Large sample approximation of the distribution for smooth monotone frontier estimation

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Abstract

This paper deals with nonparametric estimation of the upper boundary of a multivariate support under monotonicity constraint. This estimation problem arises in various contexts such as efficiency and frontier analysis in econometrics and portfolio management. The traditional estimators based on envelopment techniques are very non-robust. To reduce this defect, previous works have rather concentrated on estimation of a concept of a partial frontier of order $\alpha \in (0, 1)$ lying near the full support boundary. However the resulting sample estimator is a discontinuous curve and suffers from a lack of efficiency due to the large variation of the extreme observations involved in its construction. A smoothed-kernel variant of this empirical estimator may be then preferable as shown recently in the econometric literature, but no attention was devoted to the limit distribution of the smoothed α-frontier when it estimates the true full boundary itself. In this paper, we address this problem by specifying the different limit laws of this estimator for fixed orders $\alpha \in (0, 1]$ as well as for sequences $\alpha = \alpha_n$ tending to one at different rates as the sample size n goes to infinity.

Key words : Asymptotic distribution; Conditional extremes; Integrated kernel estimator; Monotone support curve.

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1 Introduction

This paper deals with nonparametric estimation of the monotone upper boundary of the support of a random vector $(X, Y) \in \mathbb{R}_+^p \times \mathbb{R}_+$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. This frontier estimation problem arises in various contexts. It appears naturally in productivity and efficiency analysis, where X is a set of inputs (for example labor, energy or capital) used to produce an output (for example a quantity of goods produced) Y in a certain firm. The support boundary may be then viewed as the set of efficient production units (firms,...). Economic considerations lead to the assumption that this support curve is a monotone nondecreasing surface. Here the data typically consist of pairs (X_i, Y_i) lying below the boundary curve and observed for a number n of i.i.d. firms.

Until recently, the joint support of (X, Y) which is interpreted in deterministic nonparametric frontier models as the production set $\{(x, y) \in \mathbb{R}_+^p \times \mathbb{R}_+ \mid x \text{ can produce } y\}$, was often assumed to be of the form $\{(x, y) \in \mathbb{R}_+^p \times \mathbb{R}_+ \mid y \leq \varphi(x)\}$ where φ is a monotone function whose the graph defines the support curve. A probabilistic formulation of such a frontier function has been introduced by Cazals, Florens and Simar (2002):

$$\varphi(x) = F^{-1}(1|x) := \sup\{y \geq 0 | F(y|x) < 1\} = \inf\{y \geq 0 | F(y|x) = 1\} \tag{1.1}$$

where $F(y|x) = F(x, y)/F_X(x)$, $F(x, y) = \mathbb{P}(X \leq x, Y \leq y)$ and $F_X(x) = \mathbb{P}(X \leq x) > 0$. A famous nonparametric estimation technique of this frontier is the free disposal hull (FDH) estimator, the lowest step monotone curve covering all sample points:

$$\hat{\varphi}(x) = F_n^{-1}(1|x) := \sup\{y \geq 0 | F_n(y|x) < 1\} = \max\{Y_i | i : X_i \leq x\}$$

where $F_n(y|x) = \sum_{i=1}^n \mathbb{1}(X_i \leq x, Y_i \leq y) / \sum_{i=1}^n \mathbb{1}(X_i \leq x)$, with $\mathbb{1}(A)$ being the indicator function for the set A. The smallest concave curve covering the FDH frontier is the popular data envelopment analysis (DEA) estimator. A related field of application where monotone concave frontiers naturally arise is portfolio management, where X measures the volatility or variance of a portfolio and Y its average return. In Capital Assets Pricing Models, the support of (X, Y) which represents the attainable set of portfolios is naturally convex and its boundary curve is interpreted as the set of optimal portfolios.

The asymptotic theory of the traditional frontier estimators FDH and DEA is now mostly available (see *e.g.* Jeong and Park (2006) and Daouia et al. (2008) for the limit distributions), but these envelopment estimators are by construction very non-robust. The underlying idea to reduce this vexing defect is to estimate a conditional quantile-based frontier near the support boundary as suggested by Aragon et al. (2005). They extend the formulation (1.1) of Cazals et al.(2002) to a concept of a partial support curve of order $\alpha \in (0, 1]$ characterized as the graph of the αth conditional quantile function

$$q_\alpha(x) = F^{-1}(\alpha|x) := \inf\{y \geq 0 | F(y|x) \geq \alpha\}.$$

Then, Aragon et al. (2005) show that the empirical estimator

$$q_{\alpha,n}(x) = F_n^{-1}(\alpha|x) := \inf\{y \geq 0 | F_n(y|x) \geq \alpha\} \tag{1.2}$$

estimates the full frontier $\varphi(x)$ itself and converges to the same Weibull distribution as the FDH estimator by an appropriate choice of the order α as a function of the sample size. Moreover $q_{\alpha,n}(x)$ has the advantage to be more resistant to extreme values and/or outliers than the standard nonparametric FDH and DEA estimators as established theoretically in Daouia and Ruiz-Gazen (2006). Even more strongly, it is shown recently in Daouia, Florens and Simar (2008) by an elegant method using extreme-values theory that this sample estimator of $\varphi(x)$ converges to a normal distribution under quite general extreme-values conditions. However it suffers from a lack of efficiency due to the large variation of the extreme observations (X_i, Y_i), with $X_i \leq x$, involved in its construction. A smoothed estimator may be then preferable to the sample estimator $q_{\alpha,n}(x)$. Martins-Filho and Yao (2008) propose a kernel-based variant

$$\hat{q}_\alpha(x) = \hat{F}^{-1}(\alpha|x) := \inf\{y \geq 0 | \hat{F}(y|x) \geq \alpha\} \tag{1.3}$$

where $\hat{F}(y|x) = \hat{F}_n(x,y)/\hat{F}_X(x)$ with

$$\hat{F}_X(x) = \frac{1}{n}\sum_{i=1}^{n} 1\!\!1(X_i \leq x), \quad \hat{F}_n(x,y) = \frac{1}{n}\sum_{i=1}^{n} 1\!\!1(X_i \leq x) H\left(\frac{y - Y_i}{h}\right),$$

$h = h_n \to 0$ is a sequence of bandwidths and $H(\cdot) = \int_{-\infty}^{\cdot} K(u)du$, with $K(\cdot)$ being a density kernel. This smoothed estimator may also be preferable to the sample one for the following additional respect: the construction of asymptotic confidence intervals for $q_\alpha(x)$ using the asymptotic normality of $q_{\alpha,n}(x)$ requires the estimation of the derivative $F'(q_\alpha(x)|x)$, whereas smoothing gives a naturally derived estimator of this conditional quantile density function.

However, while the consideration of the notion of partial frontiers in the literature is primely motivated by the construction of a robust estimator of the full frontier function $\varphi(x)$ which is well inside the sample $\{(X_i, Y_i), i = 1, \ldots, n\}$ but near from the optimal boundary, Martins-Filho and Yao (2008) only focus on the estimation of the partial frontier function $q_\alpha(x)$ for a fixed order $\alpha \in (0,1)$, showing that $\hat{q}_\alpha(x)$ is asymptotically biased and normally distributed. Our paper gives more insights and extends their work in four directions. i) First, similarly to Martins-Filho and Yao (2008), we derive in Proposition 1 the asymptotic normality of $\hat{q}_\alpha(x)$ as an estimator of $q_\alpha(x)$, for a fixed order $\alpha \in (0,1)$, under weaker conditions and without asymptotic bias. ii) Second, we focus on the asymptotic distribution of $\hat{q}_\alpha(x)$ as an estimator of the full frontier function $\varphi(x)$ itself, for a sequence $\alpha = \alpha_n$ tending to one as $n \to \infty$. It is established in Theorem 1 that $\hat{q}_{\alpha_n}(x)$ converges to a normal distribution when $n(1-\alpha_n) \to \infty$. It appears that $\hat{q}_{\alpha_n}(x)$ is asymptotically biased as a smooth estimator of the true frontier $\varphi(x)$, but is asymptotically unbiased as an estimator of the extreme partial frontiers $q_{\alpha_n}(x)$ lying near the full boundary $\varphi(x)$. Corollaries 1 and 2 answer the question of how to construct smooth confidence intervals for the large partial frontiers $q_{\alpha_n}(x)$. iii) The case where $n(1-\alpha_n)$ tends to a constant is addressed in Theorem 4. It is shown that $\hat{q}_{\alpha_n}(x)$, once centered on the true frontier $\varphi(x)$, has asymptotically an extreme-values distribution. iv) The extreme case $\alpha = 1$ is considered in Theorem 2. It provides the necessary and sufficient condition under which the smooth estimator $\hat{q}_1(x)$ of $\varphi(x)$ converges to a non-degenerate distribution. The limit distribution and the convergence rate are both specified. We also investigate the moment convergence in Theorem 3.

The next section is organized as follows. The assumptions needed to derive the asymptotic distributions of the smooth frontier estimators in the above mentioned situations i)-iv) are introduced and motivated in Subsection 2.1. Subsection 2.2 provides some standard examples to illustrate the used extreme-values type conditions. Our main results are presented in Subsection 2.3. The proofs are postponed to Appendix.

2 Main results

2.1 Assumptions

Let us denote by $\bar{F}(\cdot|x) = 1 - F(\cdot|x)$ the conditional survival function. Assumptions 1-3 below are used to establish the different limit distributions of the smooth frontier estimators in the four situations mentionned in Section 1.

Assumption 1. *The joint cumulative distribution function $F(\cdot,\cdot)$ is differentiable.*

Under this condition, the conditional cumulative distribution function $F(\cdot|x)$ is also differentiable. In the sequel, we denote by $f(\cdot,\cdot)$ and $f(\cdot|x)$, respectively, the joint and conditional densities.

Assumption 2. *Functions $F(\cdot|x)$ and $f(\cdot|x)$ satisfy the von-Mises condition*

$$\lim_{y\uparrow\varphi(x)} \{\varphi(x) - y\} f(y|x)/\bar{F}(y|x) = -1/\gamma(x).$$

This condition implies (see for instance Proposition 1.16 in Resnick (1987, p. 63)) that $F(\cdot|x)$ is in the maximum domain of attraction of Weibull with tail index $\gamma(x) < 0$, *i.e.*,

$\varphi(x) < \infty$ and

$$\bar{F}(y|x) = \{\varphi(x) - y\}^{-1/\gamma(x)} \ell_x \left(\{\varphi(x) - y\}^{-1}\right), \tag{2.4}$$

where ℓ_x is a slowly varying function, that is $\lim_{t\to\infty} \ell_x(tz)/\ell_x(t) = 1$ for all $z > 0$. More precisely, Assumption 2 is equivalent to $\varphi(x) < \infty$ and (2.4) with ℓ_x being a normalized slowly varying function, *i.e.*, using the Karamata representation for slowly varying functions (see *e.g.* Resnick (1987), p. 17) :

$$\ell_x(z) = \kappa_x \exp\left\{\int_1^z \frac{\varepsilon(t)}{t} dt\right\} \quad \text{for} \quad z > 0,$$

where $\kappa_x > 0$ and $\lim_{t\to\infty} \varepsilon(t) = 0$. From Proposition 0.8(v) in Resnick (1987, p. 22), note that (2.4) with ℓ_x being a normalized slowly varying function, is equivalent to

$$\varphi(x) - q_\alpha(x) = (1-\alpha)^{-\gamma(x)} L_x(\{1-\alpha\}^{-1}), \tag{2.5}$$

where L_x is a normalized slowly varying function. Note also that the general assumption (2.4) is the necessary and sufficient condition under which the conventional unsmoothed FDH estimator of $\varphi(x)$ converges to a Weibull distribution as shown in Daouia et al.(2008). It is also the necessary and sufficient condition for the smoothed estimator $\hat{q}_1(x)$ of $\varphi(x)$ to converge to a non-degenerate distribution as it will be shown in Theorem 2.

Assumption 3. *The density kernel K has a bounded support, i.e., for some constant $c > 0$,*

$$\int_{-c}^{c} K(u)du = 1.$$

This condition is satisfied by commonly used kernels in nonparametric estimation such as Biweight, Triweight, Epanechnikov, etc.

Remark 1. In the particular case where $\ell_x\left(\{\varphi(x) - y\}^{-1}\right) = \ell(x)$ is a strictly positive function in x, it is shown in Daouia et al.(2008) that the joint density of $(X, Y) \in \mathbb{R}_+^p \times \mathbb{R}_+$ satisfies

$$f(x, y) = c_x \{\varphi(x) - y\}^{\beta_x} + o(\{\varphi(x) - y\}^{\beta_x}) \quad \text{as} \quad y \uparrow \varphi(x) \tag{2.6}$$

for some constant $c_x > 0$ with $\beta_x = -(1/\gamma(x)) - (p+1) > -1$, provided that the functions $\ell(x) > 0$, $\gamma(x) > -1/p$ and $\varphi(x)$ are differentiable and the partial first derivatives of $\varphi(x)$ are strictly positive (in order to ensure the existence of the joint density near its support boundary). The restrictive variant (2.6) of Assumption 2 answers the question of how the conditional tail index $\gamma(x)$ is linked to the dimension $(p+1)$ of the data and to the shape of the joint density of (X, Y) near the frontier: when $\gamma(x) > -1/(p+1)$, the joint density decays to zero at a speed of power β_x of the distance from the frontier; when $\gamma(x) = -1/(p+1)$, the density has a sudden jump at the frontier; when $\gamma(x) < -1/(p+1)$, the density rises up to infinity at a speed of power β_x of the distance from the frontier ($\gamma(x) \leq -1/(p+1)$ corresponds to sharp or fault-type boundaries).

Remark 2. Most of the recent contributions to statistical aspects of frontier estimation are rather based on the restrictive assumption (2.6) that the density $f(x,y)$ is an algebraic function of $\varphi(x)-y$. Gijbels and Peng (2000) and Hwang, Park and Ryu (2002) consider the case $p=1$; Hall, Nussbaum and Stern (1997) focus on the case $p=1$ with $\beta_x > 1$, while there has been extensive work on the case $\beta_x = 0$ (see among others Gijbels, Mammen, Park and Simar 1999, Park, Simar and Weiner 2000, Cazals et al. 2002, Aragon et al. 2005, Daouia and Simar 2007, Martins-Filho and Yao 2008); Condition (2.6) has been also considered in Hardle, Park and Tsybakov (1995) and Hall, Park and Stern (1998). Econometric considerations also often lead to the assumption (2.6), but the shape parameter β_x is supposed most of the time to be independent of x or equal to zero. This is for instance the case in parametric approaches where it is often assumed[1] that the conditional density of Y given $X=x$ is an exponential or a half-normal or a truncated normal. Greene (1980) and Deprins and Simar (1985) analyze gamma densities with free shape parameter allowing $\beta \geq 0$, but with the homoskedastic restriction that β does not depend on x. It is also the case in the nonparametric econometric literature mentioned above where $\beta_x = 0$.

In our approach, we consider the general case $\beta_x > -1$, $p \geq 1$ and not necessarily constant functions $\ell_x(\cdot)$ in (2.4).

2.2 Examples

The following three examples have been considered among others by Gijbels et al (1999), Park et al.(2000), Cazals et al.(2002), Aragon et al.(2005), Daouia and Ruiz-Gazen (2006), Daouia and Simar (2007), Martins-Filho and Yao (2008), Daouia et al.(2008). The second and third examples are more justified from an economic point of view.

Example 1. We first consider the case where the monotone frontier is linear. We choose (X,Y) uniformly distributed over the region $D = \{(x,y) | 0 \leq x \leq 1, 0 \leq y \leq x\}$. In this case, we have $\varphi(x) = x$ and $\bar{F}(y|x) = (\varphi(x)-y)^2/F_X(x)$ for all $0 \leq y \leq \varphi(x)$. Then (2.4) holds with $\ell_x(\cdot) = 1/F_X(x)$ and $\gamma(x) = -1/2$ for all x. Assumption 2 holds as well.

Example 2. We now choose a non linear monotone frontier given by the Cobb-Douglas model $Y = X^{1/2}\exp(-U)$, where X is uniform on $[0,1]$ and U, independent of X, is Exponential with parameter $\lambda = 3$. Here $\varphi(x) = x^{1/2}$ and $F(y|x) = 3x^{-1}y^2 - 2x^{-3/2}y^3$, for

[1] Formally, the model is $Y = \varphi(X) - U$, where $U > 0$ *a.s.* is an exponential, etc... independent of X and φ is a specific parametric econometric function (Cobb-Douglas, Translog,...).

$0 < x \leq 1$ and $0 \leq y \leq \varphi(x)$. Then it is not hard to verify that Assumption 2 holds with $\gamma(x) = -1/2$ and that (2.4) holds with $\ell_x(z) = \{3\varphi(x) - 2/z\}/\varphi^3(x)$ for all $x \in]0,1]$ and $z > 0$.

Example 3. Here we choose a non convex support with monotone frontier given by the model $Y = X^3 \exp(-U)$, where X is uniform on $[1,2]$ and U, independent of X, is Exponential with parameter $\lambda = 3$. We have $\varphi(x) = x^3$ and we find

$$F(y|x) = \begin{cases} 1 & y \geq x^3 \\ \frac{1}{x-1}[y^{1/3} - 1 + \frac{y^3}{8}(y^{-8/3} - x^{-8})] & 1 \leq y \leq x^3 \\ \frac{y^3}{8}\frac{1-x^{-8}}{1-x} & 0 \leq y \leq 1. \end{cases}$$

Then we show that $\bar{F}(x^3 - \frac{1}{t}|x) = t^{-2}[\frac{1}{x-1}(\frac{542}{99}x^{-5} + o(1))]$. Thus $\gamma(x) = -1/2$ and $\ell_x(z)$ tends to $\frac{542}{99}\frac{x^{-5}}{x-1}$ as $z \to \infty$.

2.3 Asymptotic distributions

2.3.1 Estimation of the partial α-frontier $q_\alpha(x)$ when $\alpha \in (0,1)$ is fixed.

We show in the following proposition that the smooth estimator $\hat{q}_\alpha(x)$ of $q_\alpha(x)$, defined in (1.3), is asymptotically unbiased and normally distributed.

Proposition 1. *Under Assumptions 1 and 3, if $f(\cdot|x)$ is continuous in a neighborhood of $q_\alpha(x)$ with $f(q_\alpha(x)|x) > 0$ and if $nh^2 \to 0$ as $n \to \infty$, then*

$$\sigma_{n,1}^{-1}(x)\,(\hat{q}_\alpha(x) - q_\alpha(x)) \xrightarrow{d} N(0,1) \quad \textit{where} \quad \sigma_{n,1}(x) = \frac{\sqrt{\alpha(1-\alpha)}}{\sqrt{nF_X(x)}f(q_\alpha(x)|x)}.$$

Note that the asymptotic normality of $\hat{q}_\alpha(x)$ has also been proved by Martins-Filho and Yao (2008, Theorem 2), but this estimator is asymptotically biased in their result. Moreover they employ some strong conditions in their technique of proof, namely the assumptions A3(c) and A4. They also need $\min_{\{i:X_i \leq x\}} Y_i \geq hc$ for all n large enough, which could be violated by certain data generating processes as pointed out in their paper. Note also that we only need the kernel K to possess a compact support $[-c,c]$ in Assumption 3 to derive the asymptotic normality, whereas the technique of proof used by Martins-Filho and Yao (2008) requires, in addition to Assumption 3, several stringent conditions on K, namely the assumptions A2(a)-A2(e).

2.3.2 Estimation of the full frontier by $\hat{q}_{\alpha_n}(x)$ where $\alpha_n \to 1$, $n(1-\alpha_n) \to \infty$.

The asymptotic distribution of the endpoint estimator $\hat{q}_{\alpha_n}(x) = \hat{F}^{-1}(\alpha_n|x)$ of $\varphi(x)$ is given in the next theorem.

Theorem 1. *Under Assumptions 1-3, if* $n(1-\alpha_n) \to \infty$ *and* $nh^2(1-\alpha_n) = o\{(\varphi(x) - q_{\alpha_n}(x))^2\}$, *then*

$$\sigma_{n,2}^{-1}(x)\left(\hat{q}_{\alpha_n}(x) - \varphi(x) - b_n(x)\right) \xrightarrow{d} N(0,1)$$

where

$$b_n(x) = q_{\alpha_n}(x) - \varphi(x) \quad \text{and} \quad \sigma_{n,2}(x) = \frac{-\gamma(x)(\varphi(x) - q_{\alpha_n}(x))}{\{n(1-\alpha_n)F_X(x)\}^{1/2}}.$$

Under the conditions of Theorem 1, $\sigma_{n,2}^{-1}(x)b_n(x) \to -\infty$ as $n \to \infty$. Thus $\hat{q}_{\alpha_n}(x)$ is an asymptotically biased estimator of $\varphi(x)$ but it is asymptotically unbiased as an estimator of the extreme partial frontier $q_{\alpha_n}(x)$ lying close to the full frontier $\varphi(x)$.

Remark 3. Note that the convergence rate in Proposition 1 and Theorem 1 does not directly depend on the bandwidth h, but that this value interferes respectively in the conditions $\sqrt{n}h \to 0$ and $(\varphi(x) - q_{\alpha_n}(x))/h\sqrt{n(1-\alpha_n)} \to \infty$. Note also that the asymptotic mean squared error of $\hat{q}_{\alpha_n}(x)$ is given by

$$\text{AMSE}(\hat{q}_{\alpha_n}(x)) \sim (\varphi(x) - q_{\alpha_n}(x))^2 \left[1 + \frac{\gamma^2(x)}{n(1-\alpha_n)F_X(x)}\right].$$

The obtention of an acceptable value of α_n by optimizing the AMSE cannot be done without imposing extra second-order regular variation conditions, as it is often the case for unconditional sample quantiles. For instance, if in (2.4) , $\ell_x\left(\{\varphi(x) - y\}^{-1}\right) = \ell(x)$ is a strictly positive function in x, and $\gamma(x) < -1/2$, it is easily shown that the AMSE is minimum if and only if $n(1-\alpha_n)F_X(x) = -\gamma(x)\left(\gamma(x) + \frac{1}{2}\right)$. Clearly, this solution is not theoretically admissible since it does not fulfill $n(1-\alpha_n) \to \infty$. Nevertheless, it provides a motivation for studying the case where $n(1-\alpha_n)$ converges to a constant, see Theorem 4 below.

Theorem 1 is probably only of a theoretical value. The following corollaries enable one to construct smooth confidence intervals for high partial frontiers $q_{\alpha_n}(x)$ when $\alpha_n \to 1$ and $n(1-\alpha_n) \to \infty$.

Corollary 1. *Let*

$$\sigma_{n,3}(x) = -\frac{\gamma(x)}{\{n(1-\alpha_n)F_X(x)\}^{1/2}}.$$

Under the Assumptions of Theorem 1

$$\sigma_{n,3}^{-1}(x)\frac{\hat{q}_{\alpha_n}(x) - q_{\alpha_n}(x)}{\hat{q}_{\alpha_n}(x) - \hat{q}_{2\alpha_n-1}(x)}$$

is asymptotically normal with mean zero and variance $(2^{-\gamma(x)} - 1)^{-2}$.

To construct asymptotic confidence bands for the extreme partial frontier estimator $\hat{q}_{\alpha_n}(x)$ of $q_{\alpha_n}(x)$, it suffices to replace the tail index $\gamma(x)$ in the asymptotic variance with a consistent estimator. One can use for example a Pickands type estimator defined as:

$$\hat{\gamma}(x) = (\log 2)^{-1} \log \{(\hat{q}_{\alpha_n}(x) - \hat{q}_{2\alpha_n - 1}(x)) \,/\, (\hat{q}_{2\alpha_n-1}(x) - \hat{q}_{4\alpha_n-3}(x))\} .$$

Corollary 2. *Under the Assumptions of Theorem 1, $\hat{\gamma}(x) \xrightarrow{p} \gamma(x)$.*

2.3.3 Estimation of the full frontier by $\hat{q}_1(x)$.

Martins-Filho and Yao (2008, Theorem 4) have shown that[2]

$$n^{1/(p+1)}(\varphi(x) - \hat{q}_1(x) + hc) \xrightarrow{d} \text{Weibull}(\mu_x^{p+1}, p+1) \tag{2.7}$$

under some very restrictive conditions, namely: (1) $\min_{i:X_i \le x} Y_i \ge hc$, (2) the joint density $f(\cdot,\cdot)$ of (X,Y) is strictly positive on the frontier $\{(x,\varphi(x)) : F_X(x) > 0\}$, etc. The constant μ_x depends on the value of $f(\cdot,\cdot)$ at the frontier and on the slope of the frontier function $\varphi(\cdot)$ which is also assumed to be continuously differentiable. The next theorem gives more insights and generalizes the result of Martins-Filho and Yao (2008) in at least four directions: we show that their condition (1) is not needed for the convergence in distribution of $\hat{q}_1(x)$ to hold, we provide the necessary and sufficient condition under which $a_n^{-1}(\varphi(x) - \hat{q}_1(x) + hc)$ converges in distribution and we specify the limit distribution with the appropriate norming constants $a_n > 0$ in a general setup. In particular we extend their restrictive condition (2) to the more general case where $f(\cdot,\cdot)$ may decrease to zero or rise up to infinity. We also provide a limit theorem of moments.

Theorem 2. *Let Assumption 3 hold and $h = h_n$ be any sequence of bandwidths. There exists $a_n(x) > 0$ such that $a_n^{-1}(x)(\hat{q}_1(x) - \varphi(x) - hc)$ converges in distribution if and only if $F(\cdot|x)$ is in the maximum domain of attraction of Weibull, or equivalently,* (2.4) *holds. In such a case, $a_n(x)$ can be chosen as*

$$a_n(x) = \varphi(x) - q_{1-\frac{1}{nF_X(x)}}(x),$$

and the limit cumulative distribution function is

$$y \mapsto \Psi_{\gamma(x)}(y) = \begin{cases} \exp\{-(-y)^{-1/\gamma(x)}\} & y < 0 \\ 1 & y \ge 0. \end{cases}$$

As an immediate consequence of Theorem 2, if Assumptions 3 and (2.4) hold and $a_n^{-1}(x)h \to 0$, then $a_n^{-1}(x)(\hat{q}_1(x) - \varphi(x)) \xrightarrow{d} \Psi_{\gamma(x)}$.

[2] Hereafter we say that a random variable X follows the distribution Weibull(λ, r) if λX^r is Exponential with parameter 1.

Remark 4. In the particular case where in (2.4),

$$\ell_x\left(\{\varphi(x)-y\}^{-1}\right) := \ell(x)$$

is a strictly positive function in x, we have $F^{-1}(\alpha|x) = \varphi(x) - (\frac{1-\alpha}{\ell(x)})^{-\gamma(x)}$ as $\alpha \uparrow 1$, whence $a_n(x) = \{nF_X(x)\ell(x)\}^{\gamma(x)}$ for all n sufficiently large. We thus have in this particular case

$$\{nF_X(x)\ell(x)\}^{-\gamma(x)}\left(\varphi(x) - \hat{q}_1(x) + hc\right) \xrightarrow{d} \text{Weibull}(1, -1/\gamma(x)) \quad \text{as} \quad n \to \infty.$$

If, in addition (2.6) holds, then the norming constant is

$$a_n(x) = \{nF_X(x)\ell(x)\}^{-1/(\beta_x+p+1)}.$$

In the restrictive case where the joint density has a jump at its support boundary, *i.e.* $\beta_x = 0$, we achieve the best convergence rate $n^{-1/(p+1)}$ as in Martins-Filho and Yao (2008), see equation (2.7). Let us mention that the case where the frontier function is β- Lipschitzian has been addressed in Girard and Menneteau (2005) and Girard and Jacob (2008). Different estimators reaching the best convergence rate $n^{-\beta/(p+\beta)}$ have been proposed. The main difference with our approach lies in the fact that those approaches as well as the vast literature on frontier estimation focus on nonparametric estimation of the right-endpoint of the conditional distribution of Y given $X = x$, whereas the distribution of Y is conditioned by $X \leq x$ in our setup due to the monotonicity constraint.

Next we show for which values of $k > 0$ the convergence in distribution $a_n^{-1}(x)(\hat{q}_1(x) - \varphi(x) - hc) \xrightarrow{d} \Psi_{\gamma(x)}$ implies the moment convergence

$$\mathbb{E}\{a_n^{-1}(\hat{q}_1(x) - \varphi(x) - hc)\}^k \longrightarrow \int_{-\infty}^{0} y^k \Psi_{\gamma(x)}(dy) \quad \text{as} \quad n \to \infty.$$

Theorem 3. *Let Assumption 3 hold. If* $a_n^{-1}(x)(\hat{q}_1(x) - \varphi(x) - hc) \xrightarrow{d} \Psi_{\gamma(x)}$ *with* $a_n(x) = \varphi(x) - q_{1-\frac{1}{nF_X(x)}}(x)$*, then for any integer* $k \geq 1$

$$\lim_{n\to\infty} \mathbb{E}\{a_n^{-1}(x)(\hat{q}_1(x) - \varphi(x) - hc)\}^k = \int_{-\infty}^{0} y^k \Psi_{\gamma(x)}(dy) = (-1)^k \Gamma(1 - k\gamma(x)),$$

where $\Gamma(\cdot)$ *denotes the gamma function.*

Remark 5. We have $\mathbb{E}\{\varphi(x)-\hat{q}_1(x)\}^2 = \mathbb{E}\{\varphi(x)-\hat{q}_1(x)+hc\}^2 - (hc)^2 - 2hc\mathbb{E}\{\varphi(x)-\hat{q}_1(x)\}$. Then by using the fact that $\mathbb{E}\{\varphi(x) - \hat{q}_1(x) + hc\}^k = \Gamma(1 - k\gamma(x))a_n^k(x) + o(a_n^k(x))$ and optimizing the AMSE $= \mathbb{E}\{\varphi(x) - \hat{q}_1(x)\}^2$, we get the following optimal value of h,

$$h_n(x) = \arg\min_h \{h^2c - 2ha_n(x)\Gamma(1 - \gamma(x))\} = a_n(x)\Gamma(1 - \gamma(x))/c.$$

In particular, when the joint density of (X, Y) has a jump at its support boundary, as it is often assumed in applications (see Remarks 2 and 4), we have $a_n(x) = \{nF_X(x)\ell(x)\}^{\gamma(x)}$ and $\gamma(x) = -1/(p+1)$, which gives

$$h_n(x) = \{c(p+1)\}^{-1}\Gamma(1/(p+1))\{nF_X(x)\ell(x)\}^{-1/(p+1)}.$$

This also reflects how the smooth estimator $\hat{q}_1(x)$ suffers from the curse of dimensionality as the number p of inputs increases. Finally, it should be clear that $a_n^{-1}(x)(\hat{q}_1(x) - \varphi(x) - ch_n(x)) \xrightarrow{d} \Psi_{\gamma(x)}$, however the optimal value $h_n(x)$ does not satisfy $a_n^{-1}(x)h_n(x) \to 0$.

Remark 6. In the particular case where $\ell_x(\cdot) = \ell(x) > 0$, it is easy to check that, for n large enough,

$$\text{AMSE}(\hat{q}_1(x)) < \text{AMSE}(\hat{q}_{\alpha_n}(x)) < \text{AMSE}(\hat{q}_\alpha(x)),$$

where $n(1-\alpha_n) \to \infty$ and $\alpha \in (0,1)$. Thus $\hat{q}_1(\cdot)$ is the best estimator for the full frontier from a theoretical point of view, but it is by construction very non-robust. The estimator $\hat{q}_{\alpha_n}(\cdot)$ may be preferable since it is less sensitive to extremes, is asymptotically Gaussian and provides a useful confidence interval.

2.3.4 Estimation of $\varphi(x)$ by $\hat{q}_{\alpha_n}(x)$ when $n(1-\alpha_n)$ converges to a constant.

Next we show that if the bandwidth h and the scaling $a_n^{-1}(x)$, described in Theorem 2, satisfy $a_n^{-1}(x)h \to 0$ as $n \to \infty$, then we can specify the asymptotic distribution of $\hat{q}_{1-k/n\hat{F}_X(x)}(x)$ when it estimates the true frontier function $\varphi(x)$, for all fixed integers $k \geq 0$.

Theorem 4. *If Assumptions 3 and* (2.4) *hold with $a_n^{-1}(x)h \to 0$ as $n \to \infty$, then for any fixed integer $k \geq 0$,*

$$a_n^{-1}(x)\left(\hat{q}_{1-\frac{k}{n\hat{F}_X(x)}}(x) - \varphi(x)\right) \xrightarrow{d} H_{k,x}$$

with the cumulative distribution function $H_{k,x}(y) = \Psi_{\gamma(x)}(y)\sum_{i=0}^{k}(-\log\Psi_{\gamma(x)}(y))^i/i!$.

This theorem states that when $a_n^{-1}(x)(\hat{q}_1(x) - \varphi(x)) \xrightarrow{d} \Psi_{\gamma(x)}$ and $a_n^{-1}(x)h \to 0$, the nonparametric partial frontier $\hat{q}_\alpha(x)$ with $\alpha = 1 - k/n\hat{F}_X(x)$ estimates $\varphi(x)$ itself and converges in distribution as well, with the same scaling but a different limit distribution $H_{k,x}$ which coincides with $\Psi_{\gamma(x)}$ only for $k = 0$. It should be also clear that $\alpha \to 1$ and $n(1-\alpha)$ converges here to a constant, almost surely as $n \to \infty$, whereas the asymptotic normality of $\hat{q}_\alpha(x)$ requires $\alpha \to 1$ slowly so that $n(1-\alpha) \to \infty$ as established in Theorem 1 and Corollary 1.

Appendix: proofs and lemmas

Proof of Proposition 1 Let Φ be the standard normal distribution function. Our aim is to show that $\mathbb{P}[\sigma_{n,1}^{-1}(x)(\hat{q}_\alpha(x) - q_\alpha(x)) \leq z] \to \Phi(z)$ as $n \to \infty$, for all $z \in \mathbb{R}$. From the definition of $\hat{q}_\alpha(x)$, we have

$$\begin{aligned}
&\mathbb{P}[\sigma_{n,1}^{-1}(x)(\hat{q}_\alpha(x) - q_\alpha(x)) \leq z] = \mathbb{P}[\hat{F}(q_\alpha(x) + \sigma_{n,1}(x)z|x) \geq \alpha] \\
&\quad = \mathbb{P}[\hat{F}(q_\alpha(x) + \sigma_{n,1}(x)z|x) - F(q_\alpha(x) + \sigma_{n,1}(x)z|x) \geq \alpha - F(q_\alpha(x) + \sigma_{n,1}(x)z|x)].
\end{aligned}$$

Let $\{v_{n,1}\}$ be the positive sequence defined by

$$v_{n,1} = \frac{1}{\sigma_{n,1}(x) f(q_\alpha(x)|x)} = \sqrt{\frac{nF_X(x)}{\alpha(1-\alpha)}}. \tag{2.8}$$

We thus have $\mathbb{P}[\sigma_{n,1}^{-1}(x)(\hat{q}_\alpha(x) - q_\alpha(x)) \leq z] = \mathbb{P}[W_{n,1} \geq a_{n,1}]$, where

$$\begin{aligned}
a_{n,1} &= v_{n,1}\{\alpha - F(q_\alpha(x) + \sigma_{n,1}(x)z|x)\}, \\
W_{n,1} &= v_{n,1}\{\hat{F}(q_\alpha(x) + \sigma_{n,1}(x)z|x) - F(q_\alpha(x) + \sigma_{n,1}(x)z|x)\}.
\end{aligned}$$

We first need to prove that $a_{n,1} \to -z$ as $n \to \infty$ and second we shall show that $W_{n,1} \xrightarrow{d} N(0,1)$. By doing so, $G_{n,1}(\cdot) = \mathbb{P}[W_{n,1} < \cdot]$ converges pointwise to the continuous distribution function Φ. Then, by Dini's Theorem, $G_{n,1}$ converges uniformly to Φ. Therefore, taking $a_{n,1} \to -z$ into account, we obtain $G_{n,1}(a_{n,1}) \to \Phi(-z)$ and thus $\mathbb{P}[\sigma_{n,1}^{-1}(x)(\hat{q}_\alpha(x) - q_\alpha(x)) \leq z] \to 1 - \Phi(-z) = \Phi(z)$, which will complete the proof. Let us first show that $a_{n,1} \to -z$. A Taylor's expansion yields

$$a_{n,1} = -v_{n,1}\sigma_{n,1}(x) z f(q_\alpha(x) + \theta\sigma_{n,1}(x)z|x) = -z\frac{f(q_\alpha(x) + \theta\sigma_{n,1}(x)z|x)}{f(q_\alpha(x)|x)},$$

with $\theta \in (0,1)$, and thus $a_{n,1} \to -z$ as $n \to \infty$ since $f(\cdot|x)$ is continuous at $q_\alpha(x)$. Now let us show that $W_{n,1} \xrightarrow{d} N(0,1)$. To this end, consider the expansion $W_{n,1} = \Delta_{1,1} + \Delta_{2,1}$, where $\Delta_{1,1} = (v_{n,1}/\hat{F}_X(x))\sum_{i=1}^n n^{-1}[Z_{n,1,i} - \mathbb{E}(Z_{n,1,i})]$ and $\Delta_{2,1} = (v_{n,1}/\hat{F}_X(x))\mathbb{E}(Z_{n,1,1})$, with

$$Z_{n,1,i} = \mathbb{1}(X_i \leq x)\left[H\left(\frac{q_\alpha(x) + \sigma_{n,1}(x)z - Y_i}{h}\right) - F(q_\alpha(x) + \sigma_{n,1}(x)z|x)\right]$$

for $i = 1, \ldots, n$. Since the frontier function $\varphi(\cdot)$ is monotone nondecreasing and the joint density of (X, Y) satisfies $f(t, y) = 0$ for all $y > \varphi(t)$, it can be easily checked that

$$\begin{aligned}
\Delta_{2,1} &= \frac{v_{n,1}}{\hat{F}_X(x)} \left\{ \int_0^x \left[\int_0^{\varphi(x)} H\left(\frac{q_\alpha(x) + \sigma_{n,1}(x)z - y}{h} \right) f(t, y) dy \right] dt \right. \\
&\qquad \left. - F(q_\alpha(x) + \sigma_{n,1}(x)z|x) F_X(x) \right\} \\
&= \frac{v_{n,1}}{\hat{F}_X(x)} \left\{ \int_0^x \left[\int_{\frac{q_\alpha(x) - \varphi(x) + \sigma_{n,1}(x)z}{h}}^{\frac{q_\alpha(x) + \sigma_{n,1}(x)z}{h}} H(w) f(t, q_\alpha(x) + \sigma_{n,1}(x)z - hw) \right. \right. \\
&\qquad \left. \left. \times h dw \right] dt - F(q_\alpha(x) + \sigma_{n,1}(x)z|x) F_X(x) \right\}.
\end{aligned}$$

Since $h \to 0$, we have $\frac{q_\alpha(x) - \varphi(x) + \sigma_{n,1}(x)z}{h} \to -\infty$ and $\frac{q_\alpha(x) + \sigma_{n,1}(x)z}{h} \to \infty$. As an immediate consequence we get $\frac{q_\alpha(x) - \varphi(x) + \sigma_{n,1}(x)z}{h} < -c$ for all n large enough. Hence, taking account of $\int_{-c}^{c} K(t) dt = 1$ and integrating by parts, we find for all n sufficiently large that

$$\begin{aligned}
\Delta_{2,1} &= \frac{v_{n,1}}{\hat{F}_X(x)} \left\{ \int_0^x \left[\int_{-c}^{\frac{q_\alpha(x) + \sigma_{n,1}(x)z}{h}} H(w) f(t, q_\alpha(x) + \sigma_{n,1}(x)z - hw) h dw \right] dt \right. \\
&\qquad \left. - F(q_\alpha(x) + \sigma_{n,1}(x)z|x) F_X(x) \right\} \\
&= \frac{v_{n,1}}{\hat{F}_X(x)} \left\{ \int_0^x \left[-H\left(\frac{q_\alpha(x) + \sigma_{n,1}(x)z}{h} \right) \frac{\partial}{\partial t} F(t, 0) \right. \right. \\
&\qquad \left. + \int_{-c}^{c} K(w) \frac{\partial}{\partial t} F(t, q_\alpha(x) + \sigma_{n,1}(x)z - hw) dw \right] dt \\
&\qquad \left. - F(q_\alpha(x) + \sigma_{n,1}(x)z|x) F_X(x) \right\} \\
&= \frac{v_{n,1}}{\hat{F}_X(x)} \left\{ \int_{-c}^{c} \left[\int_0^x \frac{\partial}{\partial t} F(t, q_\alpha(x) + \sigma_{n,1}(x)z - hw) dt \right] K(w) dw \right. \\
&\qquad \left. - F(q_\alpha(x) + \sigma_{n,1}(x)z|x) F_X(x) \right\} \\
&= \frac{v_{n,1} F_X(x)}{\hat{F}_X(x)} \int_{-c}^{c} [F(q_\alpha(x) + \sigma_{n,1}(x)z - hw|x) - F(q_\alpha(x) + \sigma_{n,1}(x)z|x)] K(w) dw.
\end{aligned}$$

Whence

$$|\Delta_{2,1}| \leq \frac{v_{n,1}}{\hat{F}_X(x)} \sup_{y \in [-c, c]} \left| \bar{F}(q_\alpha(x) + \sigma_{n,1}(x)z - hy|x) - \bar{F}(q_\alpha(x) + \sigma_{n,1}(x)z|x) \right|$$

and from the continuity of $\bar{F}(\cdot|x)$ in a neighborhood of $q_\alpha(x)$, there exists $y^* \in [-c, c]$ such that

$$
\begin{aligned}
|\Delta_{2,1}| &\leq \frac{v_{n,1}}{\hat{F}_X(x)} \left| \bar{F}(q_\alpha(x) + \sigma_{n,1}(x)z - hy^*|x) - \bar{F}(q_\alpha(x) + \sigma_{n,1}(x)z|x) \right| \\
&\leq \frac{v_{n,1}hc}{\hat{F}_X(x)} f(q_\alpha(x) + \sigma_{n,1}(x)z - hy^*\xi|x), \qquad (2.9)
\end{aligned}
$$

where $\xi \in (0,1)$, which entails $\Delta_{2,1} = O(v_{n,1}h/\hat{F}_X(x)) \xrightarrow{a.s.} 0$ in view of $\sqrt{n}h \to 0$. It remains to show that $\Delta_{1,1} \xrightarrow{d} N(0,1)$. The above calculations show that $\mathbb{E}(Z_{n,1,1}) = O(h)$ while

$$
\mathbb{E}\left\{ \mathbb{1}(X \leq x) H\left(\frac{q_\alpha(x) + \sigma_{n,1}(x)z - Y}{h} \right) \right\} = F(q_\alpha(x) + \sigma_{n,1}(x)z|x)F_X(x) + O(h).
$$

Similarly, it is easy checked that

$$
\mathbb{E}\left\{ \mathbb{1}(X \leq x) H^2\left(\frac{q_\alpha(x) + \sigma_{n,1}(x)z - Y}{h} \right) \right\} - F_X(x)F(q_\alpha(x) + \sigma_{n,1}(x)z|x) = O(h).
$$

Thus,

$$
\mathbb{E}(Z_{n,1,1}^2) = F_X(x)F(q_\alpha(x) + \sigma_{n,1}(x)z|x)[1 - F(q_\alpha(x) + \sigma_{n,1}(x)z|x)] + O(h)
$$

which implies $\mathrm{var}(Z_{n,1,1}) \to \alpha(1-\alpha)F_X(x)$ as $n \to \infty$. Moreover, since $\mathbb{E}[|Z_{n,1,1} - \mathbb{E}(Z_{n,1,1})|^3] \leq 2\mathrm{var}(Z_{n,1,1})$, we have

$$
\frac{n\mathbb{E}[|Z_{n,1,1} - \mathbb{E}(Z_{n,1,1})|^3]}{[n\mathrm{var}(Z_{n,1,1})]^{3/2}} \leq 2/\sqrt{n\mathrm{var}(Z_{n,1,1})} \to 0
$$

and Lyapounov's Theorem entails

$$
\sum_{i=1}^{n} \frac{Z_{n,1,i} - \mathbb{E}(Z_{n,1,1})}{\sqrt{n\mathrm{var}(Z_{n,1,1})}} \xrightarrow{d} N(0,1).
$$

Finally,

$$
v_{n,1}\sqrt{\mathrm{var}(Z_{n,1,1})}/\sqrt{n}\hat{F}_X(x) \sim v_{n,1}\sqrt{\alpha(1-\alpha)F_X(x)}/\sqrt{n}\hat{F}_X(x) \xrightarrow{a.s.} 1,
$$

yields $\Delta_{1,1} \xrightarrow{d} N(0,1)$ which concludes the proof. □

Proof of Theorem 1 The proof follows the same lines as that of Proposition 1. Let us define

$$
\begin{aligned}
v_{n,2} &= -\gamma(x)\sigma_{n,2}^{-1}(x)(\varphi(x) - q_{\alpha_n}(x))/(1-\alpha_n) \qquad (2.10) \\
a_{n,2} &= v_{n,2}\{\alpha_n - F(q_{\alpha_n}(x) + \sigma_{n,2}(x)z|x)\} \\
W_{n,2} &= v_{n,2}\{\hat{F}(q_{\alpha_n}(x) + \sigma_{n,2}(x)z|x) - F(q_{\alpha_n}(x) + \sigma_{n,2}(x)z|x)\}.
\end{aligned}
$$

We first need to prove that $a_{n,2} \to -z$ as $n \to \infty$. A Taylor's expansion and Assumption 2 show that there exists $\theta \in (0,1)$ such that

$$\begin{aligned} a_{n,2} &= -v_{n,2}\sigma_{n,2}(x) z f(q_{\alpha_n}(x) + \theta\sigma_{n,2}(x)z|x) \\ &\sim \frac{v_{n,2}\sigma_{n,2}(x)z}{\gamma(x)} \frac{\bar{F}(q_{\alpha_n}(x) + \theta\sigma_{n,2}(x)z|x)}{\varphi(x) - q_{\alpha_n}(x) - \theta\sigma_{n,2}(x)z}. \end{aligned}$$

Since $\sigma_{n,2}(x)/(\varphi(x) - q_{\alpha_n}(x)) \to 0$, we have

$$a_{n,2} \sim \frac{v_{n,2}\sigma_{n,2}(x)z}{\gamma(x)} \frac{\bar{F}(q_{\alpha_n}(x)|x)}{\varphi(x) - q_{\alpha_n}(x)} = \frac{v_{n,2}\sigma_{n,2}(x)z}{\gamma(x)} \frac{1-\alpha_n}{\varphi(x) - q_{\alpha_n}(x)}$$

and thus $a_{n,2} \to -z$ in view of (2.10). Now let us show that $W_{n,2} \xrightarrow{d} N(0,1)$. We consider again the expansion $W_{n,2} = \Delta_{1,2} + \Delta_{2,2}$ where

$$\begin{aligned} \Delta_{1,2} &= (v_{n,2}/\hat{F}_X(x)) \sum_{i=1}^{n} n^{-1}[Z_{n,2,i} - \mathbb{E}(Z_{n,2,i})], \\ \Delta_{2,2} &= (v_{n,2}/\hat{F}_X(x))\mathbb{E}(Z_{n,2,1}), \\ Z_{n,2,i} &= \mathbb{I}(X_i \le x)\left[H\left(\frac{q_{\alpha_n}(x) + \sigma_{n,2}(x)z - Y_i}{h}\right) - F(q_{\alpha_n}(x) + \sigma_{n,2}(x)z|x)\right] \end{aligned}$$

for $i = 1, \dots, n$. Since $\frac{q_{\alpha_n}(x) - \varphi(x)}{h} \to -\infty$ and $\frac{\sigma_{n,2}(x)}{q_{\alpha_n}(x) - \varphi(x)} \to 0$, we have $\frac{q_{\alpha_n}(x) - \varphi(x) + \sigma_{n,2}(x)z}{h} \to -\infty$ and therefore, similarly to (2.9), it can be established by making use of the continuity of $\bar{F}(\cdot|x)$ in a left neighborhood of $\varphi(x)$ that for all n large enough

$$|\Delta_{2,2}| \le \frac{v_{n,2}hc}{\hat{F}_X(x)} f(q_{\alpha_n}(x) + \sigma_{n,2}(x)z - hy^*\xi|x),$$

where $\xi \in (0,1)$ and $y^* \in [-c, c]$. Now, Assumption 2 entails

$$|\Delta_{2,2}| \le -\frac{1}{\gamma(x)} \frac{v_{n,2}hc}{\hat{F}_X(x)} \frac{\bar{F}(q_{\alpha_n}(x) + \sigma_{n,2}(x)z - hy^*\xi|x)}{\varphi(x) - q_{\alpha_n}(x) - \sigma_{n,2}(x)z + hy^*\xi}.$$

Since $h/(\varphi(x) - q_{\alpha_n}(x)) \to 0$, $\sigma_{n,2}(x)/(\varphi(x) - q_{\alpha_n}(x)) \to 0$ and $\ell_x(.)$ is a slowly varying function, it can be easily seen that $\bar{F}(q_{\alpha_n}(x) + \sigma_{n,2}(x)z - hy^*|x) \sim \bar{F}(q_{\alpha_n}(x) + \sigma_{n,2}(x)z|x) \sim 1 - \alpha_n$ and $\varphi(x) - q_{\alpha_n}(x) - \sigma_{n,2}(x)z + hy^*\xi \sim \varphi(x) - q_{\alpha_n}(x)$. Then

$$\Delta_{2,2} = O\left(\frac{v_{n,2}h}{\hat{F}_X(x)} \frac{(1-\alpha_n)}{\varphi(x) - q_{\alpha_n}(x)}\right) = O\left(\frac{h}{\sigma_{n,2}(x)\hat{F}_X(x)}\right). \tag{2.11}$$

Thus $\Delta_{2,2} \xrightarrow{a.s.} 0$ as $n \to \infty$. It remains to show that $\Delta_{1,2} \xrightarrow{d} N(0,1)$. We know from (2.11) that $\mathbb{E}(Z_{n,2,1}) = \Delta_{2,2}\hat{F}_X(x)/v_{n,2} = O((1-\alpha_n)h/(\varphi(x) - q_{\alpha_n}(x))$. Similarly,

$$\begin{aligned} \mathbb{E}\left\{\mathbb{I}(X \le x) H^2\left(\frac{q_{\alpha_n}(x) + \sigma_{n,2}(x)z - Y}{h}\right)\right\} &= F(q_{\alpha_n}(x) + \sigma_{n,2}(x)z|x)F_X(x) \\ &\quad + O\left(\frac{(1-\alpha_n)h}{\varphi(x) - q_{\alpha_n}(x)}\right), \end{aligned}$$

leading to

$$\begin{aligned}\mathrm{var}(Z_{n,2,1}) &= \bar{F}(q_{\alpha_n}(x)+\sigma_{n,2}(x)z|x)[1-\bar{F}(q_{\alpha_n}(x)+\sigma_{n,2}(x)z|x)]F_X(x)\\ &\quad +O\left(\frac{(1-\alpha_n)h}{\varphi(x)-q_{\alpha_n}(x)}\right),\end{aligned}$$

and thus $\mathrm{var}(Z_{n,2,1}) \sim F_X(x)(1-\alpha_n)$ as $n\to\infty$. Therefore

$$\frac{n\mathbb{E}[|Z_{n,2,1}-\mathbb{E}(Z_{n,2,1})|^3]}{[n\mathrm{var}(Z_{n,2,1})]^{3/2}} \le 2/\sqrt{n\mathrm{var}(Z_{n,2,1})} \sim 2/\sqrt{n(1-\alpha_n)F_X(x)} \to 0$$

as $n\to\infty$, and consequently

$$\sum_{i=1}^{n}[Z_{n,2,i}-\mathbb{E}(Z_{n,2,i})]/\sqrt{n(1-\alpha_n)F_X(x)} \xrightarrow{d} N(0,1)$$

according to Lyapounov's Theorem. Finally since

$$\sqrt{(1-\alpha_n)F_X(x)}[v_{2,n}/\sqrt{n}\hat{F}_X(x)] \xrightarrow{a.s.} 1,$$

we obtain $\Delta_{1,2} \xrightarrow{d} N(0,1)$. This yields $W_{2,n} \xrightarrow{d} N(0,1)$ and concludes the proof. $\square$

Proof of Corollary 1 From Theorem 1, we have

$$\hat{q}_{\alpha_n}(x) = q_{\alpha_n}(x) + \sigma_{n,3}(x)\xi_{\alpha_n}\{\varphi(x)-q_{\alpha_n}(x)\},$$

where $\xi_{\alpha_n} \xrightarrow{d} N(0,1)$. Let $\delta_n = 2\alpha_n - 1$. Since $n(1-\delta_n) = 2n(1-\alpha_n)\to\infty$,

$$\hat{q}_{\delta_n}(x) = q_{\delta_n}(x) + \frac{\sigma_{n,3}(x)}{\sqrt{2}}\xi_{\delta_n}\{\varphi(x)-q_{\delta_n}(x)\},$$

where $\xi_{\delta_n} \xrightarrow{d} N(0,1)$. Let us introduce the following notation :

$$R_n(x) = \sigma_{n,3}^{-1}(x)\left\{\frac{\hat{q}_{\alpha_n}(x)-q_{\alpha_n}(x)}{\hat{q}_{\alpha_n}(x)-\hat{q}_{\delta_n}(x)}\right\}.$$

Hence,

$$\begin{aligned}R_n(x) &= \frac{\xi_{\alpha_n}\{\varphi(x)-q_{\alpha_n}(x)\}}{q_{\alpha_n}(x)-q_{\delta_n}(x)+\sigma_{n,3}(x)\left\{\xi_{\alpha_n}\{\varphi(x)-q_{\alpha_n}(x)\}-\frac{\xi_{\delta_n}}{\sqrt{2}}\{\varphi(x)-q_{\delta_n}(x)\}\right\}}\\ &= \xi_{\alpha_n}\left\{\frac{\varphi(x)-q_{\delta_n}(x)}{\varphi(x)-q_{\alpha_n}(x)}-1+\sigma_{n,3}(x)\left\{\xi_{\alpha_n}-\frac{\xi_{\delta_n}}{\sqrt{2}}\frac{\varphi(x)-q_{\delta_n}(x)}{\varphi(x)-q_{\alpha_n}(x)}\right\}\right\}^{-1}.\end{aligned}$$

From (2.5), we have

$$\frac{\varphi(x)-q_{\delta_n}(x)}{\varphi(x)-q_{\alpha_n}(x)} = 2^{-\gamma(x)}\frac{L_x(\{1-\delta_n\}^{-1})}{L_x(\{1-\alpha_n\}^{-1})} \to 2^{-\gamma(x)},$$

since $L_x(.)$ is a slowly varying function. Thus,

$$R_n(x) = \{2^{-\gamma(x)} - 1\}^{-1} \xi_{\alpha_n} (1 + o_{\mathrm{P}}(1)),$$

which concludes the proof. $\square$

Proof of Corollary 2 Let us introduce the notation $\theta_n := 4\alpha_n - 3$. From Theorem 1, $\log(2)\hat{\gamma}(x)$ is equal to

$$\begin{aligned}
& \log\left\{\frac{\varphi(x) - q_{\delta_n}(x) - \{\varphi(x) - q_{\alpha_n}(x)\} + \sigma_{n,3}(x)\left\{\xi_{\alpha_n}\{\varphi(x) - q_{\alpha_n}(x)\} - \frac{\xi_{\delta_n}}{\sqrt{2}}\{\varphi(x) - q_{\delta_n}(x)\}\right\}}{\varphi(x) - q_{\theta_n}(x) - \{\varphi(x) - q_{\delta_n}(x)\} + \frac{\sigma_{n,3}(x)}{\sqrt{2}}\left\{\xi_{\delta_n}\{\varphi(x) - q_{\delta_n}(x)\} - \frac{\xi_{\theta_n}}{\sqrt{2}}\{\varphi(x) - q_{\theta_n}(x)\}\right\}}\right\} \\
= \; & \log\left\{\frac{1 - \frac{\varphi(x)-q_{\alpha_n}(x)}{\varphi(x)-q_{\delta_n}(x)} + \sigma_{n,3}(x)\left\{\xi_{\alpha_n}\frac{\varphi(x)-q_{\alpha_n}(x)}{\varphi(x)-q_{\delta_n}(x)} - \frac{\xi_{\delta_n}}{\sqrt{2}}\right\}}{-1 + \frac{\varphi(x)-q_{\theta_n}(x)}{\varphi(x)-q_{\delta_n}(x)} + \frac{\sigma_{n,3}(x)}{\sqrt{2}}\left\{\xi_{\delta_n} - \frac{\xi_{\theta_n}}{\sqrt{2}}\frac{\varphi(x)-q_{\theta_n}(x)}{\varphi(x)-q_{\delta_n}(x)}\right\}}\right\} \\
= \; & \gamma(x)\log(2)\log\left\{\frac{1 + o(1) + \frac{\sigma_{n,3}(x)}{1-2^{\gamma(x)}}\left\{2^{\gamma(x)}\xi_{\alpha_n} - \frac{\xi_{\delta_n}}{\sqrt{2}} + o_{\mathrm{P}}(1)\right\}}{1 + o(1) + \frac{\sigma_{n,3}(x)}{(1-2^{\gamma(x)})\sqrt{2}}\left\{\xi_{\delta_n} - 2^{\gamma(x)}\frac{\xi_{\theta_n}}{\sqrt{2}} + o_{\mathrm{P}}(1)\right\}}\right\}
\end{aligned}$$

Then remarking that

$$\frac{1 + o(1) + \frac{\sigma_{n,3}(x)}{1-2^{\gamma(x)}}\left\{2^{\gamma(x)}\xi_{\alpha_n} - \frac{\xi_{\delta_n}}{\sqrt{2}} + o_{\mathrm{P}}(1)\right\}}{1 + o(1) + \frac{\sigma_{n,3}(x)}{(1-2^{\gamma(x)})\sqrt{2}}\left\{\xi_{\delta_n} - 2^{\gamma(x)}\frac{\xi_{\theta_n}}{\sqrt{2}} + o_{\mathrm{P}}(1)\right\}} \xrightarrow{p} 1$$

we get the desired conclusion. $\square$

The proofs of Theorems 2-4 are based on the next lemma in which $N_x = \sum_{i=1}^n \mathbb{1}(X_i \leq x)$ and $Y_1^x, \ldots, Y_{N_x}^x$ denote the observations Y_i such that $X_i \leq x$, with $Y_{(1)}^x \leq Y_{(2)}^x \leq \ldots \leq Y_{(N_x)}^x$ being their corresponding order statistics.

Lemma 1. *Under Assumption 3, we have for all $n \geq 1$*

(i) $\hat{q}_1(x) = Y_{(N_x)}^x + hc$.

(ii) $Y_{(N_x-k)}^x - hc < \hat{q}_{\frac{N_x-k}{N_x}}(x) \leq Y_{(N_x-k)}^x + hc$, *for each $k = 0, \ldots, N_x - 1$.*

Proof. (i) We know that $\hat{q}_1(x) = \inf\{y \geq 0 | \sum_{i=1}^{N_x} H\left(\frac{y-Y_i^x}{h}\right) = N_x\}$. It is also clear that the event $\left\{\sum_{i=1}^{N_x} H\left(\frac{y-Y_i^x}{h}\right) = N_x\right\}$ holds if and only if the event

$$\left\{H\left(\frac{y - Y_i^x}{h}\right) = 1, \text{ for each } i = 1, \ldots, N_x\right\},$$

which is equivalent to $\{y \geq Y_i^x + hc, \text{ for each } i = 1, \ldots, N_x\}$, holds. Hence $\hat{q}_1(x) = Y_{(N_x)}^x + hc$.

(ii) We have by definition of $\hat{F}(\cdot|x)$,

$$\begin{aligned}\hat{F}(Y_{(N_x-k)}^x + hc|x) &= (1/N_x)\sum_{i=1}^{N_x} H\left(\frac{Y_{(N_x-k)}^x - Y_{(i)}^x}{h} + c\right)\\ &\geq (1/N_x)\sum_{i\leq N_x-k} H\left(\frac{Y_{(N_x-k)}^x - Y_{(i)}^x}{h} + c\right) = (N_x - k)/N_x = \hat{F}(\hat{q}_{\frac{N_x-k}{N_x}}(x)|x).\end{aligned}$$

Then $Y_{(N_x-k)}^x + hc \geq \hat{q}_{\frac{N_x-k}{N_x}}(x)$ by the strict monotonicity of $\hat{F}(\cdot|x)$. Likewise, we have

$$\hat{F}(Y_{(N_x-k)}^x - hc|x) = (1/N_x)\sum_{i<N_x-k} H\left(\frac{Y_{(N_x-k)}^x - Y_{(i)}^x}{h} - c\right) < (N_x - k)/N_x,$$

which gives $Y_{(N_x-k)}^x - hc < \hat{q}_{\frac{N_x-k}{N_x}}(x)$. $\square$

Proof of Theorem 2. Let us first remark that the FDH estimator $\hat{\varphi}(x) = Y_{(N_x)}^x$ is the maximum of the random variables $Z_i^x = Y_i 1\!\!1(X_i \leq x)$, $i = 1, \ldots, n$, and that $\varphi(x)$ is the right endpoint of their common distribution function $F_x(z) = \{1 - F_X(x)[1 - F(z|x)]\}1\!\!1(z \geq 0)$. Thus, from Lemma 1(i), $a_n^{-1}(x)(\hat{q}_1(x) - \varphi(x) - hc)$ converges in distribution if and only if $a_n^{-1}(x)(\hat{\varphi}(x) - \varphi(x)) = a_n^{-1}(x)(\max_{1\leq i\leq n} Z_i^x - F_x^{-1}(1))$ converges in distribution. According to the standard extreme-values theory, the necessary and sufficient condition for $a_n^{-1}(x)(\max_{1\leq i\leq n} Z_i^x - F_x^{-1}(1))$ to converge in distribution is that $F_x(\cdot)$ belongs to the maximum domain of attraction of Weibull, which is equivalent to the condition (2.4). In this case (see *e.g.* Resnick 1987, Proposition 1.13, p. 59), the limiting distribution is the Weibull distribution function $\Psi_{\gamma(x)}(\cdot)$ and $a_n(x)$ can be taken equal to $F_x^{-1}(1) - F_x^{-1}(1 - 1/n)$ which coincides with $\varphi(x) - q_{1-\frac{1}{nF_X(x)}}(x)$. $\square$

Proof of Theorem 3. As shown in the proof of Theorem 2, we have

$$a_n^{-1}(x)(\max_{1\leq i\leq n} Z_i^x - F_x^{-1}(1)) = a_n^{-1}(x)(\hat{\varphi}(x) - \varphi(x)) = a_n^{-1}(x)(\hat{q}_1(x) - \varphi(x) - hc) \xrightarrow{d} \Psi_{\gamma(x)}$$

and $a_n(x) = \varphi(x) - q_{1-\frac{1}{nF_X(x)}}(x)$ coincides with $F_x^{-1}(1) - F_x^{-1}(1 - 1/n)$. We also have $\mathbb{E}[|Z^x|^k] = F_X(x)\mathbb{E}(Y^k|X \leq x) \leq \varphi(x)^k$. On the other hand, we know according to Resnick (1987, Proposition 2.1, p.77) that, if $a_n^{-1}(x)(\max_{1\leq i\leq n} Z_i^x - F_x^{-1}(1)) \xrightarrow{d} \Psi_{\gamma(x)}$ with $a_n(x) = F_x^{-1}(1) - F_x^{-1}(1 - 1/n)$ and if $\mathbb{E}[|Z^x|^k] < \infty$, then

$$\lim_{n\to\infty} \mathbb{E}\{a_n^{-1}(x)(\max_{1\leq i\leq n} Z_i^x - F_x^{-1}(1))\}^k = \int_{-\infty}^0 y^k \Psi_{\gamma(x)}(dy) = (-1)^k\Gamma(1 - k\gamma(x)).$$

This limit coincides with $\lim_{n\to\infty} \mathbb{E}\{a_n^{-1}(x)(\hat{q}_1(x)-\varphi(x)-hc)\}^k$, which ends the proof. $\square$

Proof of Theorem 4. From Theorem 2, Assumptions 3 and (2.4) imply that $a_n^{-1}(x)(\hat{q}_1(x)-\varphi(x)-hc) \xrightarrow{d} \Psi_{\gamma(x)}$. Lemma 1(i) thus yields $a_n^{-1}(x)(Z^x_{(n)} - F_x^{-1}(1)) \xrightarrow{d} \Psi_{\gamma(x)}$, where $Z^x_{(i)}$ denotes hereafter the ith order statistic generated by the random variables $Z_1^x, \ldots, Z_n^x$. Therefore $a_n^{-1}(x)(Z^x_{(n-k)} - F_x^{-1}(1)) \xrightarrow{d} H_{k,x}$ for any integer $k \geq 0$, according to van der Vaart (1998, Theorem 21.18, p. 313). On the other hand, it is not hard to verify (see *e.g.* Lemma 2(ii) in Daouia et al.(2008)) that

$$Z^x_{(n-k)} = \inf\{y \geq 0 | \hat{F}(y|x) \geq 1 - k/N_x\} = Y^x_{(N_x-k)} \quad \text{as} \quad n \to \infty,$$

with probability 1. Hence, we get by using Lemma 1(ii),

$$\begin{aligned} a_n^{-1}(x)(Z^x_{(n-k)} - F_x^{-1}(1)) - a_n^{-1}(x)hc &< a_n^{-1}(x)\left(\hat{q}_{\frac{N_x-k}{N_x}}(x) - \varphi(x)\right) \\ &\leq a_n^{-1}(x)(Z^x_{(n-k)} - F_x^{-1}(1)) + a_n^{-1}(x)hc \end{aligned}$$

for all n large enough, with probability 1. Thus $a_n^{-1}(x)\left(\hat{q}_{1-\frac{k}{N_x}}(x) - \varphi(x)\right) \xrightarrow{d} H_{k,x}$ since $a_n^{-1}(x)h \to 0$. $\square$

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