}
\bar{\mathbf{K}} &= \mathbf{Q}_0 \\
&+ e^2 2\text{Re} \sum_{\mathbf{l k} \epsilon} \mathcal{A}_{\mathbf{k}}^2 \epsilon \frac{\langle \phi_0 | (e^{i\mathbf{k}(\mathbf{r}+\mathbf{r}_0) \frac{m_2}{M}} - e^{-i\mathbf{k}(\mathbf{r}+\mathbf{r}_0) \frac{m_1}{M}}) | \phi_l \rangle \Omega_{\mathbf{l}, \mathbf{0}}^*}{E_{\mathbf{0Q}_0 \mathbf{0}} - E_{\mathbf{l}(\mathbf{Q}_0 - \hbar\mathbf{k})_{1k}}} \\
&+ \mathbf{B}_0 \times 2\text{Re} \sum_{\mathbf{l s k} \epsilon} \frac{e^3 \mathcal{A}_{\mathbf{k}}^2 \langle \phi_0 | \mathbf{r} + \mathbf{r}_0 | \phi_l \rangle \Omega_{\mathbf{l}, \mathbf{s}} \Omega_{\mathbf{s}, \mathbf{0}}^*}{(E_{\mathbf{0Q}_0 \mathbf{0}} - E_{\mathbf{lQ}_0 \mathbf{0}})(E_{\mathbf{0Q}_0 \mathbf{0}} - E_{\mathbf{s}(\mathbf{Q}_0 - \hbar\mathbf{k})_{1k}})} \\
&+ \mathbf{B}_0 \times \sum_{\mathbf{l s k} \epsilon} \frac{e^3 \mathcal{A}_{\mathbf{k}}^2 \langle \phi_s | \mathbf{r} + \mathbf{r}_0 | \phi_l \rangle \Omega_{\mathbf{l}, \mathbf{0}}^* \Omega_{\mathbf{0}, \mathbf{s}}}{(E_{\mathbf{0Q}_0 \mathbf{0}} - E_{\mathbf{l}(\mathbf{Q}_0 - \hbar\mathbf{k})_{1k}})(E_{\mathbf{0Q}_0 \mathbf{0}} - E_{\mathbf{s}(\mathbf{Q}_0 - \hbar\mathbf{k})_{1k}})} \quad (5)
\end{aligned}$$

We introduced  $\Omega_{\mathbf{l}, \mathbf{s}} = \langle \phi_l | \Omega | \phi_s \rangle$  with

$$\begin{aligned}
\Omega &= e\epsilon \cdot [\mathbf{B}_0 \times (\mathbf{r} + \mathbf{r}_0)] \left( \frac{e^{i\mathbf{k}(\mathbf{r}+\mathbf{r}_0) \frac{m_2}{M}}}{m_1} - \frac{e^{-i\mathbf{k}(\mathbf{r}+\mathbf{r}_0) \frac{m_1}{M}}}{m_2} \right) \\
&- \frac{\mathbf{Q}_0}{M} \cdot \epsilon \left( e^{i\mathbf{k}(\mathbf{r}+\mathbf{r}_0) \frac{m_2}{M}} - e^{-i\mathbf{k}(\mathbf{r}+\mathbf{r}_0) \frac{m_1}{M}} \right) \\
&- (\mathbf{p} - \mathbf{p}_0) \cdot \epsilon \left( \frac{e^{i\mathbf{k}(\mathbf{r}+\mathbf{r}_0) \frac{m_2}{M}}}{m_1} + \frac{e^{-i\mathbf{k}(\mathbf{r}+\mathbf{r}_0) \frac{m_1}{M}}}{m_2} \right)
\end{aligned}$$

and  $\mathcal{A}_{\mathbf{k}}^2 = \hbar/2\epsilon_0 V k c$  familiar from quantum optics.

Three kinds of contributions to  $\bar{\mathbf{K}}$  can be identified. The first class is proportional to  $\mathbf{Q}_0$  (see the middle term

in  $\Omega_{\mathbf{l}, \mathbf{s}}$ ) that survives even in the absence of external fields. We will discuss this class elsewhere [12] and argue that it is a QED correction to the kinetic momentum and thus to the mass of the oscillator. Its UV divergence can be absorbed into the total mass  $M$  in the same way as discussed below for the ME divergencies.

The second class, represented by the two last contributions in Eq. (5), are actually QED contributions to the induced electrical dipole moment  $\langle \Psi_0 | e\mathbf{r} | \Psi_0 \rangle$  of the oscillator, that find their way to the total momentum via the classical expression (1). It is straightforward to calculate these corrections - they actually do not diverge and they are relatively small - but we note that if an experimental value for  $\alpha(0)$  is used to evaluate the "classical" contribution, these terms are automatically included. In this sense they do not constitute a genuine "Casimir momentum".

The term  $e\Delta\mathbf{A}$  in Eq. (4) is a genuine contribution of the vacuum radiation to the pseudo-momentum. It will be seen to generate a momentum linear in  $\mathbf{E}_0 \times \mathbf{B}_0$  by means of the second term in Eq. (5). The following calculation will focus on this contribution. It suffers from a UV divergence, that can be eliminated by exactly the same mass regularization as applied by Bethe in his calculation of the Lamb shift [7]. In particular, this procedure establishes that the reduced mass featuring in the static polarizability  $\alpha(0) = e^2/\mu\omega_0^2$  will be replaced by the "observed" reduced mass. It is straightforward to show that the so-called transverse electromagnetic momentum, represented by the last term of Eq. (4), does not contribute a net momentum. This follows from selection rules and spatial symmetry.

We apply the closure relation  $\sum_1 |\phi_l\rangle \langle \phi_l| (E(k) + E_1)^{-1} = (E(k) + H_{ho})^{-1}$  in the second contribution to  $\bar{\mathbf{K}}$ . This generates two terms involving exponentials with opposite phases, one of which is

$$\begin{aligned}
&-e^2 \sum_{\mathbf{k} \epsilon} \mathcal{A}_{\mathbf{k}}^2 \epsilon \langle \phi_0 | e^{i\mathbf{k}(\mathbf{r}+\mathbf{r}_0) \frac{m_2}{M}} \left[ \frac{\hbar^2 k^2}{2M} - \frac{\mathbf{Q}_0 \cdot \hbar\mathbf{k}}{M} + \hbar\omega_{\mathbf{k}} - E_0 + H_{ho} \right]^{-1} e^{-i\mathbf{k}(\mathbf{r}+\mathbf{r}_0) \frac{m_2}{M}} [e\mathbf{B}_0 \times (\mathbf{r} + \mathbf{r}_0)] \cdot \epsilon | \phi_0 \rangle \frac{1}{m_1} + c.c. \\
&= \alpha(0) \mathbf{E}_0 \times \mathbf{B}_0 \frac{4\alpha}{3\pi} \left[ \frac{\hbar^2}{m_1} \int \frac{k dk}{\frac{\hbar^2 k^2}{2m_1} + \hbar c k} + \mathcal{O} \left( \frac{\hbar\omega_0}{Mc^2} \right) \right]
\end{aligned}$$

The other term just involves the other mass. The first term in the second equality neglects the Doppler terms  $\mathbf{Q}_0 \cdot \hbar\mathbf{k}$  and  $\mathbf{p} \cdot \hbar\mathbf{k}$ , generated by the identity

$e^{i\mathbf{k}\mathbf{r}} H_{ho}(\mathbf{p}) e^{-i\mathbf{k}\mathbf{r}} = H_{ho}(\mathbf{p} - \hbar\mathbf{k})$ . They provide a finite correction of order  $\frac{\hbar\omega_0}{Mc^2} \sim 10^{-8}$ . The leading contribution diverges logarithmically. In the Bethe theory for

the Lamb shift [7] exactly the same kind of divergency was encountered. The two diverging mass-like terms  $\delta m_i = \frac{4\alpha}{3\pi} \hbar^2 \int k dk (\frac{\hbar^2 k^2}{2m_i} + \hbar ck)^{-1}$  - here with recoil effects included - stem from the QED coupling of the free particles 1 and 2 with the quantum vacuum and are therefore naturally interpreted to be part of their intrinsic, observable masses [8]. Adding up three diverging contributions - the third one due to the term  $\mathbf{Q}_0/M$  in the expression

for  $\Omega$  - we obtain  $\mu^{-1}(\delta m_1/m_1 + \delta m_2/m_2 - \delta M/M) = \mu^{-2}\delta\mu = -\delta(1/\mu)$ . Since the static polarizability is proportional to  $1/\mu$ , these UV-divergent corrections disappear into the factor  $\alpha(0)$  of Eq. (1), which thus becomes defined in terms of the observed reduced mass  $\mu^*$ .

All other terms generated by Eq. (5) are finite. In particular, the term involving  $\mathbf{p}_0$  generates the following contribution to  $\mathbf{K}$ :

$$\begin{aligned} \mathbf{K}_1 &= \alpha(0)\mathbf{E}_0 \times \mathbf{B}_0 \frac{m_2 - m_1}{2M} \frac{4\alpha}{3\pi} \lim_{\delta \downarrow 0} \int_{\delta}^{\infty} dk \left( \frac{k}{k^2/2 + ck m_2/\hbar} - \frac{2}{k} - \frac{k}{k^2/2 + \hbar ck m_1/\hbar} + \frac{2}{k} \right) \\ &= -\alpha(0)\mathbf{E}_0 \times \mathbf{B}_0 \frac{4\alpha}{3\pi} \frac{m_1 - m_2}{M} \log \frac{m_1}{m_2}. \end{aligned} \quad (6)$$

The UV divergency cancels out, and the integral is finite. Of course part of the  $k$ -integral enters the relativistic regime  $\hbar k > m_i c$  in which the present theory is not valid. However, by subtracting and adding  $2/k$  to both integrands in Eq. (6) reveals two terms whose range of integration is typically  $m_i c/\hbar$ . This wavenumber was used by Bethe to cut off his nonrelativistic theory for the Lamb shift [7]. We will thus adopt the final result in Eq. (6) as a “reasonable” nonrelativistic estimate for the Casimir momentum. It is nevertheless clear that a relativistic theory is required to get the complete picture for Casimir momentum.

All other cross-terms in Eq. (5) contain oscillating exponential factors and converge rapidly for  $k > 1/a$ , i.e. stay in the nonrelativistic regime. They generate a Casimir momentum that is typically a factor  $\sqrt{\hbar\omega_0/\mu c^2} \sim \alpha$  smaller. We obtain,

$$\begin{aligned} \mathbf{K}_2 &= \alpha(0)\mathbf{E}_0 \times \mathbf{B}_0 \alpha \sqrt{\frac{\hbar\omega_0}{\mu c^2}} \\ &\times \left( -\frac{14}{15\sqrt{\pi}} + \frac{2}{3\sqrt{\pi}} \left( \frac{\Delta m}{M} \right)^2 + \frac{8}{3\sqrt{\pi}} \frac{\mu}{M} \right) \end{aligned} \quad (7)$$

When both masses are equal,  $\mathbf{K}_2$  becomes the sole contribution to Casimir momentum. For  $m_1 \gg m_2$ ,  $\mathbf{K}_1$  dominates. Since it is independent on details of the force between the two particles, it is tempting to apply Eq. (6) to the hydrogen atom. With  $\hbar\omega_0 = 10$  eV,  $m_1 = m_p$  and  $m_2 = m_e$ ,  $E_0 = 10^5$  V/m and  $B_0 = 17$  T, we find for the velocity associated with the classical contribution (1)  $v_{cl} \approx 5$   $\mu\text{m/s}$ , and a QED correction of 2 % in the same direction;  $\mathbf{K}_2$  yields a negligible correction of 0.01 %.

In conclusion, we have presented a non-relativistic quantum electrodynamic theory for the total of an harmonic oscillator, subject to external classical fields, and coupled to the electromagnetic quantum vacuum. The most important conclusions of this work are that Casimir

momentum exists and that its UV divergencies are renormalizable. The theory shows it to be basically a non-relativistic quantity, but that relativistic corrections are likely to be significant, much like in the Lamb shift problem. To our knowledge nor the classical contribution to magneto-electric momentum, neither the QED correction have ever been observed.

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