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CALDERÓN INVERSE PROBLEM WITH PARTIAL DATA ON RIEMANN SURFACES

COLIN GUILLARMOU AND LEO TZOU

ABSTRACT. On a fixed smooth compact Riemann surface with boundary (M_0, g) , we show that for the Schrödinger operator $\Delta + V$ with potential $V \in W^{1,\infty}(M_0)$, the Dirichlet-to-Neumann map $\mathcal{N}|_\Gamma$ measured on an open set $\Gamma \subset \partial M_0$ determines uniquely the potential V . We also discuss briefly the corresponding consequences for potential scattering at 0 frequency on Riemann surfaces with asymptotically Euclidean or asymptotically hyperbolic ends.

1. INTRODUCTION

The problem of determining the potential in the Schrödinger operator by boundary measurement goes back to Calderón [8]. Mathematically, it amounts to ask if one can detect some data from boundary measurement in a domain (or manifold) Ω with boundary. The typical model to have in mind is the Schrödinger operator $P := \Delta_g + V$ where g is a metric and V a potential, then we define the Cauchy data space by

$$\mathcal{C} := \{(u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}); u \in H^1(\Omega), u \in \ker P\}$$

where ∂_ν is the interior pointing normal vector field to $\partial\Omega$.

The first natural question is the following *full data* inverse problem: does the Cauchy data space determine uniquely the metric g and/or the potential V ? In a sense, the most satisfying known results are when the domain $\Omega \subset \mathbb{R}^n$ is already known and g is the Euclidean metric, then the recovery of V has been proved in dimension $n > 2$ by Sylvester-Uhlmann [28] and very recently in dimension 2 by Bukhgeim [6]. A related question is the conductivity problem which consists in taking $V = 0$ and replacing Δ_g by $-\operatorname{div}\sigma\nabla$ where σ is a definite positive symmetric tensor. An elementary observation shows that the problem of recovering a sufficiently smooth isotropic conductivity (i.e. $\sigma = \sigma_0\operatorname{Id}$ for a function σ_0) is contained in the problem above of recovering a potential V . For domain of \mathbb{R}^2 , Nachman [25] used the $\bar{\partial}$ techniques to show that the Cauchy data space determines the conductivity. Recently a new approach developed by Astala and Päivärinta in [2] improved this result to assuming that the conductivity is only a L^∞ scalar function. This was later generalized to L^∞ anisotropic conductivities by Astala-Lassas-Päivärinta in [3]. We notice that there still are rather few results in the direction of recovering the Riemannian manifold (Ω, g) when $V = 0$, for instance the surface case by Lassas-Uhlmann [21] (see also [4, 15]), the real-analytic manifold case by Lassas-Taylor-Uhlmann [20] (see also [14] for the Einstein case), the case of manifolds admitting limiting Carleman weights and in a same conformal class by Dos Santos Ferreira-Kenig-Salo-Uhlmann [9].

The second natural, but harder, problem is the *partial data* inverse problem: if Γ_1 and Γ_2 are open subsets of $\partial\Omega$, does the partial Cauchy data space for P

$$\mathcal{C}_{\Gamma_1, \Gamma_2} := \{(u|_{\Gamma_1}, \partial_\nu u|_{\Gamma_2}); u \in H^1(\partial M_0), Pu = 0, u = 0 \text{ in } \partial\Omega \setminus \Gamma_1\}$$

determine the domain Ω , the metric, the potential? For a fixed domain of \mathbb{R}^n , the recovery of the potential if $n > 2$ with partial data measurements was initiated by Bukhgeim-Uhlmann [7] and later improved by Kenig-Sjöstrand-Uhlmann [18] to the case where Γ_1 and Γ_2 are respectively open subsets of the "front" and "back" ends of the domain. We refer the reader to the references for a more precise formulation of the problem. In dimension 2, Astala-Lassas-Päiväranta in [3] showed that for the conductivity equation on a simply connected domain, knowledge of Neumann-Dirichlet map in addition to the Dirichlet-Neumann map on an open neighbourhood of the boundary determines the conductivity globally. For the Schrödinger equation, recent works of Imanuvilov-Uhlmann-Yamamoto [16] solves the problem for fixed domains Ω of \mathbb{R}^2 in the case when $\Gamma_1 = \Gamma_2$ and when the potential are in $C^{2+\alpha}(\Omega)$ for some $\alpha > 0$.

In this work, we address the same question when the background domain is a fixed Riemann surface with boundary. We prove the following recovery result:

Theorem 1.1. *Let (M_0, g) be a smooth compact Riemann surface with boundary and let Δ_g be its positive Laplacian. Let $V_1, V_2 \in W^{1,\infty}(M_0)$ be two real potentials and, for $i = 1, 2$, let*

$$(1) \quad \mathcal{C}_i^\Gamma =: \{(u|_\Gamma, \partial_\nu u|_\Gamma); u \in H^1(M_0), (\Delta_g + V_i)u = 0, u = 0 \text{ on } \partial M_0 \setminus \Gamma\}$$

be the respective Cauchy partial data spaces. If $\mathcal{C}_1^\Gamma = \mathcal{C}_2^\Gamma$ then $V_1 = V_2$.

Here the space $W^{1,\infty}(M_0)$ is the Sobolev space with one derivative in L^∞ . Notice that when $\Delta_g + V_i$ do not have L^2 eigenvalues for the Dirichlet condition, the statement above can be given in terms of Dirichlet-to-Neumann operators. Since $\Delta_{\hat{g}} = e^{-2\varphi}\Delta_g$ when $\hat{g} = e^{2\varphi}g$ for some function φ , it is clear that in the statement in Theorem 1.1, we only need to fix the conformal class of g instead of the metric g (or equivalently to fix the complex structure on M_0). Observe also that Theorem 1.1 implies that, for a fixed Riemann surface with boundary (M_0, g) , the Dirichlet-to-Neumann map on Γ for the operator $u \rightarrow -\operatorname{div}_g(\gamma\nabla^g u)$ determines the isotropic conductivity γ if $\gamma \in W^{3,\infty}(M_0)$ in the sense that two conductivities giving rise to the same Dirichlet-to-Neumann are equal. This is a standard observation by transforming the conductivity problem to a potential problem with potential $V := (\Delta_g \gamma^{\frac{1}{2}})/\gamma^{\frac{1}{2}}$. So our result also extends that of Henkin-Michel [15] in the case of isotropic conductivities.

The method to reconstruct the potential follows [6, 16] and is based on the construction of a large set of special complex geometric optic solutions of $(\Delta_g + V)u = 0$, more precisely if $\Gamma_0 = \partial M_0 \setminus \Gamma$ is the set where we do not know the Dirichlet-to-Neumann operator, we construct solutions of the form $u = \operatorname{Re}(e^{\Phi/h}(a + hr_1)) + e^{\varphi/h}r_2(h)$ with $u|_{\Gamma_0} = 0$ where $h > 0$ is a small parameter, Φ and a are holomorphic functions on (M_0, g) with $\operatorname{Re}(\Phi) = \varphi$, r_1 is independent of h while $\|r_2(h)\|_{L^2} = O(h^{3/2})$ as $h \rightarrow 0$. The idea of [6] to reconstruct $V(p)$ for $p \in M_0$ is to take Φ with a non-degenerate critical point at p and then use stationary phase as $h \rightarrow 0$. In our setting, one of our main contribution is the construction of the holomorphic Carleman weights Φ which is quite more complicated since we are working on a Riemann surface instead of a domain of \mathbb{C} . In this exposition, the construction of these

weights involves ideas from symplectic and differential topology and is quite different from the method employed in [16]

A Carleman estimate on the surface for this degenerate weight needs to be proved, and we modify the ideas of [16] to adapt it to surfaces. We also simplify the construction of [16] of the geometric optic solutions. This allows us to relax the regularity assumption and deal with Lipschitz potential. We also note that we provide a proof, in an appendix, of the fact that the partial Cauchy data space \mathcal{C}^Γ determine the potential V on Γ , since it is a needed in the proof of our main theorem.

In Section 6, we obtain two inverse scattering results as corollary of Theorem 1.1: first for partial data scattering at 0 frequency for $\Delta + V$ on asymptotically hyperbolic surfaces with potential decaying at the conformal infinity, and secondly for full data scattering at 0 frequency for $\Delta + V$ with V compactly supported on an asymptotically Euclidean surface.

Another straightforward corollary in the asymptotically Euclidean case full data setting is the recovery of a compactly supported potential from the scattering operator at a positive frequency. The proof is essentially the same as for the operator $\Delta_{\mathbb{R}^n} + V$ once we know Theorem (1.1), so we omit it.

2. HARMONIC AND HOLOMORPHIC MORSE FUNCTIONS ON A RIEMANN SURFACE

2.1. Riemann surfaces. We start by recalling few elementary definitions and results about Riemann surfaces, see for instance [10] for more details. Let (M_0, g_0) be a compact connected smooth Riemannian surface with boundary ∂M_0 . The surface M_0 can be considered as a subset of a compact Riemannian surface (M, g) , for instance by taking the double of M_0 and extending smoothly the metric g_0 to M . The conformal class of g on the closed surface M induces a structure of closed Riemann surface, i.e. a closed surface equipped with a complex structure via holomorphic charts $z_\alpha : U_\alpha \rightarrow \mathbb{C}$. The Hodge star operator \star acts on the cotangent bundle T^*M , its eigenvalues are $\pm i$ and the respective eigenspace $T_{1,0}^*M := \ker(\star + i\text{Id})$ and $T_{0,1}^*M := \ker(\star - i\text{Id})$ are sub-bundle of the complexified cotangent bundle $\mathbb{C}T^*M$ and the splitting $\mathbb{C}T^*M = T_{1,0}^*M \oplus T_{0,1}^*M$ holds as complex vector spaces. Since \star is conformally invariant on 1-forms on M , the complex structure depends only on the conformal class of g . In holomorphic coordinates $z = x + iy$ in a chart U_α , one has $\star(udx + vdy) = -vdx + udy$ and

$$T_{1,0}^*M|_{U_\alpha} \simeq \mathbb{C}dz, \quad T_{0,1}^*M|_{U_\alpha} \simeq \mathbb{C}d\bar{z}$$

where $dz = dx + idy$ and $d\bar{z} = dx - idy$. We define the natural projections induced by the splitting of $\mathbb{C}T^*M$

$$\pi_{1,0} : \mathbb{C}T^*M \rightarrow T_{1,0}^*M, \quad \pi_{0,1} : \mathbb{C}T^*M \rightarrow T_{0,1}^*M.$$

The exterior derivative d defines the De Rham complex $0 \rightarrow \Lambda^0 \rightarrow \Lambda^1 \rightarrow \Lambda^2 \rightarrow 0$ where $\Lambda^k := \Lambda^k T^*M$ denotes the real bundle of k -forms on M . Let us denote $\mathbb{C}\Lambda^k$ the complexification of Λ^k , then the ∂ and $\bar{\partial}$ operators can be defined as differential operators $\partial : \mathbb{C}\Lambda^0 \rightarrow T_{1,0}^*M$ and $\bar{\partial} : \mathbb{C}\Lambda^0 \rightarrow T_{0,1}^*M$ by

$$(2) \quad \partial f := \pi_{1,0}df, \quad \bar{\partial} f := \pi_{0,1}df,$$

they satisfy $d = \partial + \bar{\partial}$ and are expressed in holomorphic coordinates by

$$\partial f = \partial_z f dz, \quad \bar{\partial} f = \partial_{\bar{z}} f d\bar{z}.$$

with $\partial_z := \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)$. Similarly, one can define the ∂ and $\bar{\partial}$ operators from $\mathbb{C}\Lambda^1$ to $\mathbb{C}\Lambda^2$ by setting

$$\partial(\omega_{1,0} + \omega_{0,1}) := d\omega_{0,1}, \quad \bar{\partial}(\omega_{1,0} + \omega_{0,1}) := d\omega_{1,0}$$

if $\omega_{0,1} \in T_{0,1}^*M$ and $\omega_{1,0} \in T_{1,0}^*M$. In coordinates this is simply

$$\partial(udz + vd\bar{z}) = \partial v \wedge d\bar{z}, \quad \bar{\partial}(udz + vd\bar{z}) = \bar{\partial}u \wedge dz.$$

There is a natural operator, the Laplacian acting on functions and defined by

$$\Delta f := -2i \star \bar{\partial}\partial f = d^*d$$

where d^* is the adjoint of d through the metric g and \star is the Hodge star operator mapping Λ^2 to Λ^0 and induced by g as well.

2.2. Maslov Index and Boundary value problem for the $\bar{\partial}$ Operator. In this subsection we consider the setting where M is an oriented Riemann surface with boundary ∂M_0 and $\Gamma \subset \partial M_0$ is an open subset and we let $\Gamma_0 = \partial M_0 \setminus \Gamma$ be its complement in ∂M_0 . Following [22], we adopt the following notations: let $E \rightarrow M_0$ be a complex line bundle with complex structure $J : E \rightarrow E$ and let $D : C^\infty(M_0, E) \rightarrow C^\infty(M_0, T_{0,1}^* \otimes E)$ be a Cauchy-Riemann operator acting on sections of the bundle E . Observe that in the case when $E = M_0 \times \mathbb{C}$ is the trivial line bundle with the natural complex structure on M_0 , then D can be taken as the operator $\bar{\partial}$ introduced in (2). Denote by

$$D_F : W_F^{\ell, \infty}(M_0, E) \rightarrow W^{\ell-1, \infty}(M_0, T_{0,1}^* M_0 \otimes E)$$

where $F \subset E|_{\partial M_0}$ is a totally real subbundle (i.e. a subbundle such that $JF \cap F$ is the zero section) and D_F is the restriction of D to the L^∞ -based Sobolev space

$$W_F^{\ell, \infty}(M_0, E) := \{\xi \in W^{\ell, \infty}(M_0, E) \mid \xi(\partial M_0) \subset F\}$$

In this setting, we have the following boundary value Riemann-Roch theorem stated in [22]:

Theorem 2.1. *Let $E \rightarrow M_0$ be a complex line bundle over an oriented compact Riemann surface with boundary and $F \subset E|_{\partial M_0}$ be a totally real subbundle. Let D be a Cauchy-Riemann operator on E acting on $W^{\ell, \infty}(M_0, E)$, where $\ell \in \mathbb{N}$. Then*

1) *The following operators are Fredholm*

$$D_F : W_F^{\ell, \infty}(M_0, E) \rightarrow W^{\ell-1, \infty}(M_0, T_{0,1}^* M_0 \otimes E)$$

$$D_F^* : W_F^{\ell, \infty}(M_0, T_{0,1}^* M_0 \otimes E) \rightarrow W^{\ell-1, \infty}(M_0, E).$$

2) *The real Fredholm index of D_F is given by*

$$\text{Ind}(D_F) = \chi(M_0) + \mu(E, F)$$

where $\chi(M_0)$ is the Euler characteristic of M_0 and $\mu(E, F)$ is the boundary Maslov index of the subbundle F .

3) *If $\mu(E, F) < 0$, then D_F is injective, while if $\mu(E, F) + 2\chi(M_0) > 0$ the operator D_F is surjective.*

In the case of a trivial bundle $E = M_0 \times \mathbb{C}$ and $\partial M_0 = \sqcup_{j=1}^m \partial_i M_0$ is a disjoint union of m circles, the Maslov index can be defined (see [22, p.554-555]) to be the degree of the map $\rho \circ \Lambda : \partial M_0 \rightarrow \partial M_0$ where

$$\Lambda|_{\partial_i M_0} : S^1 \simeq \partial_i M_0 \rightarrow \text{GL}(1, \mathbb{C})/\text{GL}(1, \mathbb{R})$$

is the natural map given by the totally real subbundle (the space $\mathrm{GL}(1, \mathbb{C})/\mathrm{GL}(1, \mathbb{R})$ being the space of totally real subbundles of \mathbb{C} over a circle) and

$$\rho : \mathrm{GL}(1, \mathbb{C})/\mathrm{GL}(1, \mathbb{R}) \rightarrow S^1, \quad \rho(A \cdot \mathrm{GL}(1, \mathbb{R})) := \det(A^2)/\det(A^*A).$$

As an application, we obtain the following

Corollary 2.2. (i) Let $\omega \in C^\infty(M_0, T_{0,1}^*M_0)$, then there exists $u \in C^\infty(M_0)$ holomorphic on M_0 , real valued on Γ_0 , such that $\bar{\partial}u = \omega$.

(ii) For $k \in \mathbb{N}$, let $f \in W^{k,\infty}(\partial M_0)$ be a real valued function, then there exists a holomorphic function $v \in W^{k,\infty}(M_0)$ such that $\mathrm{Re}(v)|_{\Gamma_0} = f$.

(iii) For $k \in \mathbb{N}$, the space of $W^{k,\infty}(M_0)$ holomorphic functions on M_0 which are real valued on Γ_0 is infinite dimensional.

Proof. (i) Let $L \in \mathbb{N}$ be arbitrary large and let us identify the boundary as a disjoint union of circles $\partial M_0 = \coprod_{i=1}^m \partial_i M_0$ where each $\partial_i M_0 \simeq S^1$. Since Γ will be the piece of the boundary where we know the Cauchy data space, it is sufficient to assume that Γ is a connected non-empty open segment of $\partial_1 M_0 = S^1$, and which can thus be defined in a coordinate θ (respecting the orientation of the boundary) by $\Gamma = \{\theta \in S^1 \mid 0 < \theta < 2\pi/k\}$ for some integer k . Define the totally real subbundle of $F \subset E|_{\partial M_0} = \coprod_{j=1}^l (\partial_j M_0 \times \mathbb{C})$ by the following: on $\partial_1 M_0 \simeq S^1$ parametrized by $\theta \in [0, 2\pi]$, define $F_\theta = e^{ia(\theta)}\mathbb{R}$. Where $a : [0, 2\pi] \rightarrow \mathbb{R}$ is a smooth nondecreasing function such that $a(\theta) = 0$ in a neighbourhood $[0, \epsilon]$ of 0, $a(2\pi/k) = 2L\pi$ for some $L \in \mathbb{N}$, and $a(\theta) = 2L\pi$ for all $\theta > 2\pi/k$. For the rest of $\partial_2 M_0, \dots, \partial_m M_0$, we just let $F|_{\partial_i M_0} = S^1 \times \mathbb{R}$. The Maslov index $\mu(E, F)$ is then given by $2L$ and so, by theorem 2.1, D_F is surjective if $2\chi(M_0) + 2L > 0$. Since L can be taken as large as we want this achieves the proof of (i).

(ii) Let $w \in W^{k,\infty}(M_0)$ be a real function with boundary value f on ∂M_0 , then by (i) there exists $R \in W^{k+1}(M_0)$ such that $\bar{\partial}R = -\bar{\partial}w$ and R purely real on Γ_0 , thus $u := R + w$ is holomorphic such that $\mathrm{Re}(u) = f$ on Γ_0 .

(iii) Taking the subbundle F as in the proof of (i), we have that $\dim \ker D_F = \chi(M_0) + 2L$ if L satisfies $2\chi(M_0) + 2L > 0$, and since L can be taken as large as we like-, this concludes the proof. \square

Lemma 2.3. Let $\{p_0, p_1, \dots, p_n\} \subset M_0$ be a set of $n+1$ disjoint points such that $p_0 \in \mathrm{int}(M_0)$. Let $c_1, \dots, c_K \in \mathbb{C}$, $N \in \mathbb{N}$, and let z be a complex coordinate near p_0 such that $p_0 = \{z = 0\}$. Then there exists a holomorphic function f on M_0 with zeros of order at least N at each p_j , such that f is real on Γ_0 and $f(z) = c_0 + c_1 z + \dots + c_K z^K + O(|z|^{K+1})$ in the coordinate z .

Proof. Consider the subbundle F as in the proof of (i) in Corollary 2.2. The Maslov index $\mu(E, F)$ is given by $2L$ and so for each $N \in \mathbb{N}$, one can take L large enough to have $\mu(F, E) + 2\chi(M_0) \geq 2N(1+n)$. Therefore by Theorem 2.1 the dimension of the kernel of $\bar{\partial}_F$ will be greater than $2(n+1)N$. Now, since for each p_j and complex coordinate z_j near p_j , the map $u \rightarrow (u(p_j), \partial_{z_j} u(p_j), \dots, \partial_{z_j}^{N-1} u(p_j)) \in \mathbb{C}^N$ is linear, this implies that there exists a non-zero element $u \in \ker D_F$ which has zeros of order at least N at all p_j .

Now we want the desired Taylor expansion at p_0 in the coordinate z . Let us start with the case $K = 0$. In the coordinate z , one has $u(z) = \alpha z^M + O(|z|^{M+1})$ for some $\alpha \neq 0$ and $M \geq N$. For each $j = 0, \dots, K$, define the function $r_j(z) = \chi(z) \frac{c_j}{\alpha} z^{-M+j}$ where $\chi(z)$ is a smooth cut-off

function supported near p_0 and which is 1 near $p_0 = \{z = 0\}$. Since $M \geq N > 1$, this function has a pole at p_0 and trivially extends smoothly to $M_0 \setminus \{p_0\}$, which we still call r . Observe that the function is holomorphic near p_0 so that $\bar{\partial}r_j$ is a smooth and compactly supported section of $T_{0,1}^*M_0$ which is supported away from p_0 and therefore trivially extends smoothly to M_0 to a one form denoted ω_j . By the surjectivity assertion Corollary 2.2, we have that there exists a smooth function R_j satisfying $\bar{\partial}R_j = -\omega_j$ and that $R_j|_{\Gamma_0} \in \mathbb{R}$. We now have that $R_j + r_j$ is a holomorphic function on $M \setminus \{p_0\}$ and meromorphic in M with a pole of order $M - j$ and in coordinate z one has $z^{M-j}(R_j(z) + r_j(z)) = c_j + O(|z|)$. Setting $f_j = u(R_j + r_j)$, we have the desired meromorphic function in case $K = j$ and $c_0 = \dots c_{j-1} = 0$. Note that f also vanishes to order N at all p_1, \dots, p_n since u does. This achieves the proof by taking appropriate linear combinations of f_j . \square

2.3. Morse holomorphic functions with prescribed critical points. The main result of this section is the following

Proposition 2.1. *Let p be an interior point of M_0 and $\epsilon > 0$ small. Then there exists a holomorphic function Φ on M_0 which is Morse on M_0 (up to the boundary) and real valued on Γ_0 , which has a critical point p' at distance less than ϵ from p and such that $\text{Im}(\Phi(p')) \neq 0$.*

Let \mathcal{O} be a connected open set of M such that $\bar{\mathcal{O}}$ is a smooth surface with boundary, with $M_0 \subset \bar{\mathcal{O}}$ and $\Gamma_0 \subset \partial\bar{\mathcal{O}}$. Fix $k > 2$ a large integer, we denote by $C^k(\bar{\mathcal{O}})$ the Banach space of C^k real valued functions on $\bar{\mathcal{O}}$. Then the set of harmonic functions on $\bar{\mathcal{O}}$ which are in the Banach space $C^k(\bar{\mathcal{O}})$ (and smooth in \mathcal{O} by elliptic regularity) is the kernel of the continuous map $\Delta : C^k(\bar{\mathcal{O}}) \rightarrow C^{k-2}(\bar{\mathcal{O}})$, and so it is a Banach subspace of $C^k(\bar{\mathcal{O}})$. The set $\mathcal{H} \subset C^k(\bar{\mathcal{O}})$ of harmonic functions u in $C^k(\bar{\mathcal{O}})$ such there exists $v \in C^k(\bar{\mathcal{O}})$ harmonic with $u + iv$ holomorphic on \mathcal{O} is a Banach subspace of $C^k(\bar{\mathcal{O}})$ of finite codimension. Indeed, let $\{\gamma_1, \dots, \gamma_N\}$ be a homology basis for \mathcal{O} , then

$$\mathcal{H} = \ker L, \text{ with } L : \ker \Delta \cap C^k(\bar{\mathcal{O}}) \rightarrow \mathbb{C}^N \text{ defined by } L(u) := \left(\frac{1}{\pi i} \int_{\gamma_j} \partial u \right)_{j=1, \dots, N}.$$

For all $\Gamma'_0 \subset \partial M_0$ such that the complement of Γ'_0 contains an open subset, we define

$$\mathcal{H}_{\Gamma'_0} := \{u \in \mathcal{H}; u|_{\Gamma'_0} = 0\}.$$

We now show

Lemma 2.4. *The set of functions $u \in \mathcal{H}_{\Gamma'_0}$ which are Morse in \mathcal{O} is residual (i.e. a countable intersection of open dense sets) in $\mathcal{H}_{\Gamma'_0}$ with respect to the $C^k(\bar{\mathcal{O}})$ topology.*

Proof. We use an argument very similar to those used by Uhlenbeck [30]. We start by defining $m : \mathcal{O} \times \mathcal{H}_{\Gamma'_0} \rightarrow T^*\mathcal{O}$ by $(p, u) \mapsto (p, du(p)) \in T_p^*\mathcal{O}$. This is clearly a smooth map, linear in the second variable, moreover $m_u := m(\cdot, u) = (\cdot, du(\cdot))$ is Fredholm since \mathcal{O} is finite dimensional. The map u is a Morse function if and only if m_u is transverse to the zero section, denoted $T_0^*\mathcal{O}$, of $T^*\mathcal{O}$, ie. if

$$\text{Image}(D_p m_u) + T_{m_u(p)}(T_0^*\mathcal{O}) = T_{m_u(p)}(T^*\mathcal{O}), \quad \forall p \in \mathcal{O} \text{ such that } m_u(p) = (p, 0),$$

which is equivalent to the fact that the Hessian of u at critical points is non-degenerate (see for instance Lemma 2.8 of [30]). We recall the following transversality theorem (see [30, Th.2] based on the Sard-Smale theorem [26])

Theorem 2.5. *Let $m : X \times \mathcal{H}_{\Gamma'_0} \rightarrow W$ be a C^k map, where X , $\mathcal{H}_{\Gamma'_0}$, and W are separable Banach manifolds with W and X of finite dimension. Let $W' \subset W$ be a submanifold such that $k > \max(1, \dim X - \dim W + \dim W')$. If m is transverse to W' then the set $\{u \in \mathcal{H}_{\Gamma'_0}; m_u \text{ is transverse to } W'\}$ is dense in $\mathcal{H}_{\Gamma'_0}$, more precisely it is a residual set.*

We want to apply it with $X := \mathcal{O}$, $W := T^*\mathcal{O}$ and $W' := T_0^*\mathcal{O}$, and the map m is defined above. We have thus proved Lemma 2.4 if one can show that m is transverse to W' . Let (p, u) such that $m(p, u) = (p, 0) \in W'$. Then identifying $T_{(p,0)}(T^*\mathcal{O})$ with $T_p\mathcal{O} \oplus T_p^*\mathcal{O}$, one has

$$D_{(p,u)}m(z, v) = (z, dv(p) + \text{Hess}_p(u)z)$$

where $\text{Hess}_p u$ is the Hessian of u at the point p , viewed as a linear map from $T_p\mathcal{O}$ to $T_p^*\mathcal{O}$. To prove that m is transverse to W' we need to show that $(z, v) \rightarrow (z, dv(p) + \text{Hess}_p(u)z)$ is onto from $T_p\mathcal{O} \oplus \mathcal{H}_{\Gamma'_0}$ to $T_p\mathcal{O} \oplus T_p^*\mathcal{O}$, which is realized for instance if the map $v \rightarrow dv(p)$ from $\mathcal{H}_{\Gamma'_0}$ to $T_p^*\mathcal{O}$ is onto. But from Lemma 2.3, we know that there exist holomorphic functions v and \tilde{v} on \mathcal{O} such that v and \tilde{v} are purely real on Γ'_0 . Clearly the imaginary parts of v and \tilde{v} belong to $\mathcal{H}_{\Gamma'_0}$. Furthermore, for a given complex coordinate z near $p = \{z = 0\}$, we can arrange them to have series expansion $v(z) = z + O(|z|^2)$ and $\tilde{v}(z) = iz + O(|z|^2)$ around the point p . We see, by coordinate computation of the exterior derivative of $\text{Im}(v)$ and $\text{Im}(\tilde{v})$, that $d\text{Im}(v)(p)$ and $d\text{Im}(\tilde{v})(p)$ are linearly independent at the point p . This shows our claim and ends the proof of Lemma 2.4 by using Theorem 2.5. \square

We now proceed to show that the set of all functions $u \in \mathcal{H}_{\Gamma'_0}$ such that u has no degenerate critical points on Γ'_0 is also residual.

Lemma 2.6. *For all $p \in \Gamma'_0$ and $k \in \mathbb{N}$, there exists a holomorphic function $u \in C^k(\bar{\mathcal{O}})$, such that $\text{Im}(u)|_{\Gamma'_0} = 0$ and $\partial u(p) \neq 0$.*

Proof. The proof is quite similar to that of Lemma 2.3. By Lemma 2.3, we can choose a holomorphic function $v \in C^k(\bar{\mathcal{O}})$ such that $v(p) = 0$ and $\text{Im}(v)|_{\Gamma'_0} = 0$, then either $\partial v(p) \neq 0$ and we are done, or $\partial v(p) = 0$. Assume now the second case and let $M \in \mathbb{N}$ be the order of p as a zero of v . By Riemann mapping theorem, there is a conformal mapping from a neighbourhood U_p of p in $\bar{\mathcal{O}}$ to a neighbourhood $\{|z| < \epsilon, \text{Im}(z) \geq 0\}$ of the real line $\text{Im}(z) = 0$ in \mathbb{C} , and one can assume that $p = \{z = 0\}$ in these complex coordinates. Take $r(z) = \chi(z)z^{-M+1}$ where $\chi \in C_0^\infty(|z| \leq \epsilon)$ is a real valued function with $\chi(z) = 1$ in $\{|z| < \epsilon/2\}$. Then $\bar{\partial}r$ vanishes in the pointed disc $0 < |z| < \epsilon/2$ and it is a compactly supported smooth section of $T_{1,0}^*\bar{\mathcal{O}}$ outside, it can thus be extended trivially to a smooth section of $T_{1,0}^*\bar{\mathcal{O}}$ denoted by ω . We can then use (i) of Corollary 2.2: there is a function R such that $\bar{\partial}R = -\omega$ and $\text{Im}(R)|_{\Gamma'_0} = 0$, and so $\bar{\partial}(R+r) = 0$ in $\mathcal{O} \setminus \{p\}$ and $R+r$ is real valued on Γ'_0 (remark that r is real valued on Γ'_0) and has a pole at p of order exactly $M-1$. We conclude that $u := v(R+r)$ satisfies the desired properties, it vanishes at p but with non zero complex derivative at p . \square

Lemma 2.7. *Let $\Gamma'_0 \subset \partial\mathcal{O}$ be an open set of the boundary. Let $\phi : \mathcal{O} \rightarrow \mathbb{R}$ be a harmonic function with $\phi|_{\Gamma'_0} = 0$. Let $p \in \Gamma'_0$ be a critical point of ϕ , then it is nondegenerate if and only if $\partial_\tau \partial_\nu u \neq 0$ where ∂_τ and ∂_ν denote respectively the tangential and normal derivatives along the boundary.*

Proof. By Riemann mapping theorem, there is a conformal transformation mapping a neighbourhood of p in $\bar{\mathcal{O}}$ to a half-disc $D := \{|z| < \epsilon, \text{Im}(z) \geq 0\}$ and $\partial\bar{\mathcal{O}} = \{\text{Im}(z) = 0\}$

near p . Denoting $z = x + iy$, one has $(\partial_x^2 + \partial_y^2)\phi = 0$ in D and $\partial_x^2\phi|_{y=0} = 0$, which implies $\partial_y^2\phi(p) = 0$. Since $\partial_\nu = e^f\partial_y$ and $\partial_\tau = e^f\partial_x$ for some smooth function f , and since $d\phi(p) = 0$, the conclusion is then straightforward. \square

Let $N^*\partial\bar{\mathcal{O}}$ be the conormal-bundle of $\partial\bar{\mathcal{O}}$ and $N^*\Gamma'_0$ be the restriction of this bundle to Γ'_0 . Denote the zero sections of these bundles respectively by $N_0^*\partial\bar{\mathcal{O}}$ and $N_0^*\Gamma'_0$. We now define the map

$$b : \Gamma'_0 \times \mathcal{H}_{\Gamma'_0} \rightarrow N^*\Gamma'_0, \quad b(p, u) := (p, \partial_\nu u).$$

For a fixed $u \in \mathcal{H}_{\Gamma'_0}$, we also define $b_u(\cdot) := b(\cdot, u)$. Simple computations yield the

Lemma 2.8. *Suppose that $p \in \Gamma'_0$ is such that $\partial_\nu u(p) = 0$, then $\partial_\tau \partial_\nu u(p) \neq 0$ if and only if*

$$\text{Image}(D_p b_u) + T_{(p,0)}(N_0^*\Gamma'_0) = T_{(p,0)}(N^*\Gamma'_0).$$

Proof. This can be seen by the fact that for all $p \in \Gamma'_0$ such that $b_u(p) = (p, 0)$,

$$D_p b_u : T_p \Gamma'_0 \rightarrow T_{(p,0)}(N^*\Gamma'_0) \simeq T_p \Gamma'_0 \oplus N_p^* \Gamma'_0$$

is given by $w \mapsto (w, \partial_\tau \partial_\nu u(p)w)$. \square

At a point (p, u) such that $b(p, u) = 0$, a simple computation yields that the differential $D_{(p,u)}b : T_p \Gamma'_0 \times \mathcal{H}_{\Gamma'_0} \rightarrow T_{(p,\partial_\nu u(p))}(N^*\Gamma'_0)$ is given by $(w, u') \mapsto (w, \partial_\tau \partial_\nu u(p)w + \partial_\nu u'(p))$. This observation combined with Lemma 2.6 shows that for all $(p, u) \in \Gamma'_0 \times \mathcal{H}_{\Gamma'_0}$ such that $b(p, u) = (p, 0)$, b is transverse to $N_0^*\Gamma'_0$ at $(p, 0)$. Now we can apply Theorem 2.5 with $X = \Gamma'_0$, $W = N^*\Gamma'_0$ and $W' = N_0^*\Gamma'_0$ we see that the set $\{u \in \mathcal{H}_{\Gamma'_0}; b_u \text{ is transverse to } N_0^*\Gamma'_0\}$ is residual in $\mathcal{H}_{\Gamma'_0}$. In view of Lemmas 2.7, we deduce the

Lemma 2.9. *The set of functions $u \in \mathcal{H}_{\Gamma'_0}$ such that u has no degenerate critical point on Γ'_0 is residual in $\mathcal{H}_{\Gamma'_0}$.*

Observing the general fact that finite intersection of residual sets remains residual, the combination of Lemma 2.9 and Lemma 2.4 yields

Corollary 2.10. *The set of functions $u \in \mathcal{H}_{\Gamma'_0}$ which are Morse in \mathcal{O} and have no degenerate critical points on Γ'_0 is residual in $\mathcal{H}_{\Gamma'_0}$ with respect to the $C^k(\bar{\mathcal{O}})$ topology. In particular, it is dense.*

We are now in a position to give a proof of the main proposition of this section.

Proof of Proposition 2.1. As explained above, choose \mathcal{O} in such a way that $\bar{\mathcal{O}}$ is a smooth surface with boundary, containing M_0 , that $\Gamma_0 \subset \partial\mathcal{O}$ and \mathcal{O} contains $\partial M_0 \setminus \bar{\Gamma}_0$. Let Γ'_0 be an open subset of the boundary of $\bar{\mathcal{O}}$ such that the closure of Γ_0 is contained in Γ'_0 and $\partial\bar{\mathcal{O}} \setminus \bar{\Gamma}_0 \neq \emptyset$. Let p be an interior point of M_0 . By lemma 2.3, there exists a holomorphic function $f = u + iv$ on $\bar{\mathcal{O}}$ such that f is purely real on Γ'_0 , $v(p) = 1$, and $df(p) = 0$ (thus $v \in \mathcal{H}_{\Gamma'_0}$).

By Corollary 2.10, there exist a sequence $(v_j)_j$ of Morse functions $v_j \in \mathcal{H}_{\Gamma'_0}$ such that $v_j \rightarrow v$ in $C^k(M_0)$ for any fixed k large. By Cauchy integral formula, there exist harmonic conjugates u_j of v_j such that $f_j := u_j + iv_j \rightarrow f$ in $C^k(M_0)$. Let $\epsilon > 0$ be small and let $U \subset \mathcal{O}$ be a neighbourhood containing p and no other critical points of f , and with boundary

a smooth circle of radius ϵ . In complex local coordinates near p , we can identify ∂f and ∂f_j to holomorphic functions on an open set of \mathbb{C} . Then by Rouché's theorem, it is clear that ∂f_j has precisely one zero in U and v_j never vanishes in U if j is large enough.

Fix Φ to be one of the f_j for j large enough. By construction, Φ is Morse in \mathcal{O} and has no degenerate critical points on $\overline{\Gamma_0} \subset \Gamma'_0$. We notice that, since the imaginary part of Φ vanishes on all of Γ'_0 , it is clear from the reflection principle applied after using the Riemann mapping theorem (as in the proof of Lemma 2.7) that no point on $\overline{\Gamma_0} \subset \Gamma'_0$ can be an accumulation point for critical points. Now $\partial M_0 \setminus \overline{\Gamma_0}$ is contained in the interior of \mathcal{O} and therefore no points on $\partial M_0 \setminus \overline{\Gamma_0}$ can be an accumulation point of critical points. Since Φ is Morse in the interior of \mathcal{O} , there are no degenerate critical points on $\partial M_0 \setminus \overline{\Gamma_0}$. This ends the proof. \square

3. CARLEMAN ESTIMATE FOR HARMONIC WEIGHTS WITH CRITICAL POINTS

In this section, we prove a Carleman estimate using harmonic weight with non-degenerate critical points, in way similar to [16]. Let us define $\Gamma_0 := \{p \in \partial M_0; \partial_\nu \varphi(p) = 0\}$ and let $\Gamma := \partial M_0 \setminus \Gamma_0$ its complement.

Proposition 3.1. *Let (M_0, g) be a smooth Riemann surface with boundary, and let $\varphi : M_0 \rightarrow \mathbb{R}$ be $C^k(M_0)$ harmonic Morse function for k large. Then for all $V \in L^\infty(M_0)$ there exists an $h_0 > 0$ such that for all $h \in (0, h_0)$ and $u \in C^\infty(M)$ with $u|_{\partial M_0} = 0$, we have*

$$(3) \quad \begin{aligned} & \frac{1}{h} \|u\|_{L^2(M_0)}^2 + \frac{1}{h^2} \|u|d\varphi\|_{L^2(M_0)}^2 + \|du\|_{L^2(M_0)}^2 + \|\partial_\nu u\|_{L^2(\Gamma_0)}^2 \\ & \leq C \left(\|e^{-\varphi/h}(\Delta_g + V)e^{\varphi/h}u\|_{L^2(M_0)}^2 + \frac{1}{h} \|\partial_\nu u\|_{L^2(\Gamma)}^2 \right) \end{aligned}$$

where ∂_ν is the exterior unit normal vector field to ∂M_0 .

Proof. We start by modifying the weight as follows: if $\varphi_0 := \varphi : M_0 \rightarrow \mathbb{R}$ is a real valued harmonic Morse function with critical points $\{p_1, \dots, p_N\}$ in the interior of M_0 , we let $\varphi_j : M_0 \rightarrow \mathbb{R}$ be harmonic functions such that p_j is not a critical point of φ_j for $j = 1, \dots, N$, their existence is insured by Lemma 2.3. For all $\epsilon > 0$, we define the convexified weight $\varphi_\epsilon := \varphi - \frac{h}{2\epsilon} (\sum_{j=0}^N |\varphi_j|^2)$.

To prove the estimate, we shall localize in charts Ω_j covering the surfaces. These charts will be taken so that if $\Omega_j \cap \partial M_0 \neq \emptyset$, then $\Omega_j \cap \partial M_0 \simeq S^1$ is a connected component of ∂M_0 . Moreover, by Riemann mapping theorem, this chart can be taken to be a neighbourhood of $|z| = 1$ in $\{z \in \mathbb{C}; |z| \leq 1\}$ and such that the metric g is conformal to the Euclidean metric $|dz|^2$.

Lemma 3.1. *Let Ω be a chart of M_0 as above and $\varphi_\epsilon : \Omega \rightarrow \mathbb{R}$ be as above. Then there are constants $C, C' > 0$ such that for all $\omega \in C^\infty(M)$ supported in Ω and $h > 0$ small enough, the following estimate holds:*

$$(4) \quad \frac{C}{\epsilon} \|\omega\|_{L^2(M_0)}^2 + C' \left(-\text{Im}(\langle \partial_\tau \omega, \omega \rangle_{L^2(\partial M_0)}) + \frac{1}{h} \int_{\partial M_0} |\omega|^2 \partial_\nu \varphi_\epsilon \, d\nu_g \right) \leq \|e^{-\varphi_\epsilon/h} \bar{\partial} e^{\varphi_\epsilon/h} \omega\|_{L^2(M_0)}^2$$

where ∂_ν and ∂_τ denote respectively the exterior pointing normal vector fields and its rotation by an angle $+\pi/2$.

Proof. We use complex coordinates $z = x + iy$ in the chart Ω where ω is supported. Observe that the Lebesgue measure $dxdy$ is bounded below and above by dv_g , g is conformal to $|dz|^2$ and the boundary terms in (4) depend only on the conformal class, it suffices to prove the estimates with respect to $dxdy$ and the Euclidean metric. We thus integrate by parts with respect to $dxdy$

(5)

$$\begin{aligned}
4\|e^{-\varphi_\epsilon/h}\bar{\partial}e^{\varphi_\epsilon/h}\omega\|^2 &= \|(\partial_x + \frac{i\partial_y\varphi_\epsilon}{h})\omega + (i\partial_y + \frac{\partial_x\varphi_\epsilon}{h})\omega\|^2 \\
&= \|(\partial_x + \frac{i\partial_y\varphi_\epsilon}{h})\omega\|^2 + \|(i\partial_y + \frac{\partial_x\varphi_\epsilon}{h})\omega\|^2 \\
&\quad + \frac{2}{h} \int_\Omega \left(\Delta\varphi_\epsilon|\omega|^2 - \frac{1}{2}\partial_x\varphi_\epsilon.\partial_x|\omega|^2 - \frac{1}{2}\partial_y\varphi_\epsilon.\partial_y|\omega|^2 \right) \\
&\quad + \frac{2}{h} \int_{\partial M_0} \partial_\nu\varphi_\epsilon|\omega|^2 - 2 \int_{M_0} \left(\partial_x\text{Re}(\omega).\partial_y\text{Im}(\omega) - \partial_x\text{Im}(\omega).\partial_y\text{Re}(\omega) \right) \\
&= \|(\partial_x + \frac{i\partial_y\varphi_\epsilon}{h})\omega\|^2 + \|(i\partial_y + \frac{\partial_x\varphi_\epsilon}{h})\omega\|^2 + \frac{1}{h} \int_\Omega \Delta\varphi_\epsilon|\omega|^2 \\
&\quad + \frac{1}{h} \int_{\partial M_0} \partial_\nu\varphi_\epsilon|\omega|^2 + 2 \int_{\partial M_0} \partial_\tau\text{Re}(\omega).\text{Im}(\omega).
\end{aligned}$$

where $\Delta := -(\partial_x^2 + \partial_y^2)$, ∂_ν is the exterior pointing normal vector field to the boundary and ∂_τ is the tangent vector field to the boundary (i.e. ∂_ν rotated with an angle $\pi/2$) for the Euclidean metric $|dz|^2$. Then $\langle \omega\Delta\varphi_\epsilon, \omega \rangle = \frac{h}{\epsilon}(|d\varphi_0|^2 + |d\varphi_1|^2 + \dots + |d\varphi_N|^2)|\omega|^2$, since φ_j are harmonic, so the proof follows from the fact that $|d\varphi_0|^2 + |d\varphi_1|^2 + \dots + |d\varphi_N|^2$ is uniformly bounded away from zero. \square

The main step to go from (4) to (3) is the following lemma which is a slight modification of the proof in [16]:

Lemma 3.2. *With the same assumptions as in Proposition 3.1, and if Ω is either an interior chart of (M, g) or a chart containing a whole boundary connected component, then there are positive constants c and C such that for all $\epsilon > 0$ small, all $0 < h \ll \epsilon$ and all smooth function u supported in Ω with $u|_{\partial M_0} = 0$, we have*

$$\begin{aligned}
(6) \quad & C \left(\|e^{-\varphi_\epsilon/h}\Delta_g e^{\varphi_\epsilon/h}u\|_{L^2(M_0)}^2 + \frac{1}{h} \|\partial_\nu u\|_{L^2(\Gamma)}^2 \right) \geq \\
& \frac{c}{\epsilon} \left(\frac{1}{h} \|u\|_{L^2(M)}^2 + \frac{1}{h^2} \|u|d\varphi|\|_{L^2(M)}^2 + \frac{1}{h^2} \|u|d\varphi_\epsilon|\|_{L^2(M)}^2 + \|du\|_{L^2(M)}^2 \right) + \|\partial_\nu u\|_{L^2(\partial M_0)}^2
\end{aligned}$$

where ∂_ν denote the unit normal vector field to ∂M_0 .

Proof. Since the norms induced by the metric g in the chart are conformal to Euclidean norms, and there is f smooth such that $\Delta_g = -e^{2f}(\partial_x^2 + \partial_y^2) = e^{2f}\Delta$ in the complex coordinate chart, it suffices to get the estimate (6) for Euclidean norms and Laplacian. Clearly, we can assume $u \in H_0^1(M)$ to be real valued without loss of generality. Now let $Q(z)$ be a holomorphic

function in Ω which has no zeros, then by (4) we have

$$\begin{aligned} \|Qe^{-\varphi_\epsilon/h}\Delta e^{\varphi_\epsilon/h}u\|^2 &= 16\|e^{-\varphi_\epsilon/h}\bar{\partial}e^{\varphi_\epsilon/h}Qe^{-\varphi_\epsilon/h}\partial e^{\varphi_\epsilon/h}u\|^2 \geq \\ &\frac{C}{\epsilon}\|\partial u + \frac{\partial\varphi_\epsilon}{h}u\|^2 - 4\text{Im}\left(\int_{\partial M_0}\partial_\tau\omega\cdot\bar{\omega}\right) + \frac{4}{h}\int_{\partial M_0}|\omega|^2\partial_\nu\varphi_\epsilon \end{aligned}$$

with $\omega := Qe^{-\varphi_\epsilon/h}\partial e^{\varphi_\epsilon/h}u$ and here everything is measured with respect to Euclidean metric $dx^2 + dy^2$ and measure $dx dy$. Since $u|_{\partial M_0} = 0$, one has $\omega|_{\partial M_0} = (A + iB)\partial_\nu u$ where $A + iB = Q(\langle\partial_\nu, \partial_x\rangle - i\langle\partial_\nu, \partial_y\rangle)$ and $-\text{Im}(\partial_\tau\omega\cdot\bar{\omega}) = (\partial_\tau A \cdot B - A\partial_\tau B)|\partial_\nu u|^2$ we deduce that for some $c > 0$

$$(7) \quad \begin{aligned} &\|Qe^{-\varphi_\epsilon/h}\Delta e^{\varphi_\epsilon/h}u\|^2 \geq \\ &\frac{c}{\epsilon}\left(\|du\|^2 + \frac{1}{h^2}\|u|d\varphi_\epsilon|\|^2 + \frac{2}{h}\langle\partial_x u, u\partial_x\varphi_\epsilon\rangle + \frac{2}{h}\langle\partial_y u, u\partial_y\varphi_\epsilon\rangle\right) \\ &\quad + 4\int_{\partial M_0}(A\partial_\tau B - \partial_\tau AB)|\partial_\nu u|^2 + \frac{4}{h}\int_{\partial M_0}|Q|^2|\partial_\nu u|^2\partial_\nu\varphi_\epsilon. \end{aligned}$$

Using the fact that u is real valued, that φ is harmonic and that $\sum_{j=0}^N|d\varphi_j|^2$ is uniformly bounded away from 0, we see that

$$\frac{2}{h}\langle\partial_x u, u\partial_x\varphi_\epsilon\rangle + \frac{2}{h}\langle\partial_y u, u\partial_y\varphi_\epsilon\rangle = \frac{1}{h}\langle u, u\Delta\varphi_\epsilon\rangle \geq \frac{C}{\epsilon}\|u\|^2$$

for some $C > 0$ and therefore,

$$(8) \quad \|Qe^{-\varphi_\epsilon/h}\Delta e^{\varphi_\epsilon/h}u\|^2 \geq \frac{c}{\epsilon}(\|du\|^2 + \frac{1}{h^2}\|u|d\varphi_\epsilon|\|^2 + \frac{C}{\epsilon}\|u\|^2) + \text{boundary terms}.$$

Now if the diameter of the support of u is chosen small (with size depending only on $|\text{Hess}\varphi_0|(p)$) with a unique critical point p of φ_0 inside, one can use integration by parts and the fact that the critical point is non-degenerate to obtain

$$(9) \quad \|\bar{\partial}u\|^2 + \frac{1}{h^2}\|u|\partial\varphi_0|\|^2 \geq \frac{1}{h}\left|\int\partial_{\bar{z}}(u^2)\bar{\partial}_z\varphi_0 dx dy\right| \geq \frac{1}{h}\left|\int u^2\bar{\partial}_z^2\varphi_0 dx dy\right| \geq \frac{C'}{h}\|u\|^2$$

for some $C' > 0$. Clearly the same estimate holds trivially if Ω does not contain critical point of φ_0 . Using a partition of unity $(\theta_j)_j$ in Ω and absorbing terms of the form $\|u\bar{\partial}\theta_j\|^2$ into the right hand side, one obtains (9) for any function u supported in Ω and vanishing at the boundary. Thus, combining with (8), there are positive constants c, c', C'' such that for h small enough

$$\begin{aligned} \frac{c}{\epsilon}(\|du\|^2 + \frac{1}{h^2}\|u|d\varphi_\epsilon|\|^2 + \frac{C}{\epsilon}\|u\|^2) &\geq \frac{c}{\epsilon}(\|du\|^2 + \frac{1}{h^2}\|u|d\varphi_0|\|^2 - \frac{C''}{\epsilon^2}\|u\|^2) \\ &\geq \frac{c'}{\epsilon}(\|du\|^2 + \frac{1}{h^2}\|u|d\varphi_0|\|^2 + \frac{1}{h}\|u\|^2). \end{aligned}$$

Combining now with (8) and using that $|Q|$ is bounded below gives

$$\|e^{-\varphi_\epsilon/h}\Delta e^{\varphi_\epsilon/h}u\|^2 \geq \frac{c'}{\epsilon}(\|du\|^2 + \frac{1}{h^2}\|u|d\varphi_0|\|^2 + \frac{1}{h}\|u\|^2) + \text{boundary terms}.$$

Let us now discuss the boundary terms in (7). If φ_j are taken so that $\partial_\nu\varphi_j = 0$ on Γ_0 , then $\partial_\nu\varphi_\epsilon = 0$ on Γ_0 and $\partial_\nu\varphi_\epsilon = \partial_\nu\varphi + O(h/\epsilon)$ on Γ and thus

$$\frac{1}{h}\int_{\partial M_0}|Q|^2|\partial_\nu u|^2|\partial_\nu\varphi_\epsilon| \leq \frac{C_1}{h}\int_\Gamma|\partial_\nu u|^2$$

for some constant C_1 . We finally claim that there exist Q with no zeros in Ω such that $A\partial_\tau B - B\partial_\tau A$ is bounded below by a positive constant on $\partial M_0 \cap \Omega$. Indeed, since the chart near a connected component can be taken to be an interior neighbourhood of the circle $|z| = 1$ in \mathbb{C} , one can take $A + iB = e^{it}$ where $t \in S^1$ parametrize the boundary component, so that $A\partial_\tau B - B\partial_\tau A = 1$ since $\partial_\tau = \partial_t$ for the Euclidean metric. Since moreover $\langle \partial_\nu, \partial_x \rangle - i\langle \partial_\nu, \partial_y \rangle = A - iB = e^{-it}$, we deduce that on the boundary $Q(t) = e^{2it}$ and so it suffices to take $Q(z) = z^2$. This achieves the proof. \square

Proof of Proposition 3.1. Using triangular inequality and absorbing the term $\|Vu\|^2$ into the left hand side of (3), it suffices to prove (3) with Δ_g instead of $\Delta_g + V$. Let $v \in C_0^\infty(M)$, we have by Lemma 3.2 that there exist constants $c, c', C, C' > 0$ such that

$$\begin{aligned} & \frac{c}{\epsilon} \left(\frac{1}{h} \|e^{-\varphi_\epsilon/h} v\|^2 + \frac{1}{h^2} \|e^{-\varphi_\epsilon/h} v |d\varphi|\|^2 + \frac{1}{h^2} \|e^{-\varphi_\epsilon/h} v |d\varphi_\epsilon|\|^2 + \|e^{-\varphi_\epsilon/h} dv\|^2 \right) + \|e^{-\varphi_\epsilon/h} \partial_\nu v\|_{\Gamma_0}^2 \\ & \leq \sum_j \frac{c'}{\epsilon} \left(\frac{1}{h} \|e^{-\varphi_\epsilon/h} \chi_j v\|^2 + \frac{1}{h^2} \|e^{-\varphi_\epsilon/h} \chi_j v |d\varphi|\|^2 + \frac{1}{h^2} \|e^{-\varphi_\epsilon/h} \chi_j v |d\varphi_\epsilon|\|^2 \right. \\ & \quad \left. + \|e^{-\varphi_\epsilon/h} d(\chi_j v)\|^2 \right) + \|e^{-\varphi_\epsilon/h} \partial_\nu v\|_{\Gamma_0}^2 \\ & \leq C \left(\sum_j \|e^{-\varphi_\epsilon/h} \Delta_g(\chi_j v)\|^2 + \|e^{-\varphi_\epsilon/h} \partial_\nu v\|_{\Gamma}^2 \right) \\ & \leq C' \left(\|e^{-\varphi_\epsilon/h} \Delta_g v\|^2 + \|e^{-\varphi_\epsilon/h} v\|^2 + \|e^{-\varphi_\epsilon/h} dv\|^2 + \|e^{-\varphi_\epsilon/h} \partial_\nu v\|_{\Gamma}^2 \right) \end{aligned}$$

where $(\chi_j)_j$ is a partition of unity associated to the complex charts Ω_j on M . Since constants on both sides are independent of ϵ and h , we can take ϵ small enough so that $C' \|e^{-\varphi_\epsilon/h} v\|^2 + C' \|e^{-\varphi_\epsilon/h} dv\|^2$ can be absorbed to the left side. Now set $v = e^{\varphi_\epsilon/h} w$ with $w|_{\partial M_0} = 0$, then we have

$$\begin{aligned} & \frac{1}{h} \|w\|^2 + \frac{1}{h^2} \|w |d\varphi|\|^2 + \frac{1}{h^2} \|w |d\varphi_\epsilon|\|^2 + \|dw\|^2 + \|\partial_\nu w\|_{\Gamma_0}^2 \\ & \leq C \left(\|e^{-\varphi_\epsilon/h} \Delta_g e^{\varphi_\epsilon/h} w\|^2 + \|\partial_\nu w\|_{\Gamma}^2 \right) \end{aligned}$$

Finally, fix $\epsilon > 0$ and set $u := e^{\frac{1}{\epsilon} \sum_{j=0}^N |\varphi_j|^2} w$ and use the fact that $e^{\frac{1}{\epsilon} \sum_{j=0}^N |\varphi_j|^2}$ is independent of h and bounded uniformly away from zero and above, we then obtain the desired estimate for $0 < h \ll \epsilon$. \square

4. COMPLEX GEOMETRIC OPTICS ON A RIEMANN SURFACE

As explained in the Introduction, the method for identifying the potential at a point p is to construct complex geometric optic solutions depending on a small parameter $h > 0$, with phase a Carleman weight (here a Morse holomorphic function), and such that the phase has a non-degenerate critical point at p , in order to apply the stationary phase method.

Choose $p \in M_0$ such that there exists a holomorphic function $\Phi = \varphi + i\psi$ which is Morse on M_0 , C^k in M_0 for large $k \in \mathbb{N}$ and such that $\partial\Phi(p) = 0$ and Φ has only finitely many critical points in M_0 . Furthermore we ask that Φ is purely real on Γ_0 . By Proposition 2.1 such points p form a dense subset of M_0 . Given such a holomorphic function, the purpose of this section is to construct solutions u on M_0 of $(\Delta + V)u = 0$ of the form

$$(10) \quad u = e^{\Phi/h} (a + ha_0 + hr_1) + \overline{e^{\Phi/h} (a + ha_0 + hr_1)} + e^{\varphi/h} r_2 \quad \text{with} \quad u|_{\Gamma_0} = 0$$

for $h > 0$ small, where a is holomorphic and $u \in C^k(M_0)$ for large $k \in \mathbb{N}$, $a_0 \in W^{2,\infty}(M_0)$ is holomorphic, moreover $a(p) \neq 0$ and a vanishes to high order at all other critical points $p' \in M_0$ of Φ . Furthermore, we ask that the holomorphic function a is purely imaginary on Γ_0 . The existence of such a holomorphic function is a consequence of Lemma 2.3. Given such a holomorphic function on M_0 we consider a compactly supported smooth extension to M , still denoted a .

The remainder terms r_1, r_2 will be controlled as $h \rightarrow 0$ and have particular properties near the critical points of Φ . More precisely, r_2 will be a $O_{L^2}(h^{3/2})$ and r_1 will be independent of h and with an explicit expression, which can be used to obtain sufficient informations from the stationary phase method.

4.1. Construction of r_1 . We shall construct r_1 to satisfy

$$e^{-\Phi/h}(\Delta_g + V)e^{\Phi/h}(a + hr_1) = O(h)$$

in L^2 and r_1 independent of h . We let G be the Green operator of the Laplacian on the smooth surface with boundary M_0 with Dirichlet condition, so that $\Delta_g G = \text{Id}$ on $L^2(M_0)$. In particular this implies that $\bar{\partial}\partial G = \frac{i}{2}\star^{-1}$ where \star^{-1} is the inverse of \star mapping functions to 2-forms. We extend a to be a compactly supported C^k function on M_0 and we will search for $r_1 \in W^{2,\infty}(M_0)$ independent of h satisfying

$$(11) \quad e^{-2i\psi/h}\partial e^{2i\psi/h}hr_1 = -\partial G(aV) + \omega + O_{W^{1,\infty}}(h)$$

where ω is a smooth holomorphic 1-form on M_0 . Indeed, using the fact that Φ is holomorphic we have

$$e^{-\Phi/h}\Delta_g e^{\Phi/h} = -2i\star\bar{\partial}e^{-\Phi/h}\partial e^{\Phi/h} = -2i\star\bar{\partial}e^{-\frac{1}{h}(\Phi-\bar{\Phi})}\partial e^{\frac{1}{h}(\Phi-\bar{\Phi})} = -2i\star\bar{\partial}e^{-2i\psi/h}\partial e^{2i\psi/h}$$

and applying $-2i\star\bar{\partial}$ to (11), we obtain (note that $\partial G(aV) \in W^{2,\infty}(M_0)$ by elliptic regularity)

$$e^{-\Phi/h}(\Delta_g + V)e^{\Phi/h}hr_1 = -aV + O_{L^\infty}(h).$$

The form ω above, will be chosen as a correction term to optimize the use of the stationary phase later, this is why we need the following

Lemma 4.1. *Let $\{p_0, \dots, p_N\}$ be finitely many points on M_0 and let g be a C^k section of $T_{1,0}^*M_0$. Fix $K \in \mathbb{N}$, then there exists a holomorphic function f on M such that $\omega = \partial f$ satisfies $(g - \omega)(m) = O(d(m, p_j)^K)$ for all $j = 0, \dots, N$ and $m \in M_0$, where here $d(\cdot, \cdot)$ denotes the distance.*

Proof. This is a direct consequence of Lemma 2.3 applied to the manifold M_0 whose interior contains $\{p_0, \dots, p_N\}$. \square

With this lemma, we choose ω to be a holomorphic 1-form on M_0 such that at all critical point p' of Φ in M_0 , we have $(\partial G(aV) - \omega)(q) = O(|q - p'|^k)$ for all $q \in M_0$; this can be done since $\partial G(aV)$ is a 1-form with value in $T_{1,0}^*M_0$.

We will construct r_1 by observing that since $\partial G(aV) - \omega$ vanishes to high order at critical points of Φ , we can define a function $r_1 \in W^{2,\infty}(M_0)$ such that

$$2ir_1\partial\psi = -\partial G(aV) + \omega.$$

This can be done since both $\partial\psi$ and the right hand side are valued in $T_{1,0}^*M_0$. Such r_1 clearly satisfies (11) and is independent of h .

We summarize the result of this section with the following

Lemma 4.2. *Let $k \in \mathbb{N}$ be large and $\Phi \in C^k(M_0)$ be a holomorphic function on M_0 which is Morse in M_0 . Let $a \in C^k(M_0)$ be a holomorphic function on M_0 . Then there exists $r_1 \in W^{2,\infty}(M_0)$ independent of h such that*

$$e^{-\Phi/h}(\Delta + V)e^{\Phi/h}(a + hr_1) = O_{L^\infty}(h).$$

4.2. Construction of a_0 . We have constructed the correction terms r_1 which solves the Schrödinger equation to order h as stated in Lemma 4.2. In this subsection, we will construct a holomorphic function a_0 which annihilates the boundary value of the solution on Γ_0 . In particular, we have the following

Lemma 4.3. *There exists a holomorphic function $a_0 \in W^{2,\infty}(M_0)$ such that*

$$e^{-\Phi/h}(\Delta + V)e^{\Phi/h}(a + hr_1 + ha_0) = O_{L^\infty}(h)$$

and

$$[e^{\Phi/h}(a + hr_1 + ha_0) + \overline{e^{\Phi/h}(a + hr_1 + ha_0)}]_{\Gamma_0} = 0.$$

Proof. Since Φ is purely real on Γ_0 and a is purely imaginary on Γ_0 , we see that this lemma amounts to construct holomorphic functions a_0 with the boundary condition

$$2\operatorname{Re}(r_1) + 2\operatorname{Re}(a_0) = 0 \text{ on } \Gamma_0.$$

To construct a_0 , it suffices to use (iii) in Corollary 2.2. \square

4.3. Construction of r_2 . The goal of this section is to complete the construction of the complex geometric optic solutions by the following proposition:

Proposition 4.1. *There exist solutions to $(\Delta + V)u = 0$ with boundary condition $u|_{\Gamma_0} = 0$ of the form (10) with r_1, a_0 constructed in the previous sections and r_2 satisfying $\|r_2\|_{L^2} \leq Ch^{3/2}$.*

This is a consequence of the following Lemma (which follows from the Carleman estimate obtained above):

Lemma 4.4. *Let $V \in L^\infty(M_0)$ and $f \in L^2(M_0)$. For all $h > 0$ small enough, there exists a solution $v \in L^2$ to the boundary value problem*

$$e^{\varphi/h}(\Delta_g + V)e^{-\varphi/h}v = f, \quad v|_{\Gamma_0} = 0,$$

satisfying the estimate

$$\|v\|_{L^2} \leq Ch^{\frac{1}{2}}\|f\|_{L^2}.$$

Proof. The proof is the same as Proposition 2.2 of [16], we repeat the argument for the convenience of the reader. Define for all $h > 0$ the real vector space $\mathcal{A} := \{u \in H_0^1(M_0); (\Delta_g + V)u \in L^2(M_0), \partial_\nu u|_{\Gamma} = 0\}$ equipped with the real scalar product

$$(u, w)_{\mathcal{A}} := \int_{M_0} e^{-2\varphi/h}(\Delta_g u + Vu)(\Delta_g w + Vw) \, dv_g.$$

Observe that since ψ is constant along Γ_0 , $\partial_\nu \varphi = 0$ on Γ_0 . Therefore, we may apply the Carleman estimate of Proposition (3.1) to the weight φ to assert that the space \mathcal{A} is a Hilbert space equipped with the scalar product above. By using the same estimate, the linear

functional $L : w \rightarrow \int_{M_0} e^{-\varphi/h} f w \, dv_g$ on \mathcal{A} is continuous and its norm is bounded by $h^{\frac{1}{2}} \|f\|_{L^2}$. By Riesz theorem, there is an element $u \in \mathcal{A}$ such that $(\cdot, u)_{\mathcal{A}} = L$ and with norm bounded by the norm of L . It remains to take $v := e^{-\varphi/h}(\Delta_g u + V u)$ which solves $(\Delta_g + V)e^{-\varphi} v = e^{-\varphi/h} f$ and which in addition satisfies the desired norm estimate. Furthermore, since

$$\int_{M_0} e^{-\varphi/h} v (\Delta_g + V) w \, dv_g = \int_{M_0} e^{-\varphi/h} f w \, dv_g$$

for all $w \in \mathcal{A}$, we have by Green's theorem

$$\int_{\partial M_0} e^{-\varphi/h} v \partial_\nu w \, dv_g = 0 = \int_{\Gamma_0} e^{-\varphi/h} v \partial_\nu w \, dv_g$$

for all $w \in \mathcal{A}$. This implies $v = 0$ on Γ_0 . \square

Proof of Proposition 4.1. We note that

$$(\Delta + V)(e^{\Phi/h}(a + hr_1 + ha_0) + \overline{e^{\Phi/h}(a + hr_1 + ha_0)} + e^{\varphi/h} r_2) = 0$$

if and only if

$$e^{-\varphi/h}(\Delta + V)e^{\varphi/h} r_2 = -e^{-\varphi/h}(\Delta + V)(e^{\Phi/h}(a + hr_1 + ha_0) + \overline{e^{\Phi/h}(a + hr_1 + ha_0)}).$$

By Lemma 4.3, the right hand side of the above equation is $O_{L^2}(h)$. Therefore, using Lemma 4.4 one can find such r_2 which satisfies

$$\|r_2\|_{L^2} \leq Ch^{3/2}, \quad r_2|_{\Gamma_0} = 0.$$

Since the ansatz $e^{\Phi/h}(a + hr_1 + ha_0) + \overline{e^{\Phi/h}(a + hr_1 + ha_0)}$ is arranged to vanish on Γ_0 , the solution

$$u = e^{\Phi/h}(a + hr_1 + ha_0) + \overline{e^{\Phi/h}(a + hr_1 + ha_0)} + e^{\varphi/h} r_2$$

vanishes on Γ_0 as well. \square

5. IDENTIFYING THE POTENTIAL

We now assume that $V_1, V_2 \in W^{1,\infty}(M_0)$ are two real valued potentials such that the respective Cauchy data spaces $\mathcal{C}_1^\Gamma, \mathcal{C}_2^\Gamma$ for the operators $\Delta_g + V_1$ and $\Delta_g + V_2$ on $\Gamma \subset \partial M_0$ are equal. Let $\Gamma_0 = \partial M_0 \setminus \Gamma$ be the complement of Γ in ∂M_0 , and possibly by taking Γ slightly smaller, we may assume that Γ_0 contains an open set. Let $p \in M_0$ be an interior point of M_0 such that, using Proposition 2.1, we can choose a holomorphic Morse function $\Phi = \varphi + i\psi$ on M_0 with Φ purely real on Γ_0 , C^k in M_0 for some large $k \in \mathbb{N}$, with a critical point at p . Note that Proposition 2.1 states that we can choose Φ such that none of its critical points on the boundary are degenerate and such that critical points do not accumulate on the boundary.

Proposition 5.1. *If the Cauchy data spaces agree, i.e. if $\mathcal{C}_1^\Gamma = \mathcal{C}_2^\Gamma$, then $V_1(p) = V_2(p)$.*

Proof. Let a be a holomorphic function on M_0 which is purely imaginary on Γ_0 with $a(p) \neq 0$ and $a(p') = 0$ to large order for all other critical point p' of Φ . The existence of a is insured by Lemma 2.3. Let u_1 and u_2 be H^2 solutions on M_0 to

$$(\Delta_g + V_j)u_j = 0$$

constructed in Section 4 with $\Phi = \phi + i\psi$ for Carleman weight for u_1 and $-\Phi$ for u_2 , thus of the form

$$u_1 = e^{\Phi/h}(a + ha_0 + hr_1) + \overline{e^{\Phi/h}(a + ha_0 + hr_1)} + e^{\varphi/h} r_2$$

$$u_2 = e^{-\Phi/h}(a + hb_0 + hs_1) + \overline{e^{-\Phi/h}(a + hb_0 + hs_1)} + e^{-\varphi/h}s_2$$

and with boundary value $u_j|_{\partial M_0} = f_j$, where f_j vanishes on Γ_0 . We can write by Green formula

$$\begin{aligned} \int_{M_0} u_1(V_1 - V_2)u_2 dv_g &= - \int_{M_0} (\Delta_g u_1 \cdot u_2 - u_1 \cdot \Delta_g u_2) dv_g \\ &= - \int_{\partial M_0} (\partial_\nu u_1 \cdot f_2 - f_1 \cdot \partial_\nu u_2) dv_g. \end{aligned}$$

Since the Cauchy data for $\Delta_g + V_1$ agrees on Γ with that of $\Delta_g + V_2$, there exists a solution v of the boundary value problem

$$(\Delta_g + V_2)v = 0, \quad v|_{\partial M_0} = f_1,$$

satisfying $\partial_\nu v = \partial_\nu u_1$ on Γ . Since $f_j = 0$ on Γ_0 , this implies that

$$\begin{aligned} (12) \quad \int_{M_0} u_1(V_1 - V_2)u_2 dv_g &= - \int_{M_0} (\Delta_g u_1 \cdot u_2 - u_1 \cdot \Delta_g u_2) dv_g = - \int_{\partial M_0} (\partial_\nu u_1 \cdot f_2 - f_1 \cdot \partial_\nu u_2) dv_g \\ &= - \int_{\partial M_0} (\partial_\nu v \cdot f_2 - v \cdot \partial_\nu u_2) dv_g = - \int_{M_0} (\Delta_g v \cdot u_2 - v \cdot \Delta_g u_2) dv_g = 0 \end{aligned}$$

since $\Delta_g + V_2$ annihilates both v and u_2 . We substitute in the full expansion for u_1 and u_2 and, setting $V := V_1 - V_2$, we have that

$$(13) \quad 0 = I_1 + I_2 + o(h),$$

where

$$(14) \quad I_1 = \int_{M_0} V(a^2 + \bar{a}^2) dv_g + \int_{M_0} (e^{2i\psi/h} + e^{-2i\psi/h})V|a|^2 dv_g,$$

$$(15) \quad I_2 = 2h \operatorname{Re} \left(\int_{M_0} V e^{2i\psi/h} a \overline{(s_1 + b_0)} + Va(b_0 + s_1 + a_0 + r_1) dv_g \right).$$

We will apply the stationary phase to these two terms in the following two Lemmas, the proofs of which are deferred to the end of this section.

Lemma 5.1. *The following estimate holds true*

$$I_1 = \int_{M_0} V(a^2 + \bar{a}^2) dv_g + hC_p V(p)|a(p)|^2 \operatorname{Re}(e^{2i\psi(p)/h}) + o(h)$$

with $C_p \neq 0$ and independent of h .

Lemma 5.2. *The following estimates holds true*

$$I_2 = 2h \operatorname{Re} \left(\int_{M_0} Va(b_0 + s_1 + a_0 + r_1) dv_g \right) + o(h).$$

With these two lemma we can write (13) as

$$0 = \int_{M_0} V(a^2 + \bar{a}^2) dv_g + O(h)$$

and thus we can conclude that

$$0 = \int_{M_0} V(a^2 + \bar{a}^2) dv_g.$$

Therefore, (13) becomes

$$0 = C_p V(p) |a(p)|^2 \operatorname{Re}(e^{2i\psi(p)/h}) + 2\operatorname{Re}\left(\int_{M_0} V a(b_0 + s_1 + a_0 + r_1) dv_g\right) + o(1).$$

Since $\psi(p) \neq 0$ we may choose a sequence of $h_j \rightarrow 0$ such that $\operatorname{Re}(e^{2i\psi(p)/h_j}) = 1$ and another sequence $\tilde{h}_j \rightarrow 0$ such that $\operatorname{Re}(e^{2i\psi(p)/\tilde{h}_j}) = -1$ for all j . Adding the expansion with $h = h_j$ and $h = \tilde{h}_j$, we deduce that

$$0 = 2C_p V(p) |a(p)|^2 + o(1)$$

as $j \rightarrow \infty$, and since $C_p \neq 0$, $a(p) \neq 0$, we conclude that $V(p) = 0$. The set of $p \in M_0$ for which we can conclude this is dense in M_0 by Proposition 2.1. Therefore we can conclude that $V(p) = 0$ for all $p \in M_0$. \square

We now prove Lemmas 5.1 and 5.2. First we will state an auxillary lemma which will be useful throughout.

Lemma 5.3. *Let $f \in L^1(M_0)$, then as $h \rightarrow 0$*

$$\int_{M_0} e^{2i\psi/h} f dv_g = o(1).$$

Proof. Since $C^k(M_0)$ is dense in $L^1(M_0)$ for all $k \in \mathbb{N}$, it suffices to prove the Lemma for $f \in C^k(M_0)$. Let $\epsilon > 0$ be small, and choose cut off function χ which is identically equal to 1 on the boundary such that

$$\int_{M_0} \chi |f| dv_g \leq \epsilon.$$

Then, splitting the integral and using stationary phase for the $1 - \chi$ term, we obtain

$$\left| \int_{M_0} e^{2i\psi/h} f dv_g \right| \leq \left| \int_{M_0} (1 - \chi) e^{2i\psi/h} f dv_g \right| + \left| \int_{M_0} \chi e^{2i\psi/h} f dv_g \right| \leq \epsilon + O_\epsilon(h)$$

which concludes the proof by taking h small enough depending on ϵ . \square

We see that Lemma 5.3 immediately implies Lemma 5.2. The principal term I_1 will require slightly more careful analysis:

Proof of Lemma 5.1. Let χ be a smooth cutoff function on M_0 which is identically 1 everywhere except outside a small ball containing p and no other critical point of Φ , and $\chi = 0$ near p . We split the oscillatory integral in two parts:

$$\begin{aligned} \int_{M_0} (e^{2i\psi/h} + e^{-2i\psi/h}) V |a|^2 dv_g &= \int_{M_0} \chi (e^{2i\psi/h} + e^{-2i\psi/h}) V |a|^2 dv_g \\ &\quad + \int_{M_0} (1 - \chi) (e^{2i\psi/h} + e^{-2i\psi/h}) V |a|^2 dv_g \end{aligned}$$

The phase ψ has nondegenerate critical points, therefore, a standard application of the stationary phase at p gives

$$\int_{M_0} (1 - \chi) (e^{2i\psi/h} + e^{-2i\psi/h}) V(p) |a|^2 dv_g = h C_p |a(p)|^2 V(p) \operatorname{Re}(e^{2i\psi(p)/h}) + o(h)$$

where C_p is a non-zero number which depends on the Hessian of ψ at the point p . Define the potential $\tilde{V}(\cdot) := V(\cdot) - V(p)$, then we show that

$$(16) \quad \int_{M_0} (1 - \chi)(e^{2i\psi/h} + e^{-2i\psi/h})\tilde{V}|a|^2 dv_g = o(h).$$

Indeed, by integration by parts and using $\Delta_g \psi = 0$ one has

$$\begin{aligned} \int_{M_0} (1 - \chi)(e^{2i\psi/h} + e^{-2i\psi/h})\tilde{V}|a|^2 dv_g &= \frac{h}{2i} \int_{M_0} \langle d(e^{2i\psi/h} - e^{-2i\psi/h}), d\psi \rangle \tilde{V} \frac{(1 - \chi)|a|^2}{|d\psi|^2} dv_g \\ &= \frac{h}{2i} \int_{M_0} (e^{2i\psi/h} - e^{-2i\psi/h}) \langle d\left(\frac{(1 - \chi)|a|^2 \tilde{V}}{|d\psi|^2}\right), d\psi \rangle dv_g \end{aligned}$$

and $\langle d((1 - \chi)|a|^2 \tilde{V}/|d\psi|^2), d\psi \rangle \in L^1(M_0)$ by the non degeneracy of $\text{Hess}(\psi)$, the fact that $\tilde{V} \in W^{1,\infty}(M_0)$ and that $\tilde{V}(p) = 0$. It then suffice to use Lemma 5.3 to conclude that (16) holds. Using similar argument, we now show that

$$\int_{M_0} \chi(e^{2i\psi/h} + e^{-2i\psi/h})V|a|^2 dv_g = o(h).$$

Indeed, since a vanishes to large order at all boundary critical points of ψ , we may write

$$\begin{aligned} \int_{M_0} \chi(e^{2i\psi/h} + e^{-2i\psi/h})V|a|^2 dv_g &= \frac{h}{2i} \int_{M_0} \langle d(e^{2i\psi/h} - e^{-2i\psi/h}), d\psi \rangle V \frac{\chi|a|^2}{|d\psi|^2} dv_g \\ &= -\frac{h}{2i} \int_{M_0} (e^{2i\psi/h} - e^{-2i\psi/h}) \text{div}_g \left(V \frac{\chi|a|^2}{|d\psi|^2} \nabla^g \psi \right) dv_g \\ &\quad + \frac{h}{2i} \int_{\partial M_0} (e^{2i\psi/h} - e^{-2i\psi/h}) V \frac{|a|^2}{|d\psi|^2} \partial_\nu \psi dv_g. \end{aligned}$$

For the interior integral we use Lemma 5.3 to conclude that

$$-\frac{h}{2i} \int_{M_0} (e^{2i\psi/h} - e^{-2i\psi/h}) \text{div}_g \left(V \frac{\chi|a|^2}{|d\psi|^2} \nabla^g \psi \right) dv_g = o(h)$$

and for the boundary integral, we write $\partial M_0 = \Gamma_0 \cup \Gamma$ and observe that on Γ_0 , $\psi = 0$ so $(e^{2i\psi/h} - e^{-2i\psi/h}) = 0$, while on Γ we have $V = 0$ from the boundary determinacy proved in Proposition 7.1 of the Appendix. Therefore

$$\int_{M_0} \chi(e^{2i\psi/h} + e^{-2i\psi/h})V|a|^2 dv_g = o(h)$$

and the proof is complete. \square

6. INVERSE SCATTERING

We first obtain, as a trivial consequence of Theorem 1.1, a result about inverse scattering for asymptotically hyperbolic surface (AH in short). Recall that an AH surface is an open complete Riemannian surface (X, g) such that X is the interior of a smooth compact surface with boundary \bar{X} , and for any smooth boundary defining function x of $\partial \bar{X}$, $\bar{g} := x^2 g$ extends as a smooth metric to \bar{X} , with curvature tending to -1 at $\partial \bar{X}$. If $V \in C^\infty(\bar{X})$ and $V = O(x^2)$, then we can define a scattering map as follows (see for instance [17, 12] or [13]): first the L^2 kernel $\ker_{L^2}(\Delta_g + V)$ is a finite dimensional subspace of $x C^\infty(\bar{X})$ and in one-to-one correspondence with $E := \{(\partial_x \psi)|_{\partial \bar{X}}; \psi \in \ker_{L^2}(\Delta_g + V)\}$ where $\partial_x := \nabla^{\bar{g}} x$ is the normal

vector field to $\partial\bar{X}$ for \bar{g} , then for $f \in C^\infty(\partial\bar{X})$, there exists a function $u \in C^\infty(\bar{X})$, unique modulo $\ker_{L^2}(\Delta_g + V)$, such that $(\Delta_g + V)u = 0$ and $u|_{\partial\bar{X}} = f$. Then one can see that the scattering map $\mathcal{S} : C^\infty(\partial\bar{X}) \rightarrow C^\infty(\partial\bar{X})/E$ is defined by $\mathcal{S}f := \partial_x u|_{\partial\bar{X}}$. We thus obtain

Corollary 6.1. *Let (X, g) be an asymptotically hyperbolic manifold and let $V_1, V_2 \in x^2 C^\infty(\bar{X})$ be two potentials and $\Gamma \subset \partial\bar{X}$ an open subset of the conformal boundary. Assume that*

$$\{\partial_x u|_{\partial\bar{X}}; u \in \ker_{L^2}(\Delta_g + V_1)\} = \{\partial_x u|_{\partial\bar{X}}; u \in \ker_{L^2}(\Delta_g + V_2)\}$$

and let \mathcal{S}_j be the scattering map for the operator $\Delta_g + V_j$ for $j = 1, 2$. If $\mathcal{S}_1 f = \mathcal{S}_2 f$ on Γ for all $f \in C_0^\infty(\Gamma)$ then $V_1 = V_2$.

Proof. Let x be a smooth boundary defining function of $\partial\bar{X}$, and let $\bar{g} = x^2 g$ be the compactified metric and define $\bar{V}_j := V_j/x^2 \in C^\infty(\bar{X})$. By conformal invariance of the Laplacian in dimension 2, one has

$$\Delta_g + V_j = x^2(\Delta_{\bar{g}} + \bar{V}_j)$$

and so if $\ker_{L^2}(\Delta_g + V_1) = \ker_{L^2}(\Delta_g + V_2)$ and $\mathcal{S}_1 = \mathcal{S}_2$ on Γ , then the Cauchy data spaces \mathcal{C}_i^Γ for the operator $\Delta_{\bar{g}} + \bar{V}_j$ are the same. Then it suffices to apply the result in Theorem 1.1. \square

Next we consider the asymptotically Euclidean scattering at 0 frequency. An asymptotically Euclidean surface is a non-compact Riemann surface (X, g) , which compactifies into \bar{X} and such that the metric in a collar $(0, \epsilon)_x \times \partial\bar{X}$ near the boundary is of the form

$$g = \frac{dx^2}{x^4} + \frac{h(x)}{x^2}$$

where $h(x)$ is a smooth one-parameter family of metrics on $\partial\bar{X}$ with $h(0) = d\theta_{S^1}^2$ is the metric with length 2π on each copy of S^1 that forms the connected components of $\partial\bar{X}$. Notice that using the coordinates $r := 1/x$, g is asymptotic to $dr^2 + r^2 d\theta_{S^1}^2$ near $r \rightarrow \infty$. A particular case is given by the surfaces with Euclidean ends, ie. ends isometric to $\mathbb{R}^2 \setminus B(0, R)$ where $B(0, R) = \{z \in \mathbb{R}^2; |z| \leq R\}$. Note that g is conformal to an asymptotically cylindrical metric, or 'b-metric' in the sense of Melrose [23],

$$g_b := x^2 g = \frac{dx^2}{x^2} + h(x)$$

and the Laplacian satisfies $\Delta_g = x^2 \Delta_{g_b}$. Each end of X is of the form $(0, \epsilon)_x \times S_\theta^1$ and the operator Δ_{g_b} has the expression in the ends

$$\Delta_{g_b} = -(x\partial_x)^2 + \Delta_{\partial\bar{X}} + xP(x, \theta; x\partial_x, \partial_\theta)$$

for some smooth differential operator $P(x, \theta; x\partial_x, \partial_\theta)$ in the vector fields $x\partial_x, \partial_\theta$ down to $x = 0$. Let us define $V_b := x^{-2}V$, which is compactly supported and

$$H_b^{2m} := \{u \in L^2(X, \text{dvol}_{g_b}); \Delta_{g_b}^m u \in L^2(X, \text{dvol}_{g_b})\}, \quad m \in \mathbb{N}_0.$$

We also define the following spaces for $\alpha \in \mathbb{R}$

$$F_\alpha := \ker_{x^\alpha H_b^2}(\Delta_{g_b} + V_b).$$

Since the eigenvalues of Δ_{S^1} are $\{j^2; j \in \mathbb{N}_0\}$, the relative Index theorem of Melrose [23, Section 6.2] shows that $\Delta_{g_b} + V_b$ is Fredholm from $x^\alpha H_b^2$ to $x^\alpha H_b^0$ if $\alpha \notin \mathbb{Z}$. Moreover,

from subsection 2.2.4 of [23], we have that any solution of $(\Delta_{g_b} + V_b)u = 0$ in $x^\alpha H_b^2$ has an asymptotic expansion of the form

$$u \sim \sum_{j > \alpha, j \in \mathbb{Z}} \sum_{\ell=0}^{\ell_j} x^j (\log x)^\ell u_{j,\ell}(\theta), \quad \text{as } x \rightarrow 0$$

for some sequence $(\ell_j)_j$ of non negative integers and some smooth function $u_{j,\ell}$ on S^1 . In particular, it is easy to check that $\ker_{L^2(X, \text{dvol}_g)}(\Delta_g + V) = F_{1+\epsilon}$ for $\epsilon \in (0, 1)$.

Theorem 6.2. *Let (X, g) be an asymptotically Euclidean surface and V_1, V_2 be two compactly supported smooth potentials and x be a boundary defining function. Let $\epsilon \in (0, 1)$ and assume that for any $j \in \mathbb{Z}$ and any function $\psi \in \ker_{x^{j-\epsilon} H_b^2}(\Delta_g + V_1)$ there is a $\varphi \in \ker_{x^{j-\epsilon} H_b^2}(\Delta_g + V_2)$ such that $\psi - \varphi = O(x^\infty)$, and conversely. Then $V_1 = V_2$.*

Proof. The idea is to reduce the problem to the compact case. First we notice that by unique continuation, $\psi = \varphi$ where $V_1 = V_2 = 0$. Now it remains to prove that, if R_η denote the restriction of smooth functions on X to $\{x \geq \eta\}$ and V is a smooth compactly supported potential in $\{x \geq \eta\}$, then the set $\cup_{j=0}^\infty R_\eta(F_{-j-\epsilon})$ is dense in the set N_V of $H^2(\{x \geq \eta\})$ solutions of $(\Delta_g + V)u = 0$. The proof is well known for positive frequency scattering (see for instance Lemma 3.2 in [24]), here it is very similar so we do not give much details. The main argument is to show that it converges in L^2 sense and then use elliptic regularity; the L^2 convergence can be shows as follows: let $f \in N_V$ such that

$$\int_{x \geq \eta} f \psi \text{dvol}_g = 0, \quad \forall \psi \in \cup_{j=0}^\infty F_{-j-\epsilon},$$

then we want to show that $f = 0$. By Proposition 5.64 in [23], there exists $k \in \mathbb{N}$ and a generalized right inverse G_b for $P_b = \Delta_{g_b} + V_b$ (here, as before, $x^2 V_b = V$) in $x^{-k-\epsilon} H_b^2$, such that $P_b G_b = \text{Id}$. This holds in $x^{-k-\epsilon} H_b^2$ for k large enough since the cokernel of P_b on this space becomes 0 for k large. Let $\omega = G_b f$ so that $(\Delta_{g_b} + V_b)\omega = f$, and in particular this function is 0 in $\{x < \eta\}$. The asymptotic behaviour of the integral kernel $G_b(z, z')$ of G_b as $z \rightarrow \infty$ is given in Proposition 5.64 of [23] uniformly in $z' \in \{x \geq \eta\}$, we have for all $J \in \mathbb{N}$ and using the radial coordinates $(x, \theta) \in (0, \epsilon) \times S^1$ for z in the ends

$$G_b(z, z') = \sum_{j=-k}^J \sum_{\ell=0}^{\ell_j} x^j (\log x)^\ell \psi_j(\theta, z') + o(x^J)$$

for some functions $\psi_{j,\ell} \in x^{k-j-\epsilon} H_b^2$ and some sequence $(\ell_j)_j$ of non-negative integers. But the fact that $(\Delta_{g_b} + V_b)G_b(z, z') = \delta(z - z')$ as distributions implies directly that $(\Delta_{g_b} + V_b)\psi_j(\theta, \cdot) = 0$. Using our assumption on f , we deduce that $\int_X \psi_j(\theta, z') f(z') \text{dvol}_{g_b} = 0$ for all $j \in \mathbb{N}_0$ and so the function ω vanish faster than all power of x at infinity. Then by unique continuation, we deduce that $\omega = 0$ in $\{x \leq \epsilon\}$. Since now $\omega \in H^2$, its Cauchy data at $x = \eta$ are 0 and $\Delta_{g_b} + V_b$ is self adjoint for the measure dvol_{g_b} , we can use the Green formula to obtain

$$\int_{x \geq \eta} |f|^2 \text{dvol}_{g_b} = \int_{x \geq \eta} \omega (\Delta_{g_b} + V_b) \bar{f} \text{dvol}_{g_b} = 0.$$

The H^2 density is easy using elliptic regularity. □

7. APPENDIX : BOUNDARY DETERMINATION

In this appendix, we give a short proof of the fact that the partial Cauchy data space on $\Gamma \subset \partial M$ determines the potential on Γ when the potential is in $W^{1,\infty}(M)$. This result is shown for the conductivity problem on a domain of \mathbb{R}^n in [19] under the assumption that the conductivity has roughly $n/2$ -derivatives, it is also shown in [29] for continuous potentials on a smooth domain of \mathbb{R}^n by using a limiting argument from the smooth case (which they analyze using micolocal analysis near the boundary). Alessandrini [1] also proved such a result under the assumption that the domain is Lipschitz and the coefficients of the operator are in $W^{1,p}$ for $p > n$, while Brown [5] studied the case of Lipschitz domains with a continuous conductivity. Since the result in our setting is not explicitly written down, but certainly known from specialists, we provide a short proof without too many details, and using the approach of [5]. We shall prove

Proposition 7.1. *Let $\Gamma \subset \partial M_0$ be an non-empty open subset of the boundary. If $V_1, V_2 \in W^{1,\infty}(M)$ and their associated Cauchy data spaces $\mathcal{C}_1^\Gamma, \mathcal{C}_2^\Gamma$ defined in (1) are equal, then $V_1|_\Gamma = V_2|_\Gamma$.*

The key to proving this proposition is the existence of solutions to $(\Delta_g + V_i)u = 0$ which concentrate near a point $p \in \Gamma$. First we need a solvability result for the equation $(\Delta_g + V_i)u = f$, which is an easy consequence of the Carleman estimate of Proposition 3.1, and follows the method of Salo-Tzou [27, Section 6]. If we fix $h > 0$ small and take $\varphi = 1$ in the Carleman estimate of Proposition 3.1, we obtain easily that there is a constant C such that for all functions in $H^2(M_0)$ satisfying $u|_{\partial M_0} = 0$

$$(17) \quad \|u\|_{H^2}^2 + \|\partial_\nu u\|_{L^2(\Gamma_0)}^2 \leq C(\|(\Delta + V_i)u\|_{L^2}^2 + \|\partial_\nu u\|_{L^2(\Gamma)}^2)$$

As a consequence, we deduce the following solvability result: let

$$\mathcal{B} := \{w \in H^2(M_0) \cap H_0^1(M_0) \mid \partial_\nu w|_\Gamma = 0\}$$

be the closed subspace of $H^2(M_0)$ under the H^2 norm and let \mathcal{B}^* be its dual space then

Corollary 7.1. *Let $i = 1, 2$, then for all $f \in L^2(M_0)$ there exists $u \in H^2(M_0)$ solving the equation*

$$(\Delta_g + V_i)u = f$$

with boundary condition $u|_{\Gamma_0} = 0$, and $\|u\|_{L^2} \leq C\|f\|_{\mathcal{B}^*}$.

Proof. Set $\mathcal{A} := \{w \in H_0^1(M_0) \mid (\Delta_g + V_i)w \in L^2, \partial_\nu w|_\Gamma = 0\}$ equipped with the inner product

$$(v, w)_\mathcal{A} := \int_{M_0} (\Delta_g + V_i)v(\Delta_g + V_i)\bar{w} \, dv_g.$$

Thanks to (17), \mathcal{A} is a Hilbert space and $\mathcal{A} = \mathcal{B}$. For each $f \in L^2(M_0)$, let us define the linear functional on \mathcal{B}

$$L_f : w \mapsto \int_{M_0} wf \, dv_g.$$

By (17), we have that for all $w \in \mathcal{B}$

$$|L_f(w)| \leq \|f\|_{\mathcal{B}^*} \|w\|_\mathcal{B} \leq \|f\|_{\mathcal{B}^*} \|w\|_\mathcal{A}.$$

Therefore, by Riesz Theorem, there exists $v_f \in \mathcal{A}$ such that

$$\int_{M_0} (\Delta_g + V)\bar{v}_f(\Delta_g + V_i)w \, dv_g = \int_{M_0} wf \, dv_g$$

for all $w \in \mathcal{B}$. Furthermore, $\|(\Delta_g + V_i)v_f\|_{L^2} \leq \|f\|_{\mathcal{B}^*}$. Setting $u := (\Delta_g + V_i)\overline{v_f}$ we have that $(\Delta_g + V_i)u = f$ and $\|u\|_{L^2} \leq \|f\|_{\mathcal{B}^*}$. To obtain the boundary condition for u , observe that since

$$\int_{M_0} u(\Delta_g + V_i)w dv_g = \int_{M_0} f w dv_g$$

for all $w \in \mathcal{B}$, by Green's theorem

$$\int_{\partial M_0} u \partial_\nu w dv_g = 0 = \int_{\Gamma_0} u \partial_\nu w dv_g$$

for all $w \in \mathcal{B}$. This implies $u = 0$ on Γ_0 . \square

Clearly, it suffices to assume that Γ is a small piece of the boundary which is contained in a single coordinate chart with complex coordinates $z = x + iy$ where $|z| \leq 1$, $\text{Im}(z) > 0$ and the boundary is given by $\{y = 0\}$. Moreover the metric is of the form $e^{2\rho}|dz|^2$ for some smooth function ρ . Let $p \in \Gamma$ and possibly by translating the coordinates, we can assume that $p = \{z = 0\}$. Let $\eta \in C^\infty(M_0)$ be a cutoff function supported in a small neighbourhood of p . For $h > 0$ small, we define the smooth function $v_h \in C^\infty(M_0)$ supported near p via the coordinate chart $Z = (x, y) \in \mathbb{R}^2$ by

$$(18) \quad v_h(Z) := \eta(Z/\sqrt{h})e^{\frac{1}{h}\alpha \cdot Z}$$

where $\alpha := (i, -1) \in \mathbb{C}^2$ is chosen such that $\alpha \cdot \alpha = 0$. We thus get $(\partial_x^2 + \partial_y^2)e^{\alpha \cdot Z} = 0$ and thus $\Delta_g e^{\alpha \cdot Z} = 0$ by conformal covariance of the Laplacian. Therefore, we have in local coordinates

$$(19) \quad \Delta_g v_h(Z) = \frac{1}{h} e^{\frac{1}{h}\alpha \cdot Z} (\Delta_g \eta)\left(\frac{Z}{\sqrt{h}}\right) + \frac{2}{h^{3/2}} e^{\frac{1}{h}\alpha \cdot Z} \langle d\eta\left(\frac{Z}{\sqrt{h}}\right), \alpha \cdot dZ \rangle_g.$$

Lemma 7.2. *If $V \in W^{1,\infty}(M_0)$, there exists a solution $u_h \in H^2$ to $(\Delta_g + V)u = 0$ of the form*

$$u_h = v_h + R_h,$$

with v_h defined in (18) and $\|R_h\|_{L^2} \leq Ch^{5/4}$, satisfying $\text{supp}(R_h|_{\partial M_0}) \subset \Gamma$.

Proof of Lemma 7.2. We need to find R_h satisfying $\|R_h\|_{L^2} \leq Ch^{5/4}$ and solving

$$(\Delta_g + V)R_h = -(\Delta_g + V)v_h =: M_h.$$

Thanks to Corollary 7.1, it suffices to show that $\|M_h\|_{\mathcal{B}^*} \leq Ch^{5/4}$. Thus, let $w \in \mathcal{B}$, then we have by (19)

$$\int_{M_0} w M_h dv_g = I_1 + I_2 + I_3$$

where

$$I_1 := \int_{|Z| \leq \sqrt{h}} w V_i v_h e^{2\rho} dZ, \quad I_2 := \frac{1}{h} \int_{|Z| \leq \sqrt{h}} w e^{\frac{1}{h}\alpha \cdot Z} \chi_1\left(\frac{Z}{\sqrt{h}}\right) e^{2\rho} dZ,$$

$$I_3 := \frac{2}{h^{3/2}} \int_{|Z| \leq \sqrt{h}} w \chi_2\left(\frac{Z}{\sqrt{h}}\right) e^{\frac{1}{h}\alpha \cdot Z} dZ$$

and $\chi_1 = \Delta_g \eta$, $\chi_2 = i\partial_x \eta - \partial_y \eta$. In the above equation the third term I_3 has the worst growth when $h \rightarrow 0$. We will analyze its behavior and the preceding terms can be treated in similar

fashion. One has

$$\begin{aligned} I_3 &= -h^{1/2} \int_{|Z| \leq \sqrt{h}} w \chi_2\left(\frac{Z}{\sqrt{h}}\right) (\partial_x - i\partial_y)^2 e^{\frac{1}{h}\alpha \cdot Z} dZ \\ &= -h^{1/2} \int_{|Z| \leq \sqrt{h}} (\partial_x - i\partial_y)^2 (w \chi_2\left(\frac{Z}{\sqrt{h}}\right)) e^{\frac{1}{h}\alpha \cdot Z} dZ. \end{aligned}$$

Notice that the boundary term in the integration by parts vanishes because $w \in H_0^1$ and $\partial_\nu w|_{\partial M_0}$ vanishes on the support of η . The term $(\partial_x - i\partial_y)^2 (w \chi_2(\frac{Z}{\sqrt{h}}))$ has derivatives hitting both $\chi_2(\frac{Z}{\sqrt{h}})$ and w . The worst growth in h would occur when both derivatives hit $\chi_2(\frac{Z}{\sqrt{h}})$ in which case a h^{-1} factor would come out. Combined with the $h^{1/2}$ term in front of the integral this gives a total of a $h^{-1/2}$ in front. By this observation we have improved the growth from $h^{-3/2}$ to $h^{-1/2}$. Repeating this line of argument and using Cauchy-Schwarz inequality, we can see that $|I_3| \leq Ch^{5/4} \|w\|_{H^2}$ (an elementary computation shows that functions of the form $\chi(Z/\sqrt{h})e^{\frac{1}{h}\alpha \cdot Z}$ have L^2 norm bounded by $Ch^{3/4}$). Therefore, $\|\frac{1}{h^{3/2}} \chi_2(\frac{Z}{\sqrt{h}}) e^{\frac{1}{h}\alpha \cdot Z}\|_{\mathcal{B}'} \leq Ch^{5/4}$ and we are done. \square

Proof of Proposition 7.1. It suffices to plug the solutions u_h^1, u_h^2 from Proposition 7.2 into the boundary integral identity (12). A simple calculation using the fact that $V_1 - V_2$ is in $W^{1,\infty}$ yields that

$$0 = \int_{M_0} u_h^1 (V_1 - V_2) u_h^2 dv_g = Ch^{3/2} (V_1(p) - V_2(p)) + o(h^{3/2})$$

and we are done. \square

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