



HAL
open science

Radon needlet thresholding

Gerard Kerkyacharian, Erwan Le Pennec, Dominique Picard

► **To cite this version:**

Gerard Kerkyacharian, Erwan Le Pennec, Dominique Picard. Radon needlet thresholding. 2009.
hal-00409903v1

HAL Id: hal-00409903

<https://hal.science/hal-00409903v1>

Preprint submitted on 13 Aug 2009 (v1), last revised 8 May 2012 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Radon needlet thresholding

G erard Kerkyacharian, Erwan Le Pennec and Dominique Picard
LPMA / Universit Paris Diderot, France

August 13, 2009

Abstract

We provide a new algorithm for the treatment of the noisy inversion of the radon transform using an appropriate thresholding technique adapted to a well chosen new localized basis. We establish minimax results and prove their optimality. In particular we prove that the procedures provided here are able to attain minimax bounds for any \mathbb{L}_p loss. It is important to notice that most of the minimax bounds obtained here are new to our knowledge. It is also important to emphasize the adaptation properties of our procedures with respect to the regularity (sparsity) of the object to recover as well as to inhomogeneous smoothness. We also perform a numerical study which is of importance since we especially have to discuss the cubature problems and propose an averaging procedure which is mostly in the spirit of the cycle spinning performed for periodic signals.

1 Introduction

We consider the problem of inverting noisy observations of the d -dimensional Radon transform. Obviously the most immediate examples occur for $d = 2$ or 3 . However no major differences arise from considering the general case.

There is a considerable literature on the problem of reconstructing structures from their Radon transforms which is a fundamental problem in medical imaging and more generally in tomography. In our approach, we focus on several important points. We produce a procedure which is efficient from a L_2 point of view, since this loss function mimics quite well in many situations the preferences of the human eye. On the other hand, we have at the same time the requirement of clearly identifying the local bumps, of being able to well estimate the different level sets. We also want the procedure to enjoy good adaptation properties. In addition, we require the procedure to be simple to implement.

At the heart of such a problem there is a notable conflict between the inversion part which in presence of noise creates an instability reasonably handled by a Singular Value Decomposition (SVD) approach and the fact that the SVD basis very rarely is localized and capable of representing local features of images, which are especially important to recover. Our strategy is to follow the approach started in [10] which utilizes the

construction borrowed from [20] (see also [12]) of localized frames based on orthogonal polynomials on the ball, which are closely related to the Radon transform SVD basis.

To achieve the goals presented above, and especially adaptation to different regularities and local inhomogeneous smoothness, we add a fine tuning subsequent thresholding process to the estimation performed in [10].

This improves considerably the performances of the algorithm, both from a theoretical point of view and a numerical point of view. In effect, the new algorithm provides a much better spatial adaptation, as well as adaptation to the classes of regularity. We prove here that the bounds obtained by the procedure are minimax over a large class of Besov spaces and any \mathbb{L}_p losses: we provide upper bounds for the performance of our algorithm as well as lower bounds for the associated minimax rate.

It is important to notice that especially because we consider different \mathbb{L}_p losses, we provide rates of convergence of new types attained by our procedure. Those rates are minimax since confirmed by lower bounds inequalities.

The problem of choosing appropriate spaces of regularity on the ball reflecting the standard objects analyzed in tomography is a highly non trivial problem. We decided to consider the spaces which seems to stay the closest to our natural intuition, those which generalize to the ball the approximation properties by polynomials.

The procedure gives very promising results in the simulation study. We show that the estimates obtained by thresholding the needlets outperform those obtained either by thresholding the SVD or by the linear needlet estimate proposed in [10]. An important issue in the needlet scheme is the choice of the quadrature in the needlet construction. We discussed the possibilities proposed in the literature and considered a cubature formula based on the full tensorial grid on the sphere introducing an averaging which principle is close to the cycle-spinning method.

Among others, one amazing result is the fact that to attain minimax rates in \mathbb{L}_∞ norm, we need in this case to modify the estimator, which is also corroborated by the numerical results: see Theorem 2 and Figure 4.

In the first section, we introduce the Radon transform and the associated SVD basis. The following section summaries the construction of the localized basis, the needlets. Section 4 introduces our procedures and states the theoretical results: upper bounds and lower bounds. Section 5 details the simulation study. Section 6 details important properties of the needlet basis. The proof of the two main results stated in section 4 are postponed in the two last sections.

2 Radon transform and white noise model

2.1 Radon transform

Here we recall the definition and some basic facts about the Radon transform (cf. [9], [17], [13]). Denote by \mathbb{B}^d the unit ball in \mathbb{R}^d , i.e. $\mathbb{B}^d = \{\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d : |\mathbf{x}| \leq 1\}$ with $|\mathbf{x}| = (\sum_{i=1}^d x_i^2)^{1/2}$ and by \mathbb{S}^{d-1} the unit sphere in \mathbb{R}^d . The Lebesgue measure on \mathbb{B}^d will be denoted by $d\mathbf{x}$ and the usual surface measure on \mathbb{S}^{d-1} by $d\sigma(\mathbf{x})$ (sometimes

we will also deal with the surface measure on \mathbb{S}^d which will be denoted by $d\sigma_d$). We let $|A|$ denote the measure $|A| = \int_A dx$ if $A \subset \mathbb{B}^d$ as well as $|A| = \int_A d\sigma(x)$ if $A \subset \mathbb{S}^{d-1}$.

The Radon transform of a function f is defined by

$$\text{Rf}(\theta, s) = \int_{\substack{\mathbf{y} \in \theta^\perp \\ s\theta + \mathbf{y} \in \mathbb{B}^d}} f(s\theta + \mathbf{y}) d\mathbf{y}, \quad \theta \in \mathbb{S}^{d-1}, s \in [-1, 1],$$

where $d\mathbf{y}$ is the Lebesgue measure of dimension $d - 1$ and $\theta^\perp = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \theta \rangle = 0\}$. With a slight abuse of notation, we will rewrite this integral as

$$\text{Rf}(\theta, s) = \int_{\langle \mathbf{y}, \theta \rangle = s} f(\mathbf{y}) d\mathbf{y}.$$

By Fubini's theorem, we have

$$\int_{-1}^1 \text{Rf}(\theta, s) ds = \int_{\mathbb{B}^d} f(\mathbf{x}) d\mathbf{x}.$$

It is easy to see (cf. e.g. [17]) that the Radon transform is a bounded linear operator mapping $\mathbb{L}^2(\mathbb{B}^d, d\mathbf{x})$ into $\mathbb{L}^2(\mathbb{S}^{d-1} \times [-1, 1], d\mu(\theta, s))$, where

$$d\mu(\theta, s) = d\sigma(\theta) \frac{ds}{(1 - s^2)^{(d-1)/2}}.$$

2.2 Noisy observation of the Radon transform

We consider observations of the form

$$dY(\theta, s) = \text{Rf}(\theta, s) d\mu(\theta, s) + \varepsilon dW(\theta, s),$$

where the unknown function f belongs to $\mathbb{L}^2(\mathbb{B}^d, d\mathbf{x})$. The meaning of this equation is that for any $\varphi(\theta, s)$ in $\mathbb{L}^2(\mathbb{S}^{d-1} \times [-1, 1], d\mu(\theta, s))$ one can observe

$$\begin{aligned} Y_\varphi &= \int \varphi(\theta, s) dY(\theta, s) = \int_{\mathbb{S}^{d-1} \times [-1, 1]} \text{Rf}(\theta, s) \varphi(\theta, s) d\mu(\theta, s) + \varepsilon \int \varphi(\theta, s) dW(\theta, s) \\ &= \langle \text{Rf}, \varphi \rangle_\mu + \varepsilon W_\varphi. \end{aligned}$$

Here $W_\varphi = \int \varphi(\theta, s) dW(\theta, s)$ is a Gaussian field of zero mean and covariance

$$\mathbb{E}(W_\varphi, W_\psi) = \int_{\mathbb{S}^{d-1} \times [-1, 1]} \varphi(\theta, s) \psi(\theta, s) d\sigma(\theta) \frac{ds}{(1 - s^2)^{(d-1)/2}} = \langle \varphi, \psi \rangle_\mu.$$

The goal is to recover the unknown function f from the observation of Y . Our idea is to refine the algorithms proposed in [10] using thresholding methods. In [10] it is derived estimation schemes combining the stability and computability of SVD decompositions with the localization and multiscale structure of wavelets. To this end a frame was used (essentially following the construction from [12]) with elements of nearly exponential localization which is in addition compatible with the SVD basis of the Radon transform.

2.3 Singular Value Decomposition of the Radon transform

The SVD of the Radon transform was first established in [5, 14]. In this regard we also refer the reader to [17, 24].

2.3.1 Jacobi and Gegenbauer polynomials

The Radon SVD bases are defined in terms of Jacobi and Gegenbauer polynomials. The Jacobi polynomials $P_n^{(\alpha, \beta)}$, $n \geq 0$, constitute an orthogonal basis for the space $\mathbb{L}^2([-1, 1], w_{\alpha, \beta}(t) dt)$ with weight $w_{\alpha, \beta}(t) = (1-t)^\alpha(1+t)^\beta$, $\alpha, \beta > -1$. They are standardly normalized by $P_n^{(\alpha, \beta)}(1) = \binom{n+\alpha}{n}$ and then [1, 7, 22]

$$\int_{-1}^1 P_n^{(\alpha, \beta)}(t) P_m^{(\alpha, \beta)}(t) w_{\alpha, \beta}(t) dt = \delta_{n, m} h_n^{(\alpha, \beta)},$$

where

$$h_n^{(\alpha, \beta)} = \frac{2^{\alpha+\beta+1}}{(2n+\alpha+\beta+1)} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}.$$

The Gegenbauer polynomials C_n^λ are a particular case of Jacobi polynomials and are traditionally defined by

$$C_n^\lambda(t) = \frac{(2\lambda)_n}{(\lambda+1/2)_n} P_n^{(\lambda-1/2, \lambda-1/2)}(t), \quad \lambda > -1/2,$$

where by definition $(a)_n = a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$ (note that in [22] the Gegenbauer polynomial C_n^λ is denoted by P_n^λ). It is readily seen that $C_n^\lambda(1) = \binom{n+2\lambda-1}{n} = \frac{\Gamma(n+2\lambda)}{n!\Gamma(2\lambda)}$ and

$$\int_{-1}^1 C_n^\lambda(t) C_m^\lambda(t) (1-t^2)^{\lambda-1/2} dt = \delta_{n, m} h_n^{(\lambda)} \quad \text{with} \quad h_n^{(\lambda)} = \frac{2^{1-2\lambda}\pi}{\Gamma(\lambda)^2} \frac{\Gamma(n+2\lambda)}{(n+\lambda)\Gamma(n+1)}.$$

2.3.2 Polynomials on B^d and S^{d-1}

Let $\Pi_n(\mathbb{R}^d)$ be the space of all polynomials in d variables of degree $\leq n$. We denote by $\mathcal{P}_n(\mathbb{R}^d)$ the space of all homogeneous polynomials of degree n and by $\mathcal{V}_n(\mathbb{R}^d)$ the space of all polynomials of degree n which are orthogonal to lower degree polynomials with respect to the Lebesgue measure on B^d . Of course $\mathcal{V}_0(\mathbb{R}^d)$ will be the set of all constants. We have the following orthogonal decomposition:

$$\Pi_n(\mathbb{R}^d) = \bigoplus_{k=0}^n \mathcal{V}_k(\mathbb{R}^d).$$

Also, denote by $\mathbb{H}_n(\mathbb{R}^d)$ the subspace of all harmonic homogeneous polynomials of degree n (i.e. $Q \in \mathbb{H}_n(\mathbb{R}^d)$ if $Q \in \mathcal{P}_n(\mathbb{R}^d)$ and $\Delta Q = 0$) and by $\mathbb{H}_n(S^{d-1})$ the (injective)

restriction of the polynomials from $\mathbb{H}_n(\mathbb{R}^d)$ to \mathbb{S}^{d-1} . It is well known that

$$N_{d-1}(n) = \dim(\mathbb{H}_n(\mathbb{S}^{d-1})) = \binom{n+d-1}{d-1} - \binom{n+d-3}{d-1} \sim n^{d-2}.$$

Let $\Pi_n(\mathbb{S}^{d-1})$ be the space of restrictions to \mathbb{S}^{d-1} of polynomials of degree $\leq n$ on \mathbb{R}^d . As is well known

$$\Pi_n(\mathbb{S}^{d-1}) = \bigoplus_{m=0}^n \mathbb{H}_m(\mathbb{S}^{d-1})$$

(the orthogonality is with respect of the surface measure $d\sigma$ on \mathbb{S}^{d-1}). $\mathbb{H}_l(\mathbb{S}^{d-1})$ is called the space of spherical Harmonics of degree l on the sphere \mathbb{S}^{d-1} .

Let $Y_{l,i}$, $1 \leq i \leq N_{d-1}(l)$, be an orthonormal basis of $\mathbb{H}_l(\mathbb{S}^{d-1})$, i.e.

$$\int_{\mathbb{S}^{d-1}} Y_{l,i}(\xi) \overline{Y_{l,i'}(\xi)} d\sigma(\xi) = \delta_{i,i'}.$$

Then the natural extensions of $Y_{l,i}$ on B^d are defined by $Y_{l,i}(x) = |x|^l Y_{l,i}\left(\frac{x}{|x|}\right)$ and satisfy

$$\begin{aligned} \int_{B^d} Y_{l,i}(x) \overline{Y_{l,i'}(x)} dx &= \int_0^1 r^{d-1} \int_{\mathbb{S}^{d-1}} Y_{l,i}(r\xi) \overline{Y_{l,i'}(r\xi)} d\sigma(\xi) dr \\ &= \int_0^1 r^{d+2l-1} \int_{\mathbb{S}^{d-1}} Y_{l,i}(\xi) \overline{Y_{l,i'}(\xi)} d\sigma(\xi) dr = \delta_{i,i'} \frac{1}{2l+d}. \end{aligned}$$

For more details we refer the reader to [6].

The spherical harmonics on \mathbb{S}^{d-1} and orthogonal polynomials on B^d are naturally related to Gegenbauer polynomials. Thus the kernel of the orthogonal projector onto $\mathbb{H}_n(\mathbb{S}^{d-1})$ can be written as (see e.g. [21]):

$$\sum_{i=1}^{N_{d-1}(n)} Y_{l,i}(\xi) \overline{Y_{l,i}(\theta)} = \frac{2n+d-2}{(d-2)|\mathbb{S}^{d-1}|} C_n^{\frac{d-2}{2}}(\langle \xi, \theta \rangle). \quad (1)$$

The ‘‘ridge’’ Gegenbauer polynomials $C_n^{d/2}(\langle x, \xi \rangle)$ are orthogonal to $\Pi_{n-1}(B^d)$ in $\mathbb{L}^2(B^d)$ and the kernel $L_n(x, y)$ of the orthogonal projector onto $\mathcal{V}_n(B^d)$ can be written in the form (see e.g. [18, 24])

$$\begin{aligned} L_n(x, y) &= \frac{2n+d}{|\mathbb{S}^{d-1}|^2} \int_{\mathbb{S}^{d-1}} C_n^{d/2}(\langle x, \xi \rangle) C_n^{d/2}(\langle y, \xi \rangle) d\sigma(\xi) \\ &= \frac{(n+1)_{d-1}}{2^d \pi^{d-1}} \int_{\mathbb{S}^{d-1}} \frac{C_n^{d/2}(\langle x, \xi \rangle) C_n^{d/2}(\langle y, \xi \rangle)}{\|C_n^{d/2}\|^2} d\sigma(\xi). \end{aligned} \quad (2)$$

The following important identities are valid for ‘‘ridge’’ Gegenbauer polynomials:

$$\int_{B^d} C_n^{d/2}(\langle \xi, x \rangle) C_n^{d/2}(\langle \eta, x \rangle) dx = \frac{h_n^{(d/2)}}{C_n^{d/2}(1)} C_n^{d/2}(\langle \xi, \eta \rangle), \quad \xi, \eta \in \mathbb{S}^{d-1}, \quad (3)$$

and, for $\mathbf{x} \in \mathbb{B}^d$, $\boldsymbol{\eta} \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{S}^{d-1}} C_n^{d/2}(\langle \boldsymbol{\xi}, \mathbf{x} \rangle) C_n^{d/2}(\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle) d\boldsymbol{\sigma}(\boldsymbol{\xi}) = |\mathbb{S}^{d-1}| C_n^{d/2}(\langle \boldsymbol{\eta}, \mathbf{x} \rangle), \quad (4)$$

see e.g. [18]. By (2) and (4)

$$L_n(\mathbf{x}, \boldsymbol{\xi}) = \frac{(2n+d)}{|\mathbb{S}^{d-1}|} C_n^{d/2}(\langle \mathbf{x}, \boldsymbol{\xi} \rangle), \quad \boldsymbol{\xi} \in \mathbb{S}^{d-1},$$

and again by (2)

$$\int_{\mathbb{S}^{d-1}} L_n(\mathbf{x}, \boldsymbol{\xi}) L_n(\mathbf{y}, \boldsymbol{\xi}) d\boldsymbol{\sigma}(\boldsymbol{\xi}) = (2n+d) L_n(\mathbf{x}, \mathbf{y}).$$

2.3.3 The SVD of the Radon transform

Assume that $\{Y_{l,i} : 1 \leq i \leq N_{d-1}(l)\}$ is an orthonormal basis for $\mathbb{H}_l(\mathbb{S}^{d-1})$. Then it is standard and easy to see that the family of polynomials

$$f_{k,l,i}(\mathbf{x}) = (2k+d)^{1/2} p_j^{(0, l+d/2-1)}(2|\mathbf{x}|^2-1) Y_{l,i}(\mathbf{x}), \quad 0 \leq l \leq k, \quad k-l = 2j, \quad 1 \leq i \leq N_{d-1}(l),$$

form an orthonormal basis of $\mathcal{V}_k(\mathbb{B}^d)$, see e.g. [6]. Here as before $Y_{l,i}(\mathbf{x}) = |\mathbf{x}|^l Y_{l,i}(\mathbf{x}/|\mathbf{x}|)$. On the other hand the collection

$$g_{k,l,i}(\boldsymbol{\theta}, s) = [h_k^{(d/2)}]^{-1/2} (1-s^2)^{(d-1)/2} C_k^{d/2}(s) Y_{l,i}(\boldsymbol{\theta}), \quad k \geq 0, \quad l \geq 0, \quad 1 \leq i \leq N_{d-1}(l),$$

is apparently an orthonormal basis of $\mathbb{L}^2(\mathbb{S}^{d-1} \times [-1, 1], d\boldsymbol{\mu}(\boldsymbol{\theta}, s))$. Most importantly, the Radon transform $R : \mathbb{L}^2(\mathbb{B}^d) \mapsto \mathbb{L}^2(\mathbb{S}^{d-1} \times [-1, 1], d\boldsymbol{\mu}(\boldsymbol{\theta}, s))$ is a one-to-one mapping and

$$Rf_{k,l,i} = \lambda_k g_{k,l,i}, \quad R^* g_{k,l,i} = \lambda_k f_{k,l,i}, \quad \text{where}$$

$$\lambda_k^2 = \frac{2^d \pi^{d-1}}{(k+1)(k+2) \dots (k+d-1)} = \frac{2^d \pi^{d-1}}{(k+1)_{d-1}} \sim k^{-d+1}.$$

More precisely, we have: For any $f \in \mathbb{L}^2(\mathbb{B}^d)$

$$Rf = \sum_{k \geq 0} \lambda_k \sum_{0 \leq l \leq k, k-l \equiv 0 \pmod{2}} \sum_{1 \leq i \leq N_{d-1}(l)} \langle f, f_{k,l,i} \rangle g_{k,l,i}$$

Furthermore, for $f \in \mathbb{L}^2(\mathbb{B}^d)$

$$f = \sum_{k \geq 0} \lambda_k^{-1} \sum_{0 \leq l \leq k, k-l \equiv 0 \pmod{2}} \sum_{1 \leq i \leq N_{d-1}(l)} \langle Rf, g_{k,l,i} \rangle_{\boldsymbol{\mu}} f_{k,l,i}.$$

In the above identities the convergence is in \mathbb{L}^2 .

For the Radon SVD we refer the reader to [17] and [24] and [10].

3 Construction of needlets on the ball

In this section we briefly recall the construction of the needlets on the ball. This construction is due to [20]. Its aim is essentially to build a very well localized tight frame constructed using the eigenvectors of the Radon transform. For more precision we refer to [20], [11], [10]

Let $\{f_{k,l,i}\}$ be the orthonormal basis of $\mathcal{V}_k(\mathbb{B}^d)$ defined in §2.3.3. Denote by T_k the index set of this basis, i.e. $T_k = \{(l, i) : 0 \leq l \leq k, l \equiv k \pmod{2}, 0 \leq i \leq N_{d-1}(l)\}$. Then the orthogonal projector of $\mathbb{L}^2(\mathbb{B}^d)$ onto $\mathcal{V}_k(\mathbb{B}^d)$ can be written in the form

$$L_k f = \int_{\mathbb{B}^d} f(\mathbf{y}) L_k(\mathbf{x}, \mathbf{y}) d\mathbf{y} \quad \text{with} \quad L_k(\mathbf{x}, \mathbf{y}) = \sum_{l,i \in T_k} f_{k,l,i}(\mathbf{x}) f_{k,l,i}(\mathbf{y}).$$

Using (1) $L_k(\mathbf{x}, \mathbf{y})$ can be written in the form

$$\begin{aligned} L_k(\mathbf{x}, \mathbf{y}) &= (2k + d) \sum_{l \leq k, k-l \equiv 0 \pmod{2}} P_j^{(0, l+d/2-1)}(2|\mathbf{x}|^2 - 1) |\mathbf{x}|^l P_j^{(0, l+d/2-1)}(2|\mathbf{y}|^2 - 1) |\mathbf{y}|^l \\ &\quad \times \sum_i Y_{l,i}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) Y_{l,i}\left(\frac{\mathbf{y}}{|\mathbf{y}|}\right) \\ &= \frac{(2k + d)}{|\mathbb{S}^{d-1}|} \sum_{l \leq k, k-l \equiv 0 \pmod{2}} P_j^{(0, l+d/2-1)}(2|\mathbf{x}|^2 - 1) |\mathbf{x}|^l P_j^{(0, l+d/2-1)}(2|\mathbf{y}|^2 - 1) |\mathbf{y}|^l \\ &\quad \times \left(1 + \frac{l}{d/2 - 1}\right) C_l^{d/2-1}\left(\left\langle \frac{\mathbf{x}}{|\mathbf{x}|}, \frac{\mathbf{y}}{|\mathbf{y}|} \right\rangle\right). \end{aligned}$$

Another representation of $L_k(\mathbf{x}, \mathbf{y})$ has already be given in (2). Clearly

$$\int_{\mathbb{B}^d} L_m(\mathbf{x}, \mathbf{z}) L_k(\mathbf{z}, \mathbf{y}) d\mathbf{z} = \delta_{m,k} L_m(\mathbf{x}, \mathbf{y}) \quad (5)$$

and for $f \in \mathbb{L}^2(\mathbb{B}^d)$

$$f = \sum_{k \geq 0} L_k f \quad \text{and} \quad \|f\|_2^2 = \sum_k \|L_k f\|_2^2 = \sum_k \langle L_k f, f \rangle. \quad (6)$$

The construction of the needlets is based on the classical Littlewood-Paley decomposition and a subsequent discretization.

Let $\mathbf{a} \in C^\infty[0, \infty)$ be a cut-off function such that $0 \leq \mathbf{a} \leq 1$, $\mathbf{a}(t) = 1$ for $t \in [0, 1/2]$ and $\text{supp } \mathbf{a} \subset [0, 1]$. We next use this function to introduce a sequence of operators on $\mathbb{L}^2(\mathbb{B}^d)$. For $j \geq 0$ write

$$A_j f(\mathbf{x}) = \sum_{k \geq 0} \mathbf{a}\left(\frac{k}{2^j}\right) L_k f(\mathbf{x}) = \int_{\mathbb{B}^d} A_j(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y} \quad \text{with} \quad A_j(\mathbf{x}, \mathbf{y}) = \sum_k \mathbf{a}\left(\frac{k}{2^j}\right) L_k(\mathbf{x}, \mathbf{y}).$$

Also, we define $B_j f = A_{j+1} f - A_j f$. Then setting $b(t) = a(t/2) - a(t)$ we have

$$B_j f(x) = \sum_k b\left(\frac{k}{2^j}\right) L_k f(x) = \int_{B^d} B_j(x, y) f(y) dy \quad \text{with} \quad B_j(x, y) = \sum_k b\left(\frac{k}{2^j}\right) L_k(x, y).$$

Obviously, for $f \in \mathbb{L}^2(B^d)$

$$\langle A_j f, f \rangle = \sum_k a\left(\frac{k}{2^j}\right) \langle L_k f, f \rangle \leq \|f\|_2^2$$

An important result from [20] (see also [12]) asserts that the kernels $A_j(x, y)$, $B_j(x, y)$ have nearly exponential localization, namely, for any $M > 0$ there exists a constant $C_M > 0$ such that

$$|A_j(x, y)|, |B_j(x, y)| \leq C_M \frac{2^{jd}}{(1 + 2^j d(x, y))^M \sqrt{W_j(x)} \sqrt{W_j(y)}}, \quad x, y \in B^d, \quad (7)$$

where

$$W_j(x) = 2^{-j} + \sqrt{1 - |x|^2}, \quad |x|^2 = |x|_d^2 = \sum_{i=1}^d x_i^2, \quad (8)$$

and

$$d(x, y) = \text{Arccos}(\langle x, y \rangle + \sqrt{1 - |x|^2} \sqrt{1 - |y|^2}), \quad \langle x, y \rangle = \sum_{i=1}^d x_i y_i.$$

Let us define

$$C_j(x, y) = \sum_k \sqrt{a\left(\frac{k}{2^j}\right)} L_k(x, z) \quad \text{and} \quad D_j(x, y) = \sum_k \sqrt{b\left(\frac{k}{2^j}\right)} L_k(x, z).$$

Note that C_j and D_j have the same localization as the localization of A_j , B_j in (7) (cf. [20]). Using (5), we get,

$$A_j(x, y) = \int_{B^d} C_j(x, z) C_j(z, y) dz, \quad B_j(x, y) = \int_{B^d} D_j(x, z) D_j(z, y) dz. \quad (9)$$

And, obviously $z \mapsto C_j(x, z) C_j(z, y)$ (resp. $D_j(x, z) D_j(z, y)$) are polynomial of degrees $< 2^{j+1}$.

The following proposition follows from results in [20] and [23] and establishes a cubature formula.

Proposition 1. Let $\{B(\tilde{\xi}_i, \rho) : i \in I\}$ be a maximal family of disjoint spherical caps of radius $\rho = \tau 2^{-j}$ with centers on the hemisphere \mathbb{S}_+^d . Then for sufficiently small $0 < \tau \leq 1$ the set of points $\chi_j = \{\xi_i : i \in I\}$ obtained by projecting the set $\{\tilde{\xi}_i : i \in I\}$ on B^d is a set of nodes of a cubature formula which is exact for $\Pi_{2j+2}(B^d)$: for any $P \in \Pi_{2j+2}(B^d)$,

$$\int_{B^d} P(u) du = \sum_{\xi \in \chi_j} \omega_{j, \xi} P(\xi)$$

where, moreover, the coefficients $\omega_{j, \xi}$ of this cubature are positive and satisfy $\omega_{j, \xi} \sim W_j(\xi) 2^{-jd}$, and the cardinality of the set χ_j is of order 2^{jd} .

3.0.4 Needlets

Going back to identities (9) and applying the cubature formula described in Proposition 1, we get

$$\begin{aligned} A_j(x, y) &= \int_{\mathbb{B}^d} C_j(x, z)C_j(z, y)dz = \sum_{\xi \in \chi_j} \omega_{j,\xi} C_j(x, \xi)C_j(y, \xi) \quad \text{and} \\ B_j(x, y) &= \int_{\mathbb{B}^d} D_j(x, z)D_j(z, y)dz = \sum_{\xi \in \chi_j} \omega_{j,\xi} D_j(x, \xi)D_j(y, \xi). \end{aligned}$$

We define the father needlets $\varphi_{j,\xi}$ and the mother needlets $\psi_{j,\xi}$ by

$$\varphi_{j,\xi}(x) = \sqrt{\omega_{j,\xi}}C_j(x, \xi) \quad \text{and} \quad \psi_{j,\xi}(x) = \sqrt{\omega_{j,\xi}}D_j(x, \xi), \quad \xi \in \chi_j, \quad j \geq 0.$$

We also set $\psi_{-1,0} = \frac{1}{|\mathbb{B}^d|}$ and $\chi_{-1} = \{0\}$. From above it follows that

$$A_j(x, y) = \sum_{\xi \in \chi_j} \varphi_{j,\xi}(x)\varphi_{j,\xi}(y), \quad B_j(x, y) = \sum_{\xi \in \chi_j} \psi_{j,\xi}(x)\psi_{j,\xi}(y).$$

Therefore,

$$A_j f(x) = \int_{\mathbb{B}^d} A_j(x, y)f(y)dy = \sum_{\xi \in \chi_j} \langle f, \varphi_{j,\xi} \rangle \varphi_{j,\xi} = \sum_{\xi \in \chi_j} \alpha_{j,\xi} \varphi_{j,\xi}, \quad \alpha_{j,\xi} = \langle f, \varphi_{j,\xi} \rangle.$$

and

$$B_j f(x) = \int_{\mathbb{B}^d} B_j(x, y)f(y)dy = \sum_{\xi \in \chi_j} \langle f, \psi_{j,\xi} \rangle \psi_{j,\xi} = \sum_{\xi \in \chi_j} \beta_{j,\xi} \psi_{j,\xi}, \quad \beta_{j,\xi} = \langle f, \psi_{j,\xi} \rangle.$$

It is easy to prove (see [20]) that

$$\|\varphi_{j,\xi}\|_2 \leq 1.$$

From (6) and the fact that $\sum_{j \geq 0} \mathbf{b}(t2^{-j}) = 1$ for $t \in [1, \infty)$, it readily follows that

$$f = \sum_{j \geq -1} \sum_{\xi \in \chi_j} \langle f, \psi_{j,\xi} \rangle \psi_{j,\xi}, \quad f \in \mathbb{L}^2(\mathbb{B}^d),$$

and taking inner product with f this leads to

$$\|f\|_2^2 = \sum_j \sum_{\xi \in \chi_j} |\langle f, \psi_{j,\xi} \rangle|^2,$$

which in turn shows that the family $\{\psi_{j,\xi}\}$ is a tight frame for $\mathbb{L}^2(\mathbb{B}^d)$.

4 Needlet inversion of a noisy Radon transform and minimax performances

Our estimator is based on an appropriate thresholding of a needlet expansion as follows. f can be decomposed using the frame above:

$$f = \sum_{j \geq -1} \sum_{\xi \in \mathcal{X}_j} \langle f, \psi_{j,\xi} \rangle \psi_{j,\xi},$$

Our estimation procedure will be defined by the following steps

$$\widehat{\alpha}_{k,l,i} = \frac{1}{\lambda_k} \int g_{k,l,i} dY, \quad (10)$$

$$\widehat{\beta}_{j,\xi} = \sum_{k,l,i} \gamma_{k,l,i}^{j,\xi} \widehat{\alpha}_{k,l,i} \quad (11)$$

with $\gamma_{k,l,i}^{j,\xi} = \langle g_{k,l,i}, \psi_{j,\xi} \rangle$

and
$$\widehat{f} = \sum_{j=-1}^{J_\varepsilon} \sum_{\xi \in \mathcal{X}_j} \widehat{\beta}_{j,\xi} \mathbf{1}_{\{|\widehat{\beta}_{j,\xi}| \geq \kappa 2^{j\nu} c_\varepsilon\}} \psi_{j,\xi} \quad (12)$$

with

$$\nu = (d-1)/2. \quad (13)$$

Hence our procedure has 3 steps: the first one (10) corresponds to the inversion of the operator in the SVD basis, the second one (11) projects on the needlet basis, the third one (12) ends up the procedure with a final thresholding. The tuning parameters of this estimator are

- The range J_ε of resolution levels will be taken such that

$$2^{J_\varepsilon(d-\frac{1}{2})} \leq \left(\varepsilon \sqrt{\log 1/\varepsilon} \right)^{-1} < 2^{(J_\varepsilon+1)(d-\frac{1}{2})}$$

- The threshold constant κ is an important tuning of our method. The theoretical point of view asserts that for κ above a constant (for which our evaluation is probably not optimal) the minimax properties hold. Evaluations of κ from the simulations points of view are also given.
- c_ε is a constant depending on the noise level. We shall see that the following choice is appropriate

$$c_\varepsilon = \varepsilon \sqrt{\log 1/\varepsilon}.$$

- Notice that the threshold function for each coefficient contains $2^{j\nu}$. This is due to the inversion of the Radon operator, and the concentration relative to the $g_{k,l,i}$'s of the needlets.

- It is important to remark here that unlike the (linear) procedures proposed in [10], this one does not require the knowledge of the regularity, while as will be seen in the sequel, it attains bounds which are as good as the linear ones and even better since handling much wider ranges for the parameters of the Besov spaces.

We will consider the minimax properties of this estimator on the Besov bodies constructed on the needlet basis. In [12], it is proved that these spaces can also be described as approximation spaces, so they have a genuine meaning, and can be compared to standard Sobolev spaces.

We define here the Besov body $B_{\pi,r}^s$ as the space of functions $f = \sum_{j \geq -1} \sum_{\xi \in \chi_j} \beta_{j,\xi} \psi_{j,\xi}$ such that

$$\sum_j 2^{jsr} \left(\sum_{\xi \in \chi_j} (|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\pi)^\pi \right)^{r/\pi} < \infty$$

(with the obvious modifications for the cases π or $r = \infty$) as well as $B_{\pi,r}^s(M)$ the ball of radius M of this space.

Theorem 1. For $0 < r \leq \infty$, $\pi \geq 1$, $1 \leq p < \infty$ there exist some constant $c_p = c_p(s, r, p, M)$, κ_0 such that if $\kappa \geq \kappa_0$, $s > (d+1)(\frac{1}{\pi} - \frac{1}{p})_+$, in addition with if $\pi < p$, $s > \frac{d+1}{\pi} - \frac{1}{2}$

1. If $\frac{1}{p} < \frac{d}{d+1}$

$$\sup_{f \in B_{\pi,r}^s(M)} (\mathbb{E} \|\hat{f} - f\|_p^p)^{\frac{1}{p}} \leq c_p (\log 1/\varepsilon)^{\frac{p}{2}} \left(\varepsilon \sqrt{\log 1/\varepsilon} \right)^{\frac{s-(d+1)(1/\pi-1/p)}{s+d-(d+1)/\pi} \wedge \frac{s}{s+d-1/2} \wedge \frac{s-2(1/\pi-1/p)}{s+d-2/\pi}}$$

2. If $\frac{d}{d+1} \leq \frac{1}{p} < \frac{5d-1}{4d+1}$

$$\sup_{f \in B_{\pi,r}^s(M)} (\mathbb{E} \|\hat{f} - f\|_p^p)^{\frac{1}{p}} \leq c_p (\log 1/\varepsilon)^{\frac{p}{2}} \left(\varepsilon \sqrt{\log 1/\varepsilon} \right)^{\frac{s}{s+d-1/2} \wedge \frac{s-2(1/\pi-1/p)}{s+d-2/\pi}}$$

3. If $\frac{5d-1}{4d+1} \leq \frac{1}{p}$

$$\sup_{f \in B_{\pi,r}^s(M)} (\mathbb{E} \|\hat{f} - f\|_p^p)^{\frac{1}{p}} \leq c_p (\log 1/\varepsilon)^{\frac{p}{2}} \left(\varepsilon \sqrt{\log 1/\varepsilon} \right)^{\frac{s}{s+d-1/2}}$$

Remark 1. Up to logarithmic terms, the rates observed here are minimax, as will appear in the following theorem. It is known that in this kind of estimation, full adaptation yields unavoidable extra logarithmic terms. The rates of the logarithmic terms obtained in these theorems are, most of the time, suboptimal (for instance, for obvious reasons the case $p = 2$ yields much less logarithmic terms). A more detailed study could lead to optimized rates, which we decided not to include here for a sake of simplicity.

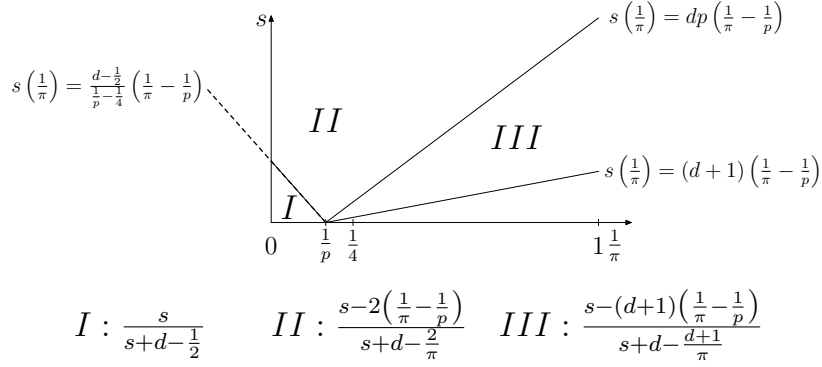


Figure 1: The three different minimax rate type zones are shown with respect to the Besov space parameters s and π for a fixed loss norm L^p with $0 < \frac{1}{p} < \frac{1}{4}$.

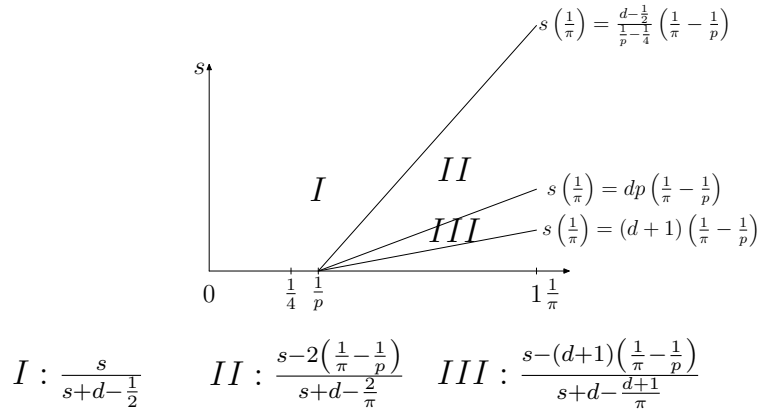


Figure 2: The three different minimax rate type zones are shown with respect to the Besov space parameters s and π for a fixed loss norm p with $\frac{1}{4} < \frac{1}{p} < \frac{d}{d+1}$.

The cumbersome comparisons of the different rates of convergence are summarized in Figures 1 and 2 for the case $0 < \frac{1}{p} < \frac{d}{d+1}$.

For the case of a L_∞ loss function, we have a slightly different result since the thresholding is here depending on the L_∞ norm of the local needlet: Let us consider the following estimate :

$$\hat{f}_\infty = \sum_{j=-1}^{J_\varepsilon} \sum_{\xi \in \mathcal{X}_j} \hat{\beta}_{j,\xi} 1_{\{\|\hat{\beta}_{j,\xi}\| \|\psi_{j,\xi}\|_\infty \geq \kappa 2^{jd} c_\varepsilon\}} \psi_{j,\xi}$$

$$2^{J_\varepsilon d} = \left(\varepsilon \sqrt{\log 1/\varepsilon} \right)^{-1}$$

Then for this estimate, we have the following results :

Theorem 2. For $0 < r \leq \infty$, $\pi \geq 1$, $s > \frac{d+1}{\pi}$,

There exist some constants $c_\infty = c_\infty(s, \pi, r, M)$ such that if $\kappa^2 \geq 4\tau_\infty$, where $\tau_\infty := \sup_{j,\xi} 2^{-j\frac{d+1}{2}} \|\psi_{j,\xi}\|_\infty$

$$\sup_{f \in B_{\pi,r}^s(M)} \mathbb{E} \|\hat{f}_\infty - f\|_\infty \leq c_\infty \left(\varepsilon \sqrt{\log 1/\varepsilon} \right)^{\frac{s-(d+1)/\pi}{s+d-(d+1)/\pi}}$$

The following theorem states lower bounds for the minimax rates over Besov spaces in this model.

Theorem 3. Let \mathcal{E} be the set of all estimators, for $0 < r \leq \infty$, $\pi \geq 1$, $s > \frac{d+1}{\pi}$,

a) There exists some constant C_∞ such ,

$$\inf_{f^* \in \mathcal{E}} \sup_{f \in B_{\pi,r}^s(M)} \mathbb{E} \|f^* - f\|_\infty \geq C_\infty \left(\varepsilon \sqrt{\log 1/\varepsilon} \right)^{\frac{s-(d+1)/\pi}{s+d-(d+1)/\pi}}$$

b) For $1 \leq p < \infty$ there exists some constant C_p such that if $s > \left(\frac{d+1}{\pi} - \frac{d+1}{p} \right)_+$,

1. If $\frac{1}{p} < \frac{d}{d+1}$

$$\inf_{f^* \in \mathcal{E}} \sup_{f \in B_{\pi,r}^s(M)} \left(\mathbb{E} \|f^* - f\|_p^p \right)^{\frac{1}{p}} \geq CM \varepsilon^{\frac{s-(d+1)(1/\pi-1/p)}{s+d-(d+1)/\pi} \wedge \frac{s}{s+d-1/2} \wedge \frac{s-2(1/\pi-1/p)}{s+d-2/\pi}}$$

2. If $\frac{d}{d+1} \leq \frac{1}{p} < \frac{5d-1}{4d+1}$

$$\inf_{f^* \in \mathcal{E}} \sup_{f \in B_{\pi,r}^s(M)} \left(\mathbb{E} \|f^* - f\|_p^p \right)^{\frac{1}{p}} \geq CM \varepsilon^{\frac{s}{s+d-1/2} \wedge \frac{s-2(1/\pi-1/p)}{s+d-2/\pi}}$$

3. If $\frac{5d-1}{4d+1} \leq \frac{1}{p}$

$$\inf_{f^* \in \mathcal{E}} \sup_{f \in B_{\pi,r}^s(M)} \left(\mathbb{E} \|f^* - f\|_p^p \right)^{\frac{1}{p}} \geq CM \varepsilon^{\frac{s}{s+d-1/2}}$$

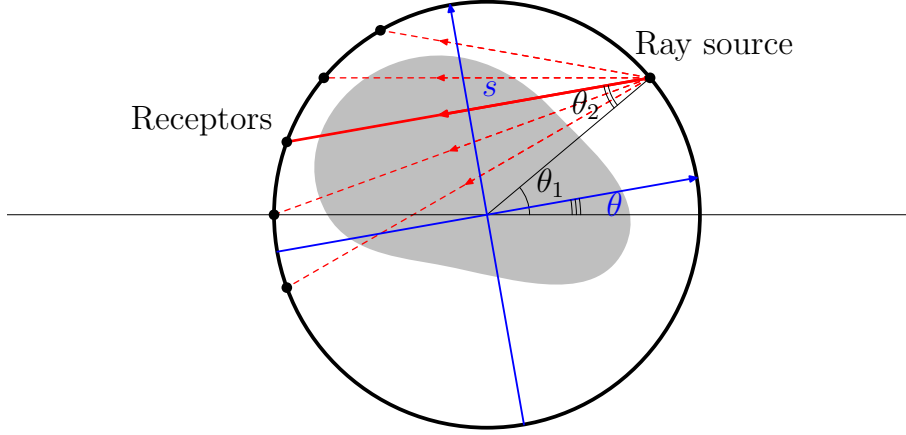


Figure 3: Simplified CAT device

5 Applications to the Fan Beam tomography

5.1 The 2D case: Fan beam tomography

When $d = 2$, the Radon transform studied in this paper is the fan beam Radon transform used in Computed Axial Tomography scanner (CAT scan). The geometry of such a device is illustrated in Figure 3. An object is placed in the middle of the scanner and X rays are sent from a pointwise source $S(\theta_1)$ making an angle θ_1 with a reference direction. Rays go through the object and the energy decay between the source and an array of receptor is measured. As the log decay along the ray is proportional to the integral of the density f of the object along the same ray, the measurements are

$$\tilde{R}f(\theta_1, \theta_2) = \int_{e_{\theta_1 + \lambda e_{(\theta_1 - \theta_2)}} \in B^2} f(x) d\lambda$$

with $e_\theta = (\cos \theta, \sin \theta)$ or equivalently the classical Radon transform

$$Rf(\theta, s) = \int_{\substack{y \in \theta^\perp \\ s\theta + y \in B^2}} f(s\theta + y) dy, \quad \theta \in S^1, s \in [-1, 1],$$

for $\theta = \theta_1 - \theta_2$ and $s = \sin \theta_2$. The ray source is then rotated to a different angle and the measurement process is repeated. In our Gaussian white noise model, we measure the continuous function $Rf(\theta, s)$ through the process $dY = Rf(\theta, s) d\theta \frac{ds}{(1-s^2)} + \varepsilon dW(\theta, s)$, where the measure $d\theta \frac{ds}{(1-s^2)}$ corresponds to the uniform measure $d\theta_1 d\theta_2$ by the change of variable that maps (θ_1, θ_2) into (θ, s) . Our goal is to recover the unknown function f from the observation of Y using the needlet thresholding mechanism described in the previous sections.

In our implementation, we exploit the tensorial structure of the SVD basis of the disk in polar coordinates:

$$f_{k,l,i}(r, \theta) = (2k+2)^{1/2} P_j^{(0,l)}(2|r|^2-1) |r|^l Y_{l,i}(\theta), \quad 0 \leq l \leq k, \quad k-l=2j, \quad 1 \leq i \leq 2,$$

where $P_j^{0,l}$ is the corresponding Jacobi polynomial, and $Y_{l,1}(\theta) = c_l \cos(l\theta)$ and $Y_{l,2}(\theta) = c_l \sin(l\theta)$ with $c_0 = \frac{1}{\sqrt{2\pi}}$ and $c_l = \frac{1}{\sqrt{\pi}}$ otherwise. The basis of $S^2 \times [-1, 1]$ has a similar tensorial structure as it is given by

$$g_{k,l,i}(\theta, s) = [h_k]^{-1/2} (1-s^2)^{1/2} C_k^1(s) Y_{l,i}(\theta), \quad k \geq 0, \quad l \geq 0, \quad 1 \leq i \leq 2,$$

where C_k^1 is the Gegenbauer of parameter 1 and degree k. We recall that the corresponding eigenvalues are

$$\lambda_k = \frac{2\sqrt{\pi}}{\sqrt{k+1}}.$$

5.2 SVD, Needlet and cubature

In our numerical studies, we compare four different type of estimators: linear SVD estimators, thresholded SVD estimators, linear needlet estimators and thresholded needlet estimators. They are defined from the measurement of the values of the Gaussian field on the SVD basis function $Y_{g_{k,l,i}}$ and the following linear estimates of respectively the SVD basis coefficients $\langle f, f_{k,l,i} \rangle$ and the needlet coefficients $\langle f, \psi_{j,\xi} \rangle$,

$$\begin{aligned} \hat{\alpha}_{k,l,i} &= \frac{1}{\lambda_k} Y_{g_{k,l,i}} = \frac{1}{\lambda_k} \int g_{k,l,i} dY \text{ and} \\ \hat{\beta}_{j,\xi} &= \sqrt{\omega_{j,\xi}} \sum_k \sqrt{b(k/2^j)} \sum_{l,i} g_{k,l,i}(\xi) \hat{\alpha}_{k,l,i} \quad . \end{aligned}$$

The estimators we consider are respectively defined as:

$$\begin{aligned} \text{linear SVD estimates} & \quad \hat{f}_J^{\text{LS}} = \sum_{k < 2^J} \sum_{l,i} \hat{\alpha}_{k,l,i} f_{k,l,i} \\ \text{linear needlet estimates} & \quad \hat{f}_J^{\text{LN}} = \sum_{j < J} \sum_{\xi} \hat{\beta}_{j,\xi} \psi_{j,\xi} \\ \text{thresholded SVD estimates} & \quad \hat{f}_T^{\text{TS}} = \sum_{k < 2^J} \sum_{l,i} \rho_{T_k}(\hat{\alpha}_{k,l,i}) f_{k,l,i} \\ \text{thresholded needlet estimates} & \quad \hat{f}_T^{\text{TN}} = \sum_{j < J} \sum_{\xi} \rho_{T_{j,\xi}}(\hat{\beta}_{j,\xi}) \psi_{j,\xi} \quad . \end{aligned}$$

where $\rho_T(\cdot)$ is the hard threshold function with threshold T:

$$\rho_T(x) = \begin{cases} x & \text{if } |x| \geq T \\ 0 & \text{otherwise} \end{cases} \quad .$$

	Linear SVD	Thresh. SVD	Linear Needlet	Thresh. Needlet
Observation	$dY = Rfd\theta \frac{ds}{\sqrt{1-s^2}} + \varepsilon dW$			
SVD Dec	$Y_{h_{k,l,i}} = \langle Y, g_{k,l,i} \rangle = \langle Rf, g_{k,l,i} \rangle + \varepsilon_{k,l,i} = \mu_k \langle f, f_{k,l,i} \rangle + \varepsilon_{k,l,i}$			
Inv. Radon	$\hat{\alpha}_{k,l,i} = \frac{1}{\mu_k} \langle Y, g_{k,l,i} \rangle = \langle f, f_{k,l,i} \rangle + \frac{1}{\mu_k} \varepsilon_{k,l,i}$			
Needlet transf.	$\hat{\beta}_{j,\xi} = \sqrt{\omega_{j,\xi}} \sum_k \sqrt{b\left(\frac{k}{2^j}\right)} \times \sum_{l,i} f_{k,l,i}(\xi) \hat{\alpha}_{k,l,i}$			
Coeff. mod.	$\hat{\alpha}_{k,l,i}^{SL} = 1_{k \leq k_{\max}} \hat{\alpha}_{k,l,i}$	$\hat{\alpha}_{k,l,i}^{ST} = \rho_{T_k}(\hat{\alpha}_{k,l,i})$	$\hat{\beta}_{j,\xi}^{NL} = 1_{j < j_{\max}} \hat{\beta}_{j,\xi}$	$\hat{\beta}_{j,\xi}^{NT} = \rho_{T_{j,\xi}}(\hat{\beta}_{j,\xi})$
Needlet inv.	$\hat{\alpha}_{k,l,i}^* = \sum_j \sqrt{b\left(\frac{k}{2^j}\right)} \sum_{\xi \in X_j} \sqrt{\omega_{j,\xi}} f_{k,l,i}(\xi) \hat{\beta}_{j,\xi}^*$			
SVD rec.	$\hat{f}^* = \sum_{k,l,i} \hat{\alpha}_{k,l,i}^* f_{k,l,i}$			
	\hat{f}^{LS}	\hat{f}^{TS}	\hat{f}^{LN}	\hat{f}^{TN}
	Linear SVD	Thresh. SVD	Linear Needlet	Thresh. Needlet

Table 1: Algorithmic description of the considered estimators

A more precise description is given in Table 1. In our experiments, the values of $Y_{g_{k,l,i}}$ have been obtained from an initial approximation of $\langle f, f_{k,l,i} \rangle$ computed with a very fine cubature to which a Gaussian i.i.d. sequence is added.

We have used in our numerical experiments thresholds of the form

$$T_k = \frac{\kappa}{\lambda_k} \varepsilon \sqrt{\log 1/\varepsilon} \quad \text{and} \quad T_{j,\xi} = \kappa \sigma_{j,\xi} \varepsilon \sqrt{\log 1/\varepsilon}$$

where $\sigma_{j,\xi}$ is the standard deviation of the noisy needlet coefficients when $f = 0$ and $\varepsilon = 1$:

$$\sigma_{j,\xi}^2 = \omega_{j,\xi} \sum_k b(k/2^j) \sum_{l,i} g_{k,l,i}(\xi)^2 .$$

Note that while the needlet threshold is different than in Theorem 1, as $\sigma_{j,\xi}$ is of order $2^{j\nu}$ its conclusions remain valid.

An important issue in the needlet scheme is the choice of the cubature in the needlet construction. Proposition 1 ensures the existence of a suitable cubature ξ_j for every level j based on a cubature $\tilde{\xi}_j$ on the sphere but does not give an explicit construction of the points on the sphere nor an explicit formula for the weights $\omega_{j,\xi}$. Those ingredients are nevertheless central in the numerical scheme and should be specified. Three possibilities have been considered: a numerical cubature deduced from an almost uniform cubature of the half sphere available, an approximate cubature deduced from the Healpix cubature on the sphere and a cubature obtained by subsampling a tensorial cubature associated to the latitude and longitude coordinates on the sphere. The first strategy has been considered by Baldi et al[2] in a slightly different context, there is however a strong limitation on the maximum degree of the cubature available and thus this solution has

been abandoned. The Healpix strategy, also considered by Baldi et al. in an other paper[3], is easily implementable but, as it is based on an approximate cubature, fails to be precise enough. The last strategy relies on the subsampling on a tensorial grid on the sphere. While such a strategy provides a simple way to construct an admissible cubature, the computation of the cubature weights is becoming an issue as not closed form are available.

To overcome those issues, we have considered a cubature formula based on the full tensorial grid appearing in the third strategy. While this cubature does not satisfy the condition of Proposition 1, its weights can be computed explicitly and we argue that, using our modified threshold, we can still control the risk of the estimator. Indeed, note first that the modified threshold is such that the thresholding of a needlet depends only on its scale parameter j and on its center ξ , and not on the corresponding cubature weight $\omega_{j,\xi}$. Assume now that we have a collection of K cubature, each satisfying conditions of Proposition 1 and thus defining a suitable estimate \hat{f}_k , the ‘‘average’’ cubature obtained by adding all the cubature points and using their average cubature weight defines a new estimate \hat{f} satisfying:

$$\hat{f} = \frac{1}{K} \sum_{k=1}^K \hat{f}_k \quad .$$

By convexity, for any $p \geq 1$,

$$\|f - \hat{f}\|_p^p = \left\| f - \frac{1}{K} \sum_{k=1}^K \hat{f}_k \right\|_p^p \leq \frac{1}{K} \sum_{k=1}^K \|f - \hat{f}_k\|_p^p$$

and thus this average estimator is as efficient as the worst estimator in the family \hat{f}_k . We argue that the full tensorial cubature is an average of suitable cubature and thus that the corresponding estimator satisfies the conclusion of Theorems 1 and 2. Remark the proximity of this principle with the cycle-spinning method introduced by Donoho et al., we claim that the same kind of numerical gain are obtained with this method. The numerical comparison of the Healpix cubature and our tensorial cubature is largely in favor of our scheme. Furthermore, the tensorial structure of the cubature leads to some simplification in the numerical implementation of the needlet estimator so that this scheme is almost as fast as the simplest Healpix based one.

5.3 Numerical results

In this section, we compare 5 ‘‘estimators’’ (linear SVD with best scale, linear needlet with best scale, thresholded SVD with best κ , thresholded needlet with best κ and thresholded needlet with $\kappa = 3$) for 7 different norms ($L_1, L_2, L_4, L_6, L_7, L_8, L_{10}$ and L_∞) and 7 noise levels ε ($2^k/1000$ for k in $0, 1, \dots, 6$). Each subfigure of Figure 4 plots the logarithm of the estimation error for a specific norm against the opposite of the logarithm of the noise level. Remark that the subfigure overall aspect is explained by the errors decay when the noise level diminishes. The good theoretical behavior of the thresholded needlet estimator is confirmed numerically: the thresholded needlet estimator with an

optimized κ appears as the best estimator for every norm while a fixed κ yields a very good estimator except for the L_∞ case, as expected by our theoretical results. This results are confirmed visually by the reconstructions of Figure 5. In the needlet ones, errors are smaller and much more localized than in their SVD counterparts. Observe also how the fine structures are much more preserved with the thresholded needlet estimate than with any other methods.

We conclude this paper with some sections devoted to the proofs of our results.

6 Needlet properties

6.1 Key inequalities

The following inequalities are true (and proved in [16],[15], [19], [12]) and will be fundamental in the sequel: In the following lines , $g_{j,\xi}$ will stand either for $\varphi_{j,\xi}$ or $\psi_{j,\xi}$,

$$\forall j \in \mathbb{N}, \forall \xi \in \chi_j, \quad 0 < c \leq \|g_{j,\xi}\|_2^2 \leq 1 \quad (14)$$

$$\forall j \in \mathbb{N}, \xi \in \chi_j, \quad \forall x \in \mathcal{X}, \quad \sum_{\xi \in \chi_j} \|g_{j,\xi}\|_1 |g_{j,\xi}(x)| \leq C < \infty \quad (15)$$

$$|g_{j,\xi}(x)| \leq C_M \frac{2^{jd/2}}{\sqrt{W_j(x)}(1 + 2^j d(x, \xi))^M} \quad (16)$$

(recall that $W_j(x)$ has been defined in (8)). From these inequalities, one can deduce the following ones (see [11]): For all $1 \leq p \leq \infty$,

$$\left(\sum_{\xi \in \chi_j} |\langle f, g_{j,\xi} \rangle|^p \|g_{j,\xi}\|_p^p \right)^{1/p} \leq C \|f\|_p \quad (17)$$

$$\left\| \sum_{\xi \in \chi_j} \lambda_\xi g_{j,\xi}(x) \right\|_p \leq \left(\frac{C}{c} \right)^2 \left(\sum_{\xi \in \chi_j} \|\lambda_\xi g_{j,\xi}\|_p^p \right)^{1/p} \quad (18)$$

6.2 Besov embeddings

It is a key point to clarify how the Besov bodies spaces defined above may be included in each others. As will be seen, the embeddings will parallel the standard embeddings of usual Besov spaces, but with important differences which in particular yield new minimax rates of convergence as detailed above.

We begin with an evaluation of the different L_p norms of the needlets. More precisely, in [12] it is shown that for $0 < p \leq \infty$

$$\|\psi_{j,\xi}\|_p \sim \|\varphi_{j,\xi}\|_p \sim \left(\frac{2^{jd}}{W_j(\xi)} \right)^{1/2-1/p}, \quad \xi \in \chi_j. \quad (19)$$

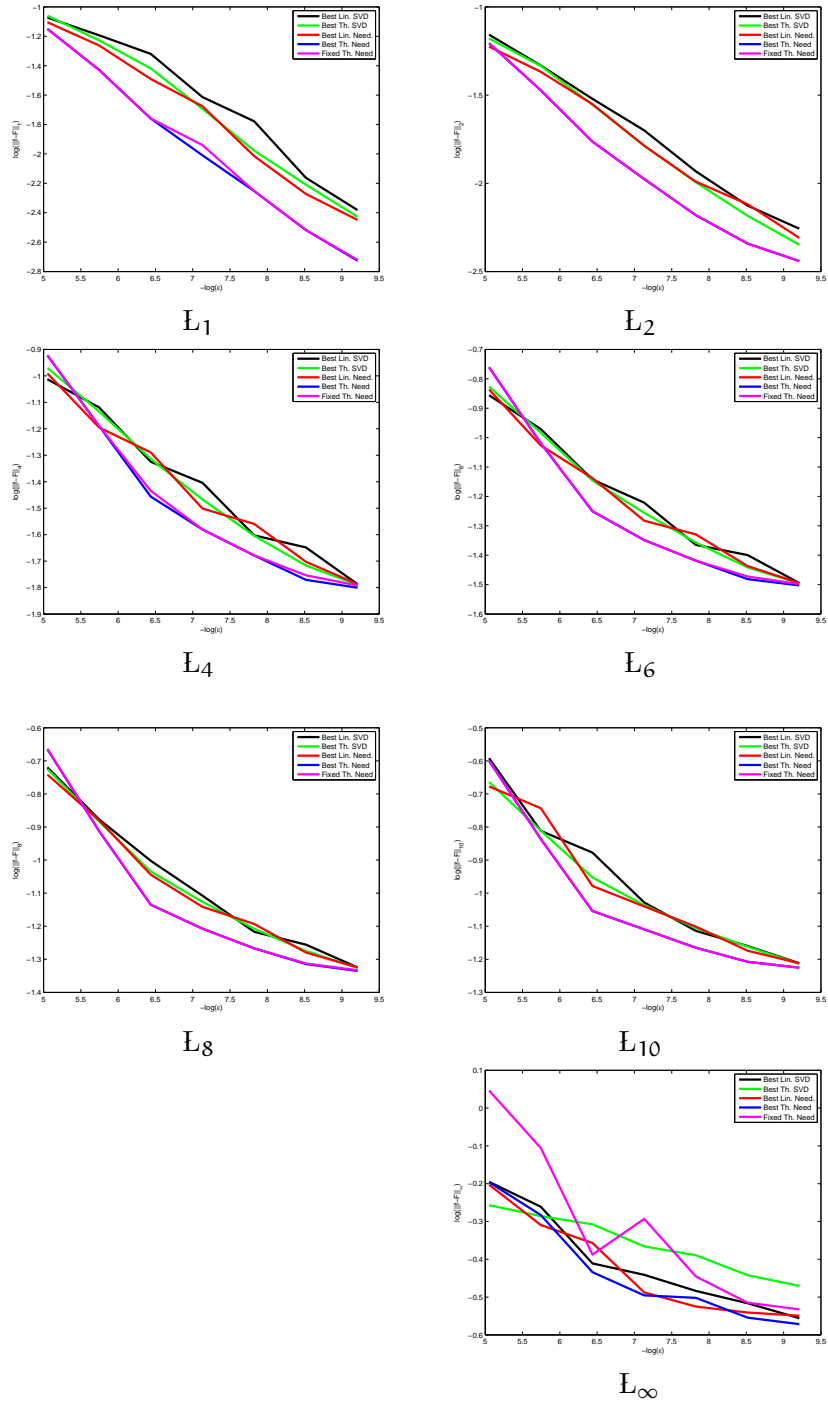


Figure 4: Estimation results in $\log L_p$ norm. Each figure shows the decay of the logarithm of the error against the logarithm of the noise parameter for the specified norm.

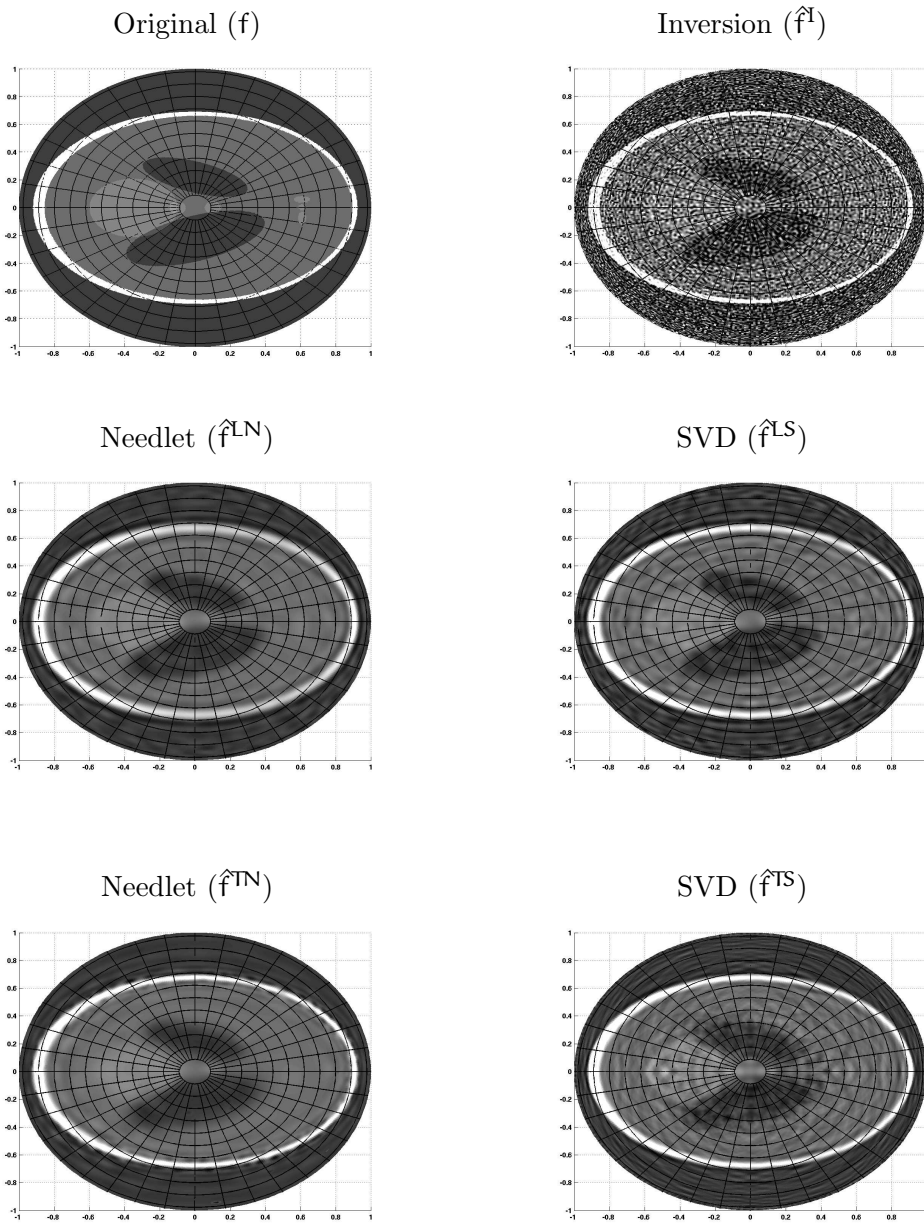


Figure 5: Visual comparison for the original Logan Shepp phantom with $\varepsilon = 8/1000$. Errors are much more localized in the needlet based estimates compared to the fully delocalized errors of the SVD based estimates. Fine structures are much more restored in the thresholded needlet estimate than in the other estimates.

The following inequalities are proved in [10]

$$\sum_{\xi \in \chi_j} \|g_{j,\xi}\|_p^p \leq c2^{j(dp/2+(p/2-2)_+)} \quad \text{if } p \neq 4, \quad (20)$$

$$\sum_{\xi \in \chi_j} \|g_{j,\xi}\|_p^p \leq cj2^{jdp/2} \quad \text{if } p = 4, \quad (21)$$

We are now able to state the embeddings results (see [11]).

Theorem 4. 1. $1 \leq p \leq \pi \leq \infty \Rightarrow B_{\pi,r}^s \subseteq B_{p,r}^s$.

2.

$$\infty \geq p \geq \pi > 0, \quad s > (d+1)(1/\pi - 1/p), \quad \Rightarrow B_{\pi,r}^s \subseteq B_{p,r}^{s-(d+1)(1/\pi-1/p)}$$

7 Proof of the upper bounds

A important tool for the proof of the upper bounds which clarifies the thresholding procedure is the following lemma.

Lemma 1. For all $j \geq -1$, $\xi \in \chi_j$, $\hat{\beta}_{j,\xi}$ has a Gaussian distribution with mean $\beta_{j,\xi}$ and variance $\sigma_{j,\xi}^2$, with

$$\sigma_{j,\xi}^2 \leq c2^{j(d-1)}\varepsilon^2$$

Proof of the lemma As we can write

$$\begin{aligned} \hat{\beta}_{j,\xi} &= \sum_{k,l,i} \gamma_{k,l,i}^{j,\xi} \int_{B^d} ff_{k,l,i} dx + \sum_{k,l,i} \gamma_{k,l,i}^{j,\xi} \frac{\varepsilon}{\lambda_k} Z_{k,l,i} \\ &= \beta_{j,\xi} + Z_{j,\xi}. \end{aligned}$$

Here the summation is over $\{(k, l, i) : 0 \leq k < 2^j, 0 \leq l \leq k, l \equiv k \pmod{2}, 1 \leq i \leq N_{d-1}(l)\}$. Since the $Z_{k,l,i}$'s are independent $N(0, 1)$ random variables, $Z_{j,\xi} \sim N(0, \sigma_{j,\xi}^2)$ we have

$$\sigma_{j,\xi}^2 = \varepsilon^2 \sum_{k,l,i} |\gamma_{k,l,i}^{j,\xi}|^2 \frac{\binom{k}{d}}{\pi^{d-1} 2^d} \leq \frac{(2^j)^{d-1}}{\pi^{d-1} 2^d} \leq c2^{j(d-1)}\varepsilon^2 \quad (22)$$

with $c = (d/2\pi)^{d-1}$. Here we used that $\{f_{k,l,i}\}$ is an orthonormal basis for \mathbb{L}^2 and hence $\sum_{k,l,i} |\gamma_{k,l,i}^{j,\xi}|^2 = \|\Psi_{j,\xi}\|_2^2 \leq 1$.

Let us now begin with the second theorem which proof is slightly simpler.

7.1 Proof of Theorem 2

We have, if we denote

$$A_J(f) := \sum_{j>J} \sum_{\xi \in \chi_j} \beta_{j,\xi} \Psi_{j,\xi}$$

$$\begin{aligned}\|\hat{f}_\infty - f\|_\infty &\leq \|\hat{f}_\infty - A_J(f)\|_\infty + \|A_J(f) - f\|_\infty \\ &\leq \|\hat{f}_\infty - A_J(f)\|_\infty + C_{\infty,\pi} \|f\|_{B_{\pi,r}^s} 2^{-J(s-(d+1)/\pi)}\end{aligned}$$

We have used as $B_{\pi,r}^s \subset B_{\infty,r}^{s-(d+1)\frac{1}{\pi}}$,

$$\|A_J f - f\|_\infty \leq C_{\infty,\pi} \|f\|_{B_{\pi,r}^s} 2^{-J(s-(d+1)\frac{1}{\pi})}.$$

Moreover

$$2^{-J(s-(d+1)/\pi)} \leq \left(\varepsilon \sqrt{\log 1/\varepsilon}\right)^{-\frac{s-(d+1)/\pi}{d}} \leq \left(\varepsilon \sqrt{\log 1/\varepsilon}\right)^{-\frac{s-(d+1)/\pi}{s-(d+1)/\pi+d}}$$

as $s > \frac{d+1}{\pi}$.

We have, using (18)

$$\begin{aligned}\|\hat{f}_\infty - A_J(f)\|_\infty &\leq \sum_{j < J} \left\| \sum_{\xi \in X_j} \left(\hat{\beta}_{j,\xi} \mathbf{1}_{\{|\hat{\beta}_{j,\xi}| \|\psi_{j,\xi}\|_\infty \geq \kappa 2^{j d} \varepsilon \sqrt{\log 1/\varepsilon}\}} - \beta_{j,\xi} \right) \psi_{j,\xi} \right\|_\infty \\ &\leq \sum_{j < J} \left(\left\| \sum_{\xi \in X_j} \left((\hat{\beta}_{j,\xi} - \beta_{j,\xi}) \psi_{j,\xi} \mathbf{1}_{\{|\hat{\beta}_{j,\xi}| \|\psi_{j,\xi}\|_\infty \geq \kappa 2^{j d} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \right\|_\infty \right. \\ &\quad \left. + \left\| \sum_{\xi \in X_j} \left(\beta_{j,\xi} \psi_{j,\xi} \mathbf{1}_{\{|\hat{\beta}_{j,\xi}| \|\psi_{j,\xi}\|_\infty < \kappa 2^{j d} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \right\|_\infty \right) \\ &\leq \frac{C}{c} \sum_{j < J} \left(\sup_{\xi \in X_j} \left(|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \mathbf{1}_{\{|\hat{\beta}_{j,\xi}| \|\psi_{j,\xi}\|_\infty \geq \kappa 2^{j d} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \right. \\ &\quad \left. + \sup_{\xi \in X_j} \left(|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \mathbf{1}_{\{|\hat{\beta}_{j,\xi}| \|\psi_{j,\xi}\|_\infty < \kappa 2^{j d} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \right)\end{aligned}$$

We decompose the first term of the last inequality,

$$\begin{aligned}&\sup_{\xi \in X_j} \left(|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \mathbf{1}_{\{|\hat{\beta}_{j,\xi}| \|\psi_{j,\xi}\|_\infty \geq \kappa 2^{j d} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \\ &= \sup_{\xi \in X_j} \left(|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \mathbf{1}_{\{|\hat{\beta}_{j,\xi}| \|\psi_{j,\xi}\|_\infty \geq \kappa 2^{j d} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right. \\ &\quad \left. \times \left(\mathbf{1}_{\{|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \geq \frac{\kappa}{2} 2^{j d} \varepsilon \sqrt{\log 1/\varepsilon}\}} + \mathbf{1}_{\{|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty < \frac{\kappa}{2} 2^{j d} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \right) \\ &\leq \sup_{\xi \in X_j} \left(|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \mathbf{1}_{\{|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \geq \frac{\kappa}{2} 2^{j d} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \\ &\quad + \sup_{\xi \in X_j} \left(|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \mathbf{1}_{\{|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty > \frac{\kappa}{2} 2^{j d} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right)\end{aligned}$$

and the second one

$$\begin{aligned}
& \sup_{\xi \in \mathcal{X}_j} \left(|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \mathbf{1}_{\{|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty < \kappa 2^{jd} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \\
&= \sup_{\xi \in \mathcal{X}_j} \left(|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \mathbf{1}_{\{|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty < \kappa 2^{jd} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right. \\
&\quad \left. \times \left(\mathbf{1}_{\{|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \geq 2\kappa 2^{jd} \varepsilon \sqrt{\log 1/\varepsilon}\}} + \mathbf{1}_{\{|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty < \kappa 2^{jd} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \right) \\
&\leq \sup_{\xi \in \mathcal{X}_j} \left(|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \mathbf{1}_{\{|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty > \kappa 2^{jd} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \\
&\quad + \sup_{\xi \in \mathcal{X}_j} \left(|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \mathbf{1}_{\{|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty < \kappa 2^{jd} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right)
\end{aligned}$$

Now we will bound each of the four terms coming from the last two inequalities. Since for $X \sim \mathcal{N}(0, \sigma^2)$, we have

$$\mathbb{E}(|Y| \mathbf{1}_{\{|Y| > \lambda \sigma\}}) = \sigma \frac{2}{\sqrt{2\pi}} \int_{\lambda}^{\infty} y e^{-y^2/2} dy = e^{-\lambda^2/2} \frac{2}{\sqrt{2\pi}} \leq e^{-\lambda^2/2}.$$

Noticing that the standard deviation of $(\hat{\beta}_{j,\xi} - \beta_{j,\xi}) \|\psi_{j,\xi}\|_\infty$ is smaller than $\tau_\infty 2^{jd} \varepsilon$ (using lemma 1 and (16)), we have:

$$\begin{aligned}
& \sum_{j \leq J} \mathbb{E} \left(\sup_{\xi \in \mathcal{X}_j} \left(|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \mathbf{1}_{\{|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \geq \frac{\kappa}{2} 2^{jd} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \right) \\
&\leq \sum_{j \leq J} \sum_{\xi \in \mathcal{X}_j} \mathbb{E} \left(|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \mathbf{1}_{\{|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \geq \frac{\kappa}{2} 2^{jd} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \leq c 2^{jd} \varepsilon^{\kappa^2/2\tau_\infty^2} \\
&\leq C \varepsilon^{\kappa^2/2\tau_\infty^2 - 1} \sqrt{\log 1/\varepsilon}^{-1} \leq C \varepsilon \sqrt{\log 1/\varepsilon}
\end{aligned}$$

if $\kappa^2 \geq 4\tau_\infty^2$, where we have used $\text{Card } \mathcal{X}_j \leq c 2^{jd}$. This proves that this term will be of the right order.

For the second term, let us observe that we have, using theorem 4

$$|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \leq C_{\infty,\pi} \|f\|_{B_{\pi,r}^s} 2^{-j(s-(d+1)/\pi)},$$

so only the j 's indexes such that $j \leq j_1$ will verify this inequality

$$2^{j_1} \sim \left(\frac{2C_{\infty,\pi} \|f\|_{B_{\pi,r}^s}}{\kappa} \left(\varepsilon \sqrt{\log 1/\varepsilon} \right) \right)^{-\frac{1}{s+d-(d+1)\pi}}$$

On the other side, using Pisier Lemma

$$\mathbb{E} \left(\sup_{\xi \in \mathcal{X}_j} (|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty) \right) \leq \tau_\infty 2^{jd} \varepsilon \sqrt{2 \log 2c 2^{jd}}.$$

So

$$\begin{aligned} & \sum_{j \leq j_1} \mathbb{E} \left(\sup_{\xi \in \mathcal{X}_j} \left(|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \mathbf{1}_{\{|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty > \frac{\kappa}{2} 2^{jd} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \right) \\ & \leq \tau_\infty \sum_{j \leq j_1} 2^{jd} \varepsilon \sqrt{2 \log 2c 2^{jd}} \leq C \varepsilon j^{\frac{1}{2}} 2^{j_1 d} \lesssim C (\|f\|_{\mathbb{B}_{\pi,r}^s})^{\frac{s-(d+1)\pi}{s+d-(d+1)\pi}} \left(\varepsilon \sqrt{\log 1/\varepsilon} \right)^{\frac{s-(d+1)\pi}{s+d-(d+1)\pi}} \end{aligned}$$

This proves that this term will be of the right order. Concerning the first term of the second inequality,

$$\begin{aligned} & \mathbb{E} \left(\sup_{\xi \in \mathcal{X}_j} \left(|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \mathbf{1}_{\{|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty > \kappa 2^{jd} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \right) \\ & \leq C_{\infty,\pi} \|f\|_{\mathbb{B}_{\pi,r}^s} \sum_{\xi \in \mathcal{X}_j} \mathbb{P} \left(|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty > \kappa 2^{jd} \varepsilon \sqrt{\log 1/\varepsilon} \right) \end{aligned}$$

but

$$\mathbb{P} \left(|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty > \kappa 2^{jd} \varepsilon \log 1/\varepsilon \right) \leq e^{-\frac{(\kappa 2^{jd} \varepsilon \sqrt{\log 1/\varepsilon})^2}{2(\varepsilon 2^{j(d-1)/2} \|\psi_{j,\xi}\|_\infty)^2}} \leq \varepsilon^{\kappa^2/2\tau_\infty^2}$$

So

$$\begin{aligned} & \sum_{j \leq J} \mathbb{E} \left(\sup_{\xi \in \mathcal{X}_j} \left(|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \mathbf{1}_{\{|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty > \kappa 2^{jd} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \right) \\ & \leq C_{\infty,\pi} \|f\|_{\mathbb{B}_{\pi,r}^s} J \varepsilon^{\kappa^2/2\tau_\infty^2} \leq C \|f\|_{\mathbb{B}_{\pi,r}^s} \left(\varepsilon \sqrt{\log 1/\varepsilon} \right)^{\frac{s-(d+1)\pi}{s+d-(d+1)\pi}} \end{aligned}$$

if $\kappa^2/2 \geq \tau_\infty^2$. This proves that this term will be of the right order. Concerning the second term of the second inequality,

$$\begin{aligned} & \sup_{\xi \in \mathcal{X}_j} \left(|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \mathbf{1}_{\{|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty < 2\kappa 2^{jd} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \\ & \leq 2\kappa 2^{jd} \varepsilon \sqrt{\log 1/\varepsilon} \wedge C_{\infty,\pi} \|f\|_{\mathbb{B}_{\pi,r}^s} 2^{-j(s-(d+1)/\pi)} \end{aligned}$$

let us again take

$$2^{j_1} \sim \left(\varepsilon \sqrt{\log 1/\varepsilon} \right)^{-\frac{1}{s+d-(d+1)\pi}}$$

$$\begin{aligned} & \sum_{j \leq J} \sup_{\xi \in \mathcal{X}_j} \left(|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty \mathbf{1}_{\{|\beta_{j,\xi}| \|\psi_{j,\xi}\|_\infty < 2\kappa 2^{jd} \varepsilon \log 1/\varepsilon\}} \right) \\ & \leq C \|f\|_{\mathbb{B}_{\pi,r}^s} \left(\varepsilon \log 1/\varepsilon \sum_{j \leq j_1} 2^{jd} + \sum_{j_1 < j \leq J} 2^{-j(s-(d+1)/\pi)} \right) \lesssim C \|f\|_{\mathbb{B}_{\pi,r}^s} \left(\varepsilon \sqrt{\log 1/\varepsilon} \right)^{\frac{s-(d+1)\pi}{s+d-(d+1)\pi}} \end{aligned}$$

This ends the proof of Theorem 2

7.2 Proof of Theorem 1

As in the previous proof we begin with the same decomposition,

$$\|\hat{f}_\varepsilon - f\|_p^p \leq 2^{p-1} \|\hat{f}_\varepsilon - \mathbf{A}_J(f)\|_p^p + \|\mathbf{A}_J(f) - f\|_p^p$$

But

$$\|\mathbf{A}_J(f) - f\|_p^p \leq C \|f\|_{\mathbf{B}_{\pi,r}^s}^p 2^{-Jsp} \quad \text{if } \pi \geq p$$

and

$$\|\mathbf{A}_J(f) - f\|_p^p \leq C \|f\|_{\mathbf{B}_{\pi,r}^s}^p 2^{-J(s-(d+1)(1/\pi-1/p))p} \quad \text{if } \pi \leq p$$

if we use the fact that for $\pi \geq p$, $\mathbf{B}_{\pi,r}^s \subset \mathbf{B}_{p,\infty}^s$, and for $\pi \leq p$, $\mathbf{B}_{\pi,r}^s \subset \mathbf{B}_{p,\infty}^{s-(d+1)(1/\pi-1/p)}$.

We have $2^{-Jsp} \leq (\varepsilon \log 1/\varepsilon)^{\frac{sp}{d-\frac{1}{2}}} \leq (\varepsilon \sqrt{\log 1/\varepsilon})^{\frac{sp}{d-\frac{1}{2}}} \leq (\varepsilon \sqrt{\log 1/\varepsilon})^{\frac{sp}{s+d-\frac{1}{2}}}$. Obviously, this term has the right rate for $\pi \geq p$. For $\pi < p$,

$$2^{-J(s-(d+1)(1/\pi-1/p))} \leq (\varepsilon \log 1/\varepsilon)^{\frac{(s-(d+1)(1/\pi-1/p))}{d-\frac{1}{2}}} \leq (\varepsilon \log 1/\varepsilon)^{\frac{(s-(d+1)(1/\pi-1/p))}{s+d-(d+1)/\pi}},$$

thanks to $s \geq (d+1)/\pi - 1/2$. This gives the right rate for $dp > d+1$. For $dp \leq d+1$, we have (again as $s \geq (d+1)/\pi - 1/2$), $s - (d+1)(\frac{1}{\pi} - \frac{1}{p}) \geq d - \frac{1}{2}$, so: $2^{-J(s-(d+1)(1/\pi-1/p))} \leq (\varepsilon \log 1/\varepsilon)^{\frac{(s-(d+1)(1/\pi-1/p))}{d-\frac{1}{2}}} \leq (\varepsilon \log 1/\varepsilon)^{\frac{s}{s+d-\frac{1}{2}}}$. Finally this proves that the bias term above always has the right rate.

Let us now investigate the stochastic term:

$$\mathbb{E} \|\hat{f} - \mathbf{A}_J(f)\|_p^p \leq C J^{p-1} \sum_{j < J} \mathbb{E} \left\| \sum_{\xi \in \mathcal{X}_j} \left(\hat{\beta}_{j,\xi} \mathbf{1}_{\{|\hat{\beta}_{j,\xi}| \geq \kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} - \hat{\beta}_{j,\xi} \right) \psi_{j,\xi} \right\|_p^p$$

But

$$\begin{aligned} & \left\| \sum_{\xi \in \mathcal{X}_j} \left(\hat{\beta}_{j,\xi} \mathbf{1}_{\{|\hat{\beta}_{j,\xi}| \geq \kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} - \hat{\beta}_{j,\xi} \right) \psi_{j,\xi} \right\|_p^p \\ & \leq C \left(\left\| \sum_{\xi \in \mathcal{X}_j} \left((\hat{\beta}_{j,\xi} - \beta_{j,\xi}) \psi_{j,\xi} \mathbf{1}_{\{|\hat{\beta}_{j,\xi}| \geq \kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \right\|_p^p \right. \\ & \quad \left. + \left\| \sum_{\xi \in \mathcal{X}_j} \left(\beta_{j,\xi} \psi_{j,\xi} \mathbf{1}_{\{|\hat{\beta}_{j,\xi}| < \kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \right\|_p^p \right) \\ & \leq C \left(\sum_{\xi \in \mathcal{X}_j} (\|\hat{\beta}_{j,\xi} - \beta_{j,\xi}\|^p \|\psi_{j,\xi}\|_p^p \mathbf{1}_{\{|\hat{\beta}_{j,\xi}| \geq \kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}}) \right. \\ & \quad \left. + \sum_{\xi \in \mathcal{X}_j} |\beta_{j,\xi}|^p \|\psi_{j,\xi}\|_p^p \mathbf{1}_{\{|\hat{\beta}_{j,\xi}| < \kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \end{aligned}$$

In turn,

$$\begin{aligned}
& |\hat{\beta}_{j,\xi} - \beta_{j,\xi}|^p \mathbf{1}_{\{|\hat{\beta}_{j,\xi}| \geq \kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} \\
&= |\hat{\beta}_{j,\xi} - \beta_{j,\xi}|^p \mathbf{1}_{\{|\hat{\beta}_{j,\xi}| \geq \kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} \left(\mathbf{1}_{\{|\beta_{j,\xi}| \geq \frac{\kappa}{2} 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} + \mathbf{1}_{\{|\beta_{j,\xi}| < \frac{\kappa}{2} 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \\
&\leq |\hat{\beta}_{j,\xi} - \beta_{j,\xi}|^p \mathbf{1}_{\{|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \geq \frac{\kappa}{2} 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} + |\hat{\beta}_{j,\xi} - \beta_{j,\xi}|^p \mathbf{1}_{\{|\beta_{j,\xi}| > \frac{\kappa}{2} 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}}
\end{aligned}$$

and

$$\begin{aligned}
|\beta_{j,\xi}|^p \mathbf{1}_{\{|\hat{\beta}_{j,\xi}| < \kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} &= |\beta_{j,\xi}|^p \mathbf{1}_{\{|\hat{\beta}_{j,\xi}| < \kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} \\
&\quad \times \left(\mathbf{1}_{\{|\beta_{j,\xi}| \geq 2\kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} + \mathbf{1}_{\{|\beta_{j,\xi}| < 2\kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \\
&\leq |\beta_{j,\xi}|^p \mathbf{1}_{\{|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| > \kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} + |\beta_{j,\xi}|^p \mathbf{1}_{\{|\beta_{j,\xi}| < \kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}}
\end{aligned}$$

We now have the following bounds: Using Pisier lemma

$$\mathbb{E} \left(|\hat{\beta}_{j,\xi} - \beta_{j,\xi}|^p \mathbf{1}_{\{|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \geq \frac{\kappa}{2} 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \leq C_p 2^{jp\nu} \varepsilon^p \left(\kappa \sqrt{\log 1/\varepsilon} \right)^{p-1} \varepsilon^{\kappa^2/2}$$

Hence,

$$\begin{aligned}
& J^{p-1} \sum_{j \leq J} \sum_{\xi \in \mathcal{X}_j} \mathbb{E} \left(|\hat{\beta}_{j,\xi} - \beta_{j,\xi}|^p \|\psi_{j,\xi}\|_p^p \mathbf{1}_{\{|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| \geq \frac{\kappa}{2} 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \\
&\leq C J^{p-1} \sum_{j \leq J} \varepsilon^p \left(\kappa \sqrt{\log 1/\varepsilon} \right)^{p-1} \varepsilon^{\kappa^2/2} \sum_{\xi \in \mathcal{X}_j} \|\psi_{j,\xi}\|_p^p \\
&\leq C \sqrt{\log 1/\varepsilon}^{p-1} \varepsilon^p \left(\kappa \sqrt{\log 1/\varepsilon} \right)^{p-1} \varepsilon^{\kappa^2/2} \sum_{j \leq J} 2^{jp\nu} 2^{j(dp/2 + (p/2 - 2))} \\
&\leq C \varepsilon^p
\end{aligned}$$

if $\kappa \geq \sqrt{2p}$ is large enough (we have used (19)).

Using the forthcoming inequality (24),

$$\begin{aligned}
& \sum_{\xi \in \mathcal{X}_j} \mathbb{E} \left(|\hat{\beta}_{j,\xi} - \beta_{j,\xi}|^p \|\psi_{j,\xi}\|_p^p \mathbf{1}_{\{|\beta_{j,\xi}| > \frac{\kappa}{2} 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \\
&\leq C \varepsilon^p \sum_{\xi \in \mathcal{X}_j} 2^{j\nu p} \|\psi_{j,\xi}\|_p^p \mathbf{1}_{\{|\beta_{j,\xi}| > \frac{\kappa}{2} 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} \\
&\leq C \varepsilon^p \left(\frac{\kappa}{2} \varepsilon \sqrt{\log 1/\varepsilon} \right)^{-q}.
\end{aligned}$$

Hence,

$$\begin{aligned}
& J^{p-1} \sum_{j \leq J} \sum_{\xi \in \mathcal{X}_j} \mathbb{E}(|\hat{\beta}_{j,\xi} - \beta_{j,\xi}|^p \|\Psi_{j,\xi}\|_p^p \mathbf{1}_{\{|\beta_{j,\xi}| > \frac{\kappa}{2} 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}}) \\
& \leq C J^p \sqrt{\log 1/\varepsilon}^{-p} \left(\frac{\kappa}{2}\right) \left(\varepsilon \sqrt{\log 1/\varepsilon}\right)^{p-q} \\
& \leq C \left(\varepsilon \sqrt{\log 1/\varepsilon}\right)^{p-q} (\log 1/\varepsilon)^{p/2}
\end{aligned}$$

Hence this term is of the right rate. Let us now turn to:

$$\begin{aligned}
& \sum_{\xi \in \mathcal{X}_j} \mathbb{E} \left(|\beta_{j,\xi}|^p \|\Psi_{j,\xi}\|_p^p \mathbf{1}_{\{|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| > \kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \\
& = \sum_{\xi \in \mathcal{X}_j} |\beta_{j,\xi}|^p \|\Psi_{j,\xi}\|_p^p \mathbb{P} \left(|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| > \kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon} \right)
\end{aligned}$$

but as the standard deviation of $\hat{\beta}_{j,\xi} - \beta_{j,\xi}$ is smaller than $\varepsilon 2^{j(d-1)/2}$

$$\mathbb{P} \left(|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| > \kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon} \right) \leq \varepsilon^{\kappa^2/2}$$

So

$$\begin{aligned}
& J^{p-1} \sum_{j \leq J} \mathbb{E} \left(\sum_{\xi \in \mathcal{X}_j} |\beta_{j,\xi}|^p \|\Psi_{j,\xi}\|_p^p \mathbf{1}_{\{|\hat{\beta}_{j,\xi} - \beta_{j,\xi}| > \kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} \right) \\
& \leq C \|f\|_{B_{\pi,r}^s}^p J^{p-1} \varepsilon^{\kappa^2/2} \leq C \left(\varepsilon \sqrt{\log 1/\varepsilon}\right)^p
\end{aligned}$$

if κ^2 is large enough (where we have used that $B_{\pi,r}^s \subset B_{p,r}^{s'} \subset \mathbb{L}^p$). Hence, this term also is of the right order.

Let us turn now to the last one: using (24):

$$\sum_{\xi \in \mathcal{X}_j} |\beta_{j,\xi}|^p \|\Psi_{j,\xi}\|_p^p \mathbf{1}_{\{|\beta_{j,\xi}| < 2\kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}} \leq \left(2\kappa \varepsilon \sqrt{\log 1/\varepsilon}\right)^{p-q}$$

Hence,

$$\begin{aligned}
& J^{p-1} \sum_{j \leq J} \sup_{\xi \in \mathcal{X}_j} (|\beta_{j,\xi}|^p \|\Psi_{j,\xi}\|_p^p \mathbf{1}_{\{|\beta_{j,\xi}| < 2\kappa 2^{j\nu} \varepsilon \sqrt{\log 1/\varepsilon}\}}) \\
& \leq C \|f\|_{B_{\pi,r}^s}^p J^p \left(\varepsilon \sqrt{\log 1/\varepsilon}\right)^{p-q} \leq C \|f\|_{B_{\pi,r}^s} \sqrt{\log 1/\varepsilon}^p \left(\varepsilon \sqrt{\log 1/\varepsilon}\right)^{p-q}.
\end{aligned}$$

This proves that all the terms have the proper rate. It remains now to state and prove the following lemma.

Lemma 2. Let $A = \{(s, \pi), \quad s > (d+1)(\frac{1}{\pi} - \frac{1}{p}) \cap (s > 0)\}$, and $f \in B_{\pi, r}^s$, $1 \leq \pi \leq \infty$, $1 \leq p < \infty$. $f = \sum_j \sum_{\xi \in \mathcal{X}_j} \beta_{j, \xi} \psi_{j, \xi}$, $\beta_{j, \xi} = \langle f, \psi_{j, \xi} \rangle$. Suppose that $\sum_{\xi \in \mathcal{X}_j} (|\beta_{j, \xi}| \|\psi_{j, \xi}\|_\pi)^\pi = \rho_j^\pi 2^{-js\pi}$, $\rho \in l_r(\mathbb{N})$, then, if $\nu = \frac{d-1}{2}$

$$\sum_{\xi \in \mathcal{X}_j} \left(\frac{|\beta_{j, \xi}|}{2^{j\nu}} \right)^q \left(2^{j\nu} \|\psi_{j, \xi}\|_p \right)^p \leq C \rho_j^q$$

where $q < p$ is as follows:

1. $p - q = \frac{sp}{s+d-1/2}$ ($q = \frac{(d-1/2)p}{s+d-1/2}$) in the following domain I:

$$\{(s, \pi), \quad (s(1/p - 1/4) \geq (d-1/2)(1/\pi - 1/p)) \cap A\}.$$

Moreover we have the following slight modification at the frontier: the domain becomes

$$\{(s, \pi), \quad (s(1/p - 1/4) = (d-1/2)(1/\pi - 1/p)) \cap A\}$$

and the inequality

$$\sum_{\xi \in \mathcal{X}_j} \left(\frac{|\beta_{j, \xi}|}{2^{j\nu}} \right)^q \left(2^{j\nu} \|\psi_{j, \xi}\|_p \right)^p \leq C \rho_j^q j^{1-q/\pi}$$

2. $p - q = \frac{(s-2(1/\pi-1/p))p}{s+d-2/\pi}$ ($q = \frac{dp+2}{s+d-2/\pi}$) in the following domain II:

$$\{(s, \pi) \quad (s > dp(1/\pi - 1/p)) \cap (s(1/p - 1/4) < (d-1/2)(1/\pi - 1/p)) \cap A\}.$$

3. $p - q = \frac{(s-(d+1)(1/\pi-1/p))p}{s+d-(d+1)/\pi}$, ($q = \frac{dp-(d+1)}{s+d-(d+1)/\pi}$) in the following domain III:

$$\{(s, \pi), \quad (dp(\frac{1}{\pi} - \frac{1}{p}) \geq s) \cap A, \quad \text{for } \frac{1}{p} < \frac{d}{d+1}, \}.$$

This lemma is to be used essentially through the following corollary:

Corollary 1. Respectively in the domains I, II, III, we have, for q described in the lemma, and $f \in B_{\pi, r}^s$

$$\sum_{\xi \in \mathcal{X}_j} 1_{\left\{ \frac{|\beta_{j, \xi}|}{2^{j\nu}} \geq \lambda \right\}} \left(2^{j\nu} \|\psi_{j, \xi}\|_p \right)^p \leq C \rho_j^q \lambda^{-q} \quad (23)$$

$$\sum_{\xi \in \mathcal{X}_j} 1_{\left\{ \frac{|\beta_{j, \xi}|}{2^{j\nu}} \leq 2^{j\nu} \lambda \right\}} |\beta_{j, \xi}|^p \|\psi_{j, \xi}\|_p^p \leq C \rho_j^q \lambda^{p-q} \quad (24)$$

with an obvious modification for

$$\{(s, \pi), \quad (s(1/p - 1/4) = (d-1/2)(1/\pi - 1/p)) \cap A\}$$

Proof of the corollary let us recall that on a measure space (X, μ) we have, if $h \in \mathbb{L}_q(\mu)$ then $\mu(|h| \geq \lambda) \leq \frac{\|h\|_q^q}{\lambda^q}$ and, as $q < p$,

$$\int_{|h| \leq \lambda} |h|^p d\mu \leq \int (|h| \wedge \lambda)^p d\mu = \int_0^\lambda px^{p-1} \mu(|h| \geq x) dx \leq \int_0^\lambda px^{p-1} \frac{\|h\|_q^q}{x^q} dx = \frac{p\|h\|_q^q}{p-q} \lambda^{p-q}.$$

For the corollary we take $X = \chi_j$, $\mu(\xi) = (2^{j\nu} \|\psi_{j,\xi}\|_p)^p$ and $h(\xi) = \frac{|\beta_{j,\xi}|}{2^{j\nu}}$. \square

Proof of lemma 2

Let us fix q (chosen later) and investigate separately the two cases $q \geq \pi$ and $q < \pi$. For $q \geq \pi$, we have using (19)

$$\begin{aligned} I_j(f, q, p) &= \sum_{\xi \in \chi_j} \left| \frac{\beta_{j,\xi}}{2^{j\nu}} \right|^q \|2^{j\nu} \psi_{j,\xi}\|_p^p \sim 2^{j\nu(p-q)} \sum_{\xi \in \chi_j} |\beta_{j,\xi}|^q \left(\frac{2^{jd}}{W_j(\xi)} \right)^{p/2-1} \\ &\leq 2^{j\nu(p-q)} \left(\sum_{\xi \in \chi_j} \left(|\beta_{j,\xi}|^q \left(\frac{2^{jd}}{W_j(\xi)} \right)^{p/2-1} \right)^{\pi/q} \right)^{q/\pi} \\ &= 2^{j\nu(p-q)} \left(\sum_{\xi \in \chi_j} |\beta_{j,\xi}|^\pi \left(\frac{2^{jd}}{W_j(\xi)} \right)^{(p/2-1)\pi/q} \right)^{q/\pi} \\ &= 2^{j\nu(p-q)} \left(\sum_{\xi \in \chi_j} |\beta_{j,\xi}|^\pi \left(\frac{2^{jd}}{W_j(\xi)} \right)^{\pi/2-1} \left(\frac{2^{jd}}{W_j(\xi)} \right)^{\frac{\pi}{q}(p/2-1)-(\pi/2-1)} \right)^{q/\pi} \\ &\leq 2^{j\nu(p-q)} \left(\sum_{\xi \in \chi_j} |\beta_{j,\xi}|^\pi \left(\frac{2^{jd}}{W_j(\xi)} \right)^{\pi/2-1} \right)^{q/\pi} 2^{j(d+1)(\frac{p-q}{2} + q(\frac{1}{\pi} - \frac{1}{q}))} \end{aligned}$$

Choosing q such that $(sq + d(p-q) + (d+1)q(\frac{1}{\pi} - \frac{1}{q})) = 0$ gives $q = \frac{pd-(d+1)}{s+d-(d+1)\pi}$. Hence $p-q = \frac{s-(d+1)(1/\pi-1/p)}{s+d-(d+1)\pi}p$; $q-\pi = -\pi \frac{s-pd(1/\pi-1/p)}{s+d-(d+1)\pi}$. Thus in domain III:

$$\left\{ \left(\frac{1}{p} < \frac{d}{d+1} \right) \cap (s - (d+1)(1/\pi - 1/p) > 0) \cap (s - pd(1/\pi - 1/p) \leq 0) \right\}$$

we have $0 < q < p$, $\pi \leq q$, $\sum_{\xi \in \chi_j} \left| \frac{\beta_{j,\xi}}{2^{j\nu}} \right|^q \|2^{j\nu} \psi_{j,\xi}\|_p^p \leq \rho_j^q$.

For $q < \pi$, we have using (19)

$$I_j(f, q, p) = \sum_{\xi \in \chi_j} \left| \frac{\beta_{j,\xi}}{2^{j\nu}} \right|^q \|2^{j\nu} \psi_{j,\xi}\|_p^p \sim 2^{j\nu(p-q)} \sum_{\xi \in \chi_j} |\beta_{j,\xi}|^q \left(\frac{2^{jd}}{W_j(\xi)} \right)^{p/2-1}.$$

$$\begin{aligned}
& 2^{j\nu(p-q)} \sum_{\xi \in \mathcal{X}_j} |\beta_{j,\xi}|^q \left(\frac{2^{jd}}{W_j(\xi)} \right)^{(\pi/2-1)q/\pi} \left(\frac{2^{jd}}{W_j(\xi)} \right)^{(p/2-1)-(\pi/2-1)q/\pi} \\
& \leq 2^{j\nu(p-q)} \left(\sum_{\xi \in \mathcal{X}_j} |\beta_{j,\xi}|^\pi \left(\frac{2^{jd}}{W_j(\xi)} \right)^{\pi/2-1} \right)^{q/\pi} \left(\sum_{\xi \in \mathcal{X}_j} \left(\frac{2^{jd}}{W_j(\xi)} \right)^{\frac{\pi}{\pi-q}((p/2-1)-(\pi/2-1)q/\pi)} \right)^{1-q/\pi} \\
& \sim 2^{j\nu(p-q)} \left(\sum_{\xi \in \mathcal{X}_j} |\beta_{j,\xi}|^\pi \|\psi_{j,\xi}\|_\pi^\pi \right)^{q/\pi} \left(\sum_{\xi \in \mathcal{X}_j} \left(\frac{2^{jd}}{W_j(\xi)} \right)^{\frac{\pi(p-q)}{2(\pi-q)}-1} \right)^{1-q/\pi} \\
& \sim 2^{j\nu(p-q)} \left(\sum_{\xi \in \mathcal{X}_j} |\beta_{j,\xi}|^\pi \|\psi_{j,\xi}\|_\pi^\pi \right)^{q/\pi} \left(\sum_{\xi \in \mathcal{X}_j} \|\psi_{j,\xi}\|_\pi^{\frac{\pi(p-q)}{(\pi-q)}} \right)^{1-q/\pi}.
\end{aligned}$$

Now let us investigate separately the cases $\frac{\pi(p-q)}{(\pi-q)} <, >$, and $= 4$.

Case $\frac{\pi(p-q)}{(\pi-q)} < 4$ Using (19), (20) and (21), we have

$$I_j(f, q, p) \leq C 2^{j\nu(p-q)} \rho_j^q 2^{-jsq} 2^{jd(p-q)/2} \leq C \rho_j^q.$$

if we put $-sq + (p-q)(d-1/2) = 0$, i.e. $q = \frac{p(d-1/2)}{s+d-1/2}$, then $p-q = \frac{sp}{s+d-1/2} > 0$.

So $p-q = \pi \frac{s-(d-1/2)p(1/\pi-1/p)}{s+d-1/2} > 0 \Leftrightarrow \frac{s}{p} > (d-1/2)(1/\pi-1/p)$.

And $\frac{\pi(p-q)}{(\pi-q)} = \frac{sp}{s-(d-1/2)p(1/\pi-1/p)} < 4 \Leftrightarrow s(1/p-1/4) > (d-1/2)(1/\pi-1/p)$. Hence, we only need to impose $s(1/p-1/4) > (d-1/2)(1/\pi-1/p)$, describing domain I

$$\{(s-(d+1)(1/\pi-1/p) > 0) \cap (s > 0)\} \cap \{s(1/p-1/4) > (d-1/2)(1/\pi-1/p)\}$$

on which $I_j(f, q, p) \leq C \rho_j^{\frac{p(d-1/2)}{s+d-1/2}}$.

Case $\frac{\pi(p-q)}{(\pi-q)} > 4$

Using (19), (20) and (21), we have

$$I_j(f, q, p) \leq C 2^{j\nu(p-q)} \rho_j^q 2^{-jsq} 2^{jd(p-q)/2} 2^{j(\frac{p-q}{2}-2\frac{\pi-q}{\pi})}.$$

If we put $(p-q)d - sq - 2\frac{\pi-q}{\pi} = 0 \Leftrightarrow q = \frac{pd-2}{s+d-2/\pi}$, we have $p-q = \frac{s-2(1/\pi-1/p)}{s+d-2/\pi} p >$

$0 \Leftrightarrow s - 2(1/\pi-1/p) > 0$ and $\pi-q = \frac{s-dp(1/\pi-1/p)}{s+d-2/\pi} \pi > 0 \Leftrightarrow s - dp(1/\pi-1/p) > 0$.

Moreover $\frac{\pi(p-q)}{(\pi-q)} = \frac{s-2(1/\pi-1/p)}{s-dp(1/\pi-1/p)} p > 4 \Leftrightarrow s(1/p-1/4) < (d-1/2)(1/\pi-1/p)$. Hence, on the domain

$$\{(0 < s) \cap (s > (d+1)(1/\pi-1/p)) \cap (s > dp(1/\pi-1/p)) \cap (s(1/p-1/4) < (d-1/2)(1/\pi-1/p))\},$$

we have $I_j(f, q, p) \leq C \rho_j^{\frac{pd-2}{s+d-2/\pi}}$.

Case $\frac{\pi(p-q)}{(\pi-q)} = 4$ Using (19), (20) and (21), we have

$$I_j(f, q, p) \leq C 2^{j\nu(p-q)} \rho_j^q 2^{-jsq} j^{1-q/\pi} 2^{jd(p-q)/2} \leq C \rho_j^q j^{1-q/\pi}$$

if $(p-q)(d-1/2) - sq = 0 \Leftrightarrow q = p \frac{d-1/2}{s+d-1/2}$. This is realized either if $p = 4 = \pi$ and for $s > 0$. or if $p \neq 4, \pi \neq 4$ and $0 < q = \pi \frac{p-4}{\pi-4} = p \frac{d-1/2}{s+d-1/2}$. Moreover $q < \pi$ and $q < p \Leftrightarrow 4 < p < \pi$; or $4 < \pi < p$ and $\frac{\pi(p-q)}{(\pi-q)} = 4 \Leftrightarrow s(1/p - 1/4) = (d-1/2)(1/\pi - 1/p)$. Hence on the domain

$$\{(s, \pi), \quad (s(1/p - 1/4) = (d-1/2)(1/\pi - 1/p)) \cap (s > 0) \cap (s > (d+1)(1/\pi - 1/p))\}$$

we have $I_j(f, q, p) \leq C \rho_j^{\frac{p(d-1/2)}{s+d-1/2}} j^{\frac{s}{s+d-1/2}}$.

8 Proof of the lower bounds

In this section we prove the lower bounds. i.e. for $0 < s < \infty$, $1 \leq \pi \leq \infty$, $0 < r \leq \infty$, $0 < M < \infty$ and $B_{\pi, r}^s(M)$ the ball of radius M of the space $B_{\pi, r}^s$, and \mathcal{E} is the set of all estimators we consider

$$\begin{aligned} \omega_p(s, \pi, r, M, \varepsilon) &= \inf_{f^* \in \mathcal{E}} \sup_{f \in B_{\pi, r}^s(M)} \mathbb{E} \|f^* - f\|_p^p \\ \omega_\infty(s, \pi, r, M, \varepsilon) &= \inf_{f^* \in \mathcal{E}} \sup_{f \in B_{\pi, r}^s(M)} \mathbb{E} \|f^* - f\|_\infty. \end{aligned}$$

The main tool will be the classical lemma introduced by Fano in 1952 [8]. We will use the version of Fano's lemma introduced in [4]. Let us recall that $K(P, Q)$ denotes the Kullback information 'distance' between P and Q .

Lemma 3. Let \mathcal{A} be a sigma algebra on the space Ω , and $A_i \in \mathcal{A}$, $i \in \{0, 1, \dots, m\}$ such that $\forall i \neq j, A_i \cap A_j = \emptyset$, P_i $i \in \{0, 1, \dots, m\}$ be $m+1$ probability measures on (Ω, \mathcal{A}) . Then if

$$\begin{aligned} p &:= \sup_{i=0}^m P_i(A_i^c), \quad \kappa := \inf_{j \in \{0, 1, \dots, m\}} \frac{1}{m} \sum_{i \neq j} K(P_i, P_j), \\ p &\geq \frac{1}{2} \wedge (C \sqrt{m} \exp(-\kappa)), \quad C = \exp\left(-\frac{3}{e}\right) \end{aligned} \tag{25}$$

This inequality will be used, in the following way: Let H_ε be the Hilbert space of measurable functions on $Z = \mathbb{S}^{d-1} \times [-1, 1]$ with the scalar product

$$\langle \varphi, \psi \rangle_\varepsilon = \varepsilon^2 \int_{\mathbb{S}^{d-1}} \int_{-1}^1 \varphi(\theta, s) \psi(\theta, s) d\sigma(\theta) \frac{ds}{(1-s^2)^{(d-1)/2}}$$

It is well known that there exists a (unique) probability on $(\Omega, \mathcal{A}) : Q_f$, which density with respect to P is

$$\frac{dQ_f}{dP} = \exp(W^\varepsilon(f) - \frac{1}{2}\|f\|_{H_\varepsilon}^2).$$

Let us now choose f_0, f_1, \dots, f_m in $B_{\pi, r}^s(M)$ such that $i \neq j \implies \|f_i - f_j\|_p \geq \delta$ and denote $P_i = Q_{R(\frac{f_i}{\varepsilon^2})}$. Let f^* be an arbitrary estimator of f . obviously the sets $A_i = (\|f^* - f\| < \frac{\delta}{2})$ are disjoint sets and we have, for $i \neq j$,

$$K(P_i, P_j) = \frac{1}{2\varepsilon^2} \int_Z |R(f_i - f_j)|^2 d\mu.$$

Now

$$\omega_p(s, \pi, q, M, \varepsilon) \geq \inf_{f^* \in \mathcal{E}} \sup_{f_i, i=0,1..m} \mathbb{E} \|f^* - f_i\|_p^p \geq \left(\frac{\delta}{2}\right)^p \inf_{f^* \in \mathcal{E}} \sup_{f_i, i=0,1..m} P \left(\|f^* - f_i\|_p \geq \frac{\delta}{2} \right).$$

Likewise,

$$\omega_\infty(s, \pi, q, M, \varepsilon) \geq \left(\frac{\delta}{2}\right) \inf_{f^* \in \mathcal{E}} \sup_{f_i, i=0,1..m} P \left(\|f^* - f_i\|_\infty \geq \frac{\delta}{2} \right).$$

Using the previous Fano's lemma

$$\sup_{f_i, i=0,1..m} P \left(\|f^* - f_i\|_p \geq \frac{\delta}{2} \right) \geq \frac{1}{2} \wedge (C\sqrt{m} \exp(-\kappa))$$

with

$$\kappa = \inf_{j=0, \dots, M} \frac{1}{m} \sum_{i \neq j} \frac{1}{2\varepsilon^2} \int_Z |R(f_i - f_j)|^2 d\mu.$$

So if for a given ε we can find f_0, f_1, \dots, f_m in $B_{\pi, r}^{s,0}(M)$ such that $i \neq j \implies \|f_i - f_j\|_p \geq \delta(\varepsilon)$ and $C\sqrt{m} \exp(-\kappa) \geq 1/2$ then we have

$$\text{for } p < \infty, \quad \omega_p(s, \pi, q, M, \varepsilon) \geq \frac{1}{2} \delta(\varepsilon)^p, \text{ and } \omega_\infty(s, \pi, q, M, \varepsilon) \geq \frac{1}{2} \delta(\varepsilon)$$

In the sequel we will choose, as usual, sets of functions containing either 2 items (sparse case) or a number of order 2^{jd} or $2^{j(d-1)}$ (dense cases). We will consider sets of functions which are basically linear combinations of needlets at a fixed level $f = \sum_{\xi \in \chi_j} \beta_{j,\xi} \psi_{j,\xi}$. Because the needlets have different order of norms depending whether they are around the north pole or closer to the equator, we will have to investigate different cases. These differences will precisely yield the different minimax rates.

8.1 Reverse inequality

Because the needlets are not forming an orthonormal system, we cannot pretend that inequality (18) is an equivalence. Since precisely in the lower bounds evaluations we need to bound both sides of L_p norm for terms of the form $\sum_{\xi \in A_j} \lambda_\xi \psi_{j,\xi}$, with $A_j \subset \chi_j$, the following subsection is devoted to this problem.

Proposition 2. Let us $A_j \subset \chi_j$. Then:

$$\frac{1}{C} \left(\sum_{\xi' \in A_j} \left| \left\langle \sum_{\xi \in A_j} \lambda_\xi \psi_{j,\xi}, \psi_{j,\xi'} \right\rangle \right|^p \|\psi_{j,\xi'}\|_p^p \right)^{1/p} \leq \left\| \sum_{\xi \in A_j} \lambda_\xi \psi_{j,\xi} \right\|_p \leq C \left(\sum_{\xi \in A_j} |\lambda_\xi|^p \|\psi_{j,\xi}\|_p^p \right)^{1/p}$$

Proof

Let $f = \sum_{\xi \in A_j} \lambda_\xi \psi_{j,\xi}$. Clearly, by (18),

$$\left\| \sum_{\xi \in A_j} \lambda_\xi \psi_{j,\xi} \right\|_p \leq C \left(\sum_{\xi \in A_j} |\lambda_\xi|^p \|\psi_{j,\xi}\|_p^p \right)^{1/p}$$

and by (17)

$$\left(\sum_{\xi' \in \chi_j} \left| \left\langle \sum_{\xi \in A_j} \lambda_\xi \psi_{j,\xi}, \psi_{j,\xi'} \right\rangle \right|^p \|\psi_{j,\xi'}\|_p^p \right)^{1/p} \leq C \left\| \sum_{\xi \in A_j} \lambda_\xi \psi_{j,\xi} \right\|_p$$

so obviously

$$\frac{1}{C} \left(\sum_{\xi' \in A_j} \left| \left\langle \sum_{\xi \in A_j} \lambda_\xi \psi_{j,\xi}, \psi_{j,\xi'} \right\rangle \right|^p \|\psi_{j,\xi'}\|_p^p \right)^{1/p} \leq \left\| \sum_{\xi \in A_j} \lambda_\xi \psi_{j,\xi} \right\|_p \leq C \left(\sum_{\xi \in A_j} |\lambda_\xi|^p \|\psi_{j,\xi}\|_p^p \right)^{1/p} .$$

□

In the sequel we will look for subset A_j with the following property: There exists

$$0 < D_j, \quad \text{such that} \quad \forall \xi \in A_j, \quad \|\psi_{j,\xi}\|_p \sim D_j.$$

(Here and in all this section, $a_{j,\xi} \sim b_j$ will mean that there exists two absolute constants c_1 and c_2 -which will not be precised for a sake of simplicity- such that $c_1 b_j \leq a_{j,\xi} \leq c_2 b_j$, for all considered ξ .) As precised above, D_j may have different forms depending on the regions: using (19), we have

$$\|\psi_{j,\xi}\|_p \sim \left(\frac{2^{jd}}{(2^{-j} + \sqrt{1 - |\xi|^2})} \right)^{1/2-1/p} .$$

For this purpose, let us precise Proposition 1 by choosing the cubature points in the following way: we choose in the hemisphere \mathbb{S}_+^d strips $S_k = B(A, (2k+1)\eta) \setminus B(A, 2k\eta)$ with $\eta \sim \frac{\pi}{22^{j+1}}$, $k \in \{0, \dots, 2^j - 1\}$ (A is the north pole). In each of these strips, we choose a maximal η -net, of points $\tilde{\xi}$ which cardinal is of order k^{d-1} . Projecting these points on the ball we obtain cubature points ξ on the ball with coefficients $\omega_{j,\xi} \sim 2^{-jd} W_j(\xi)$. As a consequence, we have in the set $\{x \in \mathbb{R}^d, |x| \leq \frac{1}{\sqrt{2}}\}$, about 2^{jd} points of cubature for which

$$D_j \sim \|\psi_{j,\xi}\|_p \sim 2^{jd(1/2-1/p)}.$$

And in the corona $\{(1 - 2^{-2j} \leq |x| \leq 1\}$, we have about $2^{j(d-1)}$ points of cubature for which

$$D_j \sim \|\psi_{j,\xi}\|_p \sim 2^{j(d+1)(1/2-1/p)}.$$

Now let us consider a set A_j of cubature points included in one of the two sets considered just above (either $\{x \in \mathbb{R}^d, |x| \leq \frac{1}{\sqrt{2}}\}$ or $\{(1 - 2^{-2j} \leq |x| \leq 1)\}$, and consider the matrix (parametrized by A_j)

$$\mathbb{M}(A_j) = (\langle \psi_{j,\xi}, \psi_{j,\xi'} \rangle)_{\xi, \xi' \in A_j \times A_j}$$

we have for any $\lambda \in \mathfrak{l}_p(A_j)$, using Proposition 2

$$\|\mathbb{M}(A_j)(\lambda)\|_{\mathfrak{l}_p(A_j)} \leq C' \|\lambda\|_{\mathfrak{l}_p(A_j)}$$

On the other way, let us observe that using (14)

$$0 < c \leq \|\psi_{j,\xi}\|_2^2 = \langle \psi_{j,\xi}, \psi_{j,\xi} \rangle \leq 1$$

so

$$\mathbb{M}(A_j) = \text{Diag}(\mathbb{M}(A_j)) + \mathbb{M}'(A_j) = \text{Diag}(\mathbb{M}(A_j))(\text{Id} + [\text{Diag}(\mathbb{M}(A_j))]^{-1}\mathbb{M}'(A_j))$$

where $\text{Diag}(\mathbb{M}(A_j))$ is the diagonal matrix parametrized by A_j extracted from $\mathbb{M}(A_j)$. Clearly, each terms of $[\text{Diag}(\mathbb{M}(A_j))]^{-1}$ is bounded by c^{-1} .

So if $\|[\text{Diag}(\mathbb{M}(A_j))]^{-1}\mathbb{M}'(A_j)\|_{\mathcal{L}(\mathfrak{l}_p(A_j))} \leq \alpha < 1$, we have

$$\|\mathbb{M}(A_j)^{-1}\|_{\mathcal{L}(\mathfrak{l}_p(A_j))} \leq c^{-1} \frac{1}{1 - \alpha}.$$

Let us prove that we can chose A_j large enough and such that such an α exists. By Schur lemma

$$\|[\text{Diag}(\mathbb{M}(A_j))]^{-1}\mathbb{M}'(A_j)\|_{\mathcal{L}(\mathfrak{l}_p(A_j))} \leq c^{-1} \sup_{\xi \in A_j} \sum_{\xi' \neq \xi, \xi' \in A_j} |\langle \psi_{j,\xi}, \psi_{j,\xi'} \rangle|$$

Now using (16),

$$|\langle \psi_{j,\xi}, \psi_{j,\xi'} \rangle| \leq C_M^2 \int_{\mathbb{B}^d} \frac{1}{\sqrt{W_j(x)}} \frac{1}{(1 + 2^j d(x, \xi))^M} \frac{1}{\sqrt{W_j(x)}} \frac{1}{(1 + 2^j d(x, \xi'))^M} dx$$

By triangular inequality

$$|\langle \psi_{j,\xi}, \psi_{j,\xi'} \rangle| \leq \frac{C_M^2}{(1 + 2^j d(\xi, \xi'))^M} \int_{\mathbb{B}^d} \frac{1}{2^{-j} + \sqrt{1 - |x|^2}} dx \leq C_M^2 \frac{1}{(1 + 2^j d(\xi, \xi'))^M} |\mathbb{S}^{d-1}| \int_0^1 r^{d-2} dr$$

So

$$\forall M, \quad |\langle \psi_{j,\xi}, \psi_{j,\xi'} \rangle| \leq C'_M \frac{1}{(1 + 2^j d(\xi, \xi'))^M} \quad (26)$$

Now, let us choose A_j as a maximal $K\eta$ net in the set $\chi_j \cap \{x \in \mathbb{R}^d, |x| \leq \frac{1}{\sqrt{2}}\}$ (case 1) or as a maximal $K\eta$ net in the set $\chi_j \cap \{(1 - 2^{-2j} \leq |x| \leq 1)\}$ (case 2). Recall that $\eta \sim \frac{\pi}{22^{j+1}}$ and K will be chosen later.

As, in case 1,

$$\text{Card}\{\xi', \quad d(\xi', \xi) \sim Kl2^{-j}\} \lesssim (Kl)^d$$

$$\sum_{\xi' \neq \xi, \xi' \in A_j} |\langle \psi_{j,\xi}, \psi_{j,\xi'} \rangle| \leq \sum_{l=1}^{\frac{2^j}{K}} (Kl)^d C_M \frac{1}{(1+Kl)^M} \leq C_M \sum_{l=1}^{\frac{2^j}{K}} (Kl)^d \frac{1}{(Kl)^M} \leq \frac{2C_M}{K^{M-d}} \leq \alpha$$

if $M - d \geq 2$, and K is large enough. In case 2, again

$$\text{Card}\{\xi', \quad d(\xi', \xi) \sim Kl2^{-j}\} \lesssim (Kl)^{d-1}$$

$$\sum_{\xi' \neq \xi, \xi' \in A_j} |\langle \psi_{j,\xi}, \psi_{j,\xi'} \rangle| \leq \sum_{l=1}^{\frac{2^j}{K}} (Kl)^{d-1} C_M \frac{1}{(1+Kl)^M} \leq C_M \sum_{l=1}^{\frac{2^j}{K}} (Kl)^{d-1} \frac{1}{(Kl)^M} \leq \frac{2C_M}{K^{M-d+1}} \leq \alpha$$

if $M - d \geq 1$, and K is large enough.

Hence $M(A_j)$ is invertible in both cases and we have:

$$\begin{aligned} c^{-1} \frac{1}{1-\alpha} \left(\sum_{\xi \in A_j} |\lambda_\xi|^p \right)^{1/p} &\leq \left(\sum_{\xi' \in A_j} \left| \sum_{\xi \in A_j} \lambda_\xi \langle \psi_{j,\xi}, \psi_{j,\xi'} \rangle \right|^p \right)^{1/p} \\ \left(\sum_{\xi \in A_j} |\lambda_\xi|^p \|\psi_{j,\xi}\|_p^p \right)^{1/p} &\lesssim \left\| \sum_{\xi \in A_j} \lambda_\xi \psi_{j,\xi} \right\|_p \lesssim \left(\sum_{\xi \in A_j} |\lambda_\xi|^p \|\psi_{j,\xi}\|_p^p \right)^{1/p} \end{aligned}$$

8.2 Lower bounds associated sparse/dense cases and different choices of A_j

Let j be fixed, and choose

$$f = \sum_{\xi \in A_j} \beta_{j,\xi} \psi_{j,\xi}.$$

We have

$$f = \sum_{2^{j-1} < k < 2^{j+1}} P_k(f)$$

where P_k is the orthogonal projector on $\mathcal{V}_k(\mathbb{B}^d)$. So

$$\begin{aligned} \|\mathbf{R}(f)\|^2 &= \langle \mathbf{R}^* \mathbf{R}(f), f \rangle = \sum_{2^{j-1} < k < 2^{j+1}} \langle \lambda_k^2 P_k(f), f \rangle \\ &\leq \left(\sup_{2^{j-1} < k < 2^{j+1}} \lambda_k^2 \right) \sum_k \|P_k(f)\|^2 \leq C 2^{-j(d-1)} \sum_{\xi \in A_j} |\beta_{j,\xi}|^2 \end{aligned}$$

8.2.1 Sparse choice, case 1

Let $f_i = \gamma \varepsilon_i \psi_{j, \varepsilon_i}$, $i \in \{1, 2\}$, ε_i is $+1$ or -1 , in such a way that

$$\|f_1 - f_2\|_p = \|\gamma \psi_{j, \varepsilon_1} - \gamma \psi_{j, \varepsilon_2}\|_p = \gamma \|\psi_{j, \varepsilon_1} - \psi_{j, \varepsilon_2}\|_p \sim \gamma (\|\psi_{j, \varepsilon_1}\| + \|\psi_{j, \varepsilon_2}\|_p)$$

In case 1, $\|\psi_{j, \varepsilon}\|_r \sim 2^{jd(1/2-1/r)}$. So

$$f_i \in B_{\pi, r}^s(1) \iff \gamma 2^{jd(1/2-1/\pi)} \sim 2^{-js} \iff \gamma \sim 2^{-j(s+d(1/2-1/\pi))}$$

$$\delta = \|f_1 - f_2\|_p \sim \gamma 2^{jd(1/2-1/p)} \sim 2^{-j(s+d(1/2-1/\pi)-d(1/2-1/p))} = 2^{-j(s-d(1/\pi-1/p))}$$

On the other hand

$$K(P_1, P_2) = \frac{1}{2} \frac{1}{\varepsilon^2} 2^{-j(d-1)} \gamma^2 \sim \frac{1}{2} \frac{1}{\varepsilon^2} 2^{-j(d-1)} 2^{-2j(s+d(1/2-1/\pi))} = \frac{1}{2} \frac{1}{\varepsilon^2} 2^{-2j(s+d-1/2-d/\pi)}$$

Now by Fano inequality if j is chosen so that $\varepsilon \sim 2^{-j(s+d-1/2-d/\pi)}$

(under the constraint $s > d(1/\pi - (1 - 1/2d))$)

$$\left(\frac{2}{\delta}\right)^p \mathbb{E} \|f^* - f_i\|_p^p \geq P(\|f^* - f_i\|_p > \delta/2) \geq c$$

So necessarily

$$\mathbb{E} \|f^* - f_i\|_p^p \geq c \delta^p \sim \varepsilon^{\frac{s-d(1/\pi-1/p)}{s+d-1/2-d/\pi}}$$

Remark 2. If

$$d(1/\pi - (1 - \frac{1}{2d})) < s \leq d(1/\pi - 1/p)$$

so necessarily $\frac{1}{p} \leq 1 - \frac{1}{2d}$, then

$$\lim_{\varepsilon \rightarrow 0} \omega_p(s, \pi, q, M, \varepsilon) \geq C > 0$$

8.2.2 Sparse choice, case 2

In case 2, $\|\psi_{j, \varepsilon}\|_r \sim 2^{j(d+1)(1/2-1/r)}$, so

$$f_i \in B_{\pi, r}^s(1) \iff \gamma 2^{j(d+1)(1/2-1/\pi)} \sim 2^{-js} \iff \gamma \sim 2^{-j(s+(d+1)(1/2-1/\pi))}$$

$$\delta = \|f_1 - f_2\|_p \sim \gamma 2^{j(d+1)(1/2-1/p)} \sim 2^{-j(s+(d+1)(1/2-1/\pi)-(d+1)(1/2-1/p))} = 2^{-j(s-(d+1)(1/\pi-1/p))}$$

On the other hand

$$K(P_1, P_2) = \frac{1}{2} \frac{1}{\varepsilon^2} 2^{-j(d-1)} \gamma^2 = \frac{1}{2} \frac{1}{\varepsilon^2} 2^{-j(d-1)} 2^{-2j(s+(d+1)(1/2-1/\pi))} \sim \frac{1}{2} \frac{1}{\varepsilon^2} 2^{-2j(s+d-(d+1)/\pi)}$$

Now by Fano inequality if $\varepsilon \sim 2^{-j(s+d-(d+1)/\pi)}$

(under the constraint $s > (d+1)(\frac{1}{\pi} - \frac{d}{d+1})$)

$$\left(\frac{2}{\delta}\right)^p \mathbb{E} \|f^* - f_i\|_p^p \geq P(\|f^* - f_i\|_p > \delta/2) \geq c$$

So necessarily

$$\mathbb{E} \|f^* - f_i\|_p^p \geq c \delta^p \sim \varepsilon^{\frac{(s-(d+1)(1/\pi-1/p))p}{s+d-(d+1)/\pi}}$$

Remark 3. If

$$(d+1)(1/\pi - \frac{d}{d+1}) < s \leq (d+1)(1/\pi - 1/p)$$

(so necessarily $\frac{1}{p} < \frac{d}{d+1}$),

$$\lim_{\varepsilon \rightarrow 0} \omega_p(s, \pi, q, M, \varepsilon) \geq C > 0$$

8.2.3 Dense choice, case 1

In this case we take now

$$f_\rho = \gamma \sum_{\xi \in \mathcal{A}_j} \varepsilon_\xi \psi_{j,\xi}, \quad \varepsilon_\xi = \pm 1, \quad \rho = (\varepsilon_\xi)_{\xi \in \mathcal{A}_j}$$

As we are in case 1, we have

$$\gamma^r \left\| \sum_{\xi \in \mathcal{A}_j} \varepsilon_\xi \psi_{j,\xi} \right\|_r^r \sim \gamma^r 2^{jd(r/2-1)} \sum_{\xi \in \mathcal{A}_j} |\varepsilon_\xi|^r \sim \gamma^r 2^{jd(r/2-1)} \text{Card}(\mathcal{A}_j) \sim \gamma^r 2^{jd r/2}$$

Using Varshamov-Gilbert theorem, we consider a subset \mathcal{A} of $\{-1, +1\}^{\mathcal{A}_j}$ such that $\text{Card}(\mathcal{A}) \sim 2^{\frac{1}{8} \text{Card}(\mathcal{A}_j)}$ and for $\rho \neq \rho'$, $\rho, \rho' \in \mathcal{A}$, $\|\rho - \rho'\|_1 \geq \frac{1}{2} \text{Card}(\mathcal{A}_j)$. Let us now restrict our set to

$$f_\rho = \gamma \sum_{\xi \in \mathcal{A}_j} \varepsilon_\xi \psi_{j,\xi}, \quad \varepsilon_\xi = \pm 1, \quad \rho = (\varepsilon_\xi)_{\xi \in \mathcal{A}_j}, \quad \rho \in \mathcal{A}.$$

$$f_\rho \in B_{\pi, r}^s(1) \iff \gamma \left(\sum_{\xi \in \mathcal{A}_j} \|\psi_{j,\xi}\|_\pi^\pi \right)^{1/\pi} \sim \gamma 2^{jd/2} \sim 2^{-js}$$

So we choose

$$\gamma \sim 2^{-j(s+d/2)}.$$

Moreover

$$\begin{aligned} \delta = \|f_\rho - f_{\rho'}\|_p &= \gamma \left\| \sum_{\xi \in \mathcal{A}_j} (\varepsilon_\xi - \varepsilon'_\xi) \psi_{j,\xi} \right\|_p \sim \gamma \left(\sum_{\xi \in \mathcal{A}_j} |\varepsilon_\xi - \varepsilon'_\xi|^p \|\psi_{j,\xi}\|_p^p \right)^{1/p} \\ &\sim \gamma 2^{jd(1/2-1/p)} \|\rho - \rho'\|_1^{1/p} \sim 2^{-j(s+d/2)} 2^{jd/2} = 2^{-js} \end{aligned}$$

Let us compute the Kullback distance:

$$K(P_\rho, P_{\rho'}) = \frac{1}{2\varepsilon^2} 2^{-j(d-1)} \|f_\rho - f_{\rho'}\|_2^2 \sim \frac{1}{2\varepsilon^2} 2^{-j(d-1)} 2^{-2j(s+d/2)} 2^{jd} = \frac{1}{2\varepsilon^2} 2^{-2j(s+d/2-1/2)}$$

so by Fano inequality

$$\frac{\mathbb{E} \|\hat{f} - f\|_p^p}{\delta^p} \geq 1/2 \wedge c 2^{\frac{1}{8} 2^{jd}} e^{-\frac{1}{2\varepsilon^2} 2^{-2j(s+d/2-1/2)}} \geq 1/2$$

if

$$\varepsilon \sim 2^{-j(s+d-1/2)}$$

So

$$\inf_{f \in \mathbb{B}_{\pi, r}^{s, 0}} \mathbb{E} \|\hat{f} - f\|_p^p \geq c \varepsilon^{\frac{sp}{s+d-1/2}}$$

8.2.4 Dense choice, case 2

Similarly to the previous case, we take now (with a slight abuse of notation since the subset \mathcal{A} obtained using Varshamov-Gilbert theorem is not the same \mathcal{A} , as \mathcal{A}_j has also changed)

$$f_\rho = \gamma \sum_{\xi \in \mathcal{A}_j} \varepsilon_\xi \psi_{j, \xi}, \quad \varepsilon_\xi = \pm 1, \quad \rho = (\varepsilon_\xi)_{\xi \in \mathcal{A}_j}, \quad \rho \in \mathcal{A}.$$

As we are in case 2, we have

$$\gamma^r \left\| \sum_{\xi \in \mathcal{A}_j} \varepsilon_\xi \psi_{j, \xi} \right\|_r^r \sim \gamma^r 2^{j(d+1)(r/2-1)} \sum_{\xi \in \mathcal{A}_j} |\varepsilon_\xi|^r \sim \gamma^r 2^{j(d+1)(r/2-1)} \text{Card}(\mathcal{A}_j) \sim \gamma^r 2^{j[(d+1)r/2-2]}$$

$$f_\rho \in \mathbb{B}_{\pi, r}^s(1) \iff \gamma \left(\sum_{\xi \in \mathcal{A}_j} \|\psi_{j, \xi}\|_\pi^\pi \right)^{1/\pi} \sim \gamma \left(2^{j(d-1)} 2^{j(d+1)(\pi/2-1)} \right)^{1/\pi} \sim \gamma 2^{-j(\frac{d+1}{2} - \frac{2}{\pi})} \sim 2^{-js}.$$

So we choose

$$\gamma \sim 2^{-j(s + \frac{d+1}{2} - \frac{2}{\pi})}.$$

Moreover

$$\begin{aligned} \delta &= \|f_\rho - f_{\rho'}\|_p = \gamma \left\| \sum_{\xi \in \mathcal{A}_j} (\varepsilon_\xi - \varepsilon'_\xi) \psi_{j, \xi} \right\|_p \sim \gamma \left(\sum_{\xi \in \mathcal{A}_j} |\varepsilon_\xi - \varepsilon'_\xi|^p \|\psi_{j, \xi}\|_p^p \right)^{1/p} \\ &\sim \gamma 2^{j(d+1)(1/2-1/p)} \|\rho - \rho'\|_1^{1/p} \sim \gamma 2^{j(d+1)(1/2-1/p)} 2^{j(d-1)\frac{1}{p}} \\ &\sim 2^{-j(s + \frac{d+1}{2} - \frac{2}{\pi})} 2^{j(\frac{d+1}{2} - \frac{2}{p})} = 2^{-j(s - 2(1/\pi - 1/p))} \end{aligned}$$

Let us compute the Kullback distance:

$$K(P_\rho, P_{\rho'}) = \frac{1}{2\varepsilon^2} 2^{-j(d-1)} \|f_\rho - f_{\rho'}\|_2^2 \sim \frac{1}{2\varepsilon^2} 2^{-j(d-1)} 2^{-2j(s + \frac{d+1}{2} - \frac{2}{\pi})} 2^{j(d-1)} = \frac{1}{2\varepsilon^2} 2^{-2j(s + \frac{d+1}{2} - \frac{2}{\pi})}$$

so by Fano inequality

$$\frac{\mathbb{E} \|\hat{f} - f\|_p^p}{\delta^p} \geq 1/2 \wedge c 2^{\frac{1}{8} 2^{j(d-1)}} e^{-\frac{1}{2\varepsilon^2} 2^{-2j(s + \frac{d+1}{2} - \frac{2}{\pi})}} \geq 1/2$$

if

$$\varepsilon \sim 2^{-j(s+d-2/\pi)}$$

So

$$\inf_{f \in \mathbb{B}_{\pi, r}^s(1)} \mathbb{E} \|\hat{f} - f\|_p^p \geq c \varepsilon^{\frac{p(s-2(1/\pi-1/p))}{s+d-2/\pi}}.$$

Remark 4. The case $p = \infty$ can be treated using the same arguments, without difficulties.

References

- [1] G. Andew, R. Askey, and R. Roy. *Special Functions*. Cambridge University Press, 2006.
- [2] P. Baldi, G. Kerkyacharian, D. Marinucci, and D. Picard. Adaptive density estimation for directional data using needlets, 2008.
- [3] P. Baldi, G. Kerkyacharian, D. Marinucci, and D. Picard. Asymptotics for spherical needlets. *Ann. Statist.*, 37(3):1150–1171, 2009.
- [4] L. Birge. A new look at an old result : Fano’s lemma. Prepublication 632, LPMA., 2001.
- [5] M. Davison. A singular value decomposition for the radon transform in n-dimensional euclidian space. *Numer. Func. Anal. and Optimiz.*, 3:321–340, 1981.
- [6] C. Dunkl and Y. Xu. *Orthogonal polynomials of several variables*, volume 81 of Cambridge University Press. Cambridge University Press, New York, 2001.
- [7] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Higher Transcendental Functions*, volume 2. McGraw-Hill, New York, 1953.
- [8] R. Fano. *Class notes for transmission of information*, course. 6.574. MIT, Cambridge, MA, 1952.
- [9] S. Helgason. *The Radon transform*. Birkhäuser, Basel and Boston, 1980.
- [10] G. Kerkyacharian, G. Kyriazis, E. Le Pennec, P. Petrushev, and D. Picard. Inversion of noisy radon transform by svd based needlets. *ACHA*, 2009.
- [11] G Kerkyacharian and D. Picard. New generation wavelets associated with statistical problems. In *The 8th Workshop on Stochastic Numerics*, pages 119–146. Research Institute for Mathematical Sciences, Kyoto University, Jan. 2009.
- [12] G. Kyriazis, P. Petrushev, and Y. Xu. Decomposition of weighted triebel-lizorkin and besov spaces on the ball. *Proc. London Math. Soc.*, 2008. to appear (arXiv:math/0703403).
- [13] B. Logan and L. Shepp. Optimal reconstruction of a function from its projections. *Duke Math. J.*, 42(4):645–659, 1975.
- [14] A. Louis. Orthogonal function series expansions and the null space of the radon transform. *SIAM J. Math. Anal.*, 15:621–633, 1984.
- [15] F. Narcowich, P. Petrushev, and J. Ward. Local tight frames on spheres. *SIAM J. Math. Anal.*, 2006.

- [16] F. J. Narcowich, P. Petrushev, and J.M. Ward. Decomposition of besov and triebel-lizorkin spaces on the sphere. *J. Funct. Anal.*, 2006.
- [17] F. Natterer. The mathematics of computerized tomography. *Classical in Applied mathematics*, Philadelphia, PA, 2001.
- [18] P. Petrushev. Approximation by ridge functions and neural networks. *SIAM J. Math. Anal.*, 30:155–189, 1999.
- [19] P. Petrushev and Y. Xu. Localized polynomial frames on the interval with Jacobi weights. *J. Fourier Anal. Appl.*, 11(5):557–575, 2005.
- [20] P. Petrushev and Y. Xu. Localized polynomials frames on the ball. *Constr. Approx.*, 27:121–148, 2008.
- [21] E. Stein and G. Weiss. *Introduction to Fourier Analysis on Euclidian spaces*. Princeton University Press, 1971.
- [22] G. Szegő. *Orthogonal polynomials*. American Mathematical Society, Providence, R.I., 1975.
- [23] Y. Xu. Orthogonal polynomials and cubature formulae on spheres and on balls. *SIAM J. Math. Anal.*, 29(3):779–793 (electronic), 1998.
- [24] Y. Xu. Reconstruction from radon projections and orthogonal expansion on ball. 2007. (arXiv: 0705.1984v1).