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# HOMFLYPT Skein module of singular links

LUIS PARIS

August 27, 2009

## Abstract

This paper is a presentation, where we compute the HOMFLYPT Skein module of singular links in the 3-sphere. This calculation is based on some results previously proved by Rabenda and the author on Markov traces on singular Hecke algebras, as well as on classical techniques that allow to pass from the framework of Markov traces on Hecke algebras to the framework of HOMFLYPT Skein modules. Some open problems on singular Hecke algebras are also presented.

**AMS Subject Classification:** Primary 57M25. Secondary 20F36, 57M27.

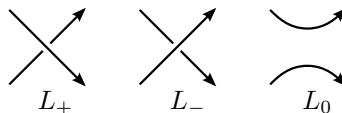
## 1 Introduction

Knot theory had a notable renewable in the 80's with the emergence of new knot invariants such as the Jones polynomial [7], [6] and the HOMFLYPT polynomial [3], [12]. The latest one is defined by the following theorem.

**Theorem 1.1** (Freyd, Yetter, Hoste, Lickorish, Millett, Ocneanu [3], Przytycki, Traczyk [12]). *Let  $\mathcal{L}$  be the set of (isotopy classes) of oriented links in the sphere  $\mathbb{S}^3$ . Then there exists a unique invariant  $I : \mathcal{L} \rightarrow \mathbb{C}(t, x)$  which is 1 on the trivial knot, and which satisfies the relation*

$$t^{-1} \cdot I(L_+) - t \cdot I(L_-) = x \cdot I(L_0), \quad (1.1)$$

for all links  $L_+, L_-, L_0 \in \mathcal{L}$  that have the same link diagram except in the neighborhood of a crossing where they are like in Figure 1.1.

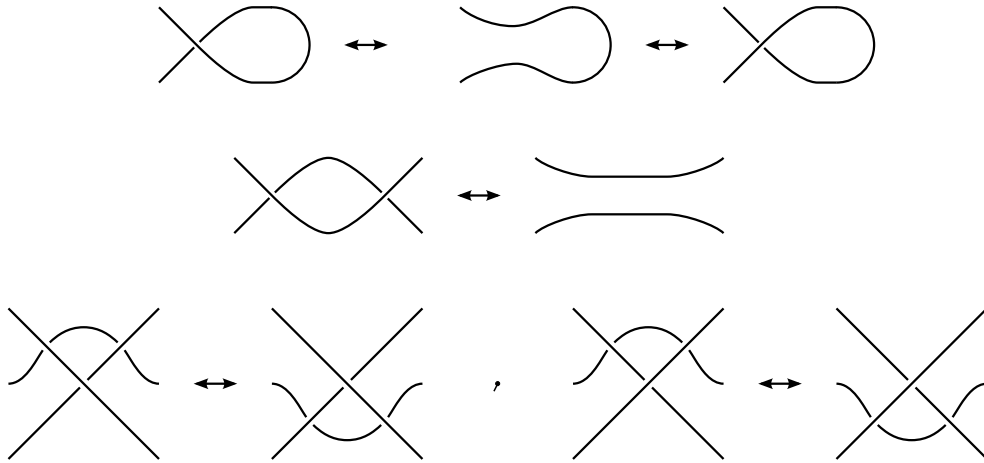


**Figure 1.1.** The links  $L_+$ ,  $L_-$ , and  $L_0$ .

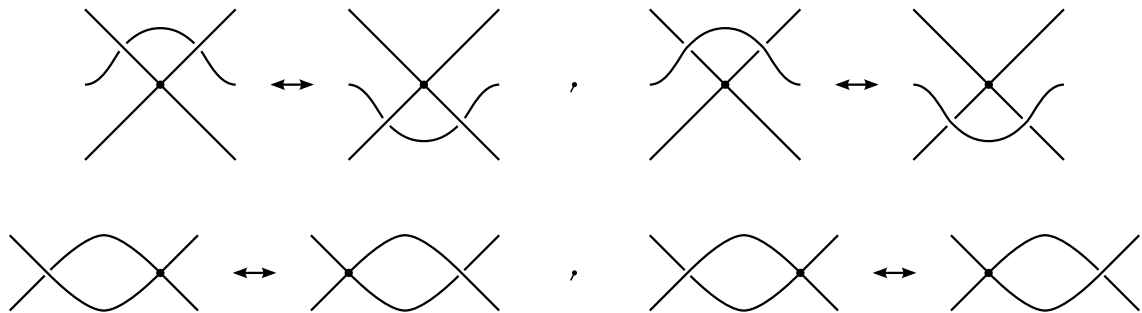
Since then, knot theorists wonder about possible extensions of this result to other sets of like-knots such as the set of links in a 3-manifold, or the set of singular links in the 3-sphere.

Recall that a *singular link* on  $n$  components is defined to be an immersion of  $n$  circles in the sphere  $\mathbb{S}^3$  which admits only finitely many singularities that are all ordinary double points. By

[8], two singular link diagrams represent the same singular link (up to isotopy) if and only if one can pass from one to the other by a finite sequence of ordinary or singular Reidemeister moves (see Figures 1.2 and 1.3).



**Figure 1.2.** Ordinary Reidemeister moves.



**Figure 1.3.** Singular Reidemeister moves.

Let  $\mathcal{L}$  be a set of like-knots. We say that an invariant  $I : \mathcal{L} \rightarrow \mathbb{C}(t, x)$  satisfies the *HOMFLYPT Skein relation* if the relation (1.1) holds for all links  $L_+, L_-, L_0 \in \mathcal{L}$  that have the same link diagram except in the neighborhood of a crossing where they are like in Figure 1.1. It has been quickly observed that, in general, there are many invariants that satisfy the HOMFLYPT Skein relation and that are 1 on the trivial knot. However, the condition that the invariant is 1 on the trivial knot is secondary, and, moreover, one can view the set of invariants that satisfy the HOMFLYPT Skein relation as a vector space over  $\mathbb{C}(x, t)$ . So, the general question is in fact to determine this vector space.

Let  $\mathcal{L}$  be a set of like-knots. Define the *HOMFLYPT Skein module* of  $\mathcal{L}$ , denoted by  $\text{Skein}(\mathcal{L})$ , to be the quotient of the vector space  $\mathbb{C}(x, t)[\mathcal{L}]$  freely spanned by  $\mathcal{L}$ , by the relations

$$t^{-1} \cdot L_+ - t \cdot L_- = x \cdot L_0,$$

for all links  $L_+, L_-, L_0 \in \mathcal{L}$  that have the same link diagram except in the neighborhood of a crossing where they are like in Figure 1.1. Note that the space of invariants of  $\mathcal{L}$  that satisfy the HOMFLYPT Skein relation is the space of linear forms on  $\text{Skein}(\mathcal{L})$ .

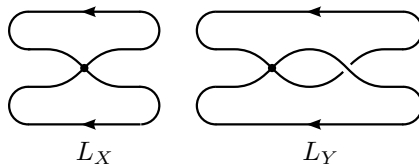
The Skein module was calculated for the set of links in a solid torus by Hoste, Kidwell [5] and, independently, Turaev [13]. Their result was extended by Przytycki [11] to the set of links in the direct product  $F \times I$  of a surface  $F$  with the interval. In this case,  $\text{Skein}(\mathcal{L})$  can be endowed with a structure of algebra. The product of two links  $L_1$  and  $L_2$  (modulo the Skein relations) is the link obtained placing  $L_2$  above  $L_1$ . Note that the Skein module of singular links can be also endowed with a structure of algebra following the same rules.

**Theorem 1.2** (Przytycki [11]). *Let  $\mathcal{L}$  be the set of links in the direct product  $F \times I$  of a surface  $F$  with the interval  $I$ . Then  $\text{Skein}(\mathcal{L})$  is isomorphic to the symmetric algebra  $SC(t, x)[\hat{\pi}^0]$  on the vector space  $\mathbb{C}(t, x)[\hat{\pi}^0]$  freely spanned by the set  $\hat{\pi}^0$  of conjugacy classes of nontrivial elements of  $\pi_1(F)$ .*

The purpose of this paper is to present an approach to the calculation of HOMFLYPT Skein modules via the study of different sorts of braid groups and monoids and their associated generalized Hecke algebras. This will be done through the study of a particular example: the singular links in the 3-sphere. However, the ideas presented here can be easily extended to other cases. In particular, a careful reading of [9] shows how to use these techniques to calculate the HOMFLYPT Skein module of the solid torus.

The main result of this paper is:

**Theorem 1.3.** *Let  $\mathcal{L}$  be the set of oriented singular links in the sphere  $\mathbb{S}^3$ . Then  $\text{Skein}(\mathcal{L})$  is isomorphic to the polynomial algebra  $\mathbb{C}(x, t)[\hat{X}, \hat{Y}]$  in the two variables  $\hat{X}$  and  $\hat{Y}$ , where  $\hat{X}$  and  $\hat{Y}$  are represented by the links  $L_X$  and  $L_Y$  drawn in Figure 1.4*



**Figure 1.4.** Generators of the Skein module of singular links.

The proof of Theorem 1.3 consists essentially in translating the main result of [10], which concerns Markov traces on singular Hecke algebras, in terms of HOMFLYPT Skein modules. On the other hand, some open questions will be presented along the text, and the proof of Theorem 1.3 will also serve as a pretext to present them.

## 2 Markov module and HOMFLYPT Skein module

Let  $\mathcal{P} = \{P_1, \dots, P_n\}$  be a set of  $n$  distinct punctures in the plane  $\mathbb{R}^2$  (except mention of the contrary, we will always assume  $P_k = (k, 0)$  for all  $1 \leq k \leq n$ ). A *singular braid* on  $n$  strands is defined to be an  $n$ -tuple  $\beta = (b_1, \dots, b_n)$  of smooth paths,  $b_k : [0, 1] \rightarrow \mathbb{R}^2 \times [0, 1]$ , such that

- there exists a permutation  $\chi \in \text{Sym}_n$  such that  $b_k(0) = (P_k, 0)$  and  $b_k(1) = (P_{\chi(k)}, 1)$  for all  $1 \leq k \leq n$ ;
- $b_k(t)$  runs monotonically on the second coordinate for all  $1 \leq k \leq n$ ;
- the image of  $b_1 \sqcup \cdots \sqcup b_n$  has finitely many singularities (called *singular points*), that are all ordinary double points.

The isotopy classes of singular braids form a monoid called *singular braid monoid* (on  $n$  strands) and denoted by  $SB_n$ . The multiplication in this monoid is the concatenation of (singular) braids.

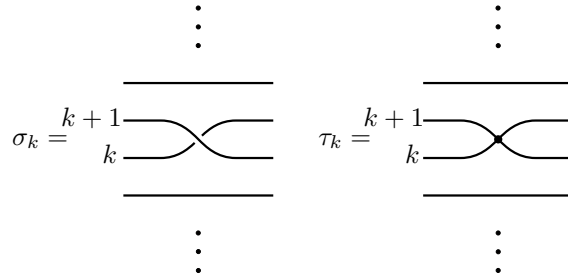
**Theorem 2.1** (Baez [1], Birman [2]). *The monoid  $SB_n$  has a monoid presentation with generators*

$$\sigma_1, \dots, \sigma_{n-1}, \sigma_1^{-1}, \dots, \sigma_{n-1}^{-1}, \tau_1, \dots, \tau_{n-1},$$

and relations

$$\begin{aligned} \sigma_k \sigma_k^{-1} &= \sigma_k^{-1} \sigma_k = 1 && \text{for } 1 \leq k \leq n-1, \\ \sigma_k \tau_k &= \tau_k \sigma_k && \text{for } 1 \leq k \leq n-1, \\ \sigma_k \sigma_l \sigma_k &= \sigma_l \sigma_k \sigma_l, && \text{if } |k-l| = 1, \\ \sigma_k \sigma_l \tau_k &= \tau_l \sigma_k \sigma_l && \text{if } |k-l| = 1, \\ \sigma_k \sigma_l &= \sigma_l \sigma_k && \text{if } |k-l| \geq 2, \\ \sigma_k \tau_l &= \tau_l \sigma_k && \text{if } |k-l| \geq 2, \\ \tau_k \tau_l &= \tau_l \tau_k && \text{if } |k-l| \geq 2. \end{aligned}$$

The braid  $\sigma_k$  in the above theorem is the standard  $k$ -th generator of the braid group  $B_n$  (see Figure 2.1). The braid  $\tau_k$  is a singular braid with a unique singular crossing between the  $k$ -th strand and the  $(k+1)$ -th strand (see Figure 2.1).

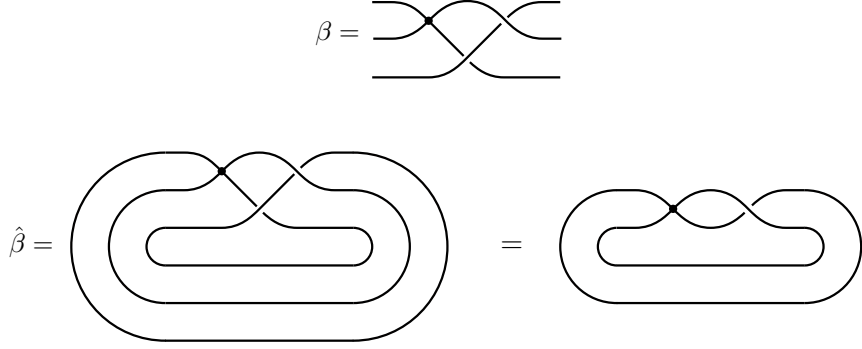


**Figure 2.1.** Generators of  $SB_n$ .

From a singular braid  $\beta$  we can construct a singular link connecting the point  $(P_k, 1)$  to the point  $(P_k, 0)$  for all  $1 \leq k \leq n$  (see Figure 2.2). This link is denoted by  $\hat{\beta}$  and is called the *closure* of  $\beta$ . By [2], every singular link is a closed singular braid.

We denote by  $\sqcup SB = \sqcup_{n=1}^{\infty} SB_n$  the disjoint union of all singular braid monoids. We use the notation  $(\beta, n)$  to denote a singular braid  $\beta$  in  $SB_n$  if we need to specify the number  $n$  of strands.

Two singular braids  $(\alpha, n)$  and  $(\beta, m)$  are said to be connected by a *Markov move* if either



**Figure 2.2.** A closed braid.

- $n = m$  and there exist  $\gamma_1, \gamma_2 \in SB_n$  such that  $\alpha = \gamma_1\gamma_2$  and  $\beta = \gamma_2\gamma_1$ ; or
- $m = n + 1$  and  $\beta = \alpha\sigma_n^{\pm 1}$ ; or
- $n = m + 1$  and  $\alpha = \beta\sigma_m^{\pm 1}$ .

**Theorem 2.2** (Gemein [4]). *Let  $(\alpha, n)$ ,  $(\beta, m)$  be two singular braids. Then  $\hat{\alpha}$  and  $\hat{\beta}$  are isotopic if and only if  $(\alpha, n)$  and  $(\beta, m)$  are connected by a finite sequence of Markov moves.*

We turn now to apply this theorem to obtain a version of the HOMFLYPT Skein module of singular links in terms of singular Hecke algebras.

The *singular Hecke algebra*, denoted by  $\mathcal{H}(SB_n)$ , is defined to be the quotient of the monoid algebra  $\mathbb{C}(q)[SB_n]$  by the relations

$$\sigma_k^2 = (q - 1)\sigma_k + q, \quad 1 \leq k \leq n - 1. \quad (2.1)$$

Note that the singular Hecke algebra is an infinite dimensional  $\mathbb{C}(q)$ -vector space (except for  $n = 1$ ). However, it can be endowed with a graduation, and each term of the graduation is of finite dimension (see [10]). This graduation is defined as follows.

For  $n \geq 2$  and  $d \geq 0$ , we denote by  $S_d B_n$  the set of singular braids with  $n$  strands and  $d$  singular points, and we denote by  $\mathbb{C}(q)[S_d B_n]$  the subspace of  $\mathbb{C}(q)[SB_n]$  spanned by  $S_d B_n$ . Note that  $S_0 B_n$  is the braid group  $B_n$  on  $n$  strands, and  $\mathbb{C}(q)[S_0 B_n] = \mathbb{C}(q)[B_n]$  is the group algebra of  $B_n$ . The monoid algebra  $\mathbb{C}(q)[SB_n]$  is naturally graded by

$$\mathbb{C}(q)[SB_n] = \bigoplus_{d=0}^{+\infty} \mathbb{C}(q)[S_d B_n].$$

Now, the relations (2.1) that define the singular Hecke algebra are all homogeneous (of degree 0), thus the graduation of  $\mathbb{C}(q)[SB_n]$  induces a graduation on  $\mathcal{H}(SB_n)$ ,

$$\mathcal{H}(SB_n) = \bigoplus_{d=0}^{+\infty} \mathcal{H}(S_d B_n),$$

where  $\mathcal{H}(S_d B_n)$  is the subspace of  $\mathcal{H}(SB_n)$  spanned by  $S_d B_n$ .

Several elementary questions on singular Hecke algebras are still open. Here are two of them.

**Question 2.3.** Note that  $\mathcal{H}(S_0 B_n) = \mathcal{H}(B_n)$  is the Hecke algebra of the symmetric group, thus  $\mathcal{H}(S_d B_n)$  is a representation of  $\mathcal{H}(B_n)$ . It would be interesting to characterize this representation. Actually, the dimension itself (over  $\mathbb{C}(q)$ ) of  $\mathcal{H}(S_d B_n)$  is unknown, even for  $d = 2$ .

**Question 2.4.** The natural inclusion  $SB_n \hookrightarrow SB_{n+1}$  induces a homomorphism  $\iota_n : \mathcal{H}(SB_n) \rightarrow \mathcal{H}(SB_{n+1})$ . We do not know whether  $\iota_n$  is injective.

Now, we introduce a new variable  $z$ , we set  $\mathcal{H}_z(SB_n) = \mathbb{C}(z, q) \otimes_{\mathbb{C}(q)} \mathcal{H}(SB_n)$  for all  $n \geq 1$ , and we consider the direct sum  $\bigoplus_{n=1}^{\infty} \mathcal{H}_z(SB_n)$ . Like for the singular braids, we use the notation  $(a, n)$  to denote an element  $a \in \mathcal{H}_z(SB_n)$  if we need to specify the number  $n$  of strands.

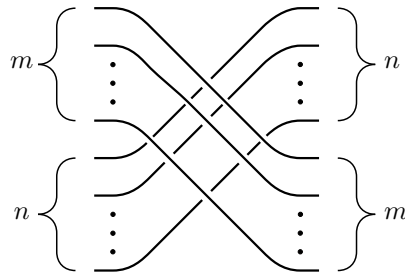
The *Markov module* of  $\sqcup SB$ , denoted by  $\text{Markov}(\sqcup SB)$ , is defined to be the quotient of the space  $\bigoplus_{n=1}^{\infty} \mathcal{H}_z(SB_n)$  by the relations

- $(ab, n) = (ba, n)$  for all  $n \geq 1$  and all  $a, b \in \mathcal{H}_z(SB_n)$ ;
- $(a, n) = (\iota_n(a), n + 1)$  for all  $n \geq 1$  and all  $a \in \mathcal{H}_z(SB_n)$ ;
- $(\iota_n(a)\sigma_n, n + 1) = z \cdot (a, n)$  for all  $n \geq 1$  and all  $a \in \mathcal{H}_z(SB_n)$ .

The space  $\text{Markov}(\sqcup SB)$  can be endowed with a structure of  $\mathbb{C}(z, q)$ -algebra as follows. Let  $[\alpha, n]$  denote the element of  $\text{Markov}(\sqcup SB)$  represented by a braid  $(\alpha, n)$ . Let  $(\alpha, n)$  and  $(\beta, m)$  be two braids. Then the product  $[\alpha, n] \cdot [\beta, m]$  is represented by the braid in  $SB_{n+m}$  obtained placing  $\beta$  above  $\alpha$ . Note that the unit for this multiplication is represented by the trivial braid in  $SB_1 = B_1 = \{1\}$ .

**Lemma 2.5.** *The above defined multiplication in  $\text{Markov}(\sqcup SB)$  is commutative.*

**Proof.** Let  $(\alpha, n), (\beta, m)$  be two singular braids. Let  $(\alpha * \beta, n + m)$  be the braid obtained placing  $\beta$  above  $\alpha$ . So,  $[\alpha, n] \cdot [\beta, m] = [\alpha * \beta, n + m]$ . Let  $\sigma_{n,m} \in B_{n+m}$  be the braid pictured in Figure 2.3. Observe that  $\sigma_{n,m}(\beta * \alpha)\sigma_{n,m}^{-1} = (\alpha * \beta)$ , thus  $[\alpha, n] \cdot [\beta, m] = [\beta, m] \cdot [\alpha, n]$ .  $\square$



**Figure 2.3.** The braid  $\sigma_{n,m}$ .

Now, the link between the HOMFLYPT Skein module of singular links and the Markov module of singular braids is given by the following.

**Theorem 2.6.** *Let  $\mathcal{L}$  be the set of singular links in the sphere  $\mathbb{S}^3$ . Set*

$$\begin{aligned} z &= \frac{q-1}{1-xy} \quad \Leftrightarrow \quad y = \frac{z-q+1}{qz}, \\ t &= \sqrt{yq}, \quad x = \sqrt{q} - \frac{1}{\sqrt{q}}. \end{aligned}$$

Let  $\mathbb{K} = \mathbb{C}(\sqrt{y}, \sqrt{q})$ . Then  $\mathbb{K} \otimes \text{Markov}(\sqcup SB)$  is isomorphic to  $\mathbb{K} \otimes \text{Skein}(\mathcal{L})$ .

**Proof.** In [7] Jones gives formulas to pass from Ocneanu's trace to the HOMFLYPT polynomial. In order to prove the above theorem, it suffices to slightly adapt these formulas to the context of the theorem.

For  $(\beta, n) \in \sqcup SB$  we denote by  $[\beta, n]$  the element of  $\text{Markov}(\sqcup SB)$  represented by  $(\beta, n)$ . Similarly, for  $L \in \mathcal{L}$  we denote by  $[L]$  the element of  $\text{Skein}(\mathcal{L})$  represented by  $L$ .

Let  $\psi_1 : \sqcup SB \rightarrow \mathbb{K} \otimes \text{Markov}(\sqcup SB)$  be the map defined by

$$\psi_1(\alpha, n) = \left( \frac{q-1}{1-xy} \right)^{-n+1} (\sqrt{y})^{\varepsilon(\alpha)-n+1} [\alpha, n],$$

where  $\varepsilon : SB_n \rightarrow \mathbb{Z}$  is the homomorphism defined by

$$\varepsilon(\sigma_k) = 1, \quad \varepsilon(\sigma_k^{-1}) = -1, \quad \varepsilon(\tau_k) = 0, \quad \text{for } 1 \leq k \leq n-1.$$

Let  $(\alpha, n), (\beta, m)$  be two singular braids. We start showing that, if  $\hat{\alpha} = \hat{\beta}$ , then  $\psi_1(\alpha, n) = \psi_1(\beta, m)$ . By Theorem 2.2, in order to do so, it suffices to consider the following three cases:

1.  $n = m$  and there exist  $\gamma_1, \gamma_2 \in SB_n$  such that  $\alpha = \gamma_1\gamma_2$  and  $\beta = \gamma_2\gamma_1$ ;
2.  $m = n+1$  and  $\beta = \alpha\sigma_n$ ;
3.  $m = n+1$  and  $\beta = \alpha\sigma_n^{-1}$ .

Suppose that  $n = m$  and there exist  $\gamma_1, \gamma_2 \in SB_n$  such that  $\alpha = \gamma_1\gamma_2$  and  $\beta = \gamma_2\gamma_1$ . By definition we have  $[\alpha, n] = [\beta, n]$  and  $\varepsilon(\alpha) = \varepsilon(\beta)$ , thus  $\psi_1(\alpha) = \psi_1(\beta)$ . Suppose that  $m = n+1$  and  $\beta = \alpha\sigma_n$ . Then

$$\begin{aligned} \psi_1(\beta, m) &= \left( \frac{q-1}{1-xy} \right)^{-m+1} (\sqrt{y})^{\varepsilon(\beta)-m+1} [\beta, m] \\ &= \left( \frac{q-1}{1-xy} \right)^{-n} (\sqrt{y})^{\varepsilon(\alpha)-n+1} [\alpha\sigma_n, n+1] \\ &= \left( \frac{q-1}{1-xy} \right)^{-n} (\sqrt{y})^{\varepsilon(\alpha)-n+1} \left( \frac{q-1}{1-xy} \right) [\alpha, n] \\ &= \psi_1(\alpha, n). \end{aligned}$$



Suppose that  $m = n + 1$  and  $\beta = \alpha\sigma_n^{-1}$ . Observe that the equality  $\sigma_n^2 = (q - 1)\sigma_n + q$  implies

$$\sigma_n^{-1} = q^{-1}\sigma_n - q^{-1}(q - 1).$$

Then

$$\begin{aligned} \psi_1(\beta, m) &= \left(\frac{q-1}{1-xy}\right)^{-m+1} (\sqrt{y})^{\varepsilon(\beta)-m+1} [\beta, m] \\ &= \left(\frac{q-1}{1-xy}\right)^{-n} (\sqrt{y})^{\varepsilon(\alpha)-n-1} [\alpha\sigma_n^{-1}, n+1] \\ &= \left(\frac{q-1}{1-xy}\right)^{-n} (\sqrt{y})^{\varepsilon(\alpha)-n-1} (q^{-1}[\alpha\sigma_n, n+1] - q^{-1}(q-1)[\alpha, n+1]) \\ &= \left(\frac{q-1}{1-xy}\right)^{-n} (\sqrt{y})^{\varepsilon(\alpha)-n+1} \left(\frac{q-1}{1-xy}\right) [\alpha, n] \\ &= \psi_1(\alpha, n). \end{aligned}$$

By the above, the map  $\psi_1$  induces a map  $\psi_2 : \mathcal{L} \rightarrow \text{Markov}(\sqcup SB)$  defined by  $\psi_2(\hat{\beta}) = \psi_1(\beta)$  for all  $\beta \in \sqcup SB$ .

Let  $L_+, L_-, L_0$  be three singular links that have the same link diagram except in the neighborhood of a crossing where they are like in Figure 1.1. It is easily deduced from [2] that there exist a singular braid  $(\beta, n)$  and an index  $1 \leq k \leq n - 1$  such that  $L_+ = \widehat{\beta\sigma_k}$ ,  $L_- = \widehat{\beta\sigma_k^{-1}}$  and  $L_0 = \widehat{\beta}$ . Then

$$\begin{aligned} &t^{-1} \cdot \psi_2(L_+) - t \cdot \psi_2(L_-) \\ &= (\sqrt{yq})^{-1} \left(\frac{q-1}{1-xy}\right)^{-n+1} (\sqrt{y})^{\varepsilon(\beta)-n+2} [\beta\sigma_k, n] - (\sqrt{yq}) \left(\frac{q-1}{1-xy}\right)^{-n+1} (\sqrt{y})^{\varepsilon(\beta)-n} [\beta\sigma_k^{-1}, n] \\ &= \left(\frac{q-1}{1-xy}\right)^{-n+1} (\sqrt{y})^{\varepsilon(\beta)-n+1} \left(\frac{1}{\sqrt{q}}(q-1)[\beta, n] + \frac{1}{\sqrt{q}}q[\beta\sigma_k^{-1}, n] - \sqrt{q}[\beta\sigma_k^{-1}, n]\right) \\ &= x \left(\frac{q-1}{1-xy}\right)^{-n+1} (\sqrt{y})^{\varepsilon(\beta)-n+1} [\beta, n] \\ &= x \cdot \psi_2(L_0). \end{aligned}$$

It follows that  $\psi_2$  induces a linear map  $\psi : \text{Skein}(\mathcal{L}) \rightarrow \text{Markov}(\sqcup SB)$ . It is easily checked that this map is an algebra homomorphism.

We turn now to construct the inverse of  $\psi$ . Let  $\phi_1 : \sqcup SB \rightarrow \text{Skein}(\mathcal{L})$  be the map defined by

$$\phi_1(\beta, n) = \left(\frac{q-1}{1-xy}\right)^{n-1} (\sqrt{y})^{n-1-\varepsilon(\beta)} [\hat{\beta}].$$

Let  $n \geq 1$ ,  $\alpha, \beta \in SB_n$ , and  $1 \leq k \leq n - 1$ . Then

$$\begin{aligned} &\phi_1(\alpha\sigma_k^2\beta, n) - (q-1) \cdot \phi_1(\alpha\sigma_k\beta, n) - q \cdot \phi_1(\alpha\beta, n) \\ &= \left(\frac{q-1}{1-xy}\right)^{n-1} (\sqrt{y})^{n-1-\varepsilon(\alpha\beta)} \left(y^{-1}[\widehat{\alpha\sigma_k^2\beta}] - (\sqrt{y})^{-1}(q-1)[\widehat{\alpha\sigma_k\beta}] - q[\widehat{\alpha\beta}]\right) \\ &= \left(\frac{q-1}{1-xy}\right)^{n-1} (\sqrt{y})^{n-2-\varepsilon(\alpha\beta)} (\sqrt{q}) \left(t^{-1}[\widehat{\alpha\sigma_k^2\beta}] - t[\widehat{\alpha\beta}] - x[\widehat{\alpha\sigma_k\beta}]\right) \\ &= 0. \end{aligned}$$

Let  $n \geq 1$  and  $\alpha, \beta \in SB_n$ . Since  $\widehat{\alpha\beta} = \widehat{\beta\alpha}$ , we have  $\phi_1(\alpha\beta) = \phi_1(\beta\alpha)$ . Let  $O$  denote the trivial link. One can easily show that

$$[L \sqcup O] = \left( \frac{t^{-1} - t}{x} \right) [L] = \left( \frac{q-1}{1-xy} \right)^{-1} (\sqrt{y})^{-1} [L],$$

where  $L$  is a link and  $L \sqcup O$  is the disjoint union of  $L$  and  $O$ . Now, let  $n \geq 1$  and  $\alpha \in SB_n$ . Observe that  $\widehat{(\alpha, n+1)} = \widehat{(\alpha, n)} \sqcup O$ , thus

$$\begin{aligned} \phi_1(\alpha, n+1) &= \left( \frac{q-1}{1-xy} \right)^n (\sqrt{y})^{n-\varepsilon(\alpha)} [\widehat{(\alpha, n+1)}] \\ &= \left( \frac{q-1}{1-xy} \right)^{n-1} (\sqrt{y})^{n-1-\varepsilon(\alpha)} [\widehat{(\alpha, n)}] \\ &= \phi_1(\alpha, n). \end{aligned}$$

We also have

$$\begin{aligned} \phi_1(\alpha\sigma_n, n+1) &= \left( \frac{q-1}{1-xy} \right)^n (\sqrt{y})^{n-\varepsilon(\alpha\sigma_n)} [\widehat{(\alpha\sigma_n, n+1)}] \\ &= z \left( \frac{q-1}{1-xy} \right)^{n-1} (\sqrt{y})^{n-1-\varepsilon(\alpha)} [\widehat{(\alpha, n)}] \\ &= z \cdot \phi_1(\alpha, n). \end{aligned}$$

We conclude that the map  $\phi_1$  induces a linear map  $\phi : \text{Markov}(\sqcup SB) \rightarrow \text{Skein}(\mathcal{L})$ . It is easily checked that  $\phi$  is the inverse of  $\psi$ , thus  $\psi$  is an isomorphism.  $\square$

We turn now to state the main result of [10] from which the calculation of the Markov module of singular braids will be deduced.

Recall that  $S_d B_n$  denotes the set of singular braids with  $n$  strands and  $d$  singular points,  $\mathcal{H}(S_d B_n)$  denotes the subspace of  $\mathcal{H}(SB_n)$  spanned by  $S_d B_n$ , and that we have the graduation

$$\mathcal{H}(SB_n) = \bigoplus_{d=0}^{+\infty} \mathcal{H}(S_d B_n).$$

Set  $\mathcal{H}_z(S_d B_n) = \mathbb{C}(z, q) \otimes_{\mathbb{C}(q)} \mathcal{H}(S_d B_n)$ . Let  $\text{Markov}(\sqcup S_d B)$  denote the quotient of  $\bigoplus_{n=1}^{\infty} \mathcal{H}_z(S_d B_n)$  by the relations

- $(ab, n) = (ba, n)$  for all  $n \geq 1$  and all  $a \in \mathcal{H}_z(S_k B_n)$  and  $b \in \mathcal{H}_z(S_l B_n)$  such that  $k+l = d$ ;
- $(a, n) = (\iota_n(a), n+1)$  for all  $n \geq 1$  and all  $a \in \mathcal{H}_z(S_d B_n)$ ;
- $(\iota_n(a)\sigma_n, n+1) = z \cdot (a, n)$  for all  $n \geq 1$  and all  $a \in \mathcal{H}_z(S_d B_n)$ .

It is clear that

$$\text{Markov}(\sqcup SB) = \bigoplus_{d=0}^{+\infty} \text{Markov}(\sqcup S_d B).$$

Let  $\mathbb{C}(q, z)[S_d B_n]$  be the vector space over  $\mathbb{C}(q, z)$  freely spanned by  $S_d B_n$ . For  $d \geq 1$ , we define the linear maps  $f_{n,0}, f_{n,1} : \mathbb{C}(q, z)[S_d B_n] \rightarrow \mathbb{C}(q, z)[S_{d-1} B_n]$  as follows. Let  $\beta \in S_d B_n$ . Then

write  $\beta$  in the form  $\beta = \alpha_0 \tau_{i_1} \alpha_1 \cdots \tau_{i_d} \alpha_d$  with  $\alpha_i \in B_n$  for  $0 \leq i \leq d$ , and set

$$f_{n,0}(\beta) = \sum_{k=0}^d \alpha_0 \tau_{i_1} \alpha_1 \cdots \tau_{i_{k-1}} \alpha_{k-1} \alpha_k \tau_{i_{k+1}} \alpha_{k+1} \cdots \tau_{i_d} \alpha_d$$

$$f_{n,1}(\beta) = \sum_{k=0}^d \alpha_0 \tau_{i_1} \alpha_1 \cdots \tau_{i_{k-1}} \alpha_{k-1} \sigma_{i_k} \alpha_k \tau_{i_{k+1}} \alpha_{k+1} \cdots \tau_{i_d} \alpha_d$$

It follows from Theorem 2.1 that this definition does not depend on the choice of the expression of  $\beta$ .

It is easily checked that the collection of linear maps  $\{f_{n,0}\}_{n \geq 1}$  induces a linear map  $g_0 : \text{Markov}(\sqcup S_d B) \rightarrow \text{Markov}(\sqcup S_{d-1} B)$ . Similarly, the collection of maps  $\{f_{n,1}\}_{n \geq 1}$  induces a linear map  $g_1 : \text{Markov}(\sqcup S_d B) \rightarrow \text{Markov}(\sqcup S_{d-1} B)$ .

Let  $\text{Markov}(\sqcup S_d B)^*$  be the dual space of  $\text{Markov}(\sqcup S_d B)$ , that is, the space of linear forms on  $\text{Markov}(\sqcup S_d B)$ . For  $d \geq 1$ , we denote by  $\Phi_{d,0} : \text{Markov}(\sqcup S_{d-1} B)^* \rightarrow \text{Markov}(\sqcup S_d B)^*$  the linear map induced by  $g_0$ , and by  $\Phi_{d,1} : \text{Markov}(\sqcup S_{d-1} B)^* \rightarrow \text{Markov}(\sqcup S_d B)^*$  the linear map induced by  $g_1$ . Note that  $\Phi_{d+1,1} \circ \Phi_{d,0} = \Phi_{d+1,0} \circ \Phi_{d,1}$  for all  $d \geq 1$ .

For  $d \geq 0$ , we define the elements  $T_{d,0}, T_{d,1}, \dots, T_{d,d} \in \text{Markov}(\sqcup S_d B)^*$  by induction on  $d$  as follows. It is proved in [7] that the space  $\text{Markov}(\sqcup S_0 B)^*$  is of dimension 1. Then we denote by  $T_{0,0}$  the generator of  $\text{Markov}(\sqcup S_0 B)^*$  whose value on the trivial braid is 1. Suppose  $d \geq 1$ . Then we set

$$\begin{aligned} T_{d,0} &= \Phi_{d,0}(T_{d-1,0}) \\ T_{d,k} &= \Phi_{d,0}(T_{d-1,k}) = \Phi_{d,1}(T_{d-1,k-1}) \quad \text{if } 1 \leq k \leq d-1 \\ T_{d,d} &= \Phi_{d,1}(T_{d-1,d-1}) \end{aligned}$$

**Theorem 2.7** (Paris, Rabenda [10]). *Let  $d \geq 0$ . Then  $\text{Markov}(\sqcup S_d B)^*$  is of dimension  $d+1$ , and  $\{T_{d,0}, T_{d,1}, \dots, T_{d,d}\}$  is a basis for  $\text{Markov}(\sqcup S_d B)^*$ .*

We can now calculate the Markov module of singular braids:

**Theorem 2.8.** *The algebra  $\text{Markov}(\sqcup SB)$  is a polynomial algebra  $\mathbb{C}(q, z)[X, Y]$  in two variables  $X$  and  $Y$ , where  $X$  and  $Y$  are the classes of  $\tau_1$  and  $\tau_1 \sigma_1$ , respectively.*

**Proof.** Let  $d \geq 0$  and  $0 \leq k \leq d$ . Observe that  $X^k Y^{d-k}$  is the class of  $\tau_1 \cdots \tau_{2k-1} (\tau_{2k+1} \sigma_{2k+1}) \cdots (\tau_{2d-1} \sigma_{2d-1})$ . In particular, we have  $X^k Y^{d-k} \in \text{Markov}(\sqcup S_d B)$ . So, in order to prove Theorem 2.8, it suffices to show that  $\{X^d, X^{d-1}Y, \dots, XY^{d-1}, Y^d\}$  is a basis for  $\text{Markov}(\sqcup S_d B)$ . Since we already know by Theorem 2.7 that  $\text{Markov}(\sqcup S_d B)$  is of dimension  $d+1$ , it actually suffices to show that  $\{X^d, X^{d-1}Y, \dots, XY^{d-1}, Y^d\}$  is linearly independent. We prove this by induction on  $d$ . The case  $d=0$  being trivial, we assume  $d \geq 1$  plus the inductive hypothesis.

A direct calculation shows that, for  $0 \leq k \leq d$ , we have

$$\begin{aligned} g_0(X^k Y^{d-k}) &= k \cdot X^{k-1} Y^{d-k} + z(d-k) \cdot X^k Y^{d-k-1} \\ g_1(X^k Y^{d-k}) &= kz \cdot X^{k-1} Y^{d-k} + (d-k)((q-1)z + q) \cdot X^k Y^{d-k-1} \end{aligned}$$

Let  $a_0, a_1, \dots, a_d \in \mathbb{C}(q, z)$  such that

$$\sum_{k=0}^d a_k X^k Y^{d-k} = 0. \quad (2.2)$$

Applying  $g_0$  and  $g_1$  to (2.2) we obtain

$$\begin{aligned} \sum_{k=0}^{d-1} ((k+1)a_{k+1} + (d-k)za_k) X^k Y^{d-k-1} &= 0 \\ \sum_{k=0}^{d-1} ((k+1)za_{k+1} + (d-k)((q-1)z+q)a_k) X^k Y^{d-k-1} &= 0 \end{aligned}$$

By induction, it follows that

$$\begin{cases} (k+1)a_{k+1} + (d-k)za_k = 0 \\ (k+1)za_{k+1} + (d-k)((q-1)z+q)a_k = 0 \end{cases}$$

for all  $0 \leq k \leq d-1$ . The determinant of this system of linear equations in the variables  $a_{k+1}, a_k$  is equal to  $-(k+1)(d-k)(z^2 - (q-1)z - q) \neq 0$ , thus  $a_{k+1} = a_k = 0$ .  $\square$

**Corollary 2.9.** *Let  $\mathcal{L}$  be the set of singular links in the sphere  $\mathbb{S}^3$ . Then the algebra  $\text{Skein}(\mathcal{L})$  is a polynomial algebra  $\mathbb{C}(t, x)[\hat{X}, \hat{Y}]$  in two variables  $\hat{X}$  and  $\hat{Y}$ , where  $\hat{X}$  and  $\hat{Y}$  are the classes of the singular links  $L_X$  and  $L_Y$  represented in Figure 1.4.*

**Question 2.10.** The proof that the set  $\mathcal{B} = \{X^a Y^b; a, b \in \mathbb{N}\}$  is linearly independent in  $\text{Markov}(\sqcup SB)$  is entirely given in the above proof of Theorem 2.8, and does not need Theorem 2.7 at all. However, the proof that  $\mathcal{B}$  spans  $\text{Markov}(\sqcup SB)$  uses the fact that the dimension of  $\text{Markov}(\sqcup S_d B)$  is (less or) equal to  $d+1$  for all  $d \geq 0$ , and the proof of this latest assertion needs long and tedious calculations. It would be interesting to find a (simplest) proof of the equivalent fact that  $\hat{\mathcal{B}} = \{\hat{X}^a \hat{Y}^b; a, b \in \mathbb{N}\}$  spans  $\text{Skein}(\mathcal{L})$ , which would directly use the Skein relations.

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