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A universal sequence of integers generating balanced Steinhaus figures modulo an odd number

Jonathan Chappelon*

LMPA Joseph Liouville

Université du Littoral Côte d'Opale

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Abstract

In this paper, we partially solve an open problem, due to J. C. Molluzzo in 1976, on the existence of balanced Steinhaus triangles modulo a positive integer n , that are Steinhaus triangles containing all the elements of $\mathbb{Z}/n\mathbb{Z}$ with the same multiplicity. For every odd number n , we build an orbit in $\mathbb{Z}/n\mathbb{Z}$, by the linear cellular automaton generating the Pascal triangle modulo n , which contains infinitely many balanced Steinhaus triangles. This orbit, in $\mathbb{Z}/n\mathbb{Z}$, is obtained from an integer sequence said to be universal. We show that there exist balanced Steinhaus triangles for at least $2/3$ of the admissible sizes, in the case where n is an odd prime power. Other balanced Steinhaus figures, as Steinhaus trapezoids, generalized Pascal triangles, Pascal trapezoids or lozenges, also appear in the orbit of the universal sequence modulo n odd. We prove the existence of balanced generalized Pascal triangles for at least $2/3$ of the admissible sizes, in the case where n is an odd prime power, and the existence of balanced lozenges for all the admissible sizes, in the case where n is a square-free odd number.

1 Introduction

Let n be a positive integer and denote by $\mathbb{Z}/n\mathbb{Z}$ the finite cyclic group of order n . Let $S = (a_j)_{j \in \mathbb{Z}}$ be a doubly infinite sequence of elements in $\mathbb{Z}/n\mathbb{Z}$. The *derived sequence* ∂S of S is the sequence obtained by pairwise adding consecutive terms of S , that is $\partial S = (a_j + a_{j+1})_{j \in \mathbb{Z}}$. This operation of derivation can be repeated and then, the i th derived sequence $\partial^i S$ is recursively defined by $\partial^0 S = S$ and $\partial^i S = \partial \partial^{i-1} S$ for all positive integers i . The sequence of all the iterated derived sequences of S is called the *orbit* $\mathcal{O}_S = (\partial^i S)_{i \in \mathbb{N}}$ of S . For every

*E-mail address : jonathan.chappelon@lmpa.univ-littoral.fr

non-negative integer i and for every integer j , we denote by $a_{i,j}$ the j th term of $\partial^i S$. Since, for every non-negative integer i and for every integer j , we have $a_{i+1,j} = a_{i,j} + a_{i,j+1}$ by the linear local rule of this cellular automaton, then the orbit of S can be seen as the multiset in $\mathbb{Z}/n\mathbb{Z}$ defined by

$$\mathcal{O}_S = \left\{ a_{i,j} = \sum_{k=0}^i \binom{i}{k} a_{j+k} \mid i \in \mathbb{N}, j \in \mathbb{Z} \right\},$$

where $\binom{i}{k}$ is the binomial coefficient $\binom{i}{k} = \frac{i!}{(i-k)!k!}$. For every non-negative integer i , the i th row of \mathcal{O}_S is the sequence $R_i = \partial^i S = (a_{i,j})_{j \in \mathbb{Z}}$ and, for every integer j , the j th diagonal and the j th anti-diagonal of \mathcal{O}_S are the sequences $D_j = (a_{i,j})_{i \in \mathbb{N}}$ and $AD_j = (a_{i,j-i})_{i \in \mathbb{N}}$ respectively. Orbits of integer sequences and the canonical projection map $\pi_n : \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ are also considered in this paper. Elementary figures appear in this linear cellular automaton. Examples of them in $\mathbb{Z}/5\mathbb{Z}$ are depicted in Figure 1.

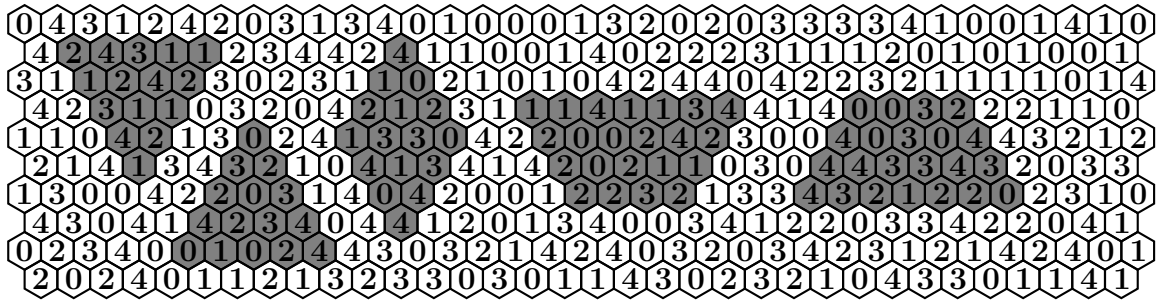


Figure 1: Examples of Steinhaus figures in $\mathbb{Z}/5\mathbb{Z}$: the Steinhaus triangle $\nabla(2, 4, 3, 1, 1)$, the Pascal triangle $\Delta(4, 2, 1, 3, 0, 2, 4, 1, 3)$, the lozenge $\diamond(4, 4, 2, 4, 1, 1, 0)$, the Steinhaus trapezoid $ST((1, 1, 4, 1, 1, 3, 4), 4)$ and the Pascal trapezoid $PT((2, 0, 3, 3, 3, 3, 4, 1, 0, 0, 1, 4, 1), 4)$.

Now, let $S_m = (a_0, \dots, a_{m-1})$ be a finite sequence of length $m \geq 1$ in $\mathbb{Z}/n\mathbb{Z}$. The *Steinhaus triangle* ∇S_m , associated with S_m , is the collection of all the iterated derived sequences of S_m , that is the finite orbit $\nabla S_m = \mathcal{O}_{S_m} = \{S_m, \partial S_m, \dots, \partial^{m-1} S_m\}$. Namely, it is the multiset in $\mathbb{Z}/n\mathbb{Z}$ defined by

$$\nabla S_m = \left\{ \sum_{k=0}^i \binom{i}{k} a_{j+k} \mid 0 \leq i \leq m-1, 0 \leq j \leq m-1-i \right\}.$$

The triangle ∇S_m is said to be of *order* m , since the set of all the Steinhaus triangles associated with a sequence of length m constitutes a $\mathbb{Z}/n\mathbb{Z}$ -module of rank m . For every positive integer m , a Steinhaus triangle of order m has a cardinality of $\binom{m+1}{2}$. These triangles have been named in honor of H. Steinhaus, who proposed this construction, for the binary case $\mathbb{Z}/2\mathbb{Z}$, in his book on elementary mathematical problems [14]. The *Steinhaus trapezoid* $ST(S_m, h)$, of order m and of height h , with $1 \leq h \leq m$, is the collection of the h first derived sequences of S_m , that is,

$$ST(S_m, h) = \bigcup_{i=0}^{h-1} \partial^i S_m = \nabla S_m \setminus \nabla \partial^h S_m.$$

A Steinhaus trapezoid of order m and of height h has a cardinality of $h(2m - h + 1)/2$. Now, let $S_{2m-1} = (a_0, \dots, a_{2m-2})$ be a finite sequence of length $2m - 1$ in $\mathbb{Z}/n\mathbb{Z}$. The *generalized Pascal triangle* (or *Pascal triangle* for short) ΔS_{2m-1} , associated with S_{2m-1} , is the triangle of height m , built from the top to the base, appearing in the center of the Steinhaus triangle ∇S_{2m-1} . Namely, it is the multiset in $\mathbb{Z}/n\mathbb{Z}$ defined by

$$\Delta S_{2m-1} = \left\{ \sum_{k=0}^i \binom{i}{k} a_{m-1-j-k} \mid 0 \leq j \leq i \leq m-1 \right\}.$$

Obviously, the generalized Pascal triangle associated with the sequence $(0, \dots, 0, 1, 0, \dots, 0)$ of length $2m - 1$ in $\mathbb{Z}/n\mathbb{Z}$, the sequence of zeros with the central term equal to 1, corresponds with the m first rows of the standard Pascal triangle modulo n . A Pascal triangle of order $2m - 1$ has a cardinality of $\binom{m+1}{2}$. The *Pascal trapezoid* $PT(S_{2m-1}, h)$, of order $2m - 1$ and of height h , is the collection of the h last rows of the Pascal triangle ΔS_{2m-1} , that is,

$$PT(S_{2m-1}, h) = \Delta S_{2m-1} \setminus \Delta(a_j)_{h \leq j \leq 2m-h-2}.$$

A Pascal trapezoid of order $2m - 1$ and of height h has a cardinality of $h(2m - h + 1)/2$. Finally, the *lozenge* $\diamond S_{2m-1}$, associated with the sequence S_{2m-1} , is the multiset union of the Pascal triangle ΔS_{2m-1} and of the Steinhaus triangle $\nabla \partial^m S_{2m-1}$. The lozenge $\diamond S_{2m-1}$ is then the multiset in $\mathbb{Z}/n\mathbb{Z}$ defined by

$$\diamond S_{2m-1} = \Delta S_{2m-1} \cup \nabla \partial^m S_{2m-1} = \left\{ \sum_{k=0}^{i+j} \binom{i+j}{k} a_{m-1-j-k} \mid 0 \leq i, j \leq m-1 \right\}.$$

A lozenge of order $2m - 1$ has a cardinality of m^2 .

In 1963 [14], H. Steinhaus posed the elementary problem which consists to determined if there exists, for every positive integer m such that $(m + 1)m/2$ is even, a binary Steinhaus triangle of order m containing as many 0's as 1's. This problem was solved, for the first time, by H. Harborth in 1972 [10]. For every positive integer $m \equiv 0$ or $3 \pmod{4}$, he explicitly build at least four such binary Steinhaus triangles of order m . Other solutions of the Steinhaus's problem appear in the literature [7, 8, 9]. A generalization of this problem in any finite cyclic group was posed by J. C. Molluzzo in 1976 [13].

A finite multiset M in $\mathbb{Z}/n\mathbb{Z}$ is said to be *balanced* if each element of $\mathbb{Z}/n\mathbb{Z}$ appears in M with the same multiplicity. Thus, if we denote by $\mathbf{m}_M : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{N}$ the multiplicity function associated with M , then the multiset M is balanced if and only if \mathbf{m}_M is the constant function on $\mathbb{Z}/n\mathbb{Z}$ equal to $|M|/n$, where $|M|$ denotes the cardinality of M .

Problem 1.1 (Molluzzo, 1976). *Let n be a positive integer. For every positive integer m such that the binomial coefficient $\binom{m+1}{2}$ is divisible by n , does there exist a balanced Steinhaus triangle of order m in $\mathbb{Z}/n\mathbb{Z}$?*

In this paper, we explicitly build balanced Steinhaus triangles of order m in $\mathbb{Z}/n\mathbb{Z}$, for every positive integer $m \equiv 0 \pmod{n}$ or $m \equiv -1 \pmod{3n}$ and for every odd number n . This

answers in the affirmative Problem 2 of [5]. In [5], the author completely and positively solved this Molluzzo's problem in $\mathbb{Z}/3^k\mathbb{Z}$, for all $k \geq 1$. Moreover, for every odd number n , he showed that there exist at least $\varphi(n)n$ balanced Steinhaus triangles of order m in $\mathbb{Z}/n\mathbb{Z}$, for every $m \equiv 0$ or $-1 \pmod{\varphi(\text{rad}(n))n}$, where φ is the *Euler totient function* and $\text{rad}(n)$ is the *radical* of n , that is the product of the distinct prime factors of n . As observed in [6], this problem of Molluzzo does not always admit a positive solution. Indeed, it can be verified, by exhaustive search, that there is no balanced Steinhaus triangle of order $m = 5$ in $\mathbb{Z}/15\mathbb{Z}$ or of order $m = 6$ in $\mathbb{Z}/21\mathbb{Z}$. Here, we are also interested in the generalization of the Molluzzo's problem on each kind of Steinhaus figures defined above, not only on Steinhaus triangles.

Problem 1.2. *Let n be a positive integer. For each kind of Steinhaus figures, do there exist balanced Steinhaus figures in $\mathbb{Z}/n\mathbb{Z}$ for all admissible sizes, i.e. for all Steinhaus figures whose cardinality is divisible by n ? In other words,*

- *for every positive integer m such that $\binom{m+1}{2}$ is divisible by n , does there exist a balanced Steinhaus triangle of order m ?*
- *for every positive integers m and h such that $h \leq m$ and $h(2m - h + 1)/2$ is divisible by n , does there exist a balanced Steinhaus trapezoid of order m and of height h ?*
- *for every positive integer m such that $\binom{m+1}{2}$ is divisible by n , does there exist a balanced Pascal triangle of order $2m - 1$?*
- *for every positive integers m and h such that $h \leq m$ and $h(2m - h + 1)/2$ is divisible by n , does there exist a balanced Pascal trapezoid of order $2m - 1$ and of height h ?*
- *for every positive integer m such that m^2 is divisible by n , does there exist a balanced lozenge of order $2m - 1$?*

For every positive integers n and k and for every k -tuples of elements $A = (a_0, \dots, a_{k-1})$ and $D = (d_0, \dots, d_{k-1})$ in $\mathbb{Z}/n\mathbb{Z}$, or in \mathbb{Z} , the k -interlaced arithmetic progression $IAP(A, D)$ is the sequence with first terms (a_0, \dots, a_{k-1}) and with common differences (d_0, \dots, d_{k-1}) , that is the doubly infinite sequence $IAP(A, D) = (a_j)_{j \in \mathbb{Z}}$ defined by $a_{j_0+jk} = a_{j_0} + jd_{j_0}$, for all integers j and for every integer j_0 , $0 \leq j_0 \leq k - 1$. For $k = 1$, we denote by $AP(a_0, d_0)$ the arithmetic progression with first element a_0 and with common difference d_0 .

Let $US = IAP((0, -1, 1), (1, -2, 1))$ be the *universal sequence* of integers. The main goal of this article is to show that, for every odd number n , the orbit of the projection $\pi_n(US)$ of the universal sequence in $\mathbb{Z}/n\mathbb{Z}$ contains infinitely many balanced Steinhaus figures. More precisely, we obtain the following theorem.

Theorem 1.3. *Let n be an odd number. Then, the orbit of the projection $\pi_n(US)$ of the universal sequence in $\mathbb{Z}/n\mathbb{Z}$ contains :*

- *balanced Steinhaus triangles of order m , for every positive integer $m \equiv 0 \pmod{n}$ or $m \equiv -1 \pmod{3n}$. This partially solves the Molluzzo's problem for 2/3 of the admissible orders m , in the case where n is an odd prime power.*

- *balanced Steinhaus trapezoids of order m and of height h , for every positive integers $m \equiv 0 \pmod{n}$ or $m \equiv -1 \pmod{3n}$ and $h \equiv m \pmod{n}$ or $h \equiv m + 1 \pmod{3n}$.*
- *balanced Pascal triangles of order $2m - 1$, for every positive integer $m \equiv -1 \pmod{n}$ or $m \equiv 0 \pmod{3n}$. This also gives a partial solution of Problem 1.2 for 2/3 of the admissible orders $2m - 1$, in the case where n is an odd prime power.*
- *balanced Pascal trapezoids of order $2m - 1$ and of height h , for every positive integers $m \equiv -1 \pmod{n}$ or $m \equiv 0 \pmod{3n}$ and $h \equiv m + 1 \pmod{n}$ or $h \equiv m \pmod{3n}$.*
- *balanced lozenges of order $2m - 1$, for every positive integer $m \equiv 0 \pmod{n}$. This completely solves Problem 1.2, for the lozenges, in the case where n is a square-free odd number.*

There also exist many results on Steinhaus figures for the binary case $\mathbb{Z}/2\mathbb{Z}$. The five smallest and the three greatest possible numbers of 1's in a binary Steinhaus triangle was determined by G. J. Chang [4]. H. Harborth and G. Glenn [11] proved that every positive integer is realizable as the number of 1's in a generalized binary Pascal triangle, that is, for every natural k , there exists a binary sequence S of length $2m_k - 1$ such that ΔS contains exactly k elements equal to 1. They also determined the minimum value for m_k . The maximum number of 1's in the binary Steinhaus figures (like Steinhaus triangles, generalized Pascal triangles, parallelograms or trapezoids) was studied by M. Bartsch in her Ph.D. Thesis [2]. Symmetries in the binary Steinhaus triangles and in the binary generalized Pascal triangles were explored in [1, 3].

This paper is organized as follows : doubly arithmetic triangles (*DAT* for short) in $\mathbb{Z}/n\mathbb{Z}$, that are triangles where all the rows are arithmetic progressions of the same common difference and where all the diagonals are also arithmetic progressions of the same common difference, are studied in Section 2. We show that these triangles constitute a source of balanced multisets in $\mathbb{Z}/n\mathbb{Z}$, for n odd, while they are never balanced in $\mathbb{Z}/n\mathbb{Z}$, for n even. Moreover, we prove that, for every positive integer n , the orbit associated with the sequence of zeros is the only doubly arithmetic orbit in $\mathbb{Z}/n\mathbb{Z}$. In Section 3, the interlaced doubly arithmetic orbits, i.e. the orbits that are an interlacing of doubly arithmetic multisets, are considered. We determine all the interlaced doubly arithmetic orbits in \mathbb{Z} and, in Section 4, we show that the projection of these particular orbits in $\mathbb{Z}/n\mathbb{Z}$, for n odd, contains infinitely many balanced Steinhaus figures. This result is refined in Section 5, by considering antisymmetric sequences. In Section 6, a particular case of this antisymmetric refinement leads to the universal sequence US and we prove Theorem 1.3. Finally, in Section 7, we recall the results, on the generalized Molluzzo's problem, obtained in this paper and we pose new open problems on the existence of balanced Steinhaus figures in the additive cellular automata of dimension 1 and in the cellular automaton of dimension 2 where the standard Pascal tetrahedron appears.

2 DAT : a source of balanced multisets

For every positive integers n and m and for every elements a , d_1 and d_2 in $\mathbb{Z}/n\mathbb{Z}$, the *doubly arithmetic triangle* $DAT(a, d_1, d_2, m)$ is the triangle of order m in $\mathbb{Z}/n\mathbb{Z}$, with first element a and where each diagonal and each row are arithmetic progressions of respective common differences d_1 and d_2 , that is the multiset in $\mathbb{Z}/n\mathbb{Z}$ defined by

$$DAT(a, d_1, d_2, m) = \{a + id_1 + jd_2 \mid 0 \leq i \leq m-1, 0 \leq j \leq m-1-i\}.$$

In this section, we show that the doubly arithmetic triangles constitute a source of balanced multisets in $\mathbb{Z}/n\mathbb{Z}$, for n odd. Obviously, we can see that the anti-diagonals of a DAT are arithmetic progressions of common difference $d_1 - d_2$. We begin by determining a necessary condition, on the common differences d_1 and d_2 , to obtain balanced DAT in $\mathbb{Z}/n\mathbb{Z}$.

Proposition 2.1. *Let n and m be two positive integers and let a , d_1 and d_2 be in $\mathbb{Z}/n\mathbb{Z}$. If the doubly arithmetic triangle $DAT(a, d_1, d_2, m)$ is balanced, then the common differences d_1 , d_2 and $d_1 - d_2$ are invertible in $\mathbb{Z}/n\mathbb{Z}$.*

Proof. For $n = 1$ or $m = 1$, it is clear. Suppose now that $n > 1$ and $m > 1$. Let $DAT(a, d_1, d_2, m)$ be a doubly arithmetic triangle in $\mathbb{Z}/n\mathbb{Z}$ where at least one of the common differences d_1 , d_2 and $d_1 - d_2$ is not invertible. Without loss of generality, suppose that it is d_2 . If not, we can consider the rotations $DAT(a, d_2, d_1, m)$ or $DAT(a + (m-1)d_2, -d_2, d_1 - d_2, m)$ of $DAT(a, d_1, d_2, m)$. Let δ_1 and δ_2 be two integers whose respective residue classes modulo n are d_1 and d_2 . We distinguish different cases upon the greatest common divisor of δ_1 , δ_2 and n .

Case 1. If $q = \gcd(\delta_1, \delta_2, n) \neq 1$, then we consider the projection map $\pi_q : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/q\mathbb{Z}$. All the elements of the triangle $\pi_q(DAT(a, d_1, d_2, m)) = DAT(\pi_q(a), 0, 0, m)$ are equal to $\pi_q(a)$. Therefore, the triangle $DAT(a, d_1, d_2, m)$ is not balanced in $\mathbb{Z}/n\mathbb{Z}$ since its projection in $\mathbb{Z}/q\mathbb{Z}$ is not.

Case 2. If $\gcd(\delta_1, \delta_2, n) = 1$, then we set $q = \gcd(\delta_2, n) \neq 1$ and we consider the projection $\pi_q(DAT(a, d_1, d_2, m)) = DAT(\pi_q(a), \pi_q(d_1), 0, m)$ in $\mathbb{Z}/q\mathbb{Z}$, where $\pi_q(d_1)$ is invertible in $\mathbb{Z}/q\mathbb{Z}$. Since the $(kq + l)$ th row of $\nabla = DAT(\pi_q(a), \pi_q(d_1), 0, m)$ is the constant sequence, of length $m - kq - l + 1$, equal to $\pi_q(a) + l\pi_q(d_1)$, for all integers l , $0 \leq l \leq q - 1$, and for all positive integers k such that $kq + l \leq m - 1$, it follows that we have

$$\mathbf{m}_{\nabla}(\pi_q(a)) > \mathbf{m}_{\nabla}(\pi_q(a) + \pi_q(d_1)) > \mathbf{m}_{\nabla}(\pi_q(a) + 2\pi_q(d_1)) \geq \dots \geq \mathbf{m}_{\nabla}(\pi_q(a) + (q-1)\pi_q(d_1)).$$

Therefore ∇ is not balanced in $\mathbb{Z}/q\mathbb{Z}$ and thus $DAT(a, d_1, d_2, m)$ is not in $\mathbb{Z}/n\mathbb{Z}$.

□

Remark. For every even number n , there is no balanced DAT in $\mathbb{Z}/n\mathbb{Z}$ since at least one element of $\{d_1, d_2, d_1 - d_2\}$ is not invertible in $\mathbb{Z}/n\mathbb{Z}$, by parity of n .

Remark. Another necessary condition for a DAT , of order m , to be balanced in $\mathbb{Z}/n\mathbb{Z}$ is that its cardinality, that is the binomial coefficient $\binom{m+1}{2}$, must be divisible by n . But these two necessary conditions are not sufficient : as depicted in Figure 2, the triangle $DAT(0, 8, 1, 5)$ is not balanced in $\mathbb{Z}/15\mathbb{Z}$, although its cardinality $\binom{6}{2} = 15$ is divisible by $n = 15$ and its common differences 8, 1 and 7 are invertible in $\mathbb{Z}/15\mathbb{Z}$.

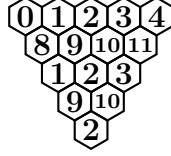


Figure 2: The doubly arithmetic triangle $DAT(0, 8, 1, 5)$ in $\mathbb{Z}/15\mathbb{Z}$.

The following theorem is the main theorem of this section.

Theorem 2.2. *Let n be an odd number and let d_1 and d_2 be two invertible elements in $\mathbb{Z}/n\mathbb{Z}$ whose difference $d_1 - d_2$ is also invertible. Then, the doubly arithmetic triangle $DAT(a, d_1, d_2, m)$ is balanced in $\mathbb{Z}/n\mathbb{Z}$, for all positive integers $m \equiv 0$ or $-1 \pmod{n}$.*

Proof. Let m be a positive integer multiple of n . For every positive integer i , $0 \leq i \leq m-1$, we denote by R_i the i th row of $DAT(a, d_1, d_2, m)$, that is $R_i = (a + id_1 + jd_2)_{0 \leq j \leq m-1-i}$. We prove that, for every integer λ such that $0 \leq \lambda \leq m/n - 2$, the n consecutive rows $\{R_{\lambda n}, R_{\lambda n+1}, \dots, R_{(\lambda+1)n-1}\}$ are balanced. Consider the permutation σ which assigns to each integer i , $0 \leq i \leq n-1$, the positive integer $\sigma(i)$, $0 \leq \sigma(i) \leq n-1$, whose residue class modulo n corresponds with $i(d_1 - d_2)d_1^{-1}$ and denote by k_i the cardinality of the permutation orbit of i following σ . Let $\nabla(i, j) = a + id_1 + jd_2$ denote the j th term in the i th row of $DAT(a, d_1, d_2, m)$. Now, we show that, for every positive integer i , $0 \leq i \leq n-1$, the concatenation $\cup_{l=0}^{l=k_i-1} R_{\lambda n + \sigma^l(i)}$ is balanced in $\mathbb{Z}/n\mathbb{Z}$. Since

$$\begin{aligned} \nabla(\lambda n + \sigma^l(i), m-1 - \lambda n - \sigma^l(i)) + d_2 &= a + (\lambda n + \sigma^l(i))d_1 + (m - \lambda n - \sigma^l(i))d_2 \\ &= a + \sigma^l(i)(d_1 - d_2) = a + \sigma^{l+1}(i)d_1 = \nabla(\lambda n + \sigma^{l+1}(i), 0), \end{aligned}$$

for all positive integers l , $0 \leq l \leq k_i - 1$, it follows that the concatenation $\cup_{l=0}^{l=k_i-1} R_{\lambda n + \sigma^l(i)}$ is an arithmetic progression with invertible common difference d_2 and of length a multiple of n . Therefore, its multiplicity function is constant on $\mathbb{Z}/n\mathbb{Z}$. Finally, since the set of the n first non-negative integers $\{0, 1, \dots, n-1\}$ is a disjoint union of orbits following σ , the multiplicity function of $\cup_{i=0}^{i=n-1} R_{\lambda n + i}$ is constant on $\mathbb{Z}/n\mathbb{Z}$ and thus the triangle $DAT(a, d_1, d_2, m)$ is balanced in $\mathbb{Z}/n\mathbb{Z}$.

For every positive integer $m \equiv -1 \pmod{n}$, the doubly arithmetic triangle $DAT(a, d_1, d_2, m)$ is obtained from the balanced triangle $DAT(a, d_1, d_2, m+1)$ by rejecting its right side. Since it is an arithmetic progression with invertible common difference $d_1 - d_2$ and of length $m+1 \equiv 0 \pmod{n}$, it follows that this right side contains all the elements of $\mathbb{Z}/n\mathbb{Z}$ with the same multiplicity. This completes the proof. \square

Remark. For every odd number n and for every invertible element d in $\mathbb{Z}/n\mathbb{Z}$, the doubly arithmetic triangles $DAT(a, d, -d, m)$, $DAT(a, d, 2d, m)$ and $DAT(a, 2d, d, m)$ are balanced in $\mathbb{Z}/n\mathbb{Z}$, for all positive integers $m \equiv 0$ or $-1 \pmod{n}$.

Let n be a positive integer and let d_1 and d_2 be two elements of $\mathbb{Z}/n\mathbb{Z}$. The orbit \mathcal{O}_S , associated with a doubly infinite sequence S in $\mathbb{Z}/n\mathbb{Z}$, is said to be (d_1, d_2) -*doubly arithmetic* if each subtriangle appearing in it is a DAT with common differences (d_1, d_2) , that is if \mathcal{O}_S is an orbit where all the diagonals are arithmetic progressions with the same common difference d_1 and where all the rows are arithmetic progressions with the same common difference d_2 .

Now, we prove that, for every positive integer n , there does not exist doubly arithmetic orbit in $\mathbb{Z}/n\mathbb{Z}$, except the trivial orbit generated by the sequence of zeros in $\mathbb{Z}/n\mathbb{Z}$.

Proposition 2.3. *Let n be a positive integer. The orbit associated with the sequence of zeros is the only doubly arithmetic orbit in $\mathbb{Z}/n\mathbb{Z}$.*

Proof. It is clear that if \mathcal{O}_S is (d_1, d_2) -doubly arithmetic, then S is an arithmetic progression with common difference d_2 . We set $S = AP(a, d_2)$. It is known [5], and easy to retrieve, that the derived sequence ∂S of S is an arithmetic progression with common difference $2d_2$. Moreover, it is also d_2 , by the doubly arithmetic structure of the orbit \mathcal{O}_S and thus, the common difference d_2 vanishes. By the local rule in \mathcal{O}_S , we obtain that $a + d_1 = 2a$ and $a + 2d_1 = 4a$. Therefore, we have $a = d_1 = 0$ and S is the sequence of zeros. This completes the proof. \square

Even if there does not exist non-trivial doubly arithmetic orbit, the results of this section will be useful in next sections, where orbits with an interlaced doubly arithmetic structure are studied.

3 Interlaced doubly arithmetic orbits of integers

For every positive integers n, k_1 and k_2 and for every doubly infinite sequence S in $\mathbb{Z}/n\mathbb{Z}$, or in \mathbb{Z} , the orbit $\mathcal{O}_S = \{a_{i,j} | a_{i+1,j} = a_{i,j} + a_{i,j+1}, i \in \mathbb{N}, j \in \mathbb{Z}\}$ is said to be (k_1, k_2) -*interlaced doubly arithmetic* if, for every integer $i_0, 0 \leq i_0 \leq k_1 - 1$, and for every integer $j_0, 0 \leq j_0 \leq k_2 - 1$, the submultiset $\{a_{i_0+ik_1, j_0+jk_2} | i \in \mathbb{N}, j \in \mathbb{Z}\}$ is doubly arithmetic, i.e. if we have

$$a_{i_0+ik_1, j_0+jk_2} = a_{i_0, j_0} + i(a_{i_0+k_1, j_0} - a_{i_0, j_0}) + j(a_{i_0, j_0+k_2} - a_{i_0, j_0}),$$

for every non-negative integer i and for every integer j .

Determine all the interlaced doubly arithmetic orbits (*IDAO* for short) in $\mathbb{Z}/n\mathbb{Z}$ seems to be very difficult. Nevertheless, the *IDAO* in \mathbb{Z} are determined in this section and their projection in $\mathbb{Z}/n\mathbb{Z}$ will be considered in next sections. First, it is clear that the sequence S , associated with a (k_1, k_2) -interlaced doubly arithmetic orbit \mathcal{O}_S , is a k_2 -interlaced arithmetic progression. We begin by showing that the interlaced arithmetic structure of a sequence is preserved under the derivation process.

Proposition 3.1. *Let n and k be two positive integers and let (a_0, \dots, a_{k-1}) and (d_0, \dots, d_{k-1}) be two k -tuples of elements in $\mathbb{Z}/n\mathbb{Z}$, or in \mathbb{Z} . Then, we have*

$$\begin{aligned} \partial IAP((a_0, \dots, a_{k-1}), (d_0, \dots, d_{k-1})) \\ = IAP((a_0 + a_1, \dots, a_{k-2} + a_{k-1}, a_{k-1} + a_0 + d_0), (d_0 + d_1, \dots, d_{k-2} + d_{k-1}, d_{k-1} + d_0)). \end{aligned}$$

Proof. Consider $S = IAP((a_0, \dots, a_{k-1}), (d_0, \dots, d_{k-1})) = (x_j)_{j \in \mathbb{N}}$ and $\partial S = (y_j)_{j \in \mathbb{N}}$. Then, for all integers l , we have

$$y_{j_0+lk} = x_{j_0+lk} + x_{j_0+lk+1} = (a_{j_0} + ld_{j_0}) + (a_{j_0+1} + ld_{j_0+1}) = (a_{j_0} + a_{j_0+1}) + l(d_{j_0} + d_{j_0+1}),$$

for all integers j_0 such that $0 \leq j_0 \leq k-2$, and

$$y_{(k-1)+lk} = x_{(k-1)+lk} + x_{(l+1)k} = (a_{k-1} + ld_{k-1}) + (a_0 + (l+1)d_0) = (a_{k-1} + a_0 + d_0) + l(d_{k-1} + d_0),$$

for $j_0 = k-1$. This completes the proof. \square

Then, we can explicitly determine all the iterated derived sequences of an interlaced arithmetic progression.

Proposition 3.2. *Let n and k be two positive integers and let A and D be two k -tuples of elements in $\mathbb{Z}/n\mathbb{Z}$, or in \mathbb{Z} . Then, for every non-negative integer i , we have*

$$\partial^i IAP(A, D) = IAP(AC_{\mathbf{i}} + D\mathbf{T}_{\mathbf{i}}, DC_{\mathbf{i}}),$$

where $\mathbf{C}_{\mathbf{i}}$ is the circulant matrix, of size k , defined by

$$\mathbf{C}_{\mathbf{i}} = \mathbf{Circ} \left(\sum_{l \geq 0} \binom{i}{lk}, \sum_{l \geq 0} \binom{i}{lk-1}, \dots, \sum_{l \geq 0} \binom{i}{lk+1} \right),$$

and where $\mathbf{T}_{\mathbf{i}}$ is the Toeplitz matrix, of size k , defined by

$$\mathbf{T}_{\mathbf{i}} = \left(\sum_{l \geq 0} l \binom{i}{r-s+lk} \right)_{1 \leq r, s \leq k}.$$

Proof. By iteration on i . Trivial for $i = 0$. For $i = 1$, Proposition 3.1 leads to

$$\mathbf{C}_{\mathbf{1}} = \mathbf{Circ}(1, 0, \dots, 0, 1) = \begin{pmatrix} 1 & 0 & \dots & 0 & 1 \\ 1 & 1 & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 1 \end{pmatrix} \text{ and } \mathbf{T}_{\mathbf{1}} = \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ \vdots & & & & 0 \\ \vdots & & 0 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}.$$

Suppose that the result is true for some positive integer i and prove it for $i+1$. First, the $(i+1)$ th derived sequence of $S = IAP(A, D)$ is equal to

$$\begin{aligned} \partial^{i+1} S &= \partial \partial^i S = \partial IAP(AC_{\mathbf{i}} + D\mathbf{T}_{\mathbf{i}}, DC_{\mathbf{i}}) \\ &= IAP(AC_{\mathbf{i}}\mathbf{C}_{\mathbf{1}} + D(\mathbf{T}_{\mathbf{i}}\mathbf{C}_{\mathbf{1}} + \mathbf{C}_{\mathbf{i}}\mathbf{T}_{\mathbf{1}}), DC_{\mathbf{i}}\mathbf{C}_{\mathbf{1}}). \end{aligned}$$

Since the product of two circulant matrices is also a circulant matrix, it follows that

$$\begin{aligned}
\mathbf{C}_{i+1} &= \mathbf{C}_i \mathbf{C}_1 = \mathbf{Circ} \left(\sum_{l \geq 0} \binom{i}{lk}, \sum_{l \geq 0} \binom{i}{lk-1}, \dots, \sum_{l \geq 0} \binom{i}{lk+1} \right) \mathbf{Circ} (1, 0, \dots, 0, 1) \\
&= \mathbf{Circ} \left(\sum_{l \geq 0} \binom{i}{lk} + \sum_{l \geq 0} \binom{i}{lk-1}, \sum_{l \geq 0} \binom{i}{lk-1} + \sum_{l \geq 0} \binom{i}{lk-2}, \dots, \dots \right. \\
&\quad \left. \dots, \sum_{l \geq 0} \binom{i}{lk} + \sum_{l \geq 0} \binom{i}{lk+1} \right) \\
&= \mathbf{Circ} \left(\sum_{l \geq 0} \binom{i+1}{lk}, \sum_{l \geq 0} \binom{i+1}{lk-1}, \dots, \sum_{l \geq 0} \binom{i+1}{lk+1} \right).
\end{aligned}$$

Moreover, we have $\mathbf{T}_{i+1} = \mathbf{T}_i \mathbf{C}_1 + \mathbf{C}_i \mathbf{T}_1 = (\beta_{r,s})_{1 \leq r, s \leq k}$, where

$$\begin{aligned}
\beta_{r,s} &= (\mathbf{T}_i \mathbf{C}_1)_{r,s} + (\mathbf{C}_i \mathbf{T}_1)_{r,s} = \sum_{u=1}^k (\mathbf{T}_i)_{r,u} (\mathbf{C}_1)_{u,s} + \sum_{v=1}^k (\mathbf{C}_i)_{r,v} (\mathbf{T}_1)_{v,s} \\
&= \begin{cases} \mathbf{T}_{i,r,s} + \mathbf{T}_{i,r,s+1} = \sum_{l \geq 0} l \binom{i}{r-s+lk} + \sum_{l \geq 0} l \binom{i}{r-s+lk-1} \\ \quad = \sum_{l \geq 0} l \binom{i+1}{r-s+lk} \text{ for } s \in \{1, \dots, k-1\}, \\ \mathbf{T}_{i,r,1} + \mathbf{T}_{i,r,k} + \mathbf{C}_{i,r,1} = \sum_{l \geq 0} l \binom{i}{r-1+lk} + \sum_{l \geq 0} l \binom{i}{r-k+lk} + \sum_{l \geq 0} \binom{i}{r-1+lk} \\ \quad = \sum_{l \geq 0} (l-1) \binom{i}{r-k+lk-1} + \sum_{l \geq 0} l \binom{i}{r-k+lk} \\ \quad + \sum_{l \geq 0} \binom{i}{r-k+lk-1} = \sum_{l \geq 0} l \binom{i+1}{r-k+lk} \text{ for } s = k. \end{cases}
\end{aligned}$$

This completes the proof. \square

The main result of this section is the complete characterization of the *IDA*O in \mathbb{Z} .

Theorem 3.3. *Every interlaced doubly arithmetic orbit \mathcal{O}_S in \mathbb{Z} is generated by an interlaced arithmetic progression of the form $S = IAP((a_0, a_1, a_2), (d, -2d - 3\Sigma, d + 3\Sigma))$, with $\Sigma = a_0 + a_1 + a_2$ and where a_0, a_1, a_2 and d are integers.*

We begin by showing that these interlaced arithmetic progressions, announced in Theorem 3.3, will generate interlaced doubly arithmetic orbits of integers.

Proposition 3.4. *Let a_0, a_1, a_2 and d be integers and let $\Sigma = a_0 + a_1 + a_2$. Then, the orbit \mathcal{O}_S , associated with $S = IAP((a_0, a_1, a_2)(d, -2d - 3\Sigma, d + 3\Sigma))$, is $(6, 3)$ -interlaced doubly arithmetic.*

Proof. Let $\mathcal{O}_S = \{a_{i,j} | a_{i+1,j} = a_{i,j} + a_{i,j+1}, i \in \mathbb{N}, j \in \mathbb{Z}\}$ be the orbit associated with S and let M_{i_0, j_0} be the submultiset $M_{i_0, j_0} = \{a_{i_0+6i, j_0+3j} | i \in \mathbb{N}, j \in \mathbb{Z}\}$, for all integers $i_0 \in \{0, 1, 2, 3, 4, 5\}$ and for all integers $j_0 \in \{0, 1, 2\}$. We can prove, by induction on i , that, for every non-negative integer i and for every integer j , we have

$$\begin{aligned}
a_{6i,3j} &= a_0 - 2i(d + 3\Sigma) + jd, \\
a_{6i,3j+1} &= a_1 - 2id - j(2d + 3\Sigma), \\
a_{6i,3j+2} &= a_2 + 2i(2d + 3\Sigma) + j(d + 3\Sigma), \\
a_{6i+1,3j} &= (a_0 + a_1) - 2i(2d + 3\Sigma) - j(d + 3\Sigma), \\
a_{6i+1,3j+1} &= (a_1 + a_2) + 2i(d + 3\Sigma) - jd, \\
a_{6i+1,3j+2} &= (a_0 + a_2 + d) + 2id + j(2d + 3\Sigma), \\
a_{6i+2,3j} &= (a_1 + \Sigma) - 2id - j(2d + 3\Sigma), \\
a_{6i+2,3j+1} &= (a_2 + \Sigma + d) + 2i(2d + 3\Sigma) + j(d + 3\Sigma), \\
a_{6i+2,3j+2} &= (a_0 - 2\Sigma) - 2i(d + 3\Sigma) + jd, \\
a_{6i+3,3j} &= (a_1 + a_2 + 2\Sigma + d) + 2i(d + 3\Sigma) - jd, \\
a_{6i+3,3j+1} &= (a_0 + a_2 - \Sigma + d) + 2id + j(2d + 3\Sigma), \\
a_{6i+3,3j+2} &= (a_0 + a_1 - 4\Sigma - 2d) - 2i(2d + 3\Sigma) - j(d + 3\Sigma), \\
a_{6i+4,3j} &= (a_2 + 2\Sigma + 2d) + 2i(2d + 3\Sigma) + j(d + 3\Sigma), \\
a_{6i+4,3j+1} &= (a_0 - 4\Sigma - d) - 2i(d + 3\Sigma) + jd, \\
a_{6i+4,3j+2} &= (a_1 - \Sigma - 2d) - 2id - j(2d + 3\Sigma), \\
a_{6i+5,3j} &= (a_0 + a_2 - 2\Sigma + d) + 2id + j(2d + 3\Sigma), \\
a_{6i+5,3j+1} &= (a_0 + a_1 - 5\Sigma - 3d) - 2i(2d + 3\Sigma) - j(d + 3\Sigma), \\
a_{6i+5,3j+2} &= (a_1 + a_2 + 4\Sigma + d) + 2i(d + 3\Sigma) - jd.
\end{aligned}$$

Thus, the 18 submultisets M_{i_0, j_0} are doubly arithmetic. This completes the proof. \square

Now, we show that there is no other sequence generating $IDA O$ in \mathbb{Z} . Since, for every positive integers k_1 and k_2 , a $(k_1, k_2) - IDAO$ is also a $(k_1 k_2, k_1 k_2) - IDAO$, we suppose that we have $k_1 = k_2 = k$ in the sequel. Then, the problem of determining all the $(k, k) - IDAO$ can be converted into a system of linear equations.

Proposition 3.5. *Let n and k be two positive integers. Let A and D be two k -tuples of elements in $\mathbb{Z}/n\mathbb{Z}$, or in \mathbb{Z} , and let $S = IAP(A, D)$ be a k -interlaced arithmetic progression. Then, the orbit \mathcal{O}_S is (k, k) -interlaced doubly arithmetic if and only if A and D verify*

$$\begin{pmatrix} \mathbf{W}_k^2 & \mathbf{W}_k \mathbf{T}_k^T \\ \mathbf{0}_k & \mathbf{W}_k \end{pmatrix} \begin{pmatrix} \mathbf{A}^T \\ \mathbf{D}^T \end{pmatrix} = \mathbf{0},$$

where $\mathbf{W}_k = \mathbf{C}_k - \mathbf{I}_k = \mathbf{Circ} \left(\binom{k}{0}, \binom{k}{1}, \dots, \binom{k}{k-1} \right)$, that is the Wendt's matrix of size k .

The proof of this proposition is based on the following two lemmas.

Lemma 3.6. *Let n and k be two positive integers and let S be a k -interlaced arithmetic progression in $\mathbb{Z}/n\mathbb{Z}$, or in \mathbb{Z} . Then, the orbit $\mathcal{O}_S = \{a_{i,j} | a_{i+1,j} = a_{i,j} + a_{i,j+1}, i \in \mathbb{N}, j \in \mathbb{Z}\}$ is (k, k) -interlaced doubly arithmetic if and only if we have (1) : for every integer i_0 , $0 \leq i_0 \leq k - 1$, and for every positive integer i , the row R_{i_0+6i} is of the same common differences than R_{i_0} , and (2) : for every integer j_0 , $0 \leq j_0 \leq k - 1$, the sequence $(a_{i_0+6i, j_0})_{i \in \mathbb{N}}$ is an arithmetic progression.*

Proof. If the orbit \mathcal{O}_S is (k, k) -interlaced doubly arithmetic, then it is clear that the assertions (1) and (2) are verified. Suppose now that (1) and (2) hold. We begin by showing (3) : for every integer j_0 , $0 \leq j_0 \leq k-1$, and for every integer j , the sequence $(a_{ik, j_0+jk})_{i \in \mathbb{N}}$ is an arithmetic progression. Indeed, for every non-negative integer i , we have

$$\begin{aligned} a_{ik, j_0+jk} &\stackrel{(1)}{=} a_{ik, j_0} + j(a_{0, j_0+k} - a_{0, j_0}) \\ &\stackrel{(2)}{=} a_{0, j_0} + i(a_{k, j_0} - a_{0, j_0}) + j(a_{0, j_0+k} - a_{0, j_0}) \\ &= a_{0, j_0+jk} + i(a_{k, j_0} - a_{0, j_0}). \end{aligned}$$

Moreover, since $a_{i_0+ik, j_0} = \sum_{l=0}^{i_0} \binom{i_0}{l} a_{ik, j_0+l}$ by the automaton's local rule, it follows that we have

$$\begin{aligned} a_{i_0+ik, j_0+jk} &\stackrel{(1)}{=} a_{i_0+ik, j_0} + j(a_{i_0, j_0+k} - a_{i_0, j_0}) = \sum_{l=0}^{i_0} \binom{i_0}{l} a_{ik, j_0+l} + j(a_{i_0, j_0+k} - a_{i_0, j_0}) \\ &\stackrel{(3)}{=} \sum_{l=0}^{i_0} \binom{i_0}{l} (a_{0, j_0+l} + i(a_{k, j_0+l} - a_{0, j_0+l})) + j(a_{i_0, j_0+k} - a_{i_0, j_0}) \\ &= \sum_{l=0}^{i_0} \binom{i_0}{l} a_{0, j_0+l} + i \left(\sum_{l=0}^{i_0} \binom{i_0}{l} a_{k, j_0+l} - \sum_{l=0}^{i_0} \binom{i_0}{l} a_{0, j_0+l} \right) + j(a_{i_0, j_0+k} - a_{i_0, j_0}) \\ &= a_{i_0, j_0} + i(a_{i_0+k, j_0} - a_{i_0, j_0}) + j(a_{i_0, j_0+k} - a_{i_0, j_0}), \end{aligned}$$

for all integers i_0 and j_0 , $0 \leq i_0, j_0 \leq k-1$, and for all integers i and j . \square

Lemma 3.7. *For all non-negative integers i and j such that $0 \leq j \leq i$, we have*

1. $\mathbf{C}_i = \mathbf{C}_1^i$,
2. $\mathbf{T}_i = \mathbf{T}_j \mathbf{C}_{i-j} + \mathbf{C}_j \mathbf{T}_{i-j}$.

Proof. The first assertion follows from the recursive definition of \mathbf{C}_i . For the second assertion, we proceed by induction on i . The result is trivial for $i = 0$ and for $i = 1$. Suppose it is true until i and prove it for $i+1$. It is clear for $j = 0$ and for $j = i+1$. Let j be an integer such that $1 \leq j \leq i$. By the recursive definition of \mathbf{T}_{i+1} and the induction hypothesis, we obtain that $\mathbf{T}_{i+1} = \mathbf{T}_i \mathbf{C}_1 + \mathbf{C}_i \mathbf{T}_1 = (\mathbf{T}_j \mathbf{C}_{i-j} + \mathbf{C}_j \mathbf{T}_{i-j}) \mathbf{C}_1 + \mathbf{C}_i \mathbf{T}_1 = \mathbf{T}_j \mathbf{C}_{i-j+1} + \mathbf{C}_j (\mathbf{T}_{i-j} \mathbf{C}_1 + \mathbf{C}_{i-j} \mathbf{T}_1) = \mathbf{T}_j \mathbf{C}_{i-j+1} + \mathbf{C}_j \mathbf{T}_{i-j+1}$. \square

We are now ready to prove Proposition 3.5.

Proof of Proposition 3.5. Let $S = IAP(A, D)$ be a k -interlaced arithmetic progression of elements in $\mathbb{Z}/n\mathbb{Z}$, or in \mathbb{Z} . We know that the orbit \mathcal{O}_S is (k, k) -interlaced doubly arithmetic if and only if the assertions (1) and (2) are verified by Lemma 3.6. We consider the equations (1') and (2') :

$$\begin{aligned} (1') &: D\mathbf{W}_k = 0, \\ (2') &: A\mathbf{W}_k^2 + D\mathbf{T}_k \mathbf{W}_k = 0. \end{aligned}$$

First, by Proposition 3.2, the assertions (1) and (1') are equivalent :

$$\begin{aligned}
(1) \quad & \stackrel{Prop.3.2}{\iff} D\mathbf{C}_{i_0+i\mathbf{k}} = D\mathbf{C}_{i_0}, \text{ for all integers } i \geq 0 \text{ and } i_0, 0 \leq i_0 \leq k-1, \\
& \iff D(\mathbf{C}_{i\mathbf{k}} - \mathbf{I}_{\mathbf{k}})\mathbf{C}_{i_0} = 0, \text{ for all integers } i \geq 0 \text{ and } i_0, 0 \leq i_0 \leq k-1, \\
& \stackrel{Lem.3.7}{\iff} D(\mathbf{C}_{\mathbf{k}} - \mathbf{I}_{\mathbf{k}}) \sum_{l=0}^{i-1} \mathbf{C}_{\mathbf{k}}^l \mathbf{C}_{i_0} = 0, \text{ for all integers } i \geq 0 \text{ and } i_0, 0 \leq i_0 \leq k-1, \\
& \iff D\mathbf{W}_{\mathbf{k}} = 0 \quad (1').
\end{aligned}$$

Proposition 3.2 also permits to put assertion (2) in equation as follows :

$$\begin{aligned}
(2) \quad & \iff (a_{ik,0}, a_{ik,1}, \dots, a_{ik,k-1})_{i \in \mathbb{N}} \text{ is arithmetic,} \\
& \stackrel{Prop.3.2}{\iff} (A\mathbf{C}_{i\mathbf{k}} + D\mathbf{T}_{i\mathbf{k}})_{i \in \mathbb{N}} \text{ is arithmetic,} \\
& \iff A(\mathbf{C}_{(i+2)\mathbf{k}} - 2\mathbf{C}_{(i+1)\mathbf{k}} + \mathbf{C}_{i\mathbf{k}}) + D(\mathbf{T}_{(i+2)\mathbf{k}} - 2\mathbf{T}_{(i+1)\mathbf{k}} + \mathbf{T}_{i\mathbf{k}}) = 0, \text{ for all } i \geq 0.
\end{aligned}$$

Moreover, Lemma 3.7 leads to

$$\mathbf{C}_{(i+2)\mathbf{k}} - 2\mathbf{C}_{(i+1)\mathbf{k}} + \mathbf{C}_{i\mathbf{k}} = (\mathbf{C}_{\mathbf{k}}^2 - 2\mathbf{C}_{\mathbf{k}} + \mathbf{I}_{\mathbf{k}})\mathbf{C}_{i\mathbf{k}} = \mathbf{W}_{\mathbf{k}}^2\mathbf{C}_{i\mathbf{k}},$$

and

$$\mathbf{T}_{(i+2)\mathbf{k}} - 2\mathbf{T}_{(i+1)\mathbf{k}} + \mathbf{T}_{i\mathbf{k}} = (\mathbf{T}_{2\mathbf{k}}\mathbf{C}_{i\mathbf{k}} + \mathbf{C}_{2\mathbf{k}}\mathbf{T}_{i\mathbf{k}}) - 2(\mathbf{T}_{\mathbf{k}}\mathbf{C}_{i\mathbf{k}} + \mathbf{C}_{\mathbf{k}}\mathbf{T}_{i\mathbf{k}}) + \mathbf{T}_{i\mathbf{k}}.$$

Finally, since $D\mathbf{C}_{\mathbf{k}} = D$ if the assertion (1') is verified, it follows that

$$\begin{aligned}
\mathbf{T}_{(i+2)\mathbf{k}} - 2\mathbf{T}_{(i+1)\mathbf{k}} + \mathbf{T}_{i\mathbf{k}} & \stackrel{(1')}{=} D(\mathbf{T}_{2\mathbf{k}} - 2\mathbf{T}_{\mathbf{k}})\mathbf{C}_{i\mathbf{k}} = D(\mathbf{T}_{\mathbf{k}}\mathbf{C}_{\mathbf{k}} + \mathbf{C}_{\mathbf{k}}\mathbf{T}_{\mathbf{k}} - 2\mathbf{T}_{\mathbf{k}})\mathbf{C}_{i\mathbf{k}} \\
& = D\mathbf{T}_{\mathbf{k}}\mathbf{W}_{\mathbf{k}}\mathbf{C}_{i\mathbf{k}},
\end{aligned}$$

and thus

$$(1) + (2) \iff (1') + A\mathbf{W}_{\mathbf{k}}^2\mathbf{C}_{i\mathbf{k}} + D\mathbf{T}_{\mathbf{k}}\mathbf{W}_{\mathbf{k}}\mathbf{C}_{i\mathbf{k}} = 0, \text{ for all } i \in \mathbb{N} \iff (1') + (2').$$

This completes the proof. □

In his Ph.D. dissertation about Fermat's Last Theorem [15], E. A. Wendt investigated the resultant of $X^k - 1$ and $(X + 1)^k - 1$, which corresponds to the determinant of $\mathbf{W}_{\mathbf{k}}$. E. Lehmer proved, for the first time, that the determinant of $\mathbf{W}_{\mathbf{k}}$ vanishes if and only if k is divisible by 6 [12]. It is also known that the Wendt's matrix $\mathbf{W}_{\mathbf{k}}$ is of rank k if k is not divisible by 6 and of rank $k - 2$ else.

Proposition 3.8.

$$\text{rank}(\mathbf{W}_{\mathbf{k}}) = \begin{cases} k & \text{if } k \not\equiv 0 \pmod{6}, \\ k - 2 & \text{if } k \equiv 0 \pmod{6}. \end{cases}$$

Sketch of proof. The eigenvalues of \mathbf{W}_k are $\lambda_l = (1 + \omega^l)^k - 1$, for all integers l , $0 \leq l \leq k - 1$, where ω is a primitive k th root of unity [12]. Then, we can show that $\lambda_l = 0$ if and only if k is divisible by 6 and then, $\omega^l = e^{2i\pi/3}$ or $\omega^l = e^{-2i\pi/3}$. The result follows. \square

We are now able to prove the main theorem of this section.

Proof of Theorem 3.3. If k is not divisible by 6, then the Wendt's matrix \mathbf{W}_k is of rank k by Proposition 3.8. This implies that $A = D = (0, \dots, 0)$ and thus S is the sequence of zeros. Otherwise, if k is divisible by 6, then Proposition 3.4 implies that the vector space of (k, k) -interlaced doubly arithmetic orbits is of dimension greater or equal to 4. Moreover, since $\text{rank}(\mathbf{W}_k^2) = \text{rank}(\mathbf{W}_k) = k - 2$ by Proposition 3.8, it follows that the matrix

$$\begin{pmatrix} \mathbf{W}_k^2 & \mathbf{W}_k \mathbf{T}_k^T \\ \mathbf{0}_k & \mathbf{W}_k \end{pmatrix}$$

is of rank greater or equal to $2k - 4$. Therefore, there is no other (k, k) -IDAO than these announced in Theorem 3.3. This completes the proof. \square

4 Balanced Steinhaus figures modulo an odd number

In this section, we show that, for every odd number n , the projection in $\mathbb{Z}/n\mathbb{Z}$ of the IDAO in \mathbb{Z} , obtained in the precedent section, contains infinitely many balanced Steinhaus figures.

Theorem 4.1. *Let n be an odd number and let a_0, a_1, a_2 and d be in $\mathbb{Z}/n\mathbb{Z}$ such that $d, d + 3\Sigma$ and $2d + 3\Sigma$, where $\Sigma = a_0 + a_1 + a_2$, are invertible. Then, the following Steinhaus figures, contained in the orbit of $S = IAP((a_0, a_1, a_2), (d, -2d - 3\Sigma, d + 3\Sigma))$, are balanced :*

- every Steinhaus triangle of order m in \mathcal{O}_S , for every integer $m \equiv 0$ or $-1 \pmod{6n}$,
- every Steinhaus trapezoid of order m and of height h in \mathcal{O}_S , for every integer $m \equiv 0$ or $-1 \pmod{6n}$ and for every integer $h \equiv m$ or $m + 1 \pmod{6n}$,
- every Pascal triangle of order $2m - 1$ in \mathcal{O}_S , for every integer $m \equiv 0$ or $-1 \pmod{6n}$,
- every Pascal trapezoid of order $2m - 1$ and of height h in \mathcal{O}_S , for every integer $m \equiv 0$ or $-1 \pmod{6n}$ and for every integer $h \equiv m$ or $m + 1 \pmod{6n}$,
- every lozenge of order $2m - 1$ in \mathcal{O}_S , for every integer $m \equiv 0 \pmod{6n}$.

Proof. Let $\mathcal{O}_S = \{a_{i,j} | a_{i+1,j} = a_{i,j} + a_{i,j+1}, i \in \mathbb{N}, j \in \mathbb{Z}\}$ be the orbit associated with S . Consider the submultisets $M_{i_0, j_0} = \{a_{i_0+6i, j_0+6j} | i \in \mathbb{N}, j \in \mathbb{Z}\}$, for i_0 and j_0 in $\{0, 1, 2, 3, 4, 5\}$. Each of these 36 submultisets is doubly arithmetic since the orbit \mathcal{O}_S is $(6, 3)$ -interlaced doubly arithmetic by Proposition 3.4. The following tabular gives their common differences $d_1, d_2, d_1 - d_2$.

M_{i_0, j_0}	d_1	d_2	$d_1 - d_2$
$M_{1,2}, M_{1,5}, M_{3,1}, M_{3,4}, M_{5,0}, M_{5,3}$	$2d$	$2(2d + 3\Sigma)$	$-2(d + 3\Sigma)$
$M_{0,1}, M_{0,4}, M_{2,0}, M_{2,3}, M_{4,2}, M_{4,5}$	$-2d$	$-2(2d + 3\Sigma)$	$2(d + 3\Sigma)$
$M_{1,1}, M_{1,4}, M_{3,0}, M_{3,3}, M_{5,2}, M_{5,5}$	$2(d + 3\Sigma)$	$-2d$	$2(2d + 3\Sigma)$
$M_{0,0}, M_{0,3}, M_{2,2}, M_{2,5}, M_{4,1}, M_{4,4}$	$-2(d + 3\Sigma)$	$2d$	$-2(2d + 3\Sigma)$
$M_{0,2}, M_{0,5}, M_{2,1}, M_{2,4}, M_{4,0}, M_{4,3}$	$2(2d + 3\Sigma)$	$2(d + 3\Sigma)$	$2d$
$M_{1,0}, M_{1,3}, M_{3,2}, M_{3,5}, M_{5,1}, M_{5,4}$	$-2(2d + 3\Sigma)$	$-2(d + 3\Sigma)$	$-2d$

Thus, each submultiset M_{i_0, j_0} is doubly arithmetic, with invertible common differences d_1 , d_2 and $d_1 - d_2$. Let λ be a positive integer and let ∇ be a Steinhaus triangle, of order $m = 6\lambda n$ or $m = 6\lambda n - 1$, that appears in \mathcal{O}_S . Since $\nabla \cap M_{i_0, j_0}$, for i_0 and j_0 in $\{0, 1, 2, 3, 4, 5\}$, is a doubly arithmetic triangle, of order λn or $\lambda n - 1$ and with invertible common differences d_1 , d_2 and $d_1 - d_2$, it follows from Theorem 2.2 that the 36 subtriangles are balanced. Therefore their union, the Steinhaus triangle ∇ , is also balanced in $\mathbb{Z}/n\mathbb{Z}$. Similarly, every Pascal triangle, of order $2m - 1$ in \mathcal{O}_S , is balanced, for all positive integers $m \equiv 0$ or $-1 \pmod{6n}$, since it can be decomposed into 36 subtriangles, which are balanced doubly arithmetic triangles by Theorem 2.2 again. For the trapezoids, a Steinhaus trapezoid (resp. Pascal trapezoid), of order m (resp. $2m - 1$) and of height h in \mathcal{O}_S , can be seen as the multiset difference between a Steinhaus triangle of order m and a Steinhaus triangle of order $m - h$ (resp. between a Pascal triangle of order $2m - 1$ and a Pascal triangle of order $2(m - h) - 1$). Therefore, these trapezoids are balanced, for all positive integers $m \equiv 0$ or $-1 \pmod{6n}$ and for all positive integers $h \equiv m$ or $m + 1 \pmod{6n}$. Finally, a lozenge, of order $2m - 1$ in \mathcal{O}_S , is balanced, for all positive integers $m \equiv 0 \pmod{6n}$, since it is the multiset union of a Steinhaus pascal of order $2m - 1$ and of a Steinhaus triangle of order $m - 1$, which are both balanced in $\mathbb{Z}/n\mathbb{Z}$. \square

The case where $a_0 = 0$, $a_1 = 1$, $a_2 = 2$ and $d = 1$ in $\mathbb{Z}/3\mathbb{Z}$, i.e. the orbit associated with the sequence $IAP((0, 1, 2), (1, 1, 1))$, is illustrated in Figure 3. In this example, balanced Steinhaus figures are depicted in gray : there are a balanced Steinhaus triangle of order 18, a balanced Pascal triangle of order 35 and a balanced lozenge of order 35.

5 The antisymmetric case

In this section, we refine Theorem 4.1 by considering the antisymmetric sequences in $\mathbb{Z}/n\mathbb{Z}$.

A finite sequence $S = (a_0, \dots, a_{m-1})$, of length $m \geq 1$ in $\mathbb{Z}/n\mathbb{Z}$, or in \mathbb{Z} , is said to be *antisymmetric* if we obtain the opposite sequence $-S$, by reading S from the right to the left, i.e. if we have $a_{m-1-j} = -a_j$, for all integers j , $0 \leq j \leq m - 1$.

For examples, the sequences $(1, 4, 0, 3, 6)$ and $(2, 6, 1, 5)$ are antisymmetric in $\mathbb{Z}/7\mathbb{Z}$. It is known, see [5], that the antisymmetry of the finite sequences is preserved by the derivation process.

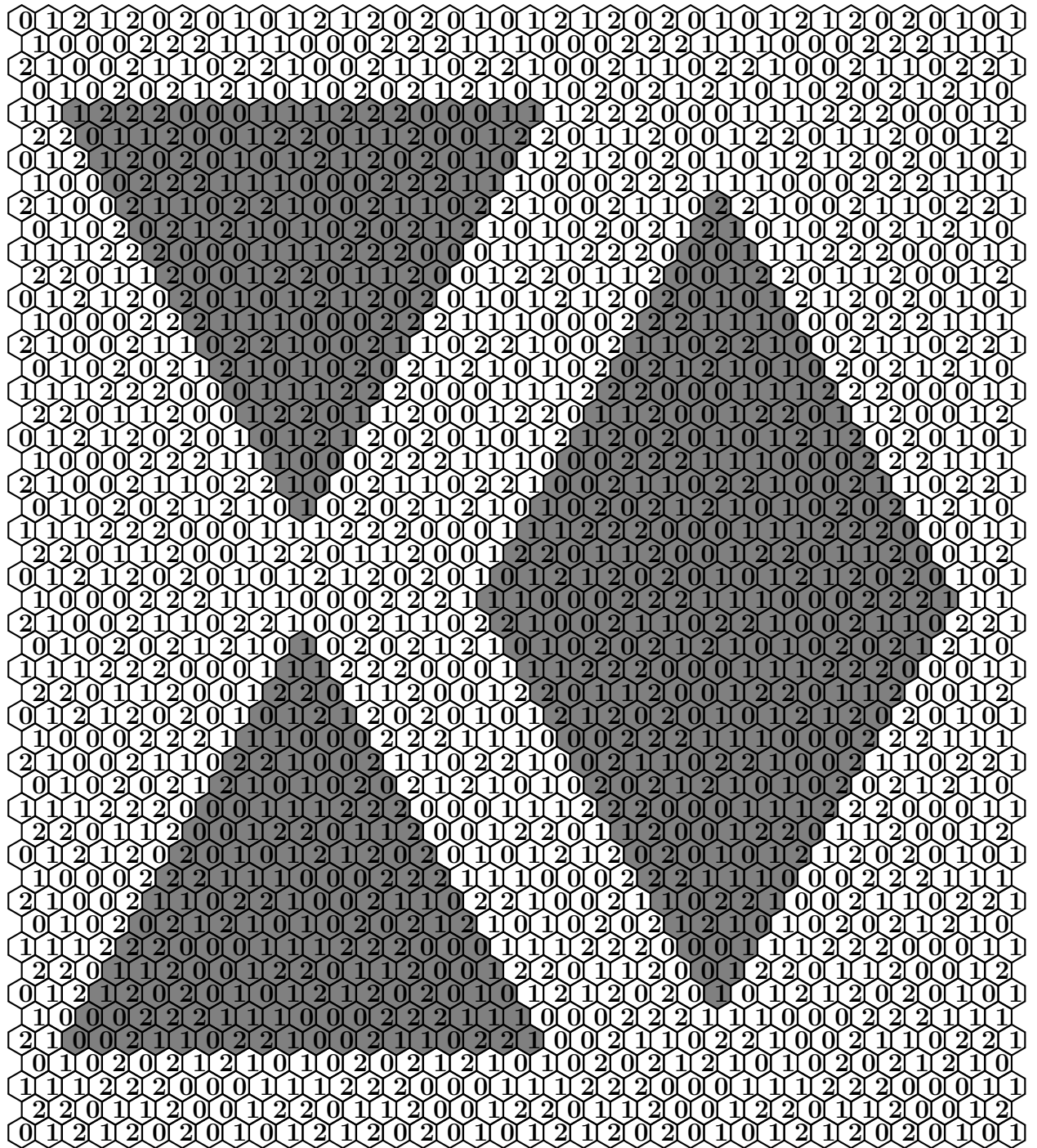


Figure 3: Balanced Steinhaus figures in the orbit of $IAP((0, 1, 2), (1, 1, 1))$ in $\mathbb{Z}/3\mathbb{Z}$.

Proposition 5.1. *Let n be a positive integer and let $S = (a_0, \dots, a_{m-1})$ be a finite sequence of length $m \geq 1$ in $\mathbb{Z}/n\mathbb{Z}$, or in \mathbb{Z} . Then, the sequence S is antisymmetric if and only if its derived sequence ∂S is also antisymmetric and we have $a_{\lfloor m/2 \rfloor} + a_{m-\lfloor m/2 \rfloor} = 0$, where $\lfloor m/2 \rfloor$ is the floor of $m/2$.*

Proof. We set $\partial S = (b_0, \dots, b_{m-2}) = (a_0 + a_1, \dots, a_{m-2} + a_{m-1})$. If S is antisymmetric, then ∂S is also antisymmetric since, for all integers j such that $0 \leq j \leq m-2$, we have $b_{m-2-j} = a_{m-2-j} + a_{m-1-j} = -a_{j+1} - a_j = -b_j$. Conversely, if ∂S is antisymmetric and $a_{\lfloor m/2 \rfloor} + a_{m-\lfloor m/2 \rfloor} = 0$, we proceed by decreasing induction on j . Since

$$\sum_{k=j}^{m-2-j} b_k = \sum_{k=j}^{m-2-j} (a_k + a_{k+1}) = a_j + 2 \sum_{k=j+1}^{m-2-j} a_k + a_{m-1-j},$$

it follows that

$$a_j + a_{m-1-j} = \sum_{k=j}^{m-2-j} b_k - 2 \sum_{k=j+1}^{m-2-j} a_k = 0,$$

by the decreasing induction hypothesis. This completes the proof. \square

The main interest of the antisymmetric sequences, in $\mathbb{Z}/n\mathbb{Z}$, is that their multiplicity function admits a certain symmetry. Indeed, it is clear that, if S is an antisymmetric sequence in $\mathbb{Z}/n\mathbb{Z}$, then its multiplicity function \mathbf{m}_S verifies $\mathbf{m}_S(x) = \mathbf{m}_S(-x)$, for all elements x in $\mathbb{Z}/n\mathbb{Z}$. The same equality appears for the multiplicity function of the Steinhaus or Pascal triangles generated by antisymmetric sequences.

Proposition 5.2. *Let n be a positive integer and let S be an antisymmetric sequence, of length $m \geq 1$ in $\mathbb{Z}/n\mathbb{Z}$. Then, we have $\mathbf{m}_{\nabla S}(x) = \mathbf{m}_{\nabla S}(-x)$, for all elements x in $\mathbb{Z}/n\mathbb{Z}$.*

Proof. Since all the derived sequences $\partial^i S$, for $0 \leq i \leq m-1$, are antisymmetric by Proposition 5.1, it follows that $\mathbf{m}_{\nabla S}(x) = \sum_{i=0}^{m-1} \mathbf{m}_{\partial^i S}(x) = \sum_{i=0}^{m-1} \mathbf{m}_{\partial^i S}(-x) = \mathbf{m}_{\nabla S}(-x)$, for all elements x in $\mathbb{Z}/n\mathbb{Z}$. \square

Proposition 5.3. *Let n be a positive integer and let S be an antisymmetric sequence, of length $2m-1 \geq 1$ in $\mathbb{Z}/n\mathbb{Z}$. Then, we have $\mathbf{m}_{\Delta S}(x) = \mathbf{m}_{\Delta S}(-x)$, for all elements x in $\mathbb{Z}/n\mathbb{Z}$.*

Now, for every odd number n , we determine all the integer sequences generating IDAO in \mathbb{Z} and whose the $3n$ first terms of the projection in $\mathbb{Z}/n\mathbb{Z}$ are antisymmetric.

For every doubly infinite sequence $S = (a_j)_{j \in \mathbb{Z}}$ in $\mathbb{Z}/n\mathbb{Z}$, or in \mathbb{Z} , and for every integers j_0 and j_1 such that $j_0 \leq j_1$, we denote by $S[j_0, j_1]$ the subsequence of S indexed between j_0 and j_1 , that is $S[j_0, j_1] = (a_{j_0}, a_{j_0+1}, \dots, a_{j_1})$.

Proposition 5.4. *Let n be an odd number. Let a_0, a_1, a_2 and d be in $\mathbb{Z}/n\mathbb{Z}$ and let $\Sigma = a_0 + a_1 + a_2$. Then, the subsequence $S_m = IAP((a_0, a_1, a_2), (d, -2d - 3\Sigma, d + 3\Sigma))[0, m-1]$, of length $m \equiv 0 \pmod{3n}$ in $\mathbb{Z}/n\mathbb{Z}$, is antisymmetric if and only if $\Sigma = 0$ and $a_1 = -d$, i.e. if we have $S_m = IAP((a, -d, d - a), (d, -2d, d))[0, m-1]$.*

Proof. Set $m = 3\lambda n$ and $S_m = IAP((a_0, a_1, a_2), (d, -2d-3\Sigma, d+3\Sigma))[0, m-1] = (a_0, \dots, a_{m-1})$ in $\mathbb{Z}/n\mathbb{Z}$. If S_m is antisymmetric, then the terms a_j must verify

$$\begin{cases} a_{3j} + a_{3(\lambda n - j - 1) + 2} = 0 \\ a_{3j+1} + a_{3(\lambda n - j - 1) + 1} = 0 \\ a_{3j+2} + a_{3(\lambda n - j - 1)} = 0 \end{cases} \iff \begin{cases} a_0 + a_2 - d - 3(j+1)\Sigma = 0 \\ 2a_1 + 2d + 3\Sigma = 0 \\ a_0 + a_2 - d + 3j\Sigma = 0 \end{cases}, \text{ for all } 0 \leq j \leq n-1.$$

This leads to $a_1 = -d$, $a_2 = d - a_0$ and $\Sigma = 0$, since n is odd, and thus $S_m = ((a_0, -d, d - a_0), (d, -2d, d))[0, m-1]$, as announced. \square

Let n be an odd number and let a and d be two elements in $\mathbb{Z}/n\mathbb{Z}$ with d invertible. We refine Theorem 4.1 by considering the orbit \mathcal{O}_S of the sequence $S = IAP((a, -d, d-a), (d, -2d, d))$. Let ∇_0 be the Steinhaus triangle, of order $3n$, generated by the $3n$ first terms of S and let Δ_0 be the Pascal triangle, of order $6n-3$, adjacent with ∇_0 , as depicted in Figure 4, that are $\nabla_0 = \nabla S[0, 3n-1]$ and $\Delta_0 = \Delta \partial S[1, 6n-3]$.

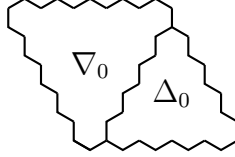


Figure 4: $\nabla_0 = \nabla S[0, 3n-1]$ and $\Delta_0 = \Delta \partial S[1, 6n-3]$.

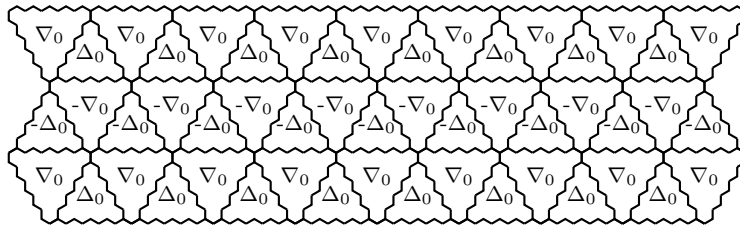
We begin by showing that these triangles are balanced in $\mathbb{Z}/n\mathbb{Z}$.

Proposition 5.5. *Let n be an odd number and let a and d be in $\mathbb{Z}/n\mathbb{Z}$ with d invertible. We consider the 3–interlaced arithmetic progression $S = IAP((a, -d, d-a)(d, -2d, d))$. Then, the triangles $\nabla_0 = \nabla S[0, 3n-1]$ and $\Delta_0 = \Delta \partial S[1, 6n-3]$ are balanced in $\mathbb{Z}/n\mathbb{Z}$.*

Proof. First, since the derived sequences of $S = IAP((a, -d, d-a), (d, -2d, d))$ are :

$$\begin{aligned} \partial^{3i} S &= (-1)^i IAP((a - id, -(i+1)d, (2i+1)d - a), (d, -2d, d)), \\ \partial^{3i+1} S &= (-1)^i IAP((a - (2i+1)d, id - a, (i+2)d), (-d, -d, 2d)), \\ \partial^{3i+2} S &= (-1)^i IAP((- (i+1)d, (2i+2)d - a, a - id), (-2d, d, d)), \end{aligned}$$

for all non-negative integers i , it follows that $\partial^{3n} S = -S$. Moreover, the orbit \mathcal{O}_S is $(6, 3)$ –interlaced doubly arithmetic and thus each row (resp. each diagonal) of \mathcal{O}_S is periodic of period $3n$ (resp. of period $6n$). This leads to the following periodic decomposition of the orbit \mathcal{O}_S into triangles ∇_0 and Δ_0 :



Particularly, the Steinhaus triangle $\nabla S[0, 6n - 1]$, of order $6n$, and the Pascal triangle $\Delta \partial S[1, 12n - 3]$, of order $12n - 3$, which are balanced by Theorem 4.1, admit the decomposition :

$$\nabla S[0, 6n - 1] = \begin{array}{c} \nabla_0 \\ \Delta_0 \\ \nabla_0 \\ -\nabla_0 \end{array} \quad \text{and} \quad \Delta \partial S[1, 12n - 3] = \begin{array}{c} \Delta_0 \\ -\nabla_0 \\ -\Delta_0 \end{array} .$$

The sequences $S[0, 3n - 1]$ and $\partial S[1, 6n - 3]$ are antisymmetric in $\mathbb{Z}/n\mathbb{Z}$, by Propositions 5.4 and 5.1, and thus we deduce, from Propositions 5.2 and 5.3, that the multiplicity functions $\mathbf{m}_{-\nabla_0}$ and $\mathbf{m}_{-\Delta_0}$ correspond with \mathbf{m}_{∇_0} and \mathbf{m}_{Δ_0} , since $\mathbf{m}_{\nabla_0}(x) = \mathbf{m}_{\nabla_0}(-x) = \mathbf{m}_{-\nabla_0}(x)$ and $\mathbf{m}_{\Delta_0}(x) = \mathbf{m}_{\Delta_0}(-x) = \mathbf{m}_{-\Delta_0}(x)$, for all x in $\mathbb{Z}/n\mathbb{Z}$. Finally, the multiplicity functions \mathbf{m}_{∇_0} and \mathbf{m}_{Δ_0} are constant because they are solutions of the following system of equations

$$\begin{aligned} 3\mathbf{m}_{\nabla_0} + \mathbf{m}_{\Delta_0} &= \mathbf{m}_{\nabla S[0, 3n-1]} = \frac{1}{n} \binom{3n+1}{2}, \\ \mathbf{m}_{\nabla_0} + 3\mathbf{m}_{\Delta_0} &= \mathbf{m}_{\Delta \partial S[1, 6n-3]} = \frac{1}{n} \binom{3n}{2}. \end{aligned}$$

Therefore, the elementary triangles ∇_0 and Δ_0 are balanced in $\mathbb{Z}/n\mathbb{Z}$. □

Finally, we obtain the refinement of Theorem 4.1 announced above.

Theorem 5.6. *Let n be an odd number and let a and d be in $\mathbb{Z}/n\mathbb{Z}$ with d invertible. Then, the following Steinhaus figures, contained in the orbit of $S = IAP((a, -d, d-a), (d, -2d, d))$, are balanced :*

- the Steinhaus triangles $\nabla S[0, 3\lambda n - 1]$, of order $3\lambda n$, and $\nabla \partial S[0, 3\lambda n - 2]$, of order $3\lambda n - 1$, for every positive integer λ ,
- the Steinhaus trapezoid $ST(S[0, 3\lambda n - 1], h)$ of order $3\lambda n$ and of height h , for every positive integer λ and for every positive integer $h \equiv 0$ or $1 \pmod{3n}$; the Steinhaus trapezoid $ST(\partial S[0, 3\lambda n - 2], h)$, of order $3\lambda n - 1$ and of height h , for every positive integer λ and for every positive integer $h \equiv -1$ or $0 \pmod{3n}$,
- the Pascal triangle $\Delta \partial S[-m, m - 2]$, of order $2m - 1$, for every positive integer $m \equiv 0$ or $-1 \pmod{3n}$,
- the Pascal trapezoid $PT(\partial S[-m, m - 2], h)$, of order $2m - 1$ and of height h , for every positive integer $m \equiv 0$ or $-1 \pmod{3n}$ and for every positive integer $h \equiv m$ or $m + 1 \pmod{3n}$,
- the lozenge $\diamond \partial S[-m, m - 2]$, of order $2m - 1$, for every positive integer $m \equiv 0 \pmod{3n}$.

Proof. For every positive integer λ , the Steinhaus triangle $\nabla S[0, 3\lambda n - 1]$ and the Pascal triangle $\Delta \partial S[-3\lambda n, 3\lambda n - 2]$ are balanced because they are multiset union of the elementary triangles $\nabla_0, -\nabla_0, \Delta_0$ and $-\Delta_0$, which are balanced in $\mathbb{Z}/n\mathbb{Z}$ by Proposition 5.5. The Steinhaus triangle $\nabla \partial S[0, 3\lambda n - 2]$ is balanced, since it is obtained from $\nabla S[0, 3\lambda n - 1]$ by rejecting the first row, which is a 3-interlaced arithmetic progression with invertible common differences and of length $3\lambda n$ and thus contains 3λ times each element of $\mathbb{Z}/n\mathbb{Z}$. Similarly, the Pascal triangle $\Delta \partial S[-3\lambda n + 1, 3\lambda n - 3]$ is balanced, since it is obtained from $\Delta \partial S[-3\lambda n, 3\lambda n - 2]$ by rejecting the last row, which is also balanced. For the trapezoids, the Steinhaus trapezoids (resp. Pascal trapezoids) announced in this theorem are multiset differences of the balanced Steinhaus triangles (resp. balanced Pascal triangles), that appeared above. Finally, the lozenge $\diamond \partial S[-3\lambda n, 3\lambda n - 2]$ is the multiset union of the Pascal triangle $\Delta \partial S[-3\lambda n, 3\lambda n - 2]$ and the Steinhaus triangle $\nabla(-1)^\lambda \partial S[-3\lambda n, -2] = \nabla(-1)^\lambda \partial S[0, 3\lambda n - 2]$, which are balanced, for all positive integers λ . \square

6 The universal sequence modulo an odd number

Let $US = IAP((0, -1, 1), (1, -2, 1))$ be the universal sequence of integers introduced in Section 1. In this section, we refine Theorem 5.6 by studying this universal sequence modulo an odd number n , namely the sequence

$$S = d\pi_n(US) = IAP((0, -d, d), (d, -2d, d)),$$

where d is invertible in $\mathbb{Z}/n\mathbb{Z}$. It corresponds to the sequence S of Theorem 5.6, with $a = 0$. First, each element of its orbit $\mathcal{O}_S = \{a_{i,j} | a_{i+1,j} = a_{i,j} + a_{i,j+1}, i \in \mathbb{N}, j \in \mathbb{Z}\}$ can be expressed as a function of d .

Proposition 6.1. *Let n be an odd number and let d be an invertible element in $\mathbb{Z}/n\mathbb{Z}$. We consider the universal sequence $S = IAP((0, -d, d), (d, -2d, d))$ in $\mathbb{Z}/n\mathbb{Z}$ and its orbit $\mathcal{O}_S = \{a_{i,j} | a_{i+1,j} = a_{i,j} + a_{i,j+1}, i \in \mathbb{N}, j \in \mathbb{Z}\}$. Then, for every non-negative integers i and j , we have*

$$a_{i,j} = (-1)^i \sum_{k>0} \binom{k}{j+2i-k} (-1)^k (k-i)d.$$

Proof. We begin by proving this equality for $i = 0$. Let $(u_j)_{j \in \mathbb{N}}$ and $(v_j)_{j \in \mathbb{N}}$ be the sequences, in $\mathbb{Z}/n\mathbb{Z}$, defined by $u_j = \sum_{k>0} \binom{k}{j-k} (-1)^k kd$ and $v_j = \sum_{k>0} \binom{k}{j-k} (-1)^k d$, for all non-negative integers j . Then, for every positive integer $j \geq 2$, we have

$$u_j = \sum_{k>0} \left(\binom{k-1}{j-k-1} + \binom{k-1}{j-k} \right) (-1)^k (k-1)d + \sum_{k>0} \binom{k}{j-k} (-1)^k d = -u_{j-2} - u_{j-1} + v_j.$$

By the same way, we can prove that the sequence $(v_j)_{j \in \mathbb{N}}$ verifies the relationship $v_j + v_{j-1} + v_{j-2} = 0$, for all positive integers $j \geq 2$. It follows that $v_{3j} = d$, $v_{3j+1} = -d$ and $v_{3j+2} = 0$, for all non-negative integers j . We complete the proof by induction on j . If

we suppose that $u_{3j} = jd$, $u_{3j+1} = -(1 + 2j)d$ and $u_{3j+2} = (1 + j)d$, then we obtain that $u_{3j+3} = -u_{3j+2} - u_{3j+1} + v_{3j+3} = (j + 1)d$, $u_{3j+4} = -u_{3j+3} - u_{3j+2} + v_{3j+4} = -(3 + 2j)d$ and $u_{3j+5} = -u_{3j+4} - u_{3j+3} + v_{3j+5} = (2 + j)d$. Therefore, we have $a_{0,j} = u_j = \sum_{k>0} \binom{k}{j-k} (-1)^k kd$, for all non-negative integers j , and this completes the proof for $i = 0$. Finally, for every positive integers i and j , we obtain

$$\begin{aligned} a_{i,j} &= \sum_{l=0}^i \binom{i}{l} a_{0,j+l} = \sum_{l=0}^i \binom{i}{l} \sum_{k>0} \binom{k}{j+l-k} (-1)^k kd \\ &= \sum_{k>0} \sum_{l=0}^i \binom{i}{k} \binom{k}{j+l-k} (-1)^k kd = \sum_{k>0} \binom{i+k}{j+i-k} (-1)^k kd \\ &= (-1)^i \sum_{k>i} \binom{k}{j+2i-k} (-1)^k (k-i)d = (-1)^i \sum_{k>0} \binom{k}{j+2i-k} (-1)^k (k-i)d. \end{aligned}$$

□

In the sequel of this section, we suppose that n is an odd number and that S is the universal sequence modulo n , that is $S = IAP((0, -d, d), (d, -2d, d))$, where d is an invertible element in $\mathbb{Z}/n\mathbb{Z}$. Let ∇_1, ∇_2 and ∇_3 be the Steinhaus triangles, of order n , associated with the sequences $S[0, n-1]$, $S[n, 2n-1]$ and $S[2n, 3n-1]$ respectively and let Δ_1, Δ_2 and Δ_3 be their adjacent Pascal triangles, of order $2n-3$, as depicted in Figure 5, that are : $\nabla_1 = \nabla S[0, n-1]$, $\nabla_2 = \nabla S[n, 2n-1]$, $\nabla_3 = \nabla S[2n, 3n-1]$, $\Delta_1 = \Delta \partial S[1, 2n-3]$, $\Delta_2 = \Delta \partial S[n+1, 3n-3]$ and $\Delta_3 = \Delta \partial S[2n+1, 4n-3]$.

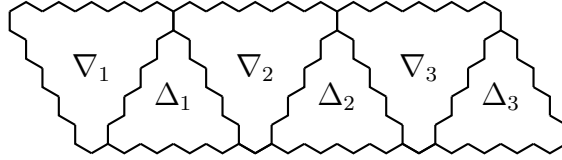


Figure 5: The elementary triangles $\nabla_1, \nabla_2, \nabla_3, \Delta_1, \Delta_2$ and Δ_3 .

We begin by showing that these triangles, or unions of them, are balanced in $\mathbb{Z}/n\mathbb{Z}$.

Proposition 6.2. *Let n be an odd number and let d be an invertible element in $\mathbb{Z}/n\mathbb{Z}$. We consider the universal sequence $S = IAP((0, -d, d), (d, -2d, d))$ modulo n and these elementary triangles $\nabla_1 = \nabla S[0, n-1]$, $\nabla_2 = \nabla S[n, 2n-1]$, $\nabla_3 = \nabla S[2n, 3n-1]$, $\Delta_1 = \Delta \partial S[1, 2n-3]$, $\Delta_2 = \Delta \partial S[n+1, 3n-3]$ and $\Delta_3 = \Delta \partial S[2n+1, 4n-3]$. Then, the multisets $\nabla_2, \nabla_1 \cup \nabla_3, \Delta_3$ and $\Delta_1 \cup \Delta_2$ are balanced in $\mathbb{Z}/n\mathbb{Z}$.*

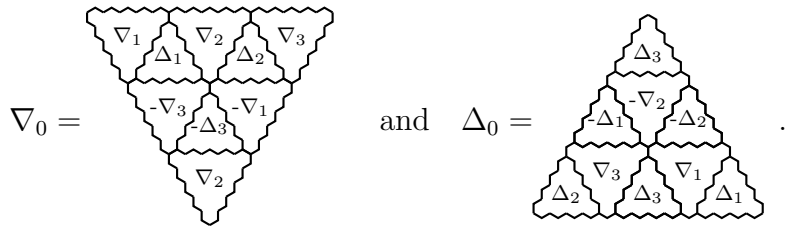
The proof of this proposition is based on the following lemma.

A finite sequence $S = (a_0, \dots, a_{m-1})$, of length $m \geq 1$ in $\mathbb{Z}/n\mathbb{Z}$, is said to be *symmetric* if it is the same sequence read from the left to the right than read from the right to the left, i.e. if we have $a_j = a_{m-1-j}$, for all non-negative integers j , $0 \leq j \leq m-1$.

Lemma 6.3. *Let n be an odd number and let $\nabla = \{a_{i,j} | 0 \leq i \leq m-1, 0 \leq j \leq m-1-i\}$ be a Steinhaus triangle, of order $m \geq 1$ in $\mathbb{Z}/n\mathbb{Z}$. Then, the anti-diagonals AD_{2j} and AD_{2j+1} are respectively antisymmetric and symmetric, for all integers j , $0 \leq 2j \leq 2j+1 \leq m-1$, if and only if we have $a_{i,i} = 0$, for all integers i , $0 \leq i \leq (m-1)/2$.*

Proof. For every non-negative integer i , $0 \leq i \leq (m-1)/2$, if the sequence $AD_{2i} = (a_{0,2i}, \dots, a_{2i,0})$ is antisymmetric, then it follows that we have $2a_{i,i} = 0$ and thus $a_{i,i} = 0$, since n is odd. Conversely, suppose now that $a_{i,i} = 0$, for all non-negative integers i , $0 \leq i \leq (m-1)/2$. We proceed by induction on j . For $j = 0$, it is clear that $AD_0 = (a_{0,0}) = (0)$ is antisymmetric and that $AD_1 = (a_{0,1}, a_{1,0}) = (a_1, a_1)$ is symmetric. Suppose that the result is true for $j-1$, i.e. that the sequences AD_{2j-2} and AD_{2j-1} are respectively antisymmetric and symmetric, and prove it for j . We begin by showing that $a_{j-k,j+k} = -a_{j+k,j-k}$, for all non-negative integers k , $0 \leq k \leq j$. For $k = 0$, it comes from hypothesis $a_{j,j} = 0$. Suppose it is true for all the integers in $\{0, \dots, k-1\}$. Since $a_{j-k,j+k-1} = a_{j+k-1,j-k}$ by symmetry of AD_{2j-1} , we obtain that $a_{j-k,j+k} = a_{j-(k-1),j+k-1} - a_{j-k,j+k-1} = -a_{j+k-1,j-(k-1)} - a_{j+k-1,j-k} = -a_{j+k,j-k}$ and thus AD_{2j} is antisymmetric. We now prove that $a_{j-k,j+1+k} = a_{j+1+k,j-k}$, for all non-negative integers k , $0 \leq k \leq j$. For $k = 0$, it follows from the equality $a_{j+1,j} = a_{j,j} + a_{j,j+1} = a_{j,j+1}$. Suppose it is true for all the integers in $\{0, \dots, k-1\}$. Since $a_{j-k,j+k} = -a_{j+k,j-k}$ by antisymmetry of AD_{2j} , we have $a_{j-k,j+k+1} = a_{j-k+1,j+k} - a_{j-k,j+k} = a_{j+k,j-(k-1)} + a_{j+k,j-k} = a_{j+k+1,j-k}$ and thus AD_{2j+1} is symmetric. This concludes the proof. \square

Proof of Proposition 6.2. First, we consider the Steinhaus triangle $\nabla_0 = \nabla S[0, 3n-1]$, of order $3n$, and the Pascal triangle $\Delta_0 = \Delta \partial S[1, 6n-3]$, of order $6n-3$, which are balanced, by Proposition 5.5. If we denote by $\mathcal{O}_S = \{a_{i,j} | a_{i+1,j} = a_{i,j} + a_{i,j+1}, i \in \mathbb{N}, j \in \mathbb{Z}\}$ the orbit associated with the universal sequence $S = IAP((0, -d, d), (d, -2d, d))$ in $\mathbb{Z}/n\mathbb{Z}$, then Proposition 6.1 implies that we have $a_{n,j} = -a_{0,2n+j}$, $a_{2n,j} = a_{n,j}$ and $a_{3n,j} = -a_{0,j}$, for all integers j . This leads to the following decomposition of ∇_0 and Δ_0 into the elementary triangles $\nabla_1, \nabla_2, \nabla_3, \Delta_1, \Delta_2$ and Δ_3 :



For every integer k in $\{0, 1, 2, 3\}$, we denote by $D_j(\nabla_k)$ and $AD_j(\nabla_k)$ the j th diagonal and the j th anti-diagonal of ∇_k , for every integer j , $0 \leq j \leq n-1$, and by $D_j(\Delta_k)$ and $AD_j(\Delta_k)$ the j th diagonal and the j th anti-diagonal of Δ_k , for every integer j , $0 \leq j \leq n-2$. Since we have $a_{i,i} = 0$, for all non-negative integers i , from the general expression of $a_{i,j}$ appearing in the proof of Proposition 3.4, it follows, from Lemma 6.3, that the sequences $AD_{2j}(\nabla_0)$ and $AD_{2j+1}(\nabla_0)$ are respectively antisymmetric and symmetric, for all non-negative integers j , $0 \leq 2j \leq 2j+1 \leq 3n-1$. This implies the following equalities on the multiplicity functions

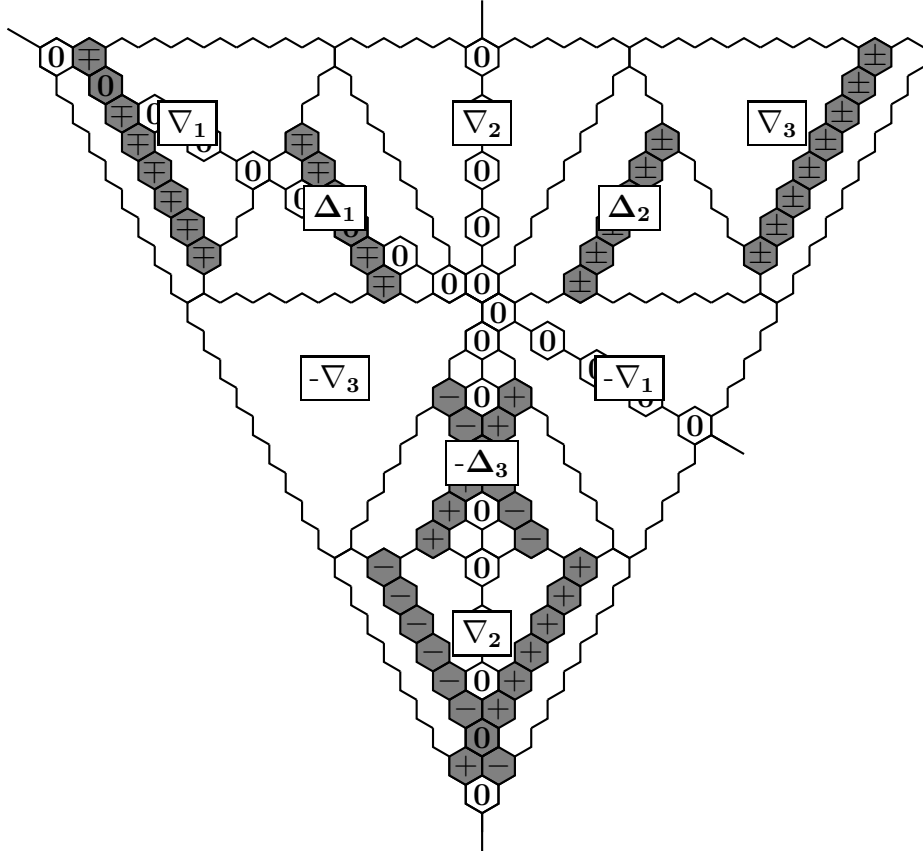


Figure 6: The Steinhaus triangle ∇_0 .

of the anti-diagonals in the triangles ∇_2 and ∇_3 :

$$\mathbf{m}_{AD_{2j}(\nabla_3)}(x) = \mathbf{m}_{AD_{2j}(\nabla_2)}(-x) \quad \text{and} \quad \mathbf{m}_{AD_{2j+1}(\nabla_3)}(x) = \mathbf{m}_{AD_{2j+1}(\nabla_2)}(x),$$

for all elements x in $\mathbb{Z}/n\mathbb{Z}$, and for all non-negative integers j , $0 \leq 2j \leq 2j+1 \leq n-1$. Moreover, we know, from Proposition 5.4, that the sequence $S[0, 3n-1]$ is antisymmetric and, thus, all the rows of ∇_0 are also antisymmetric by Proposition 5.1. Therefore, we have

$$\mathbf{m}_{AD_j(\nabla_3)}(x) = \mathbf{m}_{D_{n-1-j}(\nabla_1)}(-x) \quad \text{and} \quad \mathbf{m}_{AD_j(\nabla_2)}(x) = \mathbf{m}_{D_{n-1-j}(\nabla_2)}(-x),$$

for all elements x in $\mathbb{Z}/n\mathbb{Z}$, and for all non-negative integers j , $0 \leq j \leq n-1$. This leads to the equality

$$\begin{aligned} \mathbf{m}_{\nabla_1}(x) + \mathbf{m}_{\nabla_3}(x) &= \sum_{j=0}^{n-1} \mathbf{m}_{D_j(\nabla_1)}(x) + \sum_{j=0}^{n-1} \mathbf{m}_{AD_j(\nabla_3)}(x) = \sum_{j=0}^{n-1} \mathbf{m}_{AD_{n-1-j}(\nabla_3)}(-x) + \sum_{j=0}^{n-1} \mathbf{m}_{AD_j(\nabla_3)}(x) \\ &= \sum_{j=0}^{n-1} (\mathbf{m}_{AD_j(\nabla_3)}(-x) + \mathbf{m}_{AD_j(\nabla_3)}(x)) = \sum_{j=0}^{n-1} (\mathbf{m}_{AD_j(\nabla_2)}(-x) + \mathbf{m}_{AD_j(\nabla_2)}(x)) \\ &= \sum_{j=0}^{n-1} \mathbf{m}_{AD_j(\nabla_2)}(x) + \sum_{j=0}^{n-1} \mathbf{m}_{AD_{n-1-j}(\nabla_2)}(-x) = 2\mathbf{m}_{\nabla_2}(x), \end{aligned}$$

for all x in $\mathbb{Z}/n\mathbb{Z}$. Similarly, if we consider the diagonals and the anti-diagonals in the triangles Δ_1 , Δ_2 and $-\Delta_3$, as depicted in Figure 6, then we obtain that $\mathbf{m}_{\Delta_1} + \mathbf{m}_{\Delta_2} = 2\mathbf{m}_{-\Delta_3}$. The antisymmetry in ∇_0 also implies the following equalities : $\mathbf{m}_{\nabla_1} = \mathbf{m}_{-\nabla_3}$, $\mathbf{m}_{\nabla_3} = \mathbf{m}_{-\nabla_1}$, $\mathbf{m}_{\nabla_2} = \mathbf{m}_{-\nabla_2}$, $\mathbf{m}_{\Delta_1} = \mathbf{m}_{-\Delta_2}$, $\mathbf{m}_{\Delta_2} = \mathbf{m}_{-\Delta_1}$ and $\mathbf{m}_{\Delta_3} = \mathbf{m}_{-\Delta_3}$. Therefore, the multiplicity functions of the elementary triangles verify the following equations :

$$\begin{aligned}\mathbf{m}_{\nabla_1} + \mathbf{m}_{\nabla_3} &= 2\mathbf{m}_{\nabla_2}, \\ \mathbf{m}_{\Delta_1} + \mathbf{m}_{\Delta_2} &= 2\mathbf{m}_{\Delta_3}.\end{aligned}$$

Finally, since the triangles ∇_0 and Δ_0 are balanced in $\mathbb{Z}/n\mathbb{Z}$, it follows that the multiplicity functions \mathbf{m}_{∇_2} and \mathbf{m}_{Δ_3} are solutions of the following system of equations

$$\begin{aligned}6\mathbf{m}_{\nabla_2} + 3\mathbf{m}_{\Delta_3} &= \mathbf{m}_{\nabla_0} = \frac{1}{n} \binom{3n+1}{2}, \\ 3\mathbf{m}_{\nabla_2} + 6\mathbf{m}_{\Delta_3} &= \mathbf{m}_{\Delta_0} = \frac{1}{n} \binom{3n}{2}.\end{aligned}$$

We conclude that the triangles ∇_2 , Δ_3 and the multisets $\nabla_1 \cup \nabla_3$ and $\Delta_1 \cup \Delta_2$ are balanced. \square

We are now ready to prove Theorem 1.3, the main theorem of this paper.

Theorem 6.4. *Let n be an odd number and let d be an invertible element in $\mathbb{Z}/n\mathbb{Z}$. Then, the following Steinhaus figures, contained in the orbit associated with the universal sequence $S = IAP((0, -d, d), (d, -2d, d))$ in $\mathbb{Z}/n\mathbb{Z}$, are balanced :*

- *the Steinhaus triangles $\nabla S[m, 2m-1]$, for every positive integer $m \equiv 0 \pmod{n}$, and $\nabla \partial S[0, m-1]$, for every positive integer $m \equiv -1 \pmod{3n}$,*
- *the Steinhaus trapezoids $ST(S[m, 2m-1], h)$, for every positive integers $m \equiv 0 \pmod{n}$ and $h \equiv 0 \pmod{n}$ or $h \equiv m+1 \pmod{3n}$, and $ST(\partial S[0, m-1], h)$, for every positive integers $m \equiv -1 \pmod{3n}$ and $h \equiv -1 \pmod{n}$ or $h \equiv 0 \pmod{3n}$,*
- *the Pascal triangle $\Delta \partial S[-m, m-2]$, for every positive integer $m \equiv -1 \pmod{n}$ or $m \equiv 0 \pmod{3n}$,*
- *the Pascal trapezoid $PT(\partial S[-m, m-2], h)$, for every positive integers $m \equiv -1 \pmod{n}$ or $m \equiv 0 \pmod{3n}$ and $h \equiv m \pmod{n}$ or $h \equiv m+1 \pmod{3n}$,*
- *the lozenge $\diamond \partial S[-m, m-2]$, for every positive integer $m \equiv 0 \pmod{n}$.*

Proof. These Steinhaus figures are unions of the multisets $\pm \nabla_2$, $\pm(\nabla_1 \cup \nabla_3)$, $\pm \Delta_3$ and $\pm(\Delta_1 \cup \Delta_2)$, which are balanced in $\mathbb{Z}/n\mathbb{Z}$ by Proposition 6.2. More precisely, let λ be a positive integer. We know, from Theorem 5.6, that the Steinhaus triangles $\nabla S[3\lambda n, 6\lambda n-1]$, of order $3\lambda n$, and $\nabla \partial S[0, 3\lambda n-2]$, of order $3\lambda n-1$, are balanced. As depicted in Figure 7, the Steinhaus triangle $\nabla S[(3\lambda+1)n, (6\lambda+2)n-1]$, of order $(3\lambda+1)n$, is constituted by $\lambda+1$

triangles ∇_2 , λ multisets $\nabla_1 \cup \nabla_3$, λ triangles Δ_3 , λ multisets $\Delta_1 \cup \Delta_2$ and the Steinhaus triangle $\nabla \partial^n S[(3\lambda + 1)n, (6\lambda + 2)n - 1] = -\nabla S[3\lambda n, 6\lambda n - 1]$. This leads to the equality

$$\mathbf{m}_{\nabla S[(3\lambda+1)n, (6\lambda+2)n-1]} = (3\lambda + 1)\mathbf{m}_{\nabla_2} + 3\lambda\mathbf{m}_{\Delta_3} + \mathbf{m}_{\nabla S[3\lambda n, 6\lambda n-1]}.$$

Similarly, the Steinhaus triangle $\nabla S[(3\lambda+2)n, (6\lambda+4)n-1]$, of order $(3\lambda+2)n$, is constituted by λ triangles ∇_2 , $\lambda + 1$ multisets $\nabla_1 \cup \nabla_3$, $\lambda + 1$ triangles Δ_3 , λ multisets $\Delta_1 \cup \Delta_2$ and the Steinhaus triangle $\nabla \partial^n S[(3\lambda + 2)n, (6\lambda + 4)n - 1] = -\nabla S[(3\lambda + 1)n, (6\lambda + 2)n - 1]$. Therefore, we obtain

$$\mathbf{m}_{\nabla S[(3\lambda+2)n, (6\lambda+4)n-1]} = (3\lambda + 2)\mathbf{m}_{\nabla_2} + (3\lambda + 1)\mathbf{m}_{\Delta_3} + \mathbf{m}_{\nabla S[(3\lambda+1)n, (6\lambda+2)n-1]}.$$

This completes the proof that the Steinhaus triangle $\nabla S[m, 2m - 1]$ is balanced, for all positive integers $m \equiv 0 \pmod{n}$. A similar decomposition permits to prove that the Pascal triangle $\Delta \partial S[-m, m - 2]$ is balanced, for all positive integers $m \equiv -1 \pmod{n}$. First, we know, from Theorem 5.6, that the Pascal triangles $\Delta \partial S[-3\lambda n, 3\lambda n - 2]$, of order $6\lambda n - 1$, and $\Delta \partial S[-3\lambda n + 1, 3\lambda n - 3]$, of order $6\lambda n - 3$, are balanced in $\mathbb{Z}/n\mathbb{Z}$. The other cases come from the decomposition into elementary triangles, as depicted in Figure 8, which implies the following equalities :

$$\mathbf{m}_{\Delta \partial S[-(3\lambda+1)n+1, (3\lambda+1)n-3]} = 3\lambda\mathbf{m}_{\nabla_2} + (3\lambda + 1)\mathbf{m}_{\Delta_3} + \mathbf{m}_{\Delta \partial S[-3\lambda n+1, 3\lambda n-3]},$$

$$\mathbf{m}_{\Delta \partial S[-(3\lambda+2)n+1, (3\lambda+2)n-3]} = (3\lambda + 1)\mathbf{m}_{\nabla_2} + (3\lambda + 2)\mathbf{m}_{\Delta_3} + \mathbf{m}_{\Delta \partial S[-(3\lambda+1)n+1, (3\lambda+1)n-3]}.$$

For the trapezoids, the Steinhaus trapezoids (resp. Pascal trapezoids) announced in this theorem are multiset differences of the balanced Steinhaus triangles (resp. balanced Pascal triangles), that appeared above. Finally, the lozenge $\diamond \partial S[-m+1, m-2]$ is the multiset union of the Pascal triangle $\Delta \partial S[-m+1, m-3]$ and of the Steinhaus triangle $(-1)^m \nabla S[m, 2m-1]$, which are balanced in $\mathbb{Z}/n\mathbb{Z}$, for all positive integers $m \equiv 0 \pmod{n}$.

□

7 Conclusions and open problems

In this section, the results about the generalized Molluzzo's Problem, obtained in the precedent sections, are recalled and two possible extensions of this work are proposed.

7.1 Conclusions on generalized Molluzzo's Problem

If we summarize the results obtained, from Theorem 6.4, about Problem 1.2, then there exist, for every odd number n , infinitely many balanced Steinhaus figures in $\mathbb{Z}/n\mathbb{Z}$, for each kind of figures. More precisely, as announced in Theorem 1.3, there exist :

- balanced Steinhaus triangles of order m , for every positive integer $m \equiv 0 \pmod{n}$ or $m \equiv -1 \pmod{3n}$,

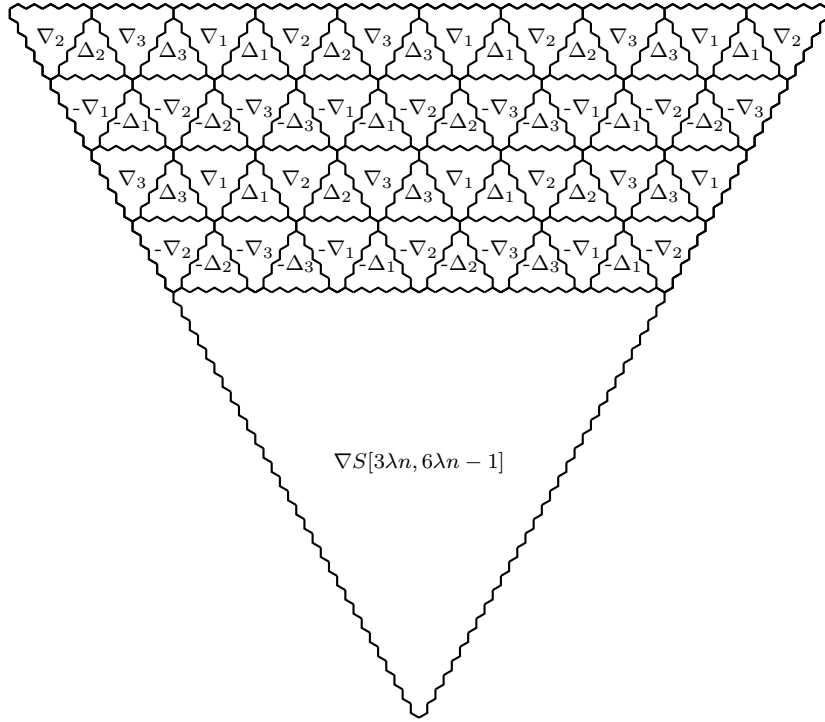


Figure 7: The Steinhaus triangle $\nabla S[m, 2m - 1]$ for $m \equiv 0 \pmod{n}$.

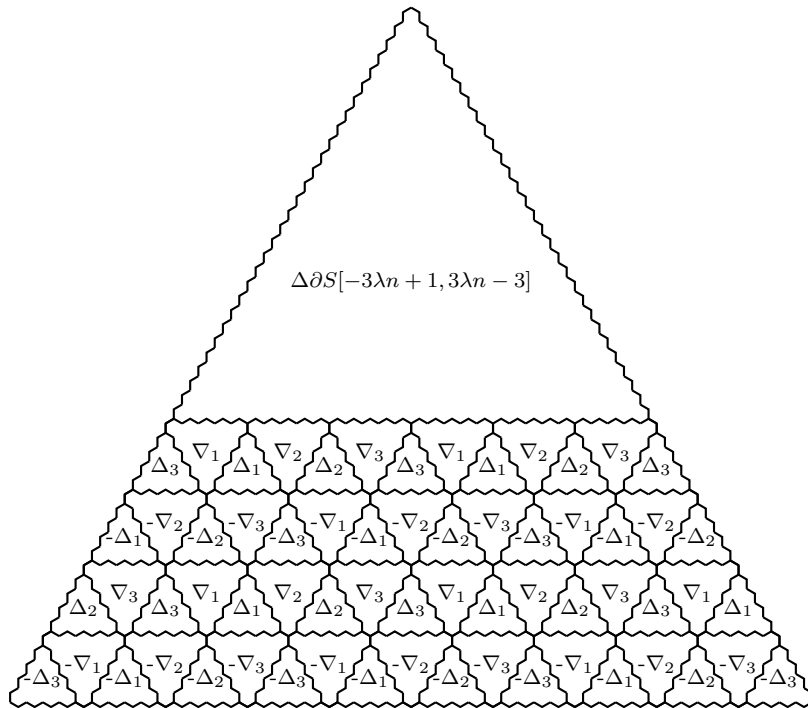


Figure 8: The Pascal triangle $\Delta \partial S[-m, m - 2]$ for $m \equiv -1 \pmod{n}$.

- balanced Steinhaus trapezoids of order m and of height h , for every positive integers $m \equiv 0 \pmod{n}$ or $m \equiv -1 \pmod{3n}$ and $h \equiv m \pmod{n}$ or $h \equiv m + 1 \pmod{3n}$,
- balanced Pascal triangles of order $2m - 1$, for every positive integer $m \equiv -1 \pmod{n}$ or $m \equiv 0 \pmod{3n}$,
- balanced Pascal trapezoids of order $2m - 1$ and of height h , for every positive integers $m \equiv -1 \pmod{n}$ or $m \equiv 0 \pmod{3n}$ and $h \equiv m + 1 \pmod{n}$ or $h \equiv m \pmod{3n}$,
- balanced lozenges of order $2m - 1$, for every positive integer $m \equiv 0 \pmod{n}$.

For the Steinhaus triangles, since a Steinhaus triangle, of order $m \geq 1$, has a cardinality of $\binom{m+1}{2}$ and since the set of all the positive integers m , such that the binomial coefficient $\binom{m+1}{2}$ is divisible by n , is an union of $2^{\omega(n)}$ classes of integers modulo n , where $\omega(n)$ is the number of distinct prime factors of n , including the classes of 0 and -1 , we have proved, in this paper, that there exist balanced Steinhaus triangles, for at least $2/(3 \cdot 2^{\omega(n)-1})$ of the admissible orders. Particularly, in the case where n is an odd prime power, this proportion becomes $2/3$. In [5], the author has proved that the arithmetic progressions, with invertible common difference, generate balanced Steinhaus triangles, for $1/(2^{\omega(n)-1}\beta(n))$ of the admissible orders, where $\beta(n)$ is the order of 2^n in the multiplicative quotient group $(\mathbb{Z}/n\mathbb{Z})^* / \{-1, 1\}$. This permitted him to prove that the Molluzzo's Problem is positively and completely solved in $\mathbb{Z}/3^k\mathbb{Z}$, for all positive integers k . A new proof of this result, shorter and based on the doubly arithmetic triangles, will appear in a forthcoming paper. For the Pascal triangles, the proportion of balanced Pascal triangles that we have highlighted is the same : $2/(3 \cdot 2^{\omega(n)-1})$ for every odd number n and, thus, $2/3$ if n is an odd prime power. Finally, for the lozenges, since a lozenge, of order $2m - 1$, has a cardinality of m^2 , the orbit of the universal sequence contains balanced lozenges, for all the admissible orders in $\mathbb{Z}/n\mathbb{Z}$, in the case where n is a square-free odd number. This completely and positively solves Problem 1.2, for the lozenges, in this case.

7.2 Additive cellular automata

Other derivation maps can be considered. For every positive integers n and k and for every k -tuple of integers $\alpha = (\alpha_0, \dots, \alpha_{k-1})$, we define the α -derivation map ∂_α by

$$\begin{aligned} \partial_\alpha : (\mathbb{Z}/n\mathbb{Z})^{\mathbb{Z}} &\longrightarrow (\mathbb{Z}/n\mathbb{Z})^{\mathbb{Z}} \\ (a_j)_{j \in \mathbb{Z}} &\longmapsto \left(\sum_{l=0}^{k-1} \alpha_l a_{j+l} \right)_{j \in \mathbb{Z}} \end{aligned} .$$

Then, the derivation map ∂ , of precedent sections, corresponds with $\partial_{(1,1)}$. Now, we naturally wonder, for every k -tuple of integers α , if there exist balanced Steinhaus figures, in the *additive cellular automaton* associated with the derivation map ∂_α in $\mathbb{Z}/n\mathbb{Z}$.

Problem 7.1. *Let n and k be two positive integers and let α be a k -tuple of integers. Does there exist balanced Steinhaus figures, in the additive cellular automaton associated with the derivation map ∂_α in $\mathbb{Z}/n\mathbb{Z}$?*

Consider the simpler case $\alpha = (\alpha_0, \alpha_1)$ in the sequel and denote by $\nabla_\alpha S$ the α -Steinhaus triangle and by $\Delta_\alpha S$ the α -Pascal triangle, associated with a finite sequence S in $\mathbb{Z}/n\mathbb{Z}$. Then, for every odd number n and for every invertible element d in $\mathbb{Z}/n\mathbb{Z}$, the universal sequence $S = IAP((0, -d, d), (d, -2d, d))$, in $\mathbb{Z}/n\mathbb{Z}$, has a $(1, 1)$ -orbit, which contains infinitely many balanced $(1, -1)$ -Steinhaus and Pascal triangles and infinitely many balanced $(-1, 1)$ -Steinhaus and Pascal triangles. Indeed, as illustrated in Figure 9, the rotation of 120 degrees defined, on the finite sequences of length $m \geq 1$ in $\mathbb{Z}/n\mathbb{Z}$, by

$$\text{rot}_{120}((a_j)_{0 \leq j \leq m-1}) = \left(\sum_{k=0}^j \binom{j}{k} a_{m-1-k} \right)_{0 \leq j \leq m-1},$$

induces an isomorphism between the $(1, 1)$ -Steinhaus triangles (resp. $(1, 1)$ -Pascal triangles) and the $(-1, 1)$ -Steinhaus triangles (resp. $(-1, 1)$ -Pascal triangles), which conserves the multiplicity. Similarly, the rotation of 240 degrees, which assigns to a sequence $(a_j)_{0 \leq j \leq m-1}$, of length m in $\mathbb{Z}/n\mathbb{Z}$, the sequence

$$\text{rot}_{240}((a_j)_{0 \leq j \leq m-1}) = \left(\sum_{k=0}^{m-1-j} \binom{m-1-j}{k} a_k \right)_{0 \leq j \leq m-1},$$

induces an isomorphism between the $(1, 1)$ -Steinhaus triangles (resp. $(1, 1)$ -Pascal triangles) and the $(1, -1)$ -Steinhaus triangles (resp. $(1, -1)$ -Pascal triangles), which conserves the multiplicity. These sequences can be seen as the right side, for $\text{rot}_{120}(S)$, and the left side, for $\text{rot}_{240}(S)$, of the $(1, 1)$ -Steinhaus triangle $\nabla_{(1,1)}S$, associated with S .

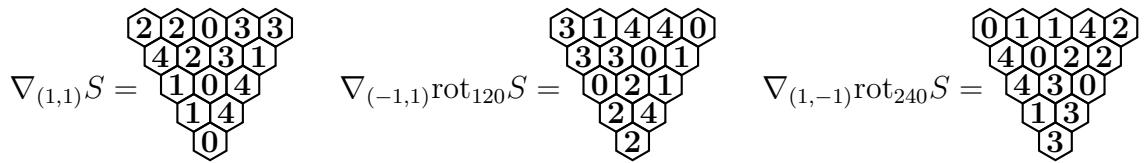


Figure 9: The Steinhaus triangles of $S = (2, 2, 0, 3, 3)$, $\text{rot}_{120}S$ and $\text{rot}_{240}S$ in $\mathbb{Z}/5\mathbb{Z}$.

Finally, since there exist balanced $(1, 1)$ -Steinhaus triangles, of order m , for every positive integer $m \equiv 0 \pmod{n}$ or $m \equiv -1 \pmod{3n}$, in $\mathbb{Z}/n\mathbb{Z}$ with n odd, then there exist balanced $(-1, 1)$ and $(1, -1)$ -Steinhaus triangles of the same orders in $\mathbb{Z}/n\mathbb{Z}$. For an odd prime power n , this corresponds to $2/3$ of the admissible orders. Similarly, there exist balanced $(-1, 1)$ and $(1, -1)$ -Pascal triangles, of order $2m - 1$, for every positive integer $m \equiv -1 \pmod{n}$ or $m \equiv 0 \pmod{3n}$, in $\mathbb{Z}/n\mathbb{Z}$ with n odd. This also corresponds to $2/3$ of the admissible orders, in the case where n is an odd prime power.

7.3 Steinhaus and Pascal tetrahedra

In this paper, we have studied balanced Steinhaus figures appearing in the cellular automaton, of dimension 1, that generates the standard Pascal triangle. We may also consider similar figures in higher dimension, in the cellular automaton, of dimension 2, generating the standard Pascal tetrahedron, for instance. Let n be a positive integer and let $S = (a_{i,j})_{i,j \in \mathbb{Z}}$ be a doubly infinite double sequence of terms in $\mathbb{Z}/n\mathbb{Z}$. The derived sequence ∂S of S is the sequence defined by $\partial S = (a_{i,j} + a_{i,j+1} + a_{i+1,j})_{i,j \in \mathbb{Z}}$ and the orbit of S is the sequence of the iterated derived sequences $\mathcal{O}_S = (\partial^k S)_{k \in \mathbb{N}}$. This orbit can also be seen as the multiset in $\mathbb{Z}/n\mathbb{Z}$, defined by

$$\mathcal{O}_S = \left\{ \sum_{i'=0}^k \sum_{j'=0}^{k-i'} \binom{k}{i',j'} a_{i+i',j+j'} \mid i \in \mathbb{Z}, j \in \mathbb{Z}, k \in \mathbb{N} \right\},$$

where $\binom{k}{i',j'}$ is the trinomial coefficient $\binom{k}{i',j'} = \frac{k!}{i'!j'!(k-i'-j)!}$. The finite orbit of a triangular submultiset $T = \{a_{i',j'} \mid 0 \leq i' \leq m-1, 0 \leq j' \leq m-1-i'\}$, of size $\binom{m+1}{2}$ in S , is called the Steinhaus tetrahedron associated with T and of order $\binom{m+1}{2}$. A Steinhaus tetrahedron of order $\binom{m+1}{2}$ has a cardinality of $\binom{m+2}{3}$. The Molluzzo's Problem on Steinhaus triangles can then be generalized as follows :

Problem 7.2. *Let n be a positive integer. For every positive integer m , such that $\binom{m+2}{3}$ is divisible by n , does there exist a balanced Steinhaus tetrahedron, of order $\binom{m+1}{2}$, in $\mathbb{Z}/n\mathbb{Z}$?*

As for the Pascal triangles of order $2m-1$, defined from the Steinhaus triangles of order $2m-1$, the Pascal tetrahedron, of order $\binom{3m-1}{2}$, is the tetrahedron of height m , built from the top to the base, that appears in the Steinhaus tetrahedron of order $\binom{3m-1}{2}$. A tetrahedron of order $\binom{3m-1}{2}$ has a cardinality of $\binom{m+2}{3}$. The Pascal tetrahedron, of order $\binom{3m-1}{2}$, associated with the triangle only composed of zeros, except its central term equal to 1, corresponds with the m first floors of the standard Pascal tetrahedron modulo n . The problem of determining the existence of balanced Pascal tetrahedra in $\mathbb{Z}/n\mathbb{Z}$ can be posed.

Problem 7.3. *Let n be a positive integer. For every positive integer m , such that $\binom{m+2}{3}$ is divisible by n , does there exist a balanced Pascal tetrahedron, of order $\binom{3m-1}{2}$, in $\mathbb{Z}/n\mathbb{Z}$?*

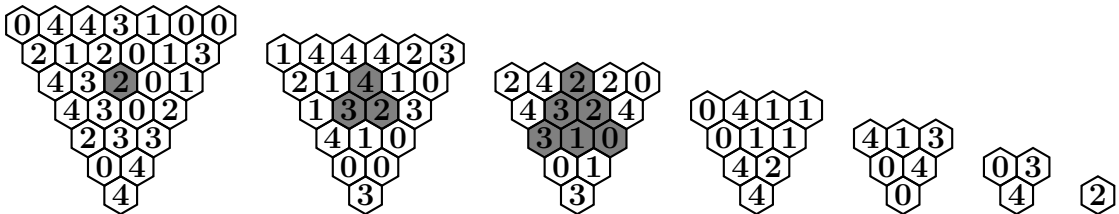


Figure 10: An example of Steinhaus tetrahedron in $\mathbb{Z}/5\mathbb{Z}$, with a Pascal tetrahedron in gray.

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Author's Address :

Jonathan Chappelon,

LMPA Joseph Liouville, FR CNRS 2956

Université du Littoral Côte d'Opale

50 rue F. Buisson, B.P. 699,

F-62228 Calais Cedex, France

e-mail: jonathan.chappelon@lmpa.univ-littoral.fr