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The Role of Complex Conjugation in Transcendental Number Theory

Michel Waldschmidt

Dedicated to Professor T.N. Shorey on the occasion of his 60th birthday.[*]

Abstract

In his two well known 1968 papers "Contributions to the theory of transcendental numbers", K. Ramachandra proved several results showing that, in certain explicit sets $\{x_1, \ldots, x_n\}$ of complex numbers, one element at least is transcendental. In specific cases the number n of elements in the set was 2 and the two numbers x_1, x_2 were both real. He then noticed that the conclusion is equivalent to saying that the complex number $x_1 + ix_2$ is transcendental.

In his 2004 paper published in the Journal de Théorie des Nombres de Bordeaux, G. Diaz investigates how complex conjugation can be used for the transcendence study of the values of the exponential function. For instance, if $\log \alpha_1$ and $\log \alpha_2$ are two non-zero logarithms of algebraic numbers, one of them being either real of purely imaginary, and not the other, then the product $(\log \alpha_1)(\log \alpha_2)$ is transcendental.

We will survey Diaz's results and produce further similar ones.

1 Theorems of Hermite–Lindemann and Gel'fond–Schneider

Denote by $\overline{\mathbb{Q}}$ the field of algebraic numbers and by $\mathcal{L}$ the $\mathbb{Q}$-vector space of logarithms of algebraic numbers:

$$\mathcal{L} = \{\lambda \in \mathbb{C} \; ; \; e^\lambda \in \overline{\mathbb{Q}}^\times\} = \exp^{-1}(\overline{\mathbb{Q}}^\times) = \{\log \alpha \; ; \; \alpha \in \overline{\mathbb{Q}}^\times\}.$$

The Theorem of Hermite and Lindemann ([13], Theorem 1.2) can be stated in several equivalent ways:

[*]Many thanks to N. Saradha for the perfect organization of this Conference DION2005 and for taking care in such an efficient way of the publication of the proceedings.

Theorem 1.1 (Hermite–Lindemann) (i) *Let α be a non-zero algebraic number and let $\log\alpha$ be any non-zero logarithm of α. Then $\log\alpha$ is transcendental.*

(ii) *There is no non-zero logarithm of algebraic number:*

$$\mathcal{L} \cap \overline{\mathbb{Q}} = \{0\}.$$

(iii) *Let β be a non-zero algebraic number. Then e^β is transcendental.*

Another classical result is the Theorem of Gel'fond and Schneider ([13, Theorem 1.4]):

Theorem 1.2 (Gel'fond-Schneider) *Let λ and β be complex numbers. Assume $\lambda \neq 0$ and $\beta \notin \mathbb{Q}$. Then one at least of the three numbers*

$$\beta, \quad e^\lambda, \quad e^{\beta\lambda}$$

is transcendental.

The proof of the Theorem 1.1 of Hermite and Lindemann rests on properties of the two functions z and e^z: they are algebraically independent, they have finite order of growth, they satisfy differential equations with rational coefficients, and for any $\beta \in \mathbb{C}$ they take values in the field $\mathbb{Q}(\beta, e^\beta)$ for all values of z of the form $s\beta$ with $s \in \mathbb{Z}$.

In a similar way, the proof by Gel'fond of Theorem 1.2 relies on the fact that, for $\beta \in \mathbb{C} \setminus \mathbb{Q}$, the two functions e^z and $e^{\beta z}$ are algebraically independent, they have finite order of growth, they satisfy differential equations in the field $\mathbb{Q}(\beta)$, and for any $\lambda \in \mathbb{C}$ they take values in the field $\mathbb{Q}(e^\lambda, e^{\beta\lambda})$ for all values of z of the form $s\lambda$ with $s \in \mathbb{Z}$.

On the other hand, Schneider does not use differential equation: in his proof he uses the fact that for $\lambda \in \mathbb{C}^\times$, the two functions z and $e^{\lambda z}$ are algebraically independent (if $\lambda \neq 0$), they have finite order of growth and they take values in the field $\mathbb{Q}(\beta, e^\lambda, e^{\beta\lambda})$ for all values of z of the form $s_1 + s_2\beta$ with $(s_1, s_2) \in \mathbb{Z} \times \mathbb{Z}$.

In connection with Theorem 1.1 of Hermite–Lindemann, G. Diaz [1] considered the following question:

- *Which are the pairs (t, β), where t is a non-zero real number and β a non-zero algebraic number, such that $e^{t\beta}$ is algebraic?*

A first example is $\beta \in \mathbb{R}$ and $t = (\log 2)/\beta$. A second example is $\beta \in i\mathbb{R}$ and $t = i\pi/\beta$. In [1], G. Diaz shows that there are no further examples.

Theorem 1.3 (G. Diaz) *Let $\beta \in \overline{\mathbb{Q}}$ and $t \in \mathbb{R}^\times$. Assume $\beta \notin \mathbb{R} \cup i\mathbb{R}$. Then $e^{t\beta}$ is transcendental. Equivalently: for $\lambda \in \mathcal{L}$ with $\lambda \notin \mathbb{R} \cup i\mathbb{R}$,*

$$\mathbb{R}\lambda \cap \overline{\mathbb{Q}} = \{0\}.$$

Proof Set $\alpha = e^{t\beta}$. The complex conjugate $\overline{\alpha}$ of α is $e^{t\overline{\beta}} = \alpha^{\overline{\beta}/\beta}$. Since $\beta \notin \mathbb{R} \cup i\mathbb{R}$, the algebraic number $\overline{\beta}/\beta$ is not real (its modulus is 1 and it is not ± 1), hence it is not rational. Gel'fond-Schneider's Theorem 1.2 implies that α and $\alpha^{\overline{\beta}/\beta}$ cannot be both algebraic. Since $\alpha^{\overline{\beta}/\beta} = \overline{\alpha}$, it follows that α and $\overline{\alpha}$ are both transcendental.

□

Another consequence that G. Diaz deduces from Gel'fond-Schneider's Theorem 1.2 is the next one.

Corollary 1.4 (G. Diaz) *There exists $\beta_0 \in \mathbb{R} \cup i\mathbb{R}$ such that*

$$\{\beta \in \overline{\mathbb{Q}}\ ;\ e^\beta \in \overline{\mathbb{Q}}\} = \mathbb{Q}\beta_0.$$

Remark 1.5 Hermite–Lindemann's Theorem 1.1 tells us that in fact $\beta_0 = 0$.

Proof From Gel'fond-Schneider's Theorem 1.2 one deduces that the $\mathbb{Q}$-vector–space $\{\beta \in \overline{\mathbb{Q}}\ ;\ e^\beta \in \overline{\mathbb{Q}}\}$ has dimension ≤ 1 and is contained in $\mathbb{R} \cup i\mathbb{R}$.

□

An interesting output of this argument is that it yields a proof of special cases of Hermite-Lindemann's Theorem 1.1 by means of Schneider's method, hence without involving derivatives. The details are provided in [1].

A challenge is to produce a transcendence proof using the function $e^{\overline{z}}$. The so-called *elementary method* in the last chapter of the book by Gelfond and Linnik [3] involves real analytic functions.

Another related challenge is to prove new results by means of the function e^{z^2}: so far it seems that all results which follow from Gel'ffond-Schneider's method involving this function can also be derived without using it.

2 The Six Exponentials Theorem and the Four Exponentials Conjecture

We first recall the statement of the six exponentials Theorem due to C.L. Siegel, S. Lang and K. Ramachandra ([13, Theorem 1.12]).

Theorem 2.1 (The six exponentials Theorem) *If x_1, x_2 are $\mathbb{Q}$–linearly independent complex numbers and y_1, y_2, y_3 are $\mathbb{Q}$–linearly independent complex numbers, then one at least of the six numbers*

$$e^{x_1y_1},\ e^{x_1y_2},\ e^{x_1y_3},\ \ e^{x_2y_1},\ e^{x_2y_2},\ e^{x_2y_3}$$

is transcendental.

Example Take $x_1 = 1$, $x_2 = \pi$, $y_1 = \log 2$, $y_2 = \pi \log 2$, $y_3 = \pi^2 \log 2$; the six exponentials are respectively

$$2,\ 2^{\pi},\ 2^{\pi^2},\ 2^{\pi},\ 2^{\pi^2},\ 2^{\pi^3};$$

hence one at least of the three numbers

$$2^{\pi},\ 2^{\pi^2},\ 2^{\pi^3}$$

is transcendental.

We refer to the papers [9] and [10] by T.N. Shorey for effective versions of this result, namely lower bounds for

$$|2^{\pi} - \alpha_1| + |2^{\pi^2} - \alpha_2| + |2^{\pi^3} - \alpha_3|$$

for algebraic $\alpha_1, \alpha_2, \alpha_3$. The estimates depend on the heights and degrees of these algebraic numbers. Another subsequent effective version of Theorem 2.1 is given in [5].

Remark 2.2 It is unknown whether one of the two numbers

$$2^{\pi},\ 2^{\pi^2}$$

is transcendental. A consequence of Schanuel's conjecture 2.3 below ([13], Conjecture 1.14) is that *each of the three numbers* $2^{\pi},\ 2^{\pi^2},\ 2^{\pi^3}$ *is transcendental* and more precisely that *the five numbers*

$$\pi,\ \log 2,\ 2^{\pi},\ 2^{\pi^2},\ 2^{\pi^3}$$

are algebraically independent.

Conjecture 2.3 (Schanuel) *Let $x_1, \ldots, x_n$ be $\mathbb{Q}$–linearly independent complex numbers. Then the transcendence degree over $\mathbb{Q}$ of the field $\mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$ is at least n.*

The four exponentials Conjecture is the next statement which has been suggested by C.L. Siegel, Th. Schneider, S. Lang and K. Ramachandra ([13, Conjecture 1.13]). It is also a consequence of Schanuel's Conjecture 2.3.

Conjecture 2.4 (The four exponentials Conjecture) *If x_1, x_2 are $\mathbb{Q}$–linearly independent complex numbers and y_1, y_2 are $\mathbb{Q}$–linearly independent complex numbers, then one at least of the four numbers*

$$e^{x_1 y_1},\ e^{x_1 y_2},\ \ e^{x_2 y_1},\ e^{x_2 y_2}$$

is transcendental.

We now describe a trick which originates in the seminal paper [6] by K. Ramachandra. As often in such circumstances, the starting point is a pretty obvious remark.

Remark 2.5 Let x and y be two real numbers. The following properties are equivalent:

(i) one at least of the two numbers x, y is transcendental.

(ii) the complex number $x + iy$ is transcendental.

We reproduce here Corollary p. 87 of [6]:

Theorem 2.6 (K. Ramachandra) *If a and b are real positive algebraic numbers different from 1 for which $\log a/\log b$ is irrational and $a < b < a^{-1}$, then one at least of the two numbers*

$$x = \left(\frac{1}{240} + \sum_{n=1}^{\infty} \frac{n^3 a^n}{1 - a^n}\right) \prod_{n=1}^{\infty} (1 - a^n)^{-8},$$

$$y = \left\{\frac{6}{(b^{1/2} - b^{-1/2})^4} - \frac{1}{(b^{1/2} - b^{-1/2})^2} - \sum_{n=1}^{\infty} \frac{n^3 a^n (b^n + b^{-n})}{1 - a^n}\right\} \prod_{n=1}^{\infty} (1 - a^n)^{-8},$$

is transcendental. Therefore the complex number $x + iy$ is transcendental.

This idea is developed by G. Diaz in ([1, Th. 1]), where he proves the following special case of the four exponentials Conjecture.

Corollary 2.7 (of the six exponentials Theorem) *Let x_1, x_2 be two elements in $\mathbb{R} \cup i\mathbb{R}$ which are $\mathbb{Q}$–linearly independent. Let y_1, y_2 be two complex numbers. Assume that the three numbers $y_1, y_2, \overline{y_2}$ are $\mathbb{Q}$–linearly independent. Then one at least of the four numbers*

$$e^{x_1y_1},\ e^{x_1y_2},\ \ e^{x_2y_1},\ e^{x_2y_2}$$

is transcendental.

Proof Set $y_3 = \overline{y_2}$. Then $e^{x_jy_3} = \overline{e^{\pm x_jy_2}}$ for $j = 1, 2$ and $\overline{\mathbb{Q}}$ is stable under complex conjugation. Hence one may use the six exponentials Theorem 2.1.

□

An alternate form (see [13, § 1.4]) of the six exponentials Theorem 2.1 (resp. the four exponentials Conjecture 2.4) is the fact that *a* 2×3 *(resp.* 2×2*) matrix with entries in* $\mathcal{L}$

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \end{pmatrix} \qquad \left(\textit{resp.} \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \right),$$

the rows of which are linearly independent over $\mathbb{Q}$ and the columns of which are also linearly independent over $\mathbb{Q}$, has maximal rank 2.

Remark 2.8 A $d \times \ell$ matrix M has rank ≤ 1 if and only if there exist $x_1, \ldots, x_d$ and $y_1, \ldots, y_\ell$ such that

$$M = \begin{pmatrix} x_1y_1 & x_1y_2 & \ldots & x_1y_\ell \\ x_2y_1 & x_2y_2 & \ldots & x_2y_\ell \\ \vdots & \vdots & \ddots & \vdots \\ x_dy_1 & x_dy_2 & \ldots & x_dy_\ell \end{pmatrix}.$$

3 The Strong Six Exponentials Theorem and the Strong Four Exponentials Conjecture

We consider now linear combinations of logarithms of algebraic numbers.

Denote by $\widetilde{\mathcal{L}}$ the $\overline{\mathbb{Q}}$-vector space spanned by 1 and $\mathcal{L}$: hence $\widetilde{\mathcal{L}}$ is the set of linear combinations with algebraic coefficients of logarithms of algebraic numbers:

$$\widetilde{\mathcal{L}} = \{\beta_0 + \beta_1\lambda_1 + \cdots + \beta_n\lambda_n\ ;\ n \geq 0, \beta_i \in \overline{\mathbb{Q}},\ \lambda_i \in \mathcal{L}\}.$$

Here is the strong six exponentials Theorem of D. Roy [8]; see also [13, Corollary 11.16].

Theorem 3.1 (Strong six exponentials Theorem) *If x_1, x_2 are $\overline{\mathbb{Q}}$-linearly independent complex numbers and y_1, y_2, y_3 are $\overline{\mathbb{Q}}$–linearly independent complex numbers, then one at least of the six numbers*

$$x_1y_1,\ x_1y_2,\ x_1y_3,\ \ x_2y_1,\ x_2y_2,\ x_2y_3$$

is not in $\widetilde{\mathcal{L}}$.

Also in [8] D. Roy proposes the strong four exponentials Conjecture ([13, Conjecture 11.17]. Again, it is a consequence of Schanuel's Conjecture 2.3.

Conjecture 3.2 (Strong four exponentials Conjecture) *If x_1, x_2 are $\overline{\mathbb{Q}}$–linearly independent complex numbers and y_1, y_2 are $\overline{\mathbb{Q}}$–linearly independent complex numbers, then one at least of the four numbers x_1y_1, x_1y_2, x_2y_1, x_2y_2 is not in $\widetilde{\mathcal{L}}$.*

An alternate form of the strong six exponentials Theorem 3.1 (resp. the strong four exponentials Conjecture 3.2) is the fact that *a 2×3 (resp. 2×2) matrix with entries in $\widetilde{\mathcal{L}}$*

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix} \qquad \left(\textit{resp.} \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \right),$$

the rows of which are linearly independent over $\overline{\mathbb{Q}}$ and the columns of which are also linearly independent over $\overline{\mathbb{Q}}$, has maximal rank 2.

Remark 3.3 Under suitable conditions ([13, Theorem 12.20]), one can show that a $d \times \ell$ matrix with entries in $\widetilde{\mathcal{L}}$ has rank $\geq d\ell/(d+\ell)$. Such statements are consequences of the *Linear Subgroup Theorem* ([13, Theorem 11.5]). Quantitative refinements exist ([13, Chap. 13]).

Here is an alternate form of the strong four exponentials Conjecture 3.2.

Conjecture 3.4 (Strong four exponentials Conjecture again) *Let Λ_1, Λ_2, Λ_3 be non-zero elements in $\widetilde{\mathcal{L}}$. Assume the numbers Λ_2/Λ_1 and Λ_3/Λ_1 are both transcendental. Then the number $\Lambda_2\Lambda_3/\Lambda_1$ is not in $\widetilde{\mathcal{L}}$.*

For the equivalence between both versions of the strong four exponentials Conjecture, notice that the matrix

$$\begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_2\Lambda_3/\Lambda_1 \end{pmatrix}$$

has rank 1 and see remark 2.8.

□

Following [1, 2], we deduce a few consequences of the strong four exponentials Conjecture 3.2 and 3.4.

Proposition 3.5 *Assume the strong four exponentials Conjecture.*

(i) *If Λ is in $\widetilde{\mathcal{L}} \setminus \overline{\mathbb{Q}}$, then the quotient $1/\Lambda$ is not in $\widetilde{\mathcal{L}}$.*

(ii) *If Λ_1 and Λ_2 are in $\widetilde{\mathcal{L}} \setminus \overline{\mathbb{Q}}$, then the product $\Lambda_1\Lambda_2$ is not in $\widetilde{\mathcal{L}}$.*

(iii) *If Λ_1 and Λ_2 are in $\widetilde{\mathcal{L}}$ with Λ_1 and Λ_2/Λ_1 transcendental, then this quotient Λ_2/Λ_1 is not in $\widetilde{\mathcal{L}}$.*

4 Recent Results

In [1, 2], G. Diaz provides examples where the strong four exponentials Conjecture 3.2 is true. For instance:

Theorem 4.1 (G. Diaz) *Let x_1 and x_2 be two elements of $\mathbb{R} \cup i\mathbb{R}$ which are $\overline{\mathbb{Q}}$–linearly independent. Let y_1, y_2 be two complex numbers such that the three numbers y_1, y_2, $\overline{y_2}$ are $\overline{\mathbb{Q}}$-linearly independent. Then one at least of the four numbers*

$$x_1y_1,\ x_1y_2,\ \ x_2y_1,\ x_2y_2$$

is not in $\widetilde{\mathcal{L}}$.

Proof Set $y_3 = \overline{y_2}$. Then $e^{x_jy_3} = \overline{e^{\pm x_jy_2}}$ for $j = 1, 2$ and $\widetilde{\mathcal{L}}$ is stable under complex conjugation.

□

An equivalent formulation of Diaz's Theorem 4.1 is the next one (part 1 of Corollaire 1 of [2]).

Corollary 4.2 (of Diaz's Theorem) *Let Λ_1, Λ_2, Λ_3 be three elements in $\widetilde{\mathcal{L}}$. Assume that the three numbers Λ_1, Λ_2, $\overline{\Lambda_2}$ are linearly independent over $\overline{\mathbb{Q}}$. Further assume $\Lambda_3/\Lambda_1 \in (\mathbb{R} \cup i\mathbb{R}) \setminus \overline{\mathbb{Q}}$. Then*

$$\Lambda_2\Lambda_3/\Lambda_1 \notin \overline{\mathbb{Q}}.$$

Proof One deduces Corollary 4.2 from Theorem 4.1 by setting $x_1 = 1$, $x_2 = \Lambda_3/\Lambda_1$, $y_1 = \Lambda_1$, $y_2 = \Lambda_2$.

□

Remark 4.3 Conversely, Theorem 4.1 follows from Corollary 4.2 with $\Lambda_1 = x_1y_1$, $\Lambda_2 = x_1y_2$, $\Lambda_3 = x_2y_1$.

One deduces examples where one can actually prove that numbers like

$$1/\Lambda, \quad \Lambda_1\Lambda_2, \quad \Lambda_2/\Lambda_1$$

(with Λ, Λ_1, Λ_2 in $\widetilde{\mathcal{L}}$) are not in $\widetilde{\mathcal{L}}$.

5 Product of Logarithms of Algebraic Numbers

No proof has been given so far that *the number* e^{π^2} *is transcendental.*

One can ask, more generally: *for* $\lambda \in \mathcal{L} \setminus \{0\}$*, is it true that* $\lambda\overline{\lambda} \notin \mathcal{L}$*?*

Again, more generally: *for* λ_1 *and* λ_2 *in* $\mathcal{L}\setminus\{0\}$*, is it true that* $\lambda_1\lambda_2 \notin \mathcal{L}$*?*

And finally, *for* λ_1 *and* λ_2 *in* $\mathcal{L} \setminus \{0\}$*, is it true that* $\lambda_1\lambda_2 \notin \widetilde{\mathcal{L}}$*?*

Clearly a positive answer to the last question (hence to the previous ones) follows from the strong four exponentials Conjecture 3.2.

One of the few transcendence results concerning the product of logarithms of algebraic numbers reads as follows (it is a special case of Theorem 3 of [1] where λ is replaced by a transcendental element of $\widetilde{\mathcal{L}}$).

Theorem 5.1 (G. Diaz) *Let* λ *and* λ' *be in* $\mathcal{L} \setminus \{0\}$*. Assume* $\lambda \in \mathbb{R} \cup i\mathbb{R}$ *and* $\lambda' \notin \mathbb{R} \cup i\mathbb{R}$*. Then* $\lambda\lambda' \notin \widetilde{\mathcal{L}}$*.*

Proof Apply Corollary 4.2 with $\Lambda_1 = 1$, $\Lambda_2 = \lambda'$ and $\Lambda_3 = \lambda$. The fact that Λ_3/Λ_1 is transcendental follows from Hermite-Lindemann Theorem, while the fact that $\Lambda_1, \Lambda_2, \overline{\Lambda_2}$ are linearly independent over $\overline{\mathbb{Q}}$ is a consequence of Baker's Theorem (see for instance [13] Theorem 1.6).

□

According to the strong four exponentials Conjecture 3.2, one expects the assumptions $\lambda \in \mathbb{R} \cup i\mathbb{R}$ and $\lambda' \notin \mathbb{R} \cup i\mathbb{R}$ in Theorem 5.1 to be unnecessary. In particular, for $\lambda \in \mathcal{L} \setminus \{0\}$, the number $|\lambda|^2 = \lambda\overline{\lambda}$ should be transcendental. In other terms (Conjecture $\mathcal{C}(|u|)$ in § 5.1 of [1]):

Conjecture 5.2 (G. Diaz) *Let* $u \in \mathbb{C}^\times$*. Assume* $|u|$ *is algebraic. Then* e^u *is transcendental.*

This would provide a nice generalization to Theorem 1.1 of Hermite and Lindemann.

6 Further Results

From the strong six exponentials Theorem 3.1, one deduces the following result ([15] Theorem 1.2), which can be viewed as a first step towards the transcendence of non-zero numbers of the form $\Lambda_1\Lambda_2 - \Lambda_3\Lambda_4$ for $\Lambda_i \in \widetilde{\mathcal{L}}$:

Theorem 6.1 *Let* M *be a* 2×3 *matrix with entries in* $\widetilde{\mathcal{L}}$*:*

$$M = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix}.$$

Assume that the five rows of the matrix

$$\begin{pmatrix} M \\ I_3 \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are linearly independent over $\overline{\mathbb{Q}}$ *and that the five columns of the matrix*

$$\big(I_2, M\big) = \begin{pmatrix} 1 & 0 & \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ 0 & 1 & \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix}$$

are linearly independent over $\overline{\mathbb{Q}}$*. Then one at least of the three numbers*

$$\Delta_1 = \begin{vmatrix} \Lambda_{12} & \Lambda_{13} \\ \Lambda_{22} & \Lambda_{23} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} \Lambda_{13} & \Lambda_{11} \\ \Lambda_{23} & \Lambda_{21} \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{vmatrix}$$

is not in $\widetilde{\mathcal{L}}$*.*

So far the conclusion of the above-mentioned statements (theorems or conjectures) were that some matrices had rank at least 2. Under suitable assumptions, it is possible to reach higher rank. Here is an example from [14] (Theorem 2.11).

Theorem 6.2 *Let* $M = (\Lambda_{ij})_{1\le i\le m; 1\le j\le \ell}$ *be a* $m \times \ell$ *matrix with entries in* $\widetilde{\mathcal{L}}$*. Denote by* I_m *the identity* $m \times m$ *matrix and assume that the* $m + \ell$ *column vectors of the matrix* (I_m, M) *are linearly independent over* $\overline{\mathbb{Q}}$*. Let* $\Lambda_1, \ldots, \Lambda_m$ *be elements of* $\widetilde{\mathcal{L}}$*. Assume that the numbers* $1, \Lambda_1, \ldots, \Lambda_m$ *are* $\overline{\mathbb{Q}}$*-linearly independent. Assume further* $\ell > m^2$*. Then one at least of the* ℓ *numbers*

$$\Lambda_1\Lambda_{1j} + \cdots + \Lambda_m\Lambda_{mj} \quad (j = 1, \ldots, \ell)$$

is not in $\widetilde{\mathcal{L}}$*.*

As pointed out in [15], the statement of Corollary 2.12, p. 347 of [14], should read as follows.

Corollary 6.3 *Let* M *be a* 2×5 *matrix with entries in* $\widetilde{\mathcal{L}}$*:*

$$M = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} & \Lambda_{15} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} & \Lambda_{25} \end{pmatrix}.$$

Assume that the three numbers 1*,* Λ_{11} *and* Λ_{21} *are linearly independent over the field of algebraic numbers and that the seven columns of the matrix*

(I_2, M) are linearly independent over $\overline{\mathbb{Q}}$. Then one at least of the four numbers

$$\Lambda_{11}\Lambda_{2j} - \Lambda_{21}\Lambda_{1j} \quad (j = 2, 3, 4, 5)$$

is not in $\widetilde{\mathcal{L}}$.

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