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Vuk Milisic

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# BLOOD-FLOW MODELLING ALONG AND THROUGH A BRAIDED MULTI-LAYER METALLIC STENT\*

VUK MILIŠIĆ<sup>1,2</sup>

**Abstract.** In this work we study the hemodynamics in a stented artery connected either to a collateral artery or to an aneurysmal sac. The blood flow is driven by the pressure drop. Our aim is to characterize the flow-rate and the pressure in the contiguous zone to the main artery: using boundary layer theory we construct a homogenized first order approximation with respect to  $\epsilon$ , the size of the stent's wires. This provides an explicit expression of the velocity profile through and along the stent. The profile depends only on the input/output pressure data of the problem and some homogenized constant quantities: it is explicit. In the collateral artery this gives the flow-rate. In the case of the aneurysm, it shows that : (i) the zeroth order term of the pressure in the sac equals the averaged pressure along the stent in the main artery, (ii) the presence of the stent inverts the rotation of the vortex. Extending the tools set up in [5,24] we prove rigorously that our asymptotic approximation is first order accurate with respect to  $\epsilon$ . We derive then new implicit interface conditions that our approximation formally satisfies, generalizing our analysis to other possible geometrical configurations. In the last part we provide numerical results that illustrate and validate the theoretical approach.

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## INTRODUCTION

Atherosclerosis and rupture of aneurysm are lethal pathologies of the cardio-vascular system. A possible therapy consists in introducing a metallic multi-layered stent (see fig. 1 right), either as a supplementary protection of the arterial wall or in order to slow vortices in the aneurysm and to favor coagulation of the sac. In this study we aim to investigate the fluid-dynamics of blood in the presence of a stent. We focus on two precise configurations in this context: (i) a stented artery is connected to the collateral artery but the aperture of the latter is partially occluded by the presence of the stent (see fig. 1 left). (ii) a saccular aneurysm is present behind a stented artery (fig. 1 middle). From the applicative point of view these two situations are of interest since they represent a dual constraint that a stent should optimize somehow: the grid generated by the wires should be coarse enough to provide blood to the collateral arteries (for instance iliac arteries in the aorta), at the same time the wires should be close enough to have a real effect on the aneurysmal zones (slow vortices, protect arterial walls, etc).

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*Keywords and phrases:* wall-laws, porous media, rough boundary, Stokes equation, multi-scale modeling, boundary layers, pressure driven flow, error estimates, vertical boundary correctors

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<sup>1</sup> Wolfgang Pauli Institute (WPI), UMI CNRS 2841, Vienna, AUSTRIA

<sup>2</sup> Laboratoire Jean Kuntzman (LJK), UMR CNRS 5523, Grenoble, FRANCE

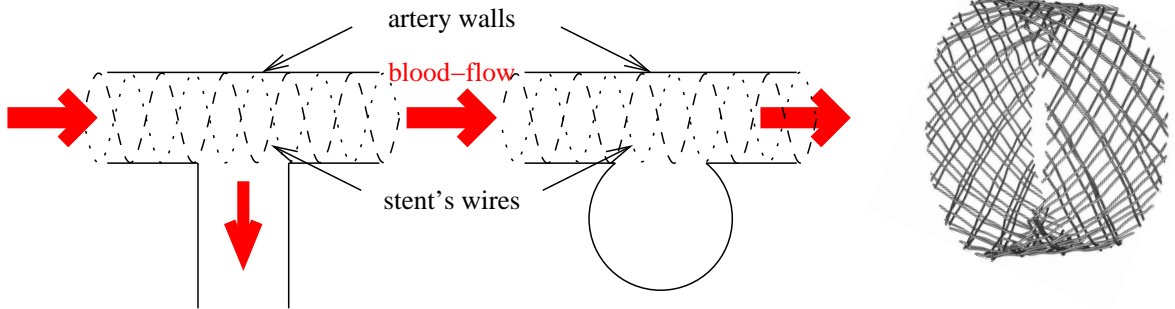


FIGURE 1. A sketch of stented arteries: with a collateral artery (left), an aneurysmal sac (middle) and a 3D example of a real metallic multi-wired stent (right)

Multi-layer metallic wired stents seem to satisfy both the constraints at the same time. Although experimentally proved [4], these facts needed better mathematical understanding. We give here results in this sense, setting a common framework for both phenomena in the case of the Stokes flow.

Inspired by homogenization techniques applied to the case of rough boundaries [1, 20, 26] we construct a first-order multi-scale approximation of the velocity and the pressure. By averaging, we get a first order accurate macroscopic description of the fluid flow. Indeed, we compute an explicit expression of the velocity through the fictitious interface supporting the stent and separating the main artery from the contiguous zone. This formula only depends on the input data of the problem and some homogenized constants obtained solving microscopic cell problems. In the case of the aneurysmal sac we show rigorously that the zero order pressure in the sac is constant and averaged with respect to the pressure in the main artery, which was not known. Then we show that formally this leads also to redefine the problem in an new and implicit way in the domain decomposition flavor. Actually we obtain a new set of interface conditions along the fictitious interface: while for the normal velocity they look similar to those presented in [2, 8, 9], the tangential conditions are new to our knowledge. They express a slip velocity in the main artery (as in [19]), but a discontinuous homotetic relationship between horizontal velocities across the interface of the stent (see system (4), p. 4).

From the mathematical point of view this paper introduces several novelties. The case of a sieve has been widely studied in a somehow different setting in [2, 8, 10, 11] and more recently in [14]. In these works, the authors considered the case of obstacles set on a surface with various dimensionalities but with a common point: the velocity was completely imposed at the boundary. Although this could seem a technicality, it influences drastically the limiting regime of the flow. Indeed a complete velocity profile is imposed as a Dirichlet condition at the inlet/outlet of the domain, so that the total flow-rate through the sieve remains constant whatever  $\epsilon$ , the size of the obstacles. In the context of blood flow such a regime seems unrealistic: occlusions of arteries occur and are even a widely spread pathology, suggesting that blood flow should be driven by pressure drop more than fixed flow-rates.

In this direction, Jäger and Mikelić considered a pressure driven fluid in [18]. But they studied an interface whose thickness was independent on  $\epsilon$ , which seemed useless for our purpose : the diameter of the wires of the stent are dependent on the radius of the artery where the stent should be implanted. It appears natural to consider roughness size that is proportional to  $\epsilon$  in any direction. Moreover in this paper we introduce both a tangential and a transverse flow along and through the stent. Indeed, in the limiting regime considered by Jäger and Mikelić [18], the velocity is zero. Here when the collateral artery or a sac are completely closed by the stent, we still expect a Poiseuille profile in the main artery.

At a more technical level, this work improves the approach developed in [5, 24] in order to correct edge oscillations introduced by periodic boundary layers. At the same time, we give an

appropriate framework to deal with this problem in the case of Stokes equations. Namely, we decompose the corrections of the superfluous boundary layer oscillations in two parts :

- on the microscopic side we use weighted Sobolev spaces to describe the behaviour at infinity of the vertical corner correctors. This provides accurate decay rates with respect to  $\epsilon$  at the macroscopic level near the corner. Indeed, using onto mappings between weighted Sobolev spaces we improve decay estimates already derived in the scalar case in [5, 24].
- a complementary macroscopic corrector is added in a second step, that handles exponentially decreasing errors far from the corners.

A first attempt to break the periodicity at the inlet/outlet of the domain was done in [19] by using a vertical corrector localized in a tiny strip near the vertical interface. But, decay estimates claimed in formula (77) p. 1123 [19] seem to work, to our knowledge, only for *a priori* estimates of the error and are not accurate enough to be used in the very weak estimates.

The paper is organized as follows: in the next section, after some basic notations and definitions, we give a detailed review of the results obtained either in the case of a collateral artery or a sacular aneurysm. Then in section 2 we give the proofs of the claims. We provide numerical results that show a first order accuracy also in the discrete case in section 3. In the Appendix we give a proof of the weighted Sobolev properties of the vertical boundary correctors.

## 1. NOTATIONS, PROBLEM SETTINGS & MAIN RESULTS

The two dimensional domains and their respective boundaries are defined as in fig. 2:  $\Omega_\epsilon := \Omega_{1,\epsilon} \cup \Gamma_0 \cup \Omega_2$  and  $\Omega := \Omega_1 \cup \Gamma_0 \cup \Omega_2$ . Each of the domains  $\Omega_{1,\epsilon}$  and  $\Omega_2$  are of height and width equal to one. The rough boundary  $\Gamma_\epsilon$  is the set of rigid regular periodic bodies  $Q_\epsilon$  whose inner diameter is proportional to  $\epsilon$  and spaced one with respect to each-other with a distance proportional to  $\epsilon$  as well. The limit fictitious interface is called  $\Gamma_0$  and separates  $\Omega_{1,\epsilon}$  from  $\Omega_2$ . On the microscopic scale we set up the infinite strip  $S := Z_+ \cup Z_-$  where  $Z_+$  contains the single  $C^\infty$  obstacle  $Q$  whose boundary is denoted  $P$  and it is separated from  $Z_-$  by the microscopic version of  $\Gamma_0$  denoted  $\Sigma$ . We define also a set of notations :  $\Gamma_N := \Gamma_{in} \cup \Gamma_{out,1} \cup \Gamma_{out,2}$  is the set of

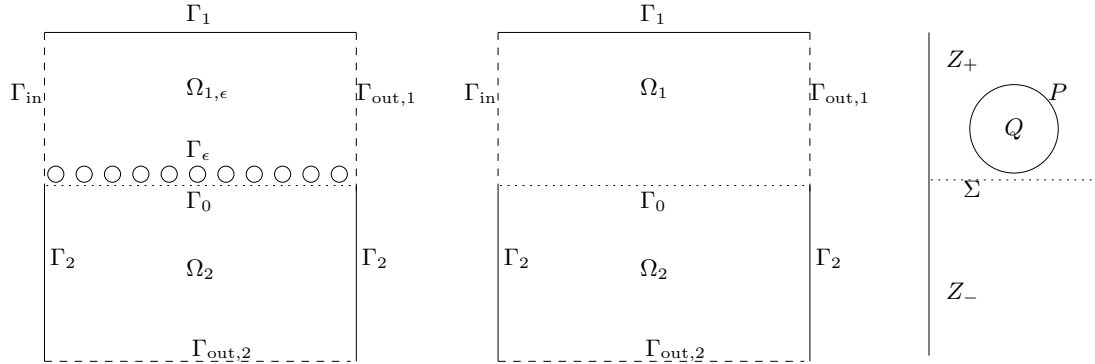


FIGURE 2. The macroscopic domains: for a fixed  $\epsilon$  (left), for  $\epsilon = 0$  (middle) and the microscopic one (right)

all boundaries on which mixed type conditions are imposed. On the contrary  $\Gamma_D := \Gamma_1 \cup \Gamma_2$  and  $\Gamma_{D,\epsilon} := \Gamma_1 \cup \Gamma_2 \cup \Gamma_\epsilon$  are the sets of edges on which Dirichlet conditions are set. The exterior normal vector to any domain is denoted by  $\mathbf{n}$ , if not stated explicitly  $\mathbf{n}$  is orientated from  $\Omega_1$  towards  $\Omega_2$  on the fictitious interface  $\Gamma_0$ . The tangent vector is defined as  $\boldsymbol{\tau}$ . We set  $B_\epsilon := \Gamma_0 \times (0, 2\epsilon)$  and  $\Omega'_1 := \Omega_1 \setminus B_\epsilon$ . The macroscopic (resp. microscopic) space variable is denoted  $x$  (resp.  $y$ ), these variables are related through the scaling:  $x = \epsilon y$ .

### 1.1. The case of a collateral artery

We study the problem : find  $(\mathbf{u}_\epsilon, p_\epsilon)$  solving the stationary Stokes equations

$$\begin{cases} -\Delta \mathbf{u}_\epsilon + \nabla p_\epsilon = 0 & \text{in } \Omega_\epsilon \\ \operatorname{div} \mathbf{u}_\epsilon = 0 & \text{in } \Omega_\epsilon \\ p_\epsilon = p_{\text{in}} \text{ on } \Gamma_{\text{in}}, \quad p_\epsilon = p_{\text{out},1} \text{ on } \Gamma_{\text{out},1}, \quad p_\epsilon = p_{\text{out},2} & \text{on } \Gamma_{\text{out},2}, \\ \mathbf{u}_\epsilon \cdot \boldsymbol{\tau} = 0 & \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out},1} \cup \Gamma_{\text{out},2} \\ \mathbf{u}_\epsilon = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_\epsilon \end{cases} \quad (1)$$

In what follows we set both  $p_{\text{out},1}$  and  $p_{\text{out},2}$  to be zero for simplicity. The results remain valid for any fixed constants  $p_{\text{out},1}$  and  $p_{\text{out},2}$  as well.

When adapted to rough boundaries, the homogenization technique decomposes in two steps : the derivation of a multi-scale asymptotic expansion and the construction of an averaged macroscopic approximation. The first part can be seen as an iterative algorithm with respect to powers of  $\epsilon$  :

- (i) pass to the limit with respect to  $\epsilon$  and obtain a limit profile, in our case, because of the very simple geometry of the artery, the Poiseuille profile is obtained in  $\Omega_1$  and a trivial solution in  $\Omega_2$  :

$$\begin{cases} \mathbf{u}_0(x) = \frac{p_{\text{in}}}{2}(1-x_2)x_2 \mathbf{e}_1 \mathbb{1}_{\Omega_1}, \quad \forall x \in \Omega \\ p_0(x) = p_{\text{in}}(1-x_1) \mathbb{1}_{\Omega_1} \end{cases} \quad (2)$$

- (ii) construct microscopic boundary layers that correct the errors made by the zeroth order approximation on  $\Gamma_\epsilon$  and  $\Gamma_0$ : we set up in the next section three boundary layers  $(\boldsymbol{\beta}, \pi)$ ,  $(\boldsymbol{\Upsilon}, \varpi)$  and  $(\boldsymbol{\chi}, \eta)$  to this purpose. These functions solve microscopic problems on the  $y_1$ -periodic  $y_2$ -infinite strip  $S$ .
- (iii) compute the constants at both + and - infinity of  $S$  that these correctors reach:  $(\overline{\boldsymbol{\beta}}^\pm, 0)$ ,  $(\overline{\boldsymbol{\Upsilon}}^\pm, 0)$  and  $(\overline{\boldsymbol{\chi}}, \overline{\eta}^\pm)$ . Then subtract these constants to the correctors. Physically,  $\overline{\boldsymbol{\beta}}^\pm$ ,  $\overline{\boldsymbol{\Upsilon}}^\pm$  provide a microscopic feed-back relative to the horizontal velocity (see the wall-law framework in [19, 26] and references therein) whereas the pressure difference  $[\overline{\eta}]$  represents a microscopic resistivity in the flavor of [2, 8].
- (iv) take into account the homogenized constants on the limit interface  $\Gamma_0$  by solving a macroscopic problem  $(\mathbf{u}_1, p_1)$

$$\begin{cases} -\Delta \mathbf{u}_1 + \nabla p_1 = 0 & \text{in } \Omega_1 \cup \Omega_2 \\ \operatorname{div} \mathbf{u}_1 = 0 \text{ in } \Omega_1 \cup \Omega_2 \\ \mathbf{u}_1 = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \\ \mathbf{u}_1 \cdot \boldsymbol{\tau} = 0 & \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out},1} \cup \Gamma_{\text{out},2}, \\ p_1 = 0 & \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out},1} \cup \Gamma_{\text{out},2} \\ \mathbf{u}_1 = \left\{ \frac{\partial u_{0,1}}{\partial x_2} \overline{\boldsymbol{\beta}}^\pm + \left[ \frac{\partial u_{0,1}}{\partial x_2} \right] \overline{\boldsymbol{\Upsilon}}^\pm \right\} \mathbf{e}_1 + \frac{[p_0]}{[\overline{\eta}]} \overline{\boldsymbol{\chi}} \mathbf{e}_2 & \text{on } \Gamma_0^\pm \end{cases} \quad (3)$$

where the brackets  $[\cdot]$  denote the jump across  $\Gamma_0$ . This macroscopic corrector depends on the zeroth order approximation and the homogenized constants. Due to the explicit form of the Poiseuille profile it is thus also explicit with respect to the data of the problem along and across  $\Gamma_0$  (but neither inside  $\Omega_1$  nor  $\Omega_2$ ).

- (v) goto (ii) and correct, on a microscopic scale, errors made by  $(\mathbf{u}_1, p_1)$  on  $\Gamma_\epsilon \cup \Gamma_0$ .

The second step consists then in averaging this ansatz and obtaining an expansion of the macroscopic solutions only. This gives, for instance, at first order :

$$\overline{\mathbf{u}}_\epsilon := \mathbf{u}_0 + \epsilon \mathbf{u}_1, \quad \overline{p}_\epsilon := p_0 + \epsilon p_1.$$

In particular as  $\mathbf{u}_1 \cdot \mathbf{n} = -[p_0]/[\bar{\eta}]$  on  $\Gamma_0$ , one gets an explicit first order velocity profile across  $\Gamma_0$ . As a consequence, we obtain a new result :

**Proposition 1.** *The flow-rate in the collateral artery  $\Omega_2$  can be computed explicitly and reads*

$$Q_{\Gamma_0} := \int_{\Gamma_0} \bar{\mathbf{u}}_\epsilon \cdot \mathbf{n} dx_1 = \frac{\epsilon}{[\bar{\eta}]} \int_{\Gamma_0} [p_0] dx_1 = \frac{\epsilon}{[\bar{\eta}]} \int_{\Gamma_0} (p_{\text{out},1} + (p_{\text{in}} - p_{\text{out},1})(1 - x_1) - p_{\text{out},2}) dx_1$$

As stated above  $[\bar{\eta}]$  depend only on the geometry of the microscopic obstacle  $Q$  and is independent on any other parameter. In the last section of this paper we give some numerical examples that illustrate the accuracy of this result. Although this is a first order correction of the flow trough the stent, we underline that in the physiological context the pressures  $(p_{\text{in}}, p_{\text{out},1})$  present in the main artery can be very important compared to  $p_{\text{out},2}$  : the first order flow rate  $Q_{\Gamma_0}$  can thus be quantitatively significant as well.

In order to validate our formal asymptotic expansion there is a large mathematical framework [2, 5, 8, 10, 11, 19, 24] that aims comparing the exact solution of (1) and our asymptotic ansatz. We have constructed in this work a sufficient framework to show

**Theorem 1.1.** *The averaged asymptotic ansatz  $(\bar{\mathbf{u}}_\epsilon, \bar{p}_\epsilon)$  belongs to  $\mathbf{L}^2(\Omega_j) \times H^{-1}(\Omega_j)$  for  $j \in \{1, 2\}$  and satisfies the convergence result*

$$\|\mathbf{u}_\epsilon - \bar{\mathbf{u}}_\epsilon\|_{L^2(\Omega_1 \cup \Omega_2)} + \sqrt{\epsilon} \|p_\epsilon - \bar{p}_\epsilon\|_{H^{-1}(\Omega'_1 \cup B_\epsilon \cup \Omega_2)} \leq k \epsilon^{\frac{3}{2}^-}$$

where  $\frac{3}{2}^-$  represent any real number strictly less than  $\frac{3}{2}$  and the constant  $k$  is independent on  $\epsilon$ .

Expressing the interface conditions satisfied by  $(\bar{\mathbf{u}}_\epsilon, \bar{p}_\epsilon)$  on  $\Gamma_0$  in an implicit way and neglecting higher order rests, we show formally that in fact  $(\bar{\mathbf{u}}_\epsilon, \bar{p}_\epsilon)$  solve at first order a new interface problem :

$$\left\{ \begin{array}{ll} -\Delta \bar{\mathbf{u}}_\epsilon + \nabla \bar{p}_\epsilon = 0 & \text{in } \Omega_1 \cup \Omega_2 \\ \operatorname{div} \bar{\mathbf{u}}_\epsilon = 0 & \text{in } \Omega_1 \cup \Omega_2 \\ \bar{\mathbf{u}}_\epsilon = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \\ \bar{\mathbf{u}}_\epsilon \cdot \boldsymbol{\tau} = 0 & \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out},1} \cup \Gamma_{\text{out},2} \\ \bar{p}_\epsilon = p_{\text{in}}, \quad \bar{p}_\epsilon = 0 & \text{on } \Gamma_{\text{out},1} \cup \Gamma_{\text{out},2} \end{array} \right. \quad (4)$$

$$\left. \begin{array}{l} \bar{\mathbf{u}}_\epsilon^+ \cdot \boldsymbol{\tau} = \epsilon (\bar{\boldsymbol{\beta}}^+ + \bar{\boldsymbol{\Upsilon}}^+) \frac{\partial \bar{u}_{\epsilon,1}^+}{\partial x_2}, \quad \frac{\bar{\mathbf{u}}_\epsilon^+ \cdot \boldsymbol{\tau}}{\bar{\boldsymbol{\beta}}^+ + \bar{\boldsymbol{\Upsilon}}^+} = \frac{\bar{\mathbf{u}}_\epsilon^- \cdot \boldsymbol{\tau}}{\bar{\boldsymbol{\beta}}^- + \bar{\boldsymbol{\Upsilon}}^-} \\ \bar{\mathbf{u}}_\epsilon^+ \cdot \mathbf{n} = \bar{\mathbf{u}}_\epsilon^- \cdot \mathbf{n} = \frac{\epsilon}{[\bar{\eta}]} ([\sigma_{\bar{\mathbf{u}}_\epsilon, \bar{p}_\epsilon}] \cdot \mathbf{n}, \mathbf{n}) \end{array} \right\} \quad \text{on } \Gamma_0$$

where  $\sigma_{\bar{\mathbf{u}}_\epsilon, \bar{p}_\epsilon} := \nabla \bar{\mathbf{u}}_\epsilon - \bar{p}_\epsilon \operatorname{Id}_2$  represents the strain tensor. Note that the interface condition on the normal velocity can be integrated in the Stokes equations as a kind of ‘‘strange term’’ in the spirit of [2, 9], but as we are at first order with respect to  $\epsilon$ , the derivation does not follow the same argumentation. In a forthcoming work we study the well-posedness of such a system as well as its consistency with respect to  $(\bar{\mathbf{u}}_\epsilon, \bar{p}_\epsilon)$  and  $(\mathbf{u}_\epsilon, p_\epsilon)$ . Because of the particular signs of the homogenized constants but also the discontinuous nature of the interface conditions in the tangential direction to  $\Gamma_0$ , this seems a challenging task.

## 1.2. The case of an aneurysm

The framework introduced above can be extended to the case of an aneurysm ; considering the same domain  $\Omega_\epsilon$  as above we define a new problem : find  $(\mathbf{u}_\epsilon, p_\epsilon)$  solving

$$\begin{cases} -\Delta \mathbf{u}_\epsilon + \nabla p_\epsilon = 0 & \text{in } \Omega_\epsilon \\ \operatorname{div} \mathbf{u}_\epsilon = 0 & \text{in } \Omega_\epsilon \\ p_\epsilon = p_{\text{in}} \text{ on } \Gamma_{\text{in}}, \quad p_\epsilon = 0 & \text{on } \Gamma_{\text{out},1}, \\ \mathbf{u}_\epsilon \cdot \boldsymbol{\tau} = 0 & \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out},1} \\ \mathbf{u}_\epsilon = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_{\text{out},2} \end{cases} \quad (5)$$

The main difference resides in the boundary condition imposed on  $\Gamma_{\text{out},2}$  : here we force a complete adherence condition on the velocity ; this closes the output  $\Gamma_{\text{out},2}$  and converts the collateral artery into an aneurysmal sac.

Again, we construct a similar multi-scale asymptotic ansatz. We extract the macroscopic part to get a homogenized expansion  $(\bar{\mathbf{u}}_\epsilon, \bar{p}_\epsilon)$

$$\bar{\mathbf{u}}_\epsilon := \mathbf{u}_0 + \epsilon \mathbf{u}_1, \quad \bar{p}_\epsilon := p_0 + \epsilon p_1$$

where  $(\mathbf{u}_0, p_0)$  is again a Poiseuille profile but it is complemented by an unknown constant pressure  $p_0^-$  inside the sac:

$$\begin{cases} \mathbf{u}_0(x) = \frac{p_{\text{in}} - p_{\text{out},1}}{2} (1 - x_2) x_2 \mathbf{e}_1 \mathbb{1}_{\Omega_1} \\ p_0(x) = p_0^+(x) \mathbb{1}_{\Omega_1} + p_0^- \mathbb{1}_{\Omega_2}, \\ p_0^+ := p_{\text{out},1} + (p_{\text{in}} - p_{\text{out},1})(1 - x_1), p_0^- \in \mathbb{R} \end{cases}, \quad \forall x \in \Omega \quad (6)$$

Then again  $(\mathbf{u}_1, p_1)$  solves a mixed Stokes problem (3) with only constant data and an explicit profile on  $\Gamma_0$ , the only difference is that  $\mathbf{u}_1 = 0$  on  $\Gamma_{\text{out},2}$ . This gives again a new result:

**Corollary 1.1.** *The zeroth order pressure is constant in  $\Omega_2$ , moreover it satisfies the following compatibility condition with respect to the data:*

$$p_0^- = \frac{1}{|\Gamma_0|} \int_{\Gamma_0} p_0^+(x_1, 0) dx_1$$

*the latter quantity depending only on the prescribed pressures at the boundaries of the domain. This gives an explicit velocity profile on  $\Gamma_0$  which reads:*

$$\bar{\mathbf{u}}_\epsilon \cdot \mathbf{n} = \frac{\epsilon}{|\bar{\eta}|} (p_0^+(x_1, 0) - p_0^-) + O(\epsilon^2)$$

The interface condition exhibited on the normal velocity shows rigorously a phenomenon already observed experimentally [4]. Set  $x_{1,\text{max}} := \max_{x \in \Gamma_0} x_1$  (resp.  $x_{1,\text{min}} := \min_{x \in \Gamma_0} x_1$ ) and  $\bar{x}_1 := (x_{1,\text{max}} + x_{1,\text{min}})/2$ , when  $x_1 < \bar{x}_1$  the pressure jump  $[p_0] := p_0^+(x) - p_0^-$  is positive, otherwise it is negative. This implies that the first order flow trough the stent is entering  $\Omega_2$  when  $x_1 < \bar{x}_1$  and leaving it otherwise. Thus the prosthesis inverts the orientation of the cavitation in  $\Omega_2$  with respect to the non-stented artery (see fig. 3).

As stated in the corollary, we will show in the next section that in fact the zero order pressure is the only constant that insures conservation of mass in  $\Omega_2$ . From the medical point of view the two claims on pressure and flow are of interest because they quantify and confirm the stabilizing effect of a porous stent: besides reducing the stress on the wall of the aneurysm because of the decrease of velocity (proportional to  $\epsilon$ ), the stent averages also the pressure inside the sac avoiding for instance corner singularities (see fig. 4).

Again one has a mathematical validation of the formal multi-scale construction

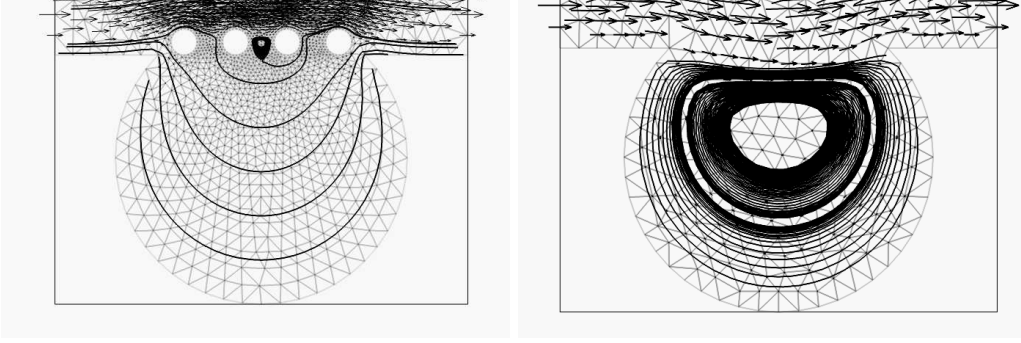


FIGURE 3. Streamlines and velocity vectors in an aneurysmal sac, with (left) and without a stent (right)

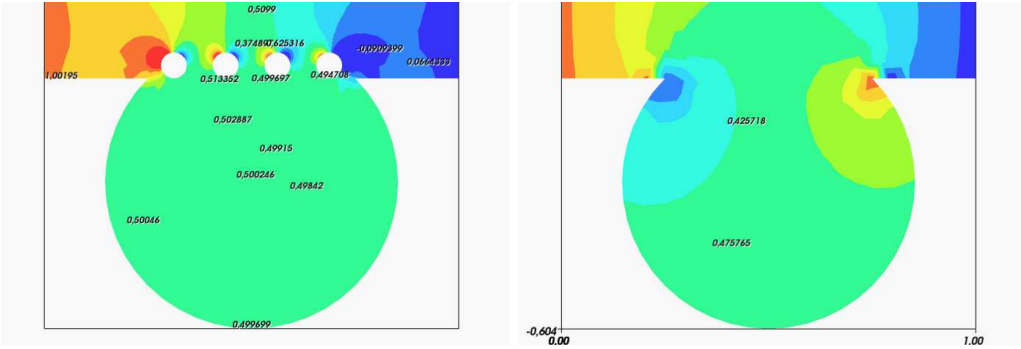


FIGURE 4. Pressure in an aneurysmal sac, with (left) and without a stent (right)

**Theorem 1.2.** *The first order approximation  $(\bar{\mathbf{u}}_\epsilon, \bar{p}_\epsilon)$  belongs to  $\mathbf{L}^2(\Omega_1) \times H^{-1}(\Omega_1)$  and  $\mathbf{L}^2(\Omega_2) \times H^{-1}(\Omega_2)/\mathbb{R}$ , moreover we have a convergence result that reads*

$$\|\mathbf{u}_\epsilon - \bar{\mathbf{u}}_\epsilon\|_{L^2(\Omega)} + \sqrt{\epsilon}(\|p_\epsilon - \bar{p}_\epsilon\|_{H^{-1}(\Omega'_1 \cup B_\epsilon)} + \|p_\epsilon - \bar{p}_\epsilon\|_{H^{-1}(\Omega_2)/\mathbb{R}}) \leq k\epsilon^{\frac{3}{2}^-}$$

where  $\frac{3}{2}^-$  represent any real number strictly less than  $\frac{3}{2}$ , the constant  $k$  is independent on  $\epsilon$ .

We show the same type of result as above :  $(\bar{\mathbf{u}}_\epsilon, \bar{p}_\epsilon)$  solve formally the same implicit problem (4) up to the second order error, but with a homogeneous Dirichlet condition on  $\Gamma_{\text{out},2}$ .

## 2. PROOF OF THE MAIN RESULTS

### 2.1. The case of a collateral artery

The zero order term. When  $\epsilon$  goes to zero we show in a first step that  $(\mathbf{u}_\epsilon, p_\epsilon)$  converges to  $(\mathbf{u}_0, p_0)$  the Poiseuille profile stated in (2), which solves in  $\Omega_\epsilon$  :

$$\begin{cases} -\Delta \mathbf{u}_0 + \nabla p_0 = [\sigma_{\mathbf{u}_0, p_0}] \cdot \mathbf{n} \delta_{\Gamma_0} & \text{in } \Omega_\epsilon \\ \operatorname{div} \mathbf{u}_0 = 0 & \text{in } \Omega_\epsilon \\ \mathbf{u}_0 = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \\ \mathbf{u}_0 \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma_N \\ p_0 = p_{\text{in}} \text{ on } \Gamma_{\text{in}}, \quad p_0 = 0 & \text{on } \Gamma_{\text{out},1} \cup \Gamma_{\text{out},2} \\ \mathbf{u}_0 \neq 0 & \text{on } \Gamma_\epsilon \end{cases}$$



**Theorem 2.1.** *For a fixed  $\epsilon$ , there exists a unique solution  $(\mathbf{u}_\epsilon, p_\epsilon) \in \mathbf{H}^1(\Omega_\epsilon)^2 \times L^2(\Omega_\epsilon)$  of the problem (1). Moreover, one has*

$$\|\mathbf{u}_\epsilon - \mathbf{u}_0\|_{\mathbf{H}^1(\Omega_\epsilon)^2} + \|p_\epsilon - p_0\|_{L^2(\Omega_\epsilon)} \leq k\sqrt{\epsilon}$$

where the constant  $k$  does not depend on  $\epsilon$ .

*Proof.* Existence and uniqueness of the solutions of problem (1) come from the standard theory of mixed problems [12, 15], so one has

$$\|\mathbf{u}_\epsilon\|_{\mathbf{H}^1(\Omega_\epsilon)^2} + \|p_\epsilon\|_{L^2(\Omega_\epsilon)} \leq \|p_{\text{in}}\|_{H^{-\frac{1}{2}}(\Gamma_{\text{in}})}.$$

As  $\mathbf{u}_0$  does not satisfy homogeneous boundary conditions on  $\Gamma_\epsilon$ , one constructs a lift denoted  $\mathcal{R}(\mathbf{u}_0) := \mathbf{u}_0\phi(x_2/\epsilon)$ , where  $\phi(y_2)$  is a cut-off function equal to one in a unit ball and zero outside the ball of radius two. We set  $\tilde{\mathbf{u}}_0 := \mathbf{u}_0 - \mathcal{R}(\mathbf{u}_0)$ . Standard *a priori* estimates give then

$$\|\mathbf{u}_\epsilon - \tilde{\mathbf{u}}_0\|_{\mathbf{H}^1(\Omega_\epsilon)} + \|p_\epsilon - p_0\|_{L^2(\Omega_\epsilon)} \leq \|\Delta\mathcal{R}(\mathbf{u}_0) - [\sigma_{\mathbf{u}_0, p_0}] \cdot \mathbf{n}\delta_{\Gamma_0}\|_{\mathbf{H}^{-1}(\Omega_\epsilon)}$$

Thanks to the vicinity of  $\Gamma_\epsilon$ , one deduces the standard Poincaré and trace inequalities

$$\|\varphi\|_{L^2(\Omega_{1,\epsilon} \cap B_\epsilon)} \leq \epsilon \|\nabla\varphi\|_{L^2(\Omega_\epsilon)}, \quad \|\varphi\|_{L^2(\Gamma_0)} \leq \sqrt{\epsilon} \|\nabla\varphi\|_{L^2(\Omega_\epsilon)}, \quad \forall \varphi \in \mathbf{H}^1(\Omega_\epsilon) \text{ s.t. } \varphi = 0 \text{ on } \Gamma_\epsilon$$

which end the proof.  $\square$

Estimates above show a threefold error: the jump of the gradient of the velocity in the horizontal direction across  $\Gamma_0$ , the Dirichlet error on  $\Gamma_\epsilon$  and the pressure jump in the normal direction across  $\Gamma_0$ . In order to correct these errors we solve three microscopic boundary layer problems.

The Dirichlet correction. The first boundary layer corrects the Dirichlet error on  $\Gamma_\epsilon$ . It is very alike to the one introduced in the wall-laws setting [19]. Namely we solve: find  $(\boldsymbol{\beta}, \pi)$  such that

$$\begin{cases} -\Delta\boldsymbol{\beta} + \nabla\pi = 0 & \text{in } S, \\ \operatorname{div}\boldsymbol{\beta} = 0 & \text{in } S, \\ \boldsymbol{\beta} = -y_2\mathbf{e}_1 & \text{on } P, \\ \boldsymbol{\beta}_2 \rightarrow 0 & |y_2| \rightarrow \infty. \end{cases} \quad (7)$$

We define the transverse section  $T(y_2) := \{z \in S \text{ s.t. } z_2 \equiv y_2\}$  and the corresponding averaging operator:

$$\bar{\boldsymbol{\beta}}(y_2) := \frac{1}{|T(y_2)|} \int_{T(y_2)} \boldsymbol{\beta}(z_1, y_2) dz_1.$$

We define as in [15] p. 56, the homogeneous Sobolev space  $\mathbf{D}^{1,2}(S) := \{\mathbf{v} \in \mathcal{D}'(S), \text{ s.t. } \nabla\mathbf{v} \in L^2(S)\}$ . Moreover we denote by  $\mathbf{D}^{1,2}(S)_0$  the subset of functions belonging to  $\mathbf{D}^{1,2}(S)_0$  and vanishing on  $P$ .

**Proposition 2.** *There exists a unique solution  $(\boldsymbol{\beta}, \pi) \in \mathbf{D}^{1,2}(S) \times L^2_{\text{loc}}(S)$ ,  $\pi$  being defined up to a constant. Moreover, one has:*

$$\boldsymbol{\beta}(y) \rightarrow \bar{\boldsymbol{\beta}}_\pm \mathbf{e}_1, \quad y_2 \rightarrow \pm\infty$$

the convergence being exponential with rate  $\gamma_\beta$  and

$$\begin{cases} \bar{\boldsymbol{\beta}}_2(y_2) = 0, & \forall y_2 \in \mathbb{R}, \\ \bar{\boldsymbol{\beta}}_1(y_2) = -|Q| - |\nabla\boldsymbol{\beta}|_{L^2(S)}^2 + \bar{\boldsymbol{\beta}}_1(0), & y_2 > y_{2,P}, \\ \bar{\boldsymbol{\beta}}_1(y_2) = \bar{\boldsymbol{\beta}}_1(0), & y_2 < 0, \end{cases}$$

where  $y_{2,P} := \max_{y \in P} y_2$  and  $|Q|$  is the 2d-volume of the obstacle  $|Q|$

*Proof.* We start by lifting the non-homogeneous Dirichlet boundary condition : we set  $\mathcal{R}(\boldsymbol{\beta}) := y_2 \phi(y_2) \mathbf{e}_1 \mathbb{1}_{Z_+}$  and  $\tilde{\boldsymbol{\beta}} := \boldsymbol{\beta} + \mathcal{R}(\boldsymbol{\beta})$ , this is still a divergence free vector. Now, there exists a unique solution s.t.

$$\left| \nabla \tilde{\boldsymbol{\beta}} \right|_{L^2(S)} \leq |\nabla \mathcal{R}(\boldsymbol{\beta})|_{L^2(S)}$$

Because in the space of  $L^2_{\text{loc}}(S)$  functions vanishing on  $P$ , the gradient norm is a norm (via Wirtinger estimates), one has proved the result for  $\boldsymbol{\beta}$ .

We apply Lemma 3.4 and Proposition 3.5 of [23] in order to recover the  $L^2_{\text{loc}}(S)$  pressure solving:

$$-\Delta \tilde{\boldsymbol{\beta}} + \nabla \pi = -\Delta \mathcal{R}(\boldsymbol{\beta})$$

and this gives existence and uniqueness of  $(\tilde{\boldsymbol{\beta}}, \pi) \in \mathbf{D}_0^{1,2}(S) \times L^2_{\text{loc}}(S)$ . On the interface located above (resp. below) the obstacle  $Q$  we apply the Fourier decomposition in modes as in Theorem 3 p. 10 [21]. One obtains the exponential convergence towards the zero modes of  $\boldsymbol{\beta}, \pi$  in an explicit way. To obtain the relationships between the constant values at infinity, one has

- (i) by the divergence free condition that  $\bar{\beta}_2(\nu) = \bar{\beta}_2(\gamma) = 0$  for all  $\nu \geq y_{2,P}$  and  $\gamma \leq 0$
- (ii) integrating the first equation of (7) in every transverse section  $\{y_2 = \delta\}$  which does not cross the obstacle  $Q$  gives

$$\frac{d^2}{dy_2^2} \left( \int_{\{y_2=\delta\}} \beta_1(y_1, y_2) dy_1 \right) = \int_{\{y_2=\delta\}} -\frac{\partial \beta_1}{\partial^2 y_1^2} + \frac{\partial \pi}{\partial y_1} dy_1 = 0$$

by  $y_1$ -periodicity. This implies that  $\bar{\beta}(\delta)$  is an affine function of  $\delta$ . As the gradient rapidly goes to zero, the linear part is zero, we conclude that only the constant remains : thus  $\bar{\beta}(\delta) = \bar{\beta}(+\infty)$  for  $\delta > y_{2,P}$ , and  $\bar{\beta}(\nu) = \bar{\beta}(0)$  for  $\nu < 0$ .

- (iii) Set  $G := y_2 \mathbb{1}_{Z_+}$  and  $F := 0$ , they satisfy:

$$\begin{cases} -\Delta G + \nabla F = \delta_\Sigma & \text{in } S \\ \operatorname{div} G = 0 & \text{in } S \end{cases} \quad (8)$$

we test the first equation in (7) by  $G$  and the first equation in (8) by  $\boldsymbol{\beta}$ , then we integrate on  $S_{\nu,\gamma} := S \cap ]0, 1[ \times ]\gamma, \nu[$  :

$$\begin{aligned} (\Delta G - \nabla F, \boldsymbol{\beta}) - (\Delta \boldsymbol{\beta} - \nabla \pi, G) &= \bar{\beta}_1(0) \\ &= (\sigma_{G,F} \cdot \mathbf{n}, \boldsymbol{\beta})_{\partial S_{\nu,\gamma}} - (\sigma_{\boldsymbol{\beta},\pi} \cdot \mathbf{n}, G)_{\partial S_{\nu,\gamma}} = (\sigma_{G,F} \cdot \mathbf{n}, \boldsymbol{\beta})_P - (\sigma_{\boldsymbol{\beta},\pi} \cdot \mathbf{n}, G)_P + \bar{\beta}_1(\nu) \\ &= -(\partial_{\mathbf{n}}(y_2 \mathbf{e}_1), y_2 \mathbf{e}_1)_P + (\sigma_{\boldsymbol{\beta},\pi} \cdot \mathbf{n}, \boldsymbol{\beta})_P + \bar{\beta}_1(\nu) = -(\partial_{\mathbf{n}}(y_2 \mathbf{e}_1), y_2 \mathbf{e}_1)_P + |\nabla \boldsymbol{\beta}|_{L^2(S)} + \bar{\beta}_1(\nu) \end{aligned}$$

where we neglected exponentially small terms on  $\{y_2 = \nu\}$  and  $\{y_2 = \gamma\}$ . Now we explicit the physical meaning of the constant  $\mathcal{Q} := (\partial_{\mathbf{n}}(y_2 \mathbf{e}_1), y_2 \mathbf{e}_1)_P$

$$\mathcal{Q} + (\partial_{\mathbf{n}}(y_2 \mathbf{e}_1), y_2 \mathbf{e}_1)_{\{y_2=\nu\} \cup \{y_2=\gamma\}} = (\Delta(y_2 \mathbf{e}_1), y_2 \mathbf{e}_1)_{S_{\nu,\gamma}} + |\nabla(y_2 \mathbf{e}_1)|_{S_{\nu,\gamma}}^2$$

which in turn gives :

$$\mathcal{Q} + \nu - \gamma = \nu - \gamma - |Q|$$

where  $|Q|$  is the volume of the obstacle  $Q$ . The quantity  $\mathcal{Q}$  represents the volume of fluid missing due to the presence of the obstacle  $Q$  above the limit interface  $\Sigma$ . If we were to consider a straight channel without a collateral artery but a roughness below the interface,  $\mathcal{Q}$  would be a positive number equal to the volume of fluid present below  $\Sigma$ .  $\square$

In what follows we should distinguish between the mean-value of a function (for instance  $\bar{\beta}(\delta)$ ) and the discontinuous piecewise-constant function taking the averaged values of the correctors at

infinity (for example denoted  $\overline{\beta}$ ) :

$$\overline{\beta}(y) := \overline{\beta}(+\infty)\mathbb{1}_{Z_+} + \overline{\beta}(-\infty)\mathbb{1}_{Z_-}, \quad \forall y \in S$$

Shear rate jump correction. The second boundary layer corrects the jump of the normal derivative of the axial velocity: we introduce a source term that accounts for a unit jump in the horizontal component but on the microscopic scale. Namely, we look for  $(\Upsilon, \varpi)$  solving:

$$\begin{cases} -\Delta \Upsilon + \nabla \varpi = -\delta_{\Sigma} \mathbf{e}_1 & \text{in } S, \\ \operatorname{div} \Upsilon = 0 & \text{in } S, \\ \Upsilon = 0 & \text{on } P, \\ \Upsilon_2 \rightarrow 0 & |y_2| \rightarrow \infty. \end{cases}$$

Again we give some basic results and the behaviour at infinity of this corrector.

**Proposition 3.** *There exists a unique  $(\Upsilon, \varpi) \in \mathbf{D}^{1,2}(S) \times L^2_{\text{loc}}(S)$ ,  $\varpi$  being defined up to a constant. Moreover, one has:*

$$\Upsilon(y) \rightarrow \overline{\Upsilon}_{\pm} \mathbf{e}_1, \quad y_2 \rightarrow \pm \infty$$

and

$$\begin{cases} \overline{\Upsilon}_2(y_2) = 0 & \forall y_2 \in \mathbb{R}, \\ \overline{\Upsilon}_1(y_2) = \overline{\Upsilon}_1(0) - \overline{\beta}_1(0) & y_2 > y_{2,P}, \\ \overline{\Upsilon}_1(y_2) = \overline{\Upsilon}_1(0) & y_2 < 0, \end{cases}$$

where  $y_{2,P} := \max_{y \in P} y_2$ .

The proof follows the same lines as in Proposition 2.

The pressure jump. In order to correct the pressure jump we use a similar the corrector to the one introduced and studied in [10] p. 25:

$$\begin{cases} -\Delta \chi + \nabla \eta = 0 & \text{in } S, \\ \operatorname{div} \chi = 0 & \text{in } S, \\ \chi = 0 & \text{on } P, \\ \chi_2 \rightarrow -1, & |y_2| \rightarrow \infty. \end{cases} \quad (9)$$

As in the proof of Proposition 2, one repeats the same arguments to study the behaviour of this boundary layer giving similarly to [10] :

**Proposition 4.** *There exists a unique solution  $(\chi, \eta) \in \mathbf{D}^{1,2}(S) \times L^2_{\text{loc}}(S)$  of system (9),  $\eta$  being defined up to a constant. Moreover, one has*

$$\chi \rightarrow \overline{\chi} \equiv -\mathbf{e}_2, \quad |y_2| \rightarrow \infty,$$

the convergence being exponential with rate  $\gamma_{\chi}$  and there exists two constants  $\overline{\eta}(+\infty)$  and  $\overline{\eta}(-\infty)$  depending only on the geometry of  $P$  such that

$$\eta(y) \rightarrow \overline{\eta}(\pm\infty), \quad |y_2| \rightarrow \infty.$$

We denote  $\overline{[\eta]} := \overline{\eta}(+\infty) - \overline{\eta}(-\infty)$ . One then proves:

$$|\nabla \chi|_{L^2(S)}^2 = \overline{[\eta]}$$

This corrector will be used in the sequel, but we already utilize it to give a first result on the average of  $\pi$  and  $\varpi$

**Corollary 2.1.** *The solution  $(\beta, \pi)$  and  $(\Upsilon, \varpi)$  satisfy :*

$$\bar{\pi}(y_2) = 0 \text{ and } \bar{\varpi}(y_2) = 0, \quad \forall y_2 \in \mathbb{R}_- \cup ]y_{2,P}, +\infty[$$

*Proof.* Setting again  $\tilde{\beta} := \beta + y_2 \mathbf{e}_1 \mathbb{1}_{Z_+}$  and writing

$$\begin{aligned} & (-\Delta \beta + \nabla \pi, \chi)_{S_{\delta, \nu}} - (-\Delta \chi + \nabla \eta, \tilde{\beta})_{S_{\delta, \nu}} = 0 \\ & = (\sigma_{\chi, \eta} \cdot \mathbf{n}, \tilde{\beta})_{\{y_2=\delta\} \cup \{y_2=\nu\}} - (\sigma_{\beta, \pi} \cdot \mathbf{n}, \chi)_{\{y_2=\delta\} \cup \{y_2=\nu\}} \rightarrow -[\overline{\eta \beta_2}]_{-\infty}^{+\infty} + [\overline{\pi \chi_2}]_{-\infty}^{+\infty} \end{aligned}$$

because  $\beta_2 \rightarrow 0$  and  $\chi_2 \rightarrow -1$ , one gets the desired result at infinity. As the pressure  $\pi$  is harmonic in  $S$ , the average  $\bar{\pi}(\delta)$  is zero in  $\mathbb{R}_- \cup ]y_{2,P}, +\infty[$ . The same proof holds for  $\varpi$ .  $\square$

In the sequel we will also need the corrector

$$\begin{cases} -\Delta \varkappa + \nabla \mu = -(\nabla \chi - (\eta - \bar{\eta}) \text{Id}_2) \cdot \mathbf{e}_1 & \text{in } S, \\ \text{div } \varkappa = 0 & \text{in } S, \\ \varkappa = 0 & \text{on } P, \\ \varkappa_2 \rightarrow 0 & |y_2| \rightarrow \infty. \end{cases}$$

**Proposition 5.** *There exists a unique solution  $(\varkappa, \mu) \in \mathbf{D}^{1,2}(S) \times L_{\text{loc}}^2(S)$ ,  $\mu$  being defined up to a constant. One has also exponential convergence towards constants with rate  $\gamma_\varkappa$*

$$[\bar{\mu}]_{-}^{+} = (\chi_1(\eta - \bar{\eta}))_S, \quad [\bar{\varkappa}_1]_{-}^{+} = -(\sigma_{\chi, (\eta - \bar{\eta})} \cdot \mathbf{e}_1, \beta)_S$$

In what follows we use the  $\epsilon$ -scaling of all boundary layers above, namely we set:

$$\beta_\epsilon(x) := \beta \left( \frac{x}{\epsilon} \right), \quad \Upsilon_\epsilon(x) := \Upsilon \left( \frac{x}{\epsilon} \right), \quad \chi_\epsilon(x) := \chi \left( \frac{x}{\epsilon} \right), \quad \varkappa_\epsilon(x) := \varkappa \left( \frac{x}{\epsilon} \right), \quad \forall x \in \Omega_\epsilon.$$

Vertical correctors on  $\Gamma_1 \cup \Gamma_2 \cup \Gamma_{\text{out},1}$ . Above boundary layers are periodic; their oscillations perturb homogeneous Dirichlet as well as Neumann stress boundary conditions on  $\Gamma_{\text{in}} \cup \Gamma_{\text{out},1} \cup \Gamma_2$ . The perturbation on these boundaries is  $O(1)$ , due to the vicinity of these edges to the geometrical perturbation  $\Gamma_\epsilon$ . In order to correct these errors, we introduce vertical boundary correctors defined on a perforated half-plane. Namely we set  $\Pi := \cup_{k=0}^{\infty} \{S + k\mathbf{e}_1\}$ , the vertical infinite boundary is divided in  $D := \{0\} \times \mathbb{R}_-$  and  $N := \{0\} \times \mathbb{R}_+$ . We denote  $B := \cup_{k=0}^{\infty} \{P + k\mathbf{e}_1\}$ . On this domain we define the vertical correctors lifting each of the periodic boundary layers above

$$\begin{cases} \begin{cases} -\Delta \mathbf{w}_\beta + \nabla \theta_\beta = 0 & \text{in } \Pi, \\ \text{div } \mathbf{w}_\beta = 0 & \text{in } \Pi, \\ \mathbf{w}_\beta = \beta - \bar{\beta} & \text{on } D, \\ w_{\beta,2} = \beta_2 & \text{on } N, \\ \theta_\beta = \pi & \text{on } N, \\ \mathbf{w}_\beta = 0 & \text{on } B, \end{cases} & \begin{cases} -\Delta \mathbf{w}_\chi + \nabla \theta_\chi = 0 & \text{in } \Pi, \\ \text{div } \mathbf{w}_\chi = 0 & \text{in } \Pi, \\ \mathbf{w}_\chi = \chi - \bar{\chi} & \text{on } D, \\ w_{\chi,2} = \chi_2 - \bar{\chi}_2 & \text{on } N, \\ \theta_\chi = \eta - \bar{\eta} & \text{on } N, \\ \mathbf{w}_\chi = 0 & \text{on } B, \end{cases} & \begin{cases} -\Delta \mathbf{w}_\Upsilon + \nabla \theta_\Upsilon = 0 & \text{in } \Pi, \\ \text{div } \mathbf{w}_\Upsilon = 0 & \text{in } \Pi, \\ \mathbf{w}_\Upsilon = \Upsilon - \bar{\Upsilon} & \text{on } D, \\ w_{\Upsilon,2} = \Upsilon_2 - \bar{\Upsilon}_2 & \text{on } N, \\ \theta_\Upsilon = \mu - \bar{\mu} & \text{on } N, \\ \mathbf{w}_\Upsilon = 0 & \text{on } B, \end{cases} \end{cases} \quad (10)$$

and  $(\mathbf{w}_\varkappa, \theta_\varkappa)$  solves a similar system lifting  $(\varkappa, \mu)$  on  $D \cup N$ . We define the usual weighted Sobolev space [3, 16]:

$$W_\alpha^{m,p}(\Pi) := \left\{ v \in \mathcal{D}'(\Pi) \text{ s.t. } |D^\lambda v| (1 + \rho^2)^{\frac{\alpha + |\lambda| - m}{2}} \in L^p(\Pi), 0 \leq |\lambda| \leq m \right\}$$

where  $\rho := (1 + |y|^2)^{\frac{1}{2}}$ . We endow this space with the corresponding weighted norm. Here we extend to the case of Stokes results obtained for mixed boundary conditions and the rough Laplace equation in [5, 24]. In the appendix we give the extensive proof of the crucial claim:

**Theorem 2.2.** *There exists a unique solution  $(\mathbf{w}_i, \theta_i) \in \mathbf{W}_\alpha^{1,2}(\Pi)^2 \times W_\alpha^{0,2}(\Pi)$  for  $i \in \{\beta, \Upsilon, \chi, \varkappa\}$ , where the constant  $\alpha$  is of modulus strictly less than 1.*

We need to localize these correctors near the vertical boundaries  $\Gamma_{\text{in}} \cup \Gamma_2 \cup \Gamma_{\text{out},1}$ . To this purpose we define two cut-off functions:

- (i) The ‘‘corner’’ cut-off functions : Set  $\psi_1 := \bar{\psi}(x)$  and  $\psi_2 := \bar{\psi}(x - \bar{x})$ , where  $\bar{x} := \bar{\Gamma}_{\text{out},1} \cap \bar{\Gamma}_2$  and  $\bar{\psi}$  is a radial monotone decreasing cut-off function such that

$$\bar{\psi}(x) := \begin{cases} 1 & \text{if } |x| \leq \frac{1}{3} \\ 0 & \text{if } |x| \geq \frac{2}{3} \end{cases}$$

Finally set  $\psi(x) := \psi_1(x) + \psi_2(x)$ . Note that with this definition  $\partial_{\mathbf{n}} \bar{\psi} = 0$  on  $\Gamma_{\text{in}} \cup \Gamma_{\text{out},1}$ .

- (ii) The ‘‘far from the corner’’ cut-off function :  $\Phi$  is defined in a complementary manner on  $\Gamma_{\text{in}} \cup \Gamma_{\text{out},1} \cup \Gamma_2$  such that

$$\begin{cases} \psi + \Phi = 1 \\ \partial_{\mathbf{n}} \Phi = 0, \end{cases} \quad \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out},1} \cup \Gamma_2$$

and one can take for instance  $\Phi(x) := 1 - \psi(0, x_2)$  for all  $x \in \Omega$ .

We set for  $i \in \{\beta, \Upsilon, \chi, \varkappa\}$ ,

$$\begin{cases} \mathbf{w}_{\epsilon,i}(x) := c_i(O) \mathbf{w}_i \left( \frac{x}{\epsilon} \right) \psi_1(x_1) + c_i(\bar{x}) \mathbf{w}_i \left( \frac{x - \bar{x}}{\epsilon} \right) \psi_2(x), \\ \theta_{\epsilon,i}(x) := c_i(O) \theta_i \left( \frac{x}{\epsilon} \right) \psi(x_1) + c_i(\bar{x}) \theta_i \left( \frac{x - \bar{x}}{\epsilon} \right) \psi_2(x), \end{cases}$$

where the constants  $c_i$  denote

$$c_\beta := \frac{\partial u_{0,1}}{\partial x_2}, \quad c_\Upsilon := \left[ \frac{\partial u_{0,1}}{\partial x_2} \right], \quad c_\chi := \frac{[p_0]}{[\bar{\eta}]}, \quad c_\varkappa := p_{\text{in}}$$

$c_\chi$  being the only constant which is different when  $x = O$  or  $x = \bar{x}$ . Then, we define the complete vertical corrector as

$$\begin{cases} \mathcal{W}_\epsilon(x) := \epsilon \sum_i \mathbf{w}_{\epsilon,i}(x) + \mathbf{W}(x), \\ \mathcal{Z}_\epsilon(x) := \sum_i \theta_{\epsilon,i} + Z(x), \end{cases} \quad \forall x \in \Omega_\epsilon$$

where  $(\mathbf{W}, Z)$  solve the system of equations on the macroscopic domain  $\Omega_\epsilon$  :

$$\begin{cases} \Delta \mathbf{W} + \nabla Z = 0, & \text{in } \Omega_\epsilon \\ \text{div } \mathbf{W} = 0 & \text{in } \Omega_\epsilon \\ \mathbf{W} \cdot \boldsymbol{\tau} = \epsilon \left\{ c_\beta(\beta_\epsilon - \bar{\beta}) + c_\Upsilon(\Upsilon_\epsilon - \bar{\Upsilon}) + c_\chi(\chi_\epsilon - \bar{\chi}) + \epsilon c_\varkappa(\varkappa_\epsilon - \bar{\varkappa}) \right\} \cdot \boldsymbol{\tau} \Phi & \text{on } \Gamma_N \\ Z = \left\{ c_\beta \pi_\epsilon + c_\Upsilon \varpi_\epsilon + c_\chi(\eta_\epsilon - \bar{\eta}) + \epsilon c_\varkappa(\mu_\epsilon - \bar{\mu}) \right\} \Phi & \\ \mathbf{W} = 0 \text{ on } \Gamma_\epsilon & \\ \mathbf{W} = \epsilon \left\{ c_\beta(\beta_\epsilon - \bar{\beta}) + c_\Upsilon(\Upsilon_\epsilon - \bar{\Upsilon}) + c_\chi(\chi_\epsilon - \bar{\chi}) + \epsilon c_\varkappa(\varkappa_\epsilon - \bar{\varkappa}) \right\} \Phi & \text{on } \Gamma_D \end{cases} \quad (11)$$

**Proposition 6.** *There exists a unique solution  $(\mathbf{W}, Z) \in \mathbf{H}^1(\Omega_\epsilon) \times L^2(\Omega_\epsilon)$  of system (11), moreover one has:*

$$\|\mathbf{W}\|_{\mathbf{H}^1(\Omega_\epsilon)} + \|Z\|_{L^2(\Omega_\epsilon)} \leq k e^{-\frac{\gamma}{\epsilon}}$$

where the exponential rate  $\gamma$  and the constant  $k$  do not depend on  $\epsilon$ .

*Proof.* By the standard theory for mixed problems [12, 15], there exists a unique solution  $(\mathbf{W}, Z)$  solving the problem above, one has the *a priori* estimates:

$$\begin{aligned} \|\mathbf{W}\|_{\mathbf{H}^1(\Omega_\epsilon)} + \|Z\|_{L^2(\Omega_\epsilon)} &\leq \left\| \left( (c_{\beta_\epsilon} \sigma_{\beta_\epsilon, \pi} + c_{\mathbf{r}} \sigma_{\mathbf{r}_\epsilon, \varpi} + c_{\chi} \sigma_{\chi_\epsilon, \eta_\epsilon} + \epsilon c_{\varkappa} \sigma_{\varkappa_\epsilon, \mu_\epsilon}) \mathbf{n}, \mathbf{n} \right) \right\|_{H^{-\frac{1}{2}}(\Gamma_N)} \\ &\quad + \epsilon \left\| \boldsymbol{\tau} \cdot \left\{ c_{\beta} (\beta_\epsilon - \bar{\beta}) + c_{\mathbf{r}} (\mathbf{r}_\epsilon - \bar{\mathbf{r}}) + c_{\chi} (\chi_\epsilon - \bar{\chi}) + \epsilon (\varkappa_\epsilon - \bar{\varkappa}) \right\} \Phi \right\|_{H^1(\Omega_\epsilon)} \end{aligned}$$

Thanks to the crucial presence of the cut-off function  $\Phi$  and the exponential decrease of rate  $\gamma := \min(\gamma_\beta, \gamma_{\mathbf{r}}, \gamma_\chi, \gamma_\varkappa)$  of all the microscopic quantities, one gets again an exponential decrease in all norms. This ends the proof.  $\square$

**Remark 2.1.** *Without the presence of the cut-off function  $\Phi$  the result above is not true, because for instance the normal derivatives are of order one and integrating them near the origin or near  $\bar{x}$ , one loses the decay property. That is the reason why we use in fact localized microscopic correctors  $\mathbf{w}_{\epsilon, i}$  near the “corners”  $(0, 0)$  and  $\bar{x}$ .*

The complete first order approximation. Having introduced every single element, we built a complete first order approximation, setting:

$$\begin{aligned} \mathcal{U}_\epsilon &:= \mathbf{u}_0 + \epsilon \left\{ \frac{\partial u_{0,1}}{\partial x_2} (\beta_\epsilon - \bar{\beta}) + \left[ \frac{\partial u_{0,1}}{\partial x_2} \right] (\mathbf{r}_\epsilon - \bar{\mathbf{r}}) + \frac{[p_0]}{[\bar{\eta}]} (\chi_\epsilon - \bar{\chi}) + u_1 \right\} \\ &\quad + \epsilon^2 \{ p_{\text{in}} (\varkappa_\epsilon - \bar{\varkappa}) + \mathbf{u}_2 \} + \mathcal{W}_\epsilon, \\ \mathcal{P}_\epsilon &:= p_0 + \left\{ \frac{\partial u_{0,1}}{\partial x_2} \pi_\epsilon + \left[ \frac{\partial u_{0,1}}{\partial x_2} \right] \varpi_\epsilon + \frac{[p_0]}{[\bar{\eta}]} (\eta_\epsilon - \bar{\eta}) + \epsilon p_1 \right\} + \epsilon p_{\text{in}} (\mu_\epsilon - \bar{\mu}) + \epsilon^2 p_2 + \mathcal{Z}_\epsilon, \end{aligned} \quad (12)$$

where the first order macroscopic corrector  $(\mathbf{u}_1, p_1)$  solves (3), while  $(\mathbf{u}_2, p_2)$  satisfies

$$\left\{ \begin{array}{ll} -\Delta \mathbf{u}_2 + \nabla p_2 = 0 & \text{in } \Omega_1 \cup \Omega_2, \\ \operatorname{div} \mathbf{u}_2 = 0 & \text{in } \Omega_1 \cup \Omega_2, \\ \mathbf{u}_2 = 0 & \text{on } \Gamma_D, \\ \mathbf{u}_2 \cdot \boldsymbol{\tau} = 0, & \text{on } \Gamma_N, \\ p_2 = 0, & \text{on } \Gamma_N, \\ \mathbf{u}_2 = p_{\text{in}} \bar{\varkappa}^\pm & \text{on } \Gamma_0^\pm. \end{array} \right. \quad (13)$$

We give two distinct values to the horizontal first order correction of the velocity on  $\Gamma_0$ . This is due to the different values of the constants whom the boundary layer correctors  $\beta$  and  $\mathbf{r}$  tend at infinity. The velocity vectors  $\mathbf{u}_1$  and  $\mathbf{u}_2$  being discontinuous across  $\Gamma_0$ , it is not possible to obtain *a priori* estimates in  $\Omega$ . But thanks to the specific mixed type boundary conditions on  $\Gamma_{\text{in}} \cup \Gamma_{\text{out}, 1}$ ,  $(\mathbf{u}_1, p_1)$  and  $(\mathbf{u}_2, p_2)$  belong to  $\mathbf{H}^1(\Omega_{1, \epsilon}) \times L^2(\Omega_{1, \epsilon})$ . Nevertheless due to the discontinuity of the Dirichlet data at the intersection between  $\Gamma_2$  and  $\Gamma_0$  (at the corners  $(0, 0)$  and  $(1, 0)$ )  $\mathbf{u}_1$  and  $\mathbf{u}_2$  cannot belong to  $\mathbf{H}^1(\Omega_2) \times L^2(\Omega_2)$ . Instead thanks to [11], one has that  $(\mathbf{u}_1, p_1)$  and  $(\mathbf{u}_2, p_2)$  exist and are unique very weak solutions in  $\mathbf{L}^2(\Omega_2) \times H^{-1}(\Omega_2)$ .

*A priori* estimates. Due to the lack of regularity of the complete ansatz  $(\mathcal{U}_\epsilon, \mathcal{P}_\epsilon)$  near the vertical boundaries  $\Gamma_{\text{in}} \cup \Gamma_{\text{out}, 1} \cup \Gamma_2$ , one cannot obtain directly *a priori* estimates in  $\Omega_\epsilon$ . Instead inspiring ourselves from ideas similar to energy methods [10] p. 39, we focus on the  $\Omega_\epsilon$ -“regular” terms of  $\mathcal{U}_\epsilon$  present on  $\Gamma_0$ . Namely we set :

$$\begin{aligned} \mathcal{V}_\epsilon &:= \mathbf{u}_\epsilon - (\mathbf{u}_0 + \mathcal{F}_\epsilon(1 - \Phi)), \quad \mathcal{Q}_\epsilon := p_\epsilon - (p_0 + \mathcal{G}_\epsilon(1 - \Phi)) \\ \mathcal{F}_\epsilon &:= \epsilon \left\{ \frac{\partial u_{0,1}}{\partial x_2} \beta_\epsilon + \left[ \frac{\partial u_{0,1}}{\partial x_2} \right] \mathbf{r}_\epsilon + \frac{[p_0]}{[\bar{\eta}]} \chi_\epsilon \right\} + \epsilon^2 \{ p_{\text{in}} \varkappa_\epsilon \} =: \epsilon \mathcal{F}_1 \left( \frac{x}{\epsilon} \right) + \epsilon^2 \mathcal{F}_2 \left( \frac{x}{\epsilon} \right) \\ \mathcal{G}_\epsilon &:= \left\{ \frac{\partial u_{0,1}}{\partial x_2} \pi_\epsilon + \left[ \frac{\partial u_{0,1}}{\partial x_2} \right] \varpi_\epsilon + \frac{[p_0]}{[\bar{\eta}]} (\eta_\epsilon - \bar{\eta}) \right\} + \epsilon p_{\text{in}} (\mu_\epsilon - \bar{\mu}) =: \mathcal{G}_1 \left( \frac{x}{\epsilon} \right) + \epsilon \mathcal{G}_2 \left( \frac{x}{\epsilon} \right) \end{aligned}$$

In fact, we do not consider neither the constants at infinity nor their macroscopic correctors. Because we localize these terms in a fixed zone near the rough layer this is also a possible more regular approximation of  $(\mathbf{u}_\epsilon, p_\epsilon)$ .

**Theorem 2.3.** *One has the a priori error estimates*

$$\|\mathcal{V}_\epsilon\|_{\mathbf{H}^1(\Omega_\epsilon)} + \|\mathcal{Q}_\epsilon\|_{L^2(\Omega_\epsilon)} \leq k\epsilon^-$$

where the constant  $k$  is independent on  $\epsilon$  and  $\epsilon^-$  is any positive number strictly less than  $\epsilon$ .

*Proof.* In order to correct vertical oscillations we define correctors similar to those presented above in (10). The major difference is that we do not take into account the constants at infinity inside the Dirichlet part of the boundary : we look for  $(\bar{\mathbf{w}}_1, \bar{\theta}_1)$  and  $(\bar{\mathbf{w}}_2, \bar{\theta}_2)$  solving respectively

$$\begin{cases} -\Delta \bar{\mathbf{w}}_i + \nabla \bar{\theta} = 0 & \text{in } \Pi, \\ \operatorname{div} \bar{\mathbf{w}}_i = 0 & \text{in } \Pi, \\ \bar{\mathbf{w}}_i = \mathcal{F}_i(0, y_2) & \text{on } D, \\ \bar{\mathbf{w}}_i \cdot \boldsymbol{\tau} = \mathcal{F}_i(0, y_2) \cdot \boldsymbol{\tau} & \text{on } N, \\ \sigma_{\bar{\mathbf{w}}_i, \bar{\theta}_i} \cdot \mathbf{n}, \mathbf{n} = \sigma_{\mathcal{F}_i, \mathcal{G}_i} \cdot \mathbf{n}, \mathbf{n} & \text{on } N, \\ \bar{\mathbf{w}}_i = 0 & \text{on } B, \end{cases} \quad (14)$$

Because  $\mathcal{F}_i$  goes to  $\bar{\mathcal{F}}_i \neq 0$  exponentially fast with respect to  $y_2$ ,  $\bar{\mathbf{w}}_i$  tends to a constant at infinity on  $N \cup D$ .

**Proposition 7.** *There exists  $(\bar{\mathbf{w}}_i, \bar{\theta}_i)$  in  $\mathbf{W}_\alpha^{1,2}(\Pi) \times W_\alpha^{0,2}(\Pi)$  a unique solution of problem (14) if the weight exponent  $\alpha$  is s.t.  $-1 < \alpha < 0$ .*

*Proof of Proposition 7.* If  $\alpha < 0$ , the boundary data  $\mathcal{F}_i(0, y_2) \in \mathbf{W}_\alpha^{\frac{1}{2},2}(D \cup N)$ , and if  $|\alpha| < 1$  by Theorem 3.1 in the appendix, there exists a unique solution in the corresponding weighted Sobolev spaces.  $\square$

We lift as well the second order error on  $\Gamma_\epsilon$  by setting

$$\mathcal{R} := \left( \mathbf{u}_0 - \frac{\partial u_{0,1}}{\partial x_2} x_2 \mathbf{e}_1 \right) \phi \left( \frac{x_2}{\epsilon} \right)$$

Finally we define:

$$\begin{cases} \tilde{\mathcal{V}}_\epsilon := \mathcal{V}_\epsilon - \epsilon^i \bar{\mathbf{w}}_i \left( \frac{x}{\epsilon} \right) \psi - \mathcal{R} =: \mathcal{V}_\epsilon - \epsilon^i \bar{\mathbf{w}}_{\epsilon,i} \psi - \mathcal{R}, \\ \tilde{\mathcal{Q}}_\epsilon := \mathcal{Q}_\epsilon - \bar{\theta}_i \left( \frac{x}{\epsilon} \right) \psi =: \mathcal{Q}_\epsilon - \bar{\theta}_{\epsilon,i} \psi \end{cases}$$

where  $\psi$  is the radial cut-off function already introduced above. These new variables solve :

$$\begin{cases} -\Delta \tilde{\mathcal{V}}_\epsilon + \nabla \tilde{\mathcal{Q}}_\epsilon = \Delta \mathcal{R} - 2\sigma_{\mathcal{F}_\epsilon, \mathcal{G}_\epsilon} \cdot \nabla \Phi - \Delta \Phi \mathcal{F}_\epsilon \\ \quad \quad \quad - 2\sigma_{\epsilon^i \bar{\mathbf{w}}_{\epsilon,i}, \bar{\theta}_{\epsilon,i}} \cdot \nabla \psi - \epsilon^i \Delta \psi \bar{\mathbf{w}}_{\epsilon,i}, \text{ in } \Omega_\epsilon \\ \operatorname{div} \tilde{\mathcal{V}}_\epsilon = \nabla \Phi \cdot \mathcal{F}_\epsilon + \epsilon^i \nabla \psi \cdot \bar{\mathbf{w}}_{\epsilon,i} \end{cases} \quad (15)$$

and they satisfy homogeneous boundary conditions on  $\partial\Omega_\epsilon$  :

$$\begin{cases} \tilde{\mathcal{V}}_\epsilon \cdot \boldsymbol{\tau} = 0 \\ \tilde{\mathcal{Q}}_\epsilon = 0 \\ \tilde{\mathcal{V}}_\epsilon = 0 \end{cases} \quad \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out},1} \cup \Gamma_{\text{out},2} \\ \quad \quad \quad \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_\epsilon$$

By standard theory we have *a priori* estimates. They provide  $\mathbf{H}^1(\Omega_\epsilon) \times L^2(\Omega_\epsilon)$  control of  $(\tilde{\mathcal{V}}_\epsilon, \tilde{\mathcal{Q}}_\epsilon)$  by the  $\mathbf{H}^{-1}(\Omega_\epsilon)$  norm of the rhs in the first two lines of system (15) and by the  $L^2(\Omega_\epsilon)$  norm of

the rhs in the divergence equation. We detail the  $\mathbf{H}^{-1}(\Omega_\epsilon)$  estimates, the  $L^2(\Omega_\epsilon)$  part follows the same way.

Thanks to the specific form of the lift  $\mathcal{R}$ , one easily writes

$$\|\Delta \mathcal{R}\|_{H^{-1}(\Omega_\epsilon)} = \|\nabla \mathcal{R}\|_{L^2(\Omega_\epsilon)} \leq 2\epsilon \|\mathbf{u}_0\|_{\mathbf{H}^2(\Omega_\epsilon)}$$

The support of  $\nabla \Phi$  and  $\Delta \Phi$  being fixed (independent on  $\epsilon$ ) and far from the roughness, the contribution of the  $\mathcal{F}_\epsilon$  part of the rhs in (15) is exponentially small with respect to  $\epsilon$ . As an example we detail the estimates of the part of  $\sigma_{\mathcal{F}_\epsilon, \mathcal{G}_\epsilon} \nabla \Phi$  that contains  $(\beta, \pi)$ , the rest follows the same

$$\int_{\Omega_\epsilon} |\nabla_y \beta_\epsilon - \pi_\epsilon \text{Id}_2|^2 |\nabla \Phi|^2 dx \leq k\epsilon \int_0^1 \int_{\frac{1}{3\epsilon}}^{\frac{2}{3\epsilon}} (|\nabla_y \beta_\epsilon|^2 + |\pi|^2) dy_2 dy_1 \leq \epsilon e^{-\frac{\gamma\beta}{\epsilon}}$$

Finally we detail the contribution of the vertical correctors  $(\epsilon \bar{\mathbf{w}}_{\epsilon, i}, \bar{\theta}_{\epsilon, i})\psi$

$$\begin{aligned} \left\| \sigma_{\epsilon \bar{\mathbf{w}}_{\epsilon, i}, \bar{\theta}_{\epsilon, i}} \cdot \nabla \psi \right\|_{L^2(\Omega_\epsilon)}^2 &= \int_{\Omega_\epsilon} \left| \sigma_{\epsilon \bar{\mathbf{w}}_{\epsilon, i}, \bar{\theta}_{\epsilon, i}} \cdot \nabla_x \psi(x) \right|^2 dx = \epsilon^2 \int_{\Pi} |(\nabla_y \bar{\mathbf{w}}_i - \bar{\theta}_i \text{Id}_2) \cdot \nabla_x \psi(\epsilon y)|^2 dy \\ &\leq k\epsilon^2 \int_{\Pi \cap ]\frac{1}{3\epsilon}, \frac{2}{3\epsilon}[ \times (0, \pi)} \{ |\nabla_y \bar{\mathbf{w}}_i|^2 + |\bar{\theta}_i|^2 \} \rho^{2\alpha} r dr d\tilde{\theta} \sup_{r \in [\frac{1}{3\epsilon}, \frac{2}{3\epsilon}]} \rho^{-2\alpha} \\ &\leq k\epsilon^{2-\alpha} \left( \|\bar{\mathbf{w}}\|_{\mathbf{W}_\alpha^{1,2}(\Pi)}^2 + \|\bar{\theta}\|_{\mathbf{W}_\alpha^{0,2}(\Pi)}^2 \right) \end{aligned}$$

In the same way

$$\left\| \epsilon^i (\Delta \psi) \bar{\mathbf{w}}_{\epsilon, i} \right\|_{L^2(\Omega_\epsilon)}^2 \leq \epsilon^4 \int_{\Pi \cap ]\frac{1}{3\epsilon}, \frac{2}{3\epsilon}[ \times (0, \pi)} \frac{|\bar{\mathbf{w}}_i|^2}{\rho^2} \rho^{2\alpha} r dr d\tilde{\theta} \sup_{r \in [\frac{1}{3\epsilon}, \frac{2}{3\epsilon}]} \rho^{2-2\alpha} \leq \epsilon^{2-2\alpha}$$

By a similar argument we obtain that

$$\left\| \nabla(\epsilon^i \bar{\mathbf{w}}_{\epsilon, i} \psi) \right\|_{L^2(\Omega_\epsilon)^4} \leq k\epsilon^{1-\alpha}$$

which by a triangular inequality ends the proof of theorem 2.3 □

**Remark 2.2.** Note that the energy method combined with the  $\mathbf{H}^1$  norm provides a similar result in formula (2.8c) of Theorem 2.4 p. 10 in [10]. Nevertheless, thanks to the vertical correctors  $(\epsilon^i \bar{\mathbf{w}}_{\epsilon, i}, \bar{\theta}_{\epsilon, i})$  presented above, we avoid the particular decomposition of the gradient of  $\mathcal{V}_\epsilon / \sqrt{\epsilon}$  in formula (5.18) p. 39. So that (i) we do not need to prove the weak convergence of  $\mathcal{V}_\epsilon / \epsilon \rightarrow 0$  on  $\Gamma_0$  and (ii) we obtain a  $\mathbf{H}^1(\Omega_\epsilon)$  error estimate of  $\mathcal{V}_\epsilon$  which is  $O(\epsilon^-)$  instead of  $o(\sqrt{\epsilon})$  as the latter approach would yield.

Very weak estimates. We use here the framework of very weak solutions, for scalar elliptic problems see [25], a similar setting is derived by duality in the case of Stokes in [11], see also [13] for more general results. The essential motivation comes from the lack of regularity of the averaged approximation  $(\bar{\mathbf{u}}_\epsilon, \bar{p}_\epsilon)$  across the interface  $\Gamma_0$  and the optimal cost of the boundary layers' velocity in the  $\mathbf{L}^2$  norm. As the obstacles are contained inside the limiting domain, we apply similar ideas as in the proof of Theorem 3.1 in [26], we decompose our domain in three parts.

**Theorem 2.4.** The full approximation  $(\mathcal{U}_\epsilon, \mathcal{P}_\epsilon)$  satisfies the error estimates:

$$\|\mathbf{u}_\epsilon - \mathcal{U}_\epsilon\|_{\mathbf{L}^2(\Omega_1 \cup \Omega_2)} + \|p_\epsilon - \mathcal{P}_\epsilon\|_{H^{-1}(\Omega'_1 \cup B_\epsilon \cup \Omega_2)} \leq k\epsilon^{\frac{3}{2}-}$$

where the constant  $k$  does not depend on  $\epsilon$  and  $\frac{3}{2}-$  is a real strictly less than  $\frac{3}{2}$ .



*Proof.* Writing down the system of equations satisfied by  $(\mathbf{u}_\epsilon - \mathcal{U}_\epsilon, p_\epsilon - \mathcal{P}_\epsilon)$  in  $\Omega'_1$ , and applying the very weak framework of Appendix A in [11], one has

$$\begin{aligned} \|\mathbf{u}_\epsilon - \mathcal{U}_\epsilon\|_{\mathbf{L}^2(\Omega'_1)} + \|p_\epsilon - \mathcal{P}_\epsilon\|_{H^{-1}(\Omega'_1)} &\leq \|-\Delta(\epsilon \mathbf{w}_{\epsilon,i}) + \nabla \theta_{\epsilon,i}\|_{\mathbf{H}^{-1}(\Omega'_1)} + \left\| \epsilon \frac{\nabla[p_0]}{[\bar{\eta}]} \cdot (\boldsymbol{\chi}_\epsilon - \bar{\boldsymbol{\chi}}) \right\|_{L^2(\Omega'_1)} \\ &\quad + \|\epsilon \operatorname{div} \mathbf{w}_{\epsilon,i}\|_{L^2(\Omega'_1)} + \|\mathbf{u}_\epsilon - \mathcal{U}_\epsilon\|_{\mathbf{L}^2(\{x_2=2\epsilon\})} \end{aligned} \quad (16)$$

The first term is estimated by the  $\mathbf{L}^2(\Omega_\epsilon)$  norm:

$$\begin{aligned} I &:= \|-\Delta(\epsilon \mathbf{w}_{\epsilon,i}) + \nabla \theta_{\epsilon,i}\|_{\mathbf{L}^2(\Omega_\epsilon)} \\ &\leq \epsilon k \|(\Delta \psi_1) \mathbf{w}_i(\cdot/\epsilon)\|_{\mathbf{L}^2(\Omega_\epsilon)} + k \|((\theta_i \operatorname{Id}_2 + \nabla_y \mathbf{w}_i) \nabla \psi_1)\|_{\mathbf{L}^2(\Omega_\epsilon)} =: \epsilon k I_1 + k I_2 \end{aligned}$$

Here as in the proof of theorem 2.3, the support of the derivatives of the cut-off function  $\psi$  plays an important role:

$$\begin{aligned} I_1^2 &= \int_{\Omega_\epsilon} |\mathbf{w}_i(x/\epsilon)|^2 (\Delta_x \psi_1(x))^2 dx = \epsilon^2 \int_{\Pi} |\mathbf{w}_i|^2 (\Delta_x \psi_1(\epsilon y))^2 dy \leq k \epsilon^2 \int_{\Pi \cap ]\frac{1}{3\epsilon}, \frac{2}{3\epsilon}[ \times ]0, \pi[} |\mathbf{w}_i|^2 r dr d\tilde{\theta} \\ &\leq k \epsilon^2 \left( \int_{\Pi \cap ]\frac{1}{3\epsilon}, \frac{2}{3\epsilon}[ \times ]0, \pi[} \left( \frac{|\mathbf{w}_i|}{\rho} \right)^2 \rho^{2\alpha} r dr d\tilde{\theta} \right) \cdot \left( \sup_{r \in ]\frac{1}{3\epsilon}, \frac{2}{3\epsilon}[} \rho^{2-2\alpha} \right) \leq \epsilon^{2\alpha} k \|\mathbf{w}_i\|_{\mathbf{W}_\alpha^{1,2}(\Pi)}^2 \end{aligned}$$

We give the computation for  $\psi_1 \mathbf{w}_i(\cdot/\epsilon)$ , obviously near  $\bar{x}$ , one proceeds identically. In the same way, we treat the other term :

$$\begin{aligned} I_2^2 &\leq \epsilon^2 \int_{\Pi} (|\nabla \mathbf{w}_i|^2 + \theta_i^2) |\nabla \psi_1|^2 dy \leq k \epsilon^2 \int_{\Pi \cap ]\frac{1}{3\epsilon}, \frac{2}{3\epsilon}[ \times ]0, \pi[} (|\nabla \mathbf{w}_i|^2 + \theta_i^2) dy \\ &\leq \epsilon^2 \| |\nabla \mathbf{w}_i|^2 + \theta_i^2 \|_{W_\alpha^{0,2}(\Pi)}^2 \sup_{r \in ]\frac{1}{3\epsilon}, \frac{2}{3\epsilon}[} \rho^{-2\alpha} \leq k \epsilon^{2(1+\alpha)} \end{aligned}$$

The second term in (16) is estimated directly, while the treatment of third term follows the same ideas as above. To handle the fourth term of the rhs in (16), we split  $\mathbf{v} := \mathbf{u}_\epsilon - \mathcal{U}_\epsilon$  in three parts:

$$\begin{aligned} \mathbf{v}_1 &:= \mathbf{u}_\epsilon - \left( u_0 + \epsilon \left\{ \frac{\partial u_{0,1}}{\partial x_2} \boldsymbol{\beta}_\epsilon + \left[ \frac{\partial u_{0,1}}{\partial x_2} \right] \boldsymbol{\Upsilon} + \frac{[p_0]}{[\bar{\eta}]} \boldsymbol{\chi}_\epsilon \right\} + \epsilon^2 \{p_{\text{in}} \boldsymbol{\varkappa}_\epsilon\} \right) \\ \mathbf{v}_2 &:= \epsilon^i (\mathbf{u}_i(x) - \mathbf{u}_i(x_1, 0)) \\ &= \epsilon \left( \mathbf{u}_1(x) - \left\{ \frac{\partial u_{0,1}}{\partial x_2} \bar{\boldsymbol{\beta}} + \left[ \frac{\partial u_{0,1}}{\partial x_2} \right] \bar{\boldsymbol{\Upsilon}} + \frac{[p_0]}{[\bar{\eta}]} \bar{\boldsymbol{\chi}} \right\} - \epsilon (\mathbf{u}_2(x) - \{p_{\text{in}} \bar{\boldsymbol{\varkappa}}\}) \right) \\ \mathbf{v}_3 &:= \epsilon \sum_{i \in \{\boldsymbol{\beta}, \boldsymbol{\Upsilon}, \boldsymbol{\chi}\}} \mathbf{w}_{\epsilon,i} + \epsilon^2 \mathbf{w}_{\epsilon, \boldsymbol{\varkappa}}, \end{aligned}$$

Because  $\mathbf{v}_1 \equiv \mathcal{V}_\epsilon$  on  $\{x_2 = 2\epsilon\}$  and  $\{x_2 = 0\}$  and thanks to Theorem 2.3, one obtains

$$\|\mathbf{v}_1\|_{\mathbf{L}^2(\{x_2=2\epsilon\})} \leq \sqrt{\epsilon} \|\nabla \mathbf{v}_1\|_{L^2(B_\epsilon)} \leq \epsilon^{\frac{3}{2}-}$$

Due to the  $\mathbf{H}^1(\Omega_1)$  regularity of  $\mathbf{u}_1$ , one has also

$$\|\mathbf{v}_2\|_{L^2(\{x_2=2\epsilon\})} \leq \sqrt{\epsilon} \|\nabla \mathbf{v}_2\|_{L^2(B_\epsilon)}$$

while on  $\Gamma_0$  this term is identically zero. For the vertical corrector, one has for both  $\{x_2 = 2\epsilon\}$  and  $\{x_2 = 0\}$  that :

$$\begin{aligned} \|\mathbf{w}_{\epsilon,i}\|_{\mathbf{L}^2(\{x_2=\epsilon\})}^2 &\leq \int_0^1 |\mathbf{w}|^2 \left( \frac{x_1}{\epsilon}, \epsilon \right) dx_1 = \epsilon \int_0^{\frac{1}{\epsilon}} |\mathbf{w}|^2(y_1, 1) dy_1 \\ &\leq \epsilon \int_0^{\frac{1}{\epsilon}} \frac{|\mathbf{w}|^2}{\rho}(y_1, 1) \rho^{2\alpha} dy_1 \sup_{y_1 \in ]0, \frac{1}{\epsilon}[} \rho^{1-2\alpha} \leq k\epsilon \|\mathbf{w}_i\|_{\mathbf{W}_\alpha^{1,2}(\Pi)}. \end{aligned}$$

So that the same arguments give also the result in  $\Omega_2$  :

$$\|\mathbf{u}_\epsilon - \mathcal{U}_\epsilon\|_{\mathbf{L}^2(\Omega_2)} + \|p_\epsilon - \mathcal{P}_\epsilon\|_{H^{-1}(\Omega_2)} \leq k\epsilon^{\frac{3}{2}-}.$$

It remains to consider the rough layer  $B_\epsilon$ . There, we have

$$\|\mathbf{u}_\epsilon - \mathcal{U}_\epsilon\|_{\mathbf{L}^2(B_\epsilon)} + \|p_\epsilon - \mathcal{P}_\epsilon\|_{H^{-1}(B_\epsilon)} \leq \epsilon \left\{ \|\mathbf{u}_\epsilon - \mathcal{V}_\epsilon\|_{\mathbf{H}^1(B_\epsilon)} + \|p_\epsilon - \mathcal{P}_\epsilon\|_{L^2(B_\epsilon)} \right\}, \quad (17)$$

where the  $\mathbf{H}^1(B_\epsilon)$  regularity is again obtained using the triple decomposition presented above. As the estimates on the velocity come using Poincaré estimates at the microscopic level as in Lemma 3.2 in [10], the pressure estimate is obtained by duality. Indeed by definition of the dual norm one has :

$$\|p_\epsilon - \mathcal{P}_\epsilon\|_{H^{-1}(B_\epsilon)} = \sup_{\varphi \in H_0^1(B_\epsilon)} \langle p_\epsilon - \mathcal{P}_\epsilon, \varphi \rangle_{H^{-1}, H_0^1}$$

where  $H_0^1(B_\epsilon)$  denotes the set of functions in  $H^1(B_\epsilon)$  vanishing on  $\Gamma_0$ . As  $(p_\epsilon - \mathcal{P}_\epsilon)$  belongs to  $L^2(B_\epsilon)$  the duality bracket can be transformed into an integral, namely

$$\begin{aligned} \langle p_\epsilon - \mathcal{P}_\epsilon, \varphi \rangle_{H^{-1}, H_0^1} &= \int_{B_\epsilon} (p_\epsilon - \mathcal{P}_\epsilon) \varphi dx \leq \|p_\epsilon - \mathcal{P}_\epsilon\|_{L^2(B_\epsilon)} \|\varphi\|_{L^2(B_\epsilon)}, \\ &\leq \epsilon \|p_\epsilon - \mathcal{P}_\epsilon\|_{L^2(B_\epsilon)} \|\varphi\|_{H^1(B_\epsilon)}, \end{aligned}$$

taking the sup over all functions in  $H_0^1(B_\epsilon)$ , one concludes the norm correspondence. Applying the estimates from Theorem 2.3 again to the rhs of (17), one obtains

$$\|\mathbf{u}_\epsilon - \mathcal{U}_\epsilon\|_{\mathbf{L}^2(B_\epsilon)} + \|p_\epsilon - \mathcal{P}_\epsilon\|_{H^{-1}(B_\epsilon)} \leq k\epsilon^{2-}$$

□

**Remark 2.3.** Note that the weak  $H^{-1}(\Omega'_1 \cup B_\epsilon \cup \Omega_2)$  norm of the pressure is smaller than the  $H^{-1}(\Omega_1 \cup \Omega_2)$  one, on the other hand the microscopic roughness make the domain connected and not convex which is a sufficient condition in order to obtain very weak estimates (cf p. 53 [11]).

Here we consider the oscillating part of our approximation. We recall that  $\bar{\mathbf{u}}_\epsilon := \mathbf{u}_0 + \epsilon \mathbf{u}_1$  and  $\bar{p}_\epsilon := p_0 + \epsilon p_1$ , and we set

$$\bar{\mathbf{v}}_\epsilon := \mathcal{U}_\epsilon - \bar{\mathbf{u}}_\epsilon, \quad \bar{q}_\epsilon := \mathcal{P}_\epsilon - \bar{p}_\epsilon.$$

These function satisfy on each sub-domain  $\Omega_j$

$$\left\{ \begin{array}{ll} -\Delta \bar{\mathbf{v}}_\epsilon + \nabla \bar{q}_\epsilon = -\epsilon \Delta \mathbf{w}_{\epsilon,i} + \nabla \theta_{\epsilon,i} & \text{in } \Omega_j \\ \operatorname{div} \bar{\mathbf{v}}_\epsilon = \epsilon \operatorname{div} \mathbf{w}_{\epsilon,i} & \text{in } \Omega_j \\ \bar{\mathbf{v}}_\epsilon \cdot \boldsymbol{\tau} = 0 & \text{on } \Gamma_N \\ \bar{q}_\epsilon = 0 & \text{on } \Gamma_N \\ \bar{\mathbf{v}}_\epsilon = \epsilon \left\{ \frac{\partial u_{0,1}}{\partial x_2} \boldsymbol{\beta}_\epsilon + \frac{\partial u_{0,1}}{\partial x_2} \bar{\boldsymbol{\beta}} + \left[ \frac{\partial u_{0,1}}{\partial x_2} \right] \bar{\mathbb{Y}} + \frac{[p_0]}{[\bar{\eta}]} \bar{\boldsymbol{\chi}} + \epsilon \{p_{\text{in}} \bar{\boldsymbol{\chi}}\} \right\} = \frac{\partial u_{0,1}}{\partial x_2} x_2 + \epsilon k_1 & \text{on } \Gamma_\epsilon \\ \bar{\mathbf{v}}_\epsilon = 0 & \text{on } \Gamma_D \end{array} \right. \quad (18)$$

At the same time  $\bar{\mathbf{v}}_\epsilon, \bar{q}_\epsilon$  are explicit functions of all the correctors.

**Theorem 2.5.** *The rapidly oscillating rest  $(\mathcal{U}_\epsilon - \bar{\mathbf{u}}_\epsilon, \mathcal{P}_\epsilon - \bar{p}_\epsilon)$  satisfies*

$$\|\mathcal{U}_\epsilon - \bar{\mathbf{u}}_\epsilon\|_{\mathbf{L}^2(\Omega)} + \sqrt{\epsilon} \|\mathcal{P}_\epsilon - \bar{p}_\epsilon\|_{H^{-1}(\Omega'_1 \cup B_\epsilon \cup \Omega_2)} \leq k\epsilon^{\frac{3}{2}}$$

where the constant  $k$  is independent on  $\epsilon$ .

*Proof.* Because  $\bar{\mathbf{v}}_\epsilon$  is explicit and reads :

$$\bar{\mathbf{v}}_\epsilon = \epsilon \left\{ c_\beta(\beta_\epsilon - \bar{\beta}) + c_\Upsilon(\Upsilon_\epsilon - \bar{\Upsilon}) + c_\chi(\chi_\epsilon - \bar{\chi}) + \epsilon c_\varkappa(\varkappa_\epsilon - \bar{\varkappa}) \right\} + \epsilon \mathbf{w}_{\epsilon,i} + \mathbf{W}$$

a direct computation of the  $L^2$  norm gives that

$$\begin{aligned} \|\bar{\mathbf{v}}_\epsilon\|_{\mathbf{L}^2(\Omega_j)} &\leq \epsilon k \left\{ \|\beta_\epsilon - \bar{\beta}\|_{\mathbf{L}^2(\Omega_j)} + \|\chi_\epsilon - \bar{\chi}\|_{\mathbf{L}^2(\Omega_j)} + \|\Upsilon_\epsilon - \bar{\Upsilon}\|_{\mathbf{L}^2(\Omega_j)} + \epsilon \|\varkappa_\epsilon - \bar{\varkappa}\|_{\mathbf{L}^2(\Omega_j)} \right\} \\ &\quad + \epsilon \|\mathbf{w}_{\epsilon,i}\|_{\mathbf{L}^2(\Omega_j)} + \|\mathbf{W}\|_{\mathbf{L}^2(\Omega_j)} \leq k\epsilon^{\frac{3}{2}}. \end{aligned}$$

in this estimate we follow the method introduced in theorem 6, p. 1123 [19], but the same approach does not provide more than  $\sqrt{\epsilon}$  convergence rate for pressures due to the different  $\epsilon$ -scaling of the pressure terms in (12), so

$$\|\bar{p}_\epsilon\|_{\mathbf{L}^2(\Omega_j)} \leq k\sqrt{\epsilon}$$

We use again the decomposition  $\Omega := \Omega'_1 \cup \bar{B}_\epsilon \cup \Omega_2$  and very weak estimates in  $\Omega'_1$  and  $\Omega_2$ ; these give

$$\|\bar{q}_\epsilon\|_{H^{-1}(\Omega'_1)} \leq \|\bar{\mathbf{v}}_\epsilon\|_{\mathbf{L}^2(\{x_2=2\epsilon\})} + O\left(\epsilon^{\frac{3}{2}-}\right),$$

where we put in the last term of the above inequality the contribution of the source terms of the rhs in (18) already computed in the proof of Theorem 2.4. But because  $\bar{\mathbf{v}}_\epsilon$  is explicit, and due to the  $y_1$ -periodic microscopic structure of  $\beta_\epsilon, \Upsilon_\epsilon, \chi_\epsilon$  and  $\varkappa_\epsilon$ , one has only

$$\|\bar{\mathbf{v}}_\epsilon\|_{\mathbf{L}^2(\{x_2=2\epsilon\})} \leq k\epsilon.$$

In  $\Omega_2$  a similar argument gives

$$\|\bar{q}_\epsilon\|_{H^{-1}(\Omega_2)} \leq \|\bar{\mathbf{v}}_\epsilon\|_{\mathbf{L}^2(\Gamma_0)} + O\left(\epsilon^{\frac{3}{2}-}\right) \leq k\epsilon$$

In  $B_\epsilon$  we use the dual estimate (17) based on the Poincaré inequality, to get

$$\|\bar{q}_\epsilon\|_{H^{-1}(B_\epsilon)} \leq k\epsilon \|\bar{q}_\epsilon\|_{L^2(B_\epsilon)} \leq k\epsilon^{\frac{3}{2}}$$

□

Combining Theorems 2.4 and 2.5 above, one gets the main result of the paper

**Theorem 2.6.** *The averaged approximation  $(\bar{\mathbf{u}}_\epsilon := \mathbf{u}_0 + \epsilon \mathbf{u}_1, \bar{p}_\epsilon := p_0 + \epsilon p_1)$  satisfies the very weak estimates:*

$$\|\mathbf{u}_\epsilon - \bar{\mathbf{u}}_\epsilon\|_{\mathbf{L}^2(\Omega_j)} + \sqrt{\epsilon} \|p_\epsilon - \bar{p}_\epsilon\|_{H^{-1}(\Omega_j)} \leq k\epsilon^{\frac{3}{2}-}, \quad j \in \{1, 2\}$$

Implicit interface conditions. We start with the horizontal velocity. We call  $\mathbf{u}_1^\pm$  (resp  $\partial_2 u_{0,1}^\pm$ ) the values across  $\Gamma_0$ . The first order interface condition derived above on  $\Gamma_0$  reads:

$$\mathbf{u}_1^\pm = \left\{ \frac{\partial u_{0,1}}{\partial x_2} \overline{\beta}^\pm + \left[ \frac{\partial u_{0,1}}{\partial x_2} \right] \overline{\Upsilon}^\pm \right\} \mathbf{e}_1 + \frac{[p_0]}{[\eta]} \overline{\chi}_2 \mathbf{e}_2$$

assembling together normal derivatives of the velocity on both sides and because  $\partial_2 u_{0,1}^- \equiv 0$ , one has also :

$$\begin{aligned} \mathbf{u}_1^+ &= \left\{ \frac{\partial u_{0,1}}{\partial x_2} (\overline{\beta}^+ + \overline{\Upsilon}^+) \right\} \mathbf{e}_1 + \frac{[p_0]}{[\eta]} \overline{\chi}_2 \mathbf{e}_2 \\ \mathbf{u}_1^- &= \left\{ \frac{\partial u_{0,1}}{\partial x_2} (\overline{\beta}^- + \overline{\Upsilon}^-) \right\} \mathbf{e}_1 + \frac{[p_0]}{[\eta]} \overline{\chi}_2 \mathbf{e}_2 \end{aligned}$$

But thanks to Proposition 3,  $\overline{\Upsilon}^+ = \overline{\Upsilon}^- + \overline{\beta}^-$  and this organizes the expression above as an implicit difference

$$\mathbf{u}_1^+ \cdot \mathbf{e}_1 = ((\overline{\beta}^+ + \overline{\Upsilon}^+) - (\overline{\beta}^- + \overline{\Upsilon}^-)) \frac{\partial u_{0,1}}{\partial x_2} \mathbf{e}_1 + \mathbf{u}_1^- \cdot \mathbf{e}_1 \quad \text{and} \quad \frac{\mathbf{u}_1^+ \cdot \mathbf{e}_1}{\overline{\beta}^+ + \overline{\Upsilon}^+} = \frac{\mathbf{u}_1^- \cdot \mathbf{e}_1}{\overline{\beta}^- + \overline{\Upsilon}^-}$$

which finally gives

$$\frac{\mathbf{u}_1^+ \cdot \mathbf{e}_1}{\overline{\beta}^+ + \overline{\Upsilon}^+} = \frac{\partial u_{0,1}}{\partial x_2} \quad \text{and} \quad \frac{\mathbf{u}_1^+ \cdot \mathbf{e}_1}{\overline{\beta}^+ + \overline{\Upsilon}^+} = \frac{\mathbf{u}_1^- \cdot \mathbf{e}_1}{\overline{\beta}^- + \overline{\Upsilon}^-}$$

Setting  $\overline{\mathbf{u}}_\epsilon := \mathbf{u}_0 + \epsilon \mathbf{u}_1$  and because  $\mathbf{u}_0 \equiv 0$  on  $\Gamma_0$ , one has also

$$\overline{\mathbf{u}}_\epsilon^+ \cdot \boldsymbol{\tau} = \epsilon (\overline{\beta}^+ + \overline{\Upsilon}^+) \frac{\partial \overline{u}_{\epsilon,1}}{\partial x_2} + O(\epsilon^2), \quad \text{and} \quad \frac{\overline{\mathbf{u}}_\epsilon^+ \cdot \boldsymbol{\tau}}{\overline{\beta}^+ + \overline{\Upsilon}^+} = \frac{\overline{\mathbf{u}}_\epsilon^- \cdot \boldsymbol{\tau}}{\overline{\beta}^- + \overline{\Upsilon}^-}$$

One recovers a slip velocity condition in the main artery and a new discontinuous relationship between the horizontal components of the velocity at the interface.

For the vertical velocity, thanks to the continuity of  $\overline{\chi}_2$  across  $\Gamma_0$ , one has that

$$u_{1,2}^+ = u_{1,2}^- = u_{1,2} = \frac{[p_0]}{[\eta]},$$

this in turn gives the implicit interface condition :

$$\overline{\mathbf{u}}_\epsilon \cdot \mathbf{n} = \frac{\epsilon}{[\eta]} ([\sigma_{\overline{\mathbf{u}}_\epsilon, \overline{p}_\epsilon}] \cdot \mathbf{n}, \mathbf{n}) + O(\epsilon^2).$$

## 2.2. The case of an aneurysmal sac

When  $\epsilon$  goes to 0, the limit solution  $(\mathbf{u}_0, p_0)$  is explicit (we set  $p_{\text{out},1} = 0$  in (6)):

$$\begin{cases} \mathbf{u}_0(x) = \frac{p_{\text{in}}}{2} (1 - x_2) x_2 \mathbf{e}_1 \mathbb{1}_{\Omega_1}, & \forall x \in \Omega \\ p_0(x) = p_{\text{in}} (1 - x_1) \mathbb{1}_{\Omega_1} + p_0^- \mathbb{1}_{\Omega_2}, \end{cases}$$

where  $p_0^-$  is any real constant. Although we could impose the uniqueness of the limit supposing that  $p_0^- \in L^2(\Omega_2)/\mathbb{R}$  and thus setting it to zero we keep it as a degree of freedom to be fixed later on.

**Theorem 2.7.** *For every fixed  $\epsilon$ , there exists a unique solution  $(\mathbf{u}_\epsilon, p_\epsilon) \in \mathbf{H}^1(\Omega_\epsilon) \times L^2(\Omega_\epsilon)$  of the problem (1). Moreover, one has*

$$\|\mathbf{u}_\epsilon - \mathbf{u}_0\|_{\mathbf{H}^1(\Omega_\epsilon)^2} + \|p_\epsilon - p_0\|_{L^2(\Omega_{1,\epsilon})} + \|p_\epsilon - p_0\|_{L^2(\Omega_2)/\mathbb{R}} \leq k\sqrt{\epsilon}$$

where the constant  $k$  depends on  $p_0^-$  but not on  $\epsilon$ .

*Proof.* Set  $\mathbf{v}_\epsilon := \mathbf{u}_\epsilon - \tilde{\mathbf{u}}_0$ ,  $q_\epsilon := p_\epsilon - p_0$ , where  $\tilde{\mathbf{u}}_0 := \mathbf{u}_0 - \mathcal{R}(\mathbf{u}_0)$  already defined in the proof of Theorem 2.1, these functions solve

$$\begin{cases} -\Delta \mathbf{v}_\epsilon + \nabla q_\epsilon = [\sigma_{\mathbf{u}_0, p_0}] \cdot \mathbf{n} \delta_{\Gamma_0} - \Delta \mathcal{R}(\mathbf{u}_0) & \text{in } \Omega \\ \operatorname{div} \mathbf{v}_\epsilon = 0 & \text{in } \Omega \\ \mathbf{v}_\epsilon = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_{\text{out},2} \\ \mathbf{v}_\epsilon \cdot \boldsymbol{\tau} = 0 & \text{on } \Gamma_N \\ p_0 = p_{\text{in}} \text{ on } \Gamma_{\text{in}}, \quad p_0 = 0 & \text{on } \Gamma_{\text{out},1} \\ \mathbf{v}_\epsilon = 0 & \text{on } \Gamma_\epsilon \end{cases}$$

The standard theory for mixed problems gives existence and uniqueness of a solution  $(\mathbf{v}_\epsilon, \tilde{q}_\epsilon)$  in  $\mathbf{H}^1(\Omega_\epsilon) \times L^2(\Omega_\epsilon)$  for every jump term on  $\Gamma_0$ , the corresponding estimates follow exactly as in Theorem 2.1

$$\|\mathbf{v}_\epsilon\|_{\mathbf{H}^1(\Omega_\epsilon)} + \|q_\epsilon\|_{L^2(\Omega_\epsilon)} \leq k(p_0^-) \sqrt{\epsilon}$$

Because of the pressure boundary condition on  $\Gamma_{\text{in}} \cup \Gamma_{\text{out},1}$ , the pressure  $\tilde{q}_\epsilon$  is uniquely defined in  $\Omega_2$ . But it may differ up to a constant from  $q_\epsilon := p_\epsilon - p_0$  in  $\Omega_2$ . As  $\Omega_2$  is now a cavity, we rewrite the problem solved by  $(\mathbf{v}_\epsilon, q_\epsilon)$  with a fixed velocity  $\mathbf{v}_\epsilon$  imposed on  $\Gamma_0$  and lifted inside the domain  $\Omega_2$ , namely  $\hat{\mathbf{v}}_\epsilon := \mathbf{v}_\epsilon - \mathcal{R}(\mathbf{v}_\epsilon)$ . Then  $(\hat{\mathbf{v}}_\epsilon, q_\epsilon)$  solve

$$\begin{cases} -\Delta \hat{\mathbf{v}}_\epsilon + \nabla q_\epsilon = \Delta \mathcal{R}(\mathbf{v}_\epsilon) & \text{in } \Omega_2 \\ \operatorname{div} \hat{\mathbf{v}}_\epsilon = \operatorname{div} \mathcal{R}(\mathbf{v}_\epsilon) & \text{in } \Omega_2 \\ \hat{\mathbf{v}}_\epsilon = 0 & \text{on } \partial\Omega_2 \end{cases}$$

and this gives

$$\|q_\epsilon\|_{L^2(\Omega_2)/\mathbb{R}} \leq \|\mathcal{R}(\mathbf{v}_\epsilon)\|_{\mathbf{H}^1(\Omega_2)} \leq \|\mathbf{v}_\epsilon\|_{\mathbf{H}(\Omega_\epsilon)} \leq k\sqrt{\epsilon}$$

□

First order approximation. Due to the presence of three kind of errors above, we construct a full boundary layer approximation  $(\mathcal{U}_\epsilon, \mathcal{P}_\epsilon)$  exactly as in (12). One has to make few minor changes in the definition of  $(\mathcal{W}_\epsilon, \mathcal{Z}_\epsilon)$  that are left to the reader. The only difference stands in the pressure jump:

$$[p_0] = p_0^+(x_1, 0) - p_0^-$$

where  $p_0^-$  is the constant pressure not yet fixed. The first order macroscopic corrector  $(\mathbf{u}_1, p_1)$  should satisfy

$$\begin{cases} -\Delta \mathbf{u}_1 + \nabla p_1 = 0 & \text{in } \Omega_1 \cup \Omega_2 \\ \operatorname{div} \mathbf{u}_1 = 0 & \text{in } \Omega_1 \cup \Omega_2 \\ \mathbf{u}_1 = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_{\text{out},2} \\ \mathbf{u}_1 \cdot \boldsymbol{\tau} = 0 & \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out},1}, \\ p_1 = 0 & \\ \mathbf{u}_1 = \left\{ \frac{\partial u_{0,1}}{\partial x_2} \overline{\boldsymbol{\beta}}^\pm + \left[ \frac{\partial u_{0,1}}{\partial x_2} \right] \overline{\boldsymbol{\Upsilon}}^\pm \right\} \mathbf{e}_1 + \frac{[p_0]}{[\overline{\eta}]} \overline{\boldsymbol{\chi}} \mathbf{e}_2 & \text{on } \Gamma_0^\pm \end{cases} \quad (19)$$

As we impose the velocity on every edge of  $\Omega_2$  there is a compatibility condition between the Dirichlet data and the divergence free condition reading

$$\int_{\Omega_2} \operatorname{div} \mathbf{u}_1 \, dx = \int_{\partial\Omega_2} \mathbf{u}_1 \cdot \mathbf{n} \, d\sigma = \int_{\Gamma_0} \mathbf{u}_1 \cdot \mathbf{n} \, dx_1 = 0,$$

and this precisely identifies the pressure  $p_0^-$  giving

$$|\Gamma_0| p_0^- = \int_{\Gamma_0} p_0^+(x_1, 0) dx_1. \quad (20)$$

The first order constants are fixed in the definition of  $(\mathcal{U}_\epsilon, \mathcal{P}_\epsilon)$ . Even if  $p_0^-$  is now well defined, the first and second order pressures  $p_1$  and  $p_2$  are again defined in  $\Omega_2$  up to a constant. This is why we always need quotiented norms on the pressure in  $\Omega_2$ .

Following the same lines as in the section above but taking into account the pressures in  $\Omega_2$  up to a constant as in the proof of Theorem 2.7, one proves Theorem 1.2.

### 3. NUMERICAL EVIDENCE

#### 3.1. Approximation of the velocity and the flow-rate in the collateral artery

We solve numerically problem (1) in 2D, for various values of  $\epsilon$ . For each  $\epsilon$ , we confront the corresponding numerical quantities with the information provided by the homogenized first-order explicit approximation : velocity profile, flow-rate, pressure errors are computed with respect to different norms evaluated above in a theoretical manner.

Discretizing the rough solution  $(\mathbf{u}_\epsilon, p_\epsilon)$ . The domain  $\Omega_\epsilon$  is discretized for  $\epsilon \in ]0, 1]$  using a triangulation. To discretize the velocity-pressure variables, a  $(\mathbb{P}_2, \mathbb{P}_1)$  finite element basis is chosen. Because of the presence of microscopic perturbations, when solving the Stokes equations, the penalty method gave instabilities. For this reason we opted for the Uzawa conjugate gradient solver (see p. 178 in [17], and references there in). The code is written in the `freefem++` language<sup>1</sup>. On the boundary we impose the following data :  $p_{\text{in}} = 2, p_{\text{out},1} = 0, p_{\text{out},2} = -1$ .

The microscopic cell problem. Using the same numerical tools, we solve the microscopic problem (9). As the domain is infinite we truncate it for  $y_2 = L$ , where  $L$  is large enough. We let natural boundary conditions on the first component of the velocity and impose  $\chi_2 = -1$  on the other component. For similar microscopic problems it was proved in [22] that the solutions of the truncated problem converge exponentially with respect to  $L$  to the solution of the unbounded problem. Then we compute the numerical value of the pressure drop  $[\overline{\eta}] := \overline{\eta}(+\infty) - \overline{\eta}(-\infty)$ . If  $Q$  is a sphere of radius  $1/4$  in a period of size 1 centered at  $(1/2, 1/4)$  this gives a value of the pressure difference equal to  $[\overline{\eta}] := 52.6961$ .

Results : In fig. 5 right, we plot various values of  $\epsilon$  on the  $x$ -axis and the flow-rate through  $\Gamma_0$  on the  $y$ -axis. Note that  $\epsilon = 0.5$  corresponds to the case of two huge obstacles in front of  $\Omega_2$  while

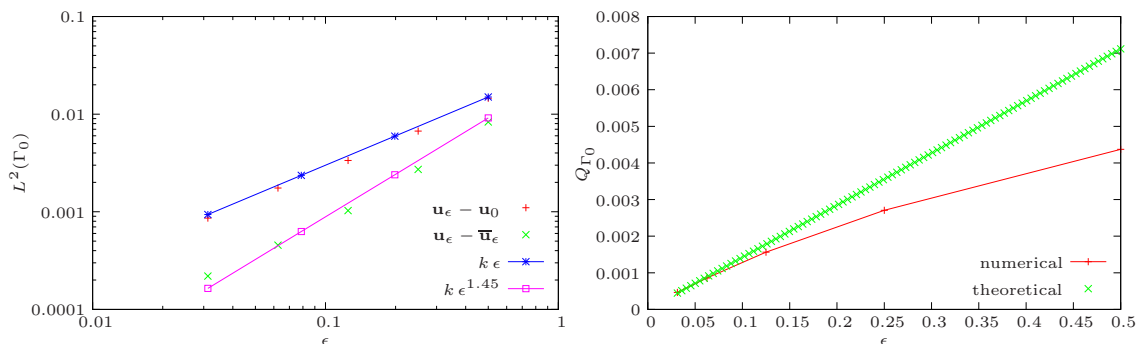


FIGURE 5. Zeroth and first order error estimates (left) and rough versus homogenized flow-rates for various values of  $\epsilon$  (right)

$\epsilon = 0.03125$  corresponds to 16 spheres. One observes that the asymptotic expansion gives the first order derivative of the flow-rate with respect to  $\epsilon$  near  $\epsilon = 0$  which was expected. One notices also that the actual rough flow-rate behaves as a square-root of  $\epsilon$ . This seems difficult to obtain using averaged interface conditions only [6, 7]. In fig. 5 left, we plot numerical  $L^2(\Gamma_0)$  error estimates : we compute  $\|(\mathbf{u}_\epsilon - \mathbf{u}_0) \cdot \mathbf{n}\|_{L^2(\Gamma_0)}$  and  $\|(\mathbf{u}_\epsilon - \overline{\mathbf{u}}_\epsilon) \cdot \mathbf{n}\|_{L^2(\Gamma_0)}$ . Our approximation  $\overline{\mathbf{u}}_\epsilon$  is explicit on

<sup>1</sup><http://www.freefem.org/ff++>

$\Gamma_0$ , so one only solves the rough problem (1). Up to discretization errors, we recover theoretical claims:

$$\|(\mathbf{u}_\epsilon - \mathbf{u}_0) \cdot \mathbf{n}\|_{L^2(\Gamma_0)} \sim k\epsilon, \quad \|(\mathbf{u}_\epsilon - \bar{\mathbf{u}}_\epsilon) \cdot \mathbf{n}\|_{L^2(\Gamma_0)} \sim k\epsilon^{1.45}.$$

In fig. 6 we display the mesh used for two different values of  $\epsilon$  (middle and right) and the mesh size  $h$  with respect to  $\epsilon$  (left).

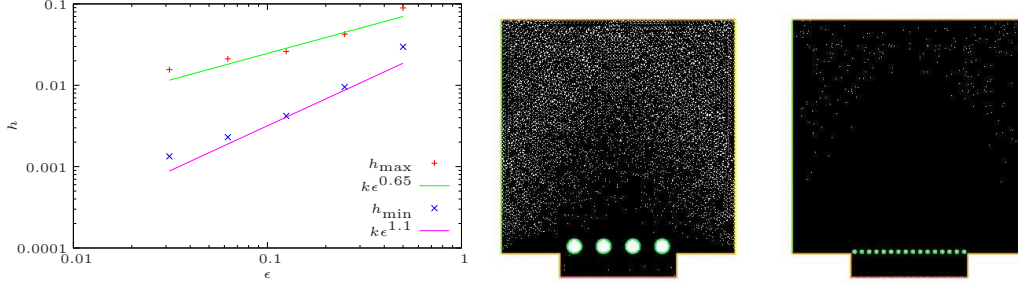


FIGURE 6. The mesh size  $h$ , meshes for  $\epsilon = 0.25$  and  $\epsilon = 0.0625$

### 3.2. Velocity profile and pressures in the aneurysmal sac

In the same numerical framework we validate the theoretical claims for the aneurysm : in order to show that our result do not only apply to the basic geometry of figure 2, we choose for  $\Omega_2$ , the geometry shown as an illustration in fig. 3 and 4 in section 1.2 :  $\Omega_2 := B((\frac{1}{2}, -\frac{1}{4}), \frac{1}{4}) \cap \mathbb{R}_{x_2}^-$ . This is supposed to give a more realistic shape to the aneurysm.

In fig. 7 we plot numerical error estimates for the zeroth and first order approximations (top), we display also velocity profiles of  $\mathbf{u}_\epsilon \cdot \mathbf{n}$  and  $\bar{\mathbf{u}}_\epsilon \cdot \mathbf{n}$  across  $\Gamma_0$  for different values of  $\epsilon$ (bottom).

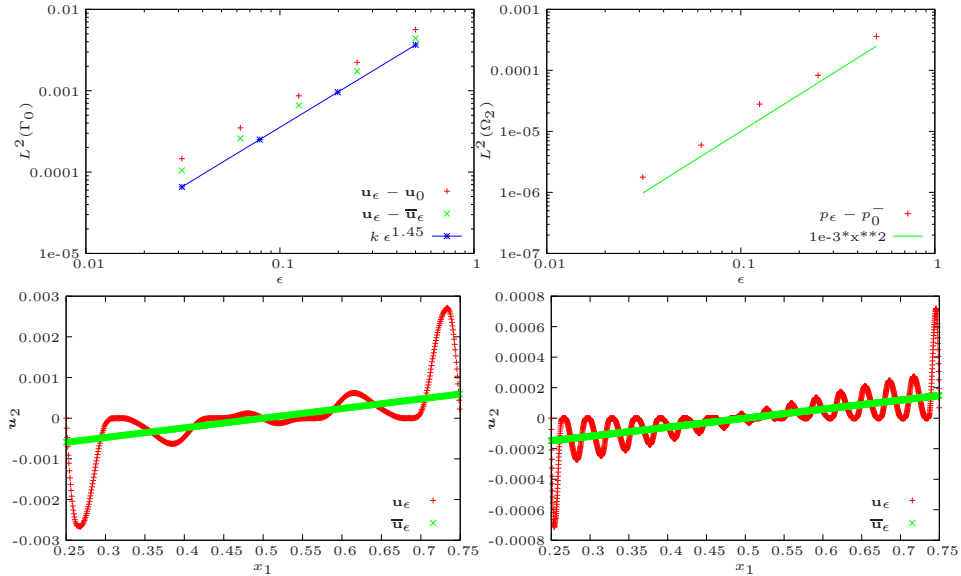
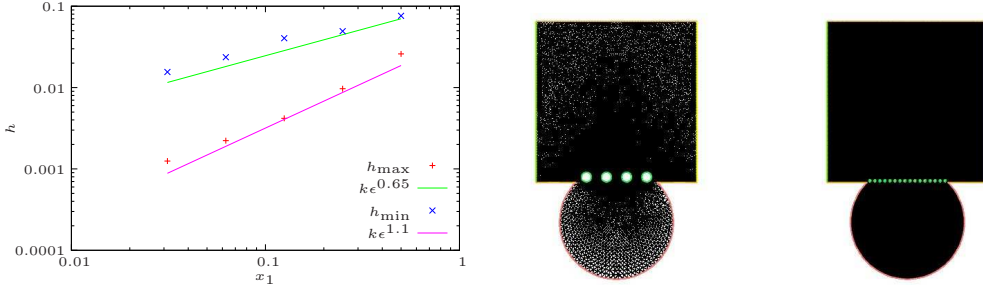


FIGURE 7. Zeroth and first order error estimates of the normal velocity across  $\Gamma_0$  (top, left) zeroth order pressure estimate in  $\Omega_2$  with respect to  $\epsilon$  (top, right) Normal velocity profiles across  $\Gamma_0$  for  $\epsilon = 0.25$  (bottom, left) and  $\epsilon = 0.0625$  (bottom, right)  $\bar{\mathbf{u}}_\epsilon$  versus  $\mathbf{u}_\epsilon$

In fig. 8 middle and right, we depict an example of two meshes built respectively for  $\epsilon = 0.25$  and  $\epsilon = 0.0625$ . On the left we give the mesh sizes as a function of  $\epsilon$

FIGURE 8. The mesh size  $h$ , meshes for  $\epsilon = 0.25$  and  $\epsilon = 0.0625$ 

Up to discretization errors we recover results shown theoretically. Our first order approximation is still better than the zeroth order one (see fig. 7 top-left). Nevertheless the convergence order seems to coincide for both approximations around  $\epsilon^{1.45}$ . We do not have an explanation for this particular fact. Nevertheless this phenomenon is not due to the change of shape of  $\Omega_2$ : similar tests on meshes presented in fig. 6 below give the same improved order of convergence for the zeroth order profile in the case of an aneurysm.

In fig. 7 (top-left) we display the  $L^2(\Omega_2)$  comparison between  $p_\epsilon$ , the numerical solution of problem (5) and the constant  $p_0^-$  provided by Corollary 1.1. We notice that the convergence order of the error is quadratic. This validates the choice of the constant  $p_0^-$  computed theoretically, and shows that the microscopic perturbations induce only second order errors for the pressure. This in turn gives some hope that the theoretical results should be improved in this sense.

## APPENDIX

We study the problem: find  $(\mathbf{w}, \theta) \in \mathbf{W}_\alpha^{1,2}(\Pi) \times W_\alpha^{0,2}(\Pi)$  solving

$$\begin{cases} -\Delta \mathbf{w} + \nabla \theta = 0 & \text{in } \Pi, \\ \operatorname{div} \mathbf{w} = 0 & \text{in } \Pi \\ \mathbf{w} = \mathbf{f} & \text{on } D, \\ \mathbf{w} \cdot \boldsymbol{\tau} = \mathbf{f} \cdot \boldsymbol{\tau}, \text{ and } \theta = h & \text{on } N, \\ \mathbf{w} = 0 & \text{on } B, \end{cases} \quad (21)$$

**Theorem 3.1.** *If the real  $\alpha$  is such that  $|\alpha| < 1$  and if*

$$\mathbf{f} \in \mathbf{W}_\alpha^{\frac{1}{2},2}(D \cup N), \quad h \in W_\alpha^{0,2}(N)$$

*there exists a unique solution  $(\mathbf{w}, \theta)$  solving problem (21).*

Before giving the proof of the theorem, we need two intermediate propositions. We define

$$X_\alpha := \{\mathbf{v} \in \mathbf{W}_\alpha^{1,2}(\Pi) \text{ s.t. } \mathbf{v} = 0 \text{ on } D \cup B, \quad \mathbf{v} \cdot \boldsymbol{\tau} = 0 \text{ on } N\}, \quad Y_\alpha := W_\alpha^{0,2}(\Pi).$$

At first, we show that the divergence operator is surjective from  $X_\alpha$  into  $Y_\alpha$

**Proposition 8.** *For any given function  $q \in Y_\alpha$  there exists a vector function  $\mathbf{v} \in X_\alpha$  such that*

$$\operatorname{div} \mathbf{v} = q, \text{ and } \|\mathbf{v}\|_{X_\alpha} \leq k(\Pi, \alpha) \|q\|_{Y_\alpha}$$

*where the constant  $k$  depends only on the geometry of the domain, and on  $\alpha$ .*

*Proof.* We define a sequence of annular domains covering  $\Pi$

$$C_n := \{y \in \Pi \text{ s.t. if } x = (r, \tilde{\theta}) \quad r \in ]2^{n-1}, 2^n[ \}, \quad n \geq 1, \quad C_0 := B(0, 1) \cap \Pi.$$



We decompose  $q$  as  $q = \sum_{n=0}^{\infty} q_n$ , with  $q_n := q \mathbb{1}_{C_n}$ . On each  $C_n$  we solve the problem: find  $\mathbf{v}_n \in X_{\alpha,n}$  s.t.  $\operatorname{div} \mathbf{v}_n = q_n$  and  $|\mathbf{v}_n|_{X_\alpha} \leq k(C_n, \alpha) |q_n|_{Y_\alpha}$ , where

$$X_{\alpha,n} := \{\mathbf{v} \in X_\alpha \text{ s.t. } \mathbf{v} = 0 \text{ on } \{|y| = 2^{n-1}\} \cup \{|y| = 2^n\} \cup (\overline{C_n} \cap B) \cup (\overline{C_n} \cap D), \\ \text{and } \mathbf{v} \cdot \boldsymbol{\tau} = 0 \text{ on } (\overline{C_n} \cap N)\}$$

But solve the latter equation in a weak sense means

$$\int_{C_n} \operatorname{div} \mathbf{v}_n \cdot \omega \, r dr d\tilde{\theta} = \int_{C_n} q_n \cdot \omega \, r dr d\tilde{\theta}, \quad \forall \omega \in W_{-\alpha}^{0,2}(C_n)$$

making the change of variables:  $(\tilde{r} = r/2^{n-1}, \tilde{\theta})$  and setting

$$\tilde{\mathbf{v}}_n(\tilde{r}, \tilde{\theta}) := \mathbf{v}(2^{n-1}\tilde{r}, \tilde{\theta}), \quad \tilde{q}_n(\tilde{r}, \tilde{\theta}) := q(2^{n-1}\tilde{r}, \tilde{\theta}), \quad \tilde{\omega}(\tilde{r}, \tilde{\theta}) := \omega(2^{n-1}\tilde{r}, \tilde{\theta})$$

the problem becomes: find  $\tilde{\mathbf{v}} \in X_{0,1}$  defined on  $C_1$  s.t.

$$\int_{C_1} \operatorname{div} \tilde{\mathbf{v}}_n \cdot \tilde{\omega} \tilde{r} d\tilde{r} d\tilde{\theta} = 2^{n-1} \int_{C_1} \tilde{q} \cdot \tilde{\omega} \tilde{r} d\tilde{r} d\tilde{\theta}, \quad \forall \omega \in L^2(C_1)$$

the test space is defined on a compact fixed domain  $C_1$ , weighted Sobolev spaces coincide with the classical ones as soon as the weight is strictly positive and bounded. In this framework the operator  $\operatorname{div} : X_{0,1} \rightarrow Y_{0,1}$  is surjective thanks to Lemma 4.9 p. 181 in [12]. Thus there exists  $\tilde{\mathbf{v}}_n \in X_{0,1}$  s.t.  $\operatorname{div} \tilde{\mathbf{v}}_n = \tilde{q}_n 2^{n-1}$ . Note that there is no need of a compatibility condition on the integral of  $\tilde{q}_n$  as in Lemma 3.1 chap. III in [15] because  $\tilde{\mathbf{v}}_n \cdot \mathbf{n} \neq 0$  on  $N \cap \overline{C_1}$ . Moreover one has that

$$|\tilde{\mathbf{v}}_n|_{\mathbf{H}^1(C_1)} \leq k(C_1, 0) \|2^{n-1} \tilde{q}_n\|_{L^2(C_1)}$$

where  $k$  depends only on the geometry of  $C_1$  and is thus independent on  $n$ . Turning back to the original variables  $(r, \theta)$  one has then that  $\operatorname{div} \mathbf{v}_n = q_n$  and

$$\int_{C_n} |\nabla \mathbf{v}_n|^2 r \, dr d\tilde{\theta} \leq k(C_1, 0) \int_{C_n} |q_n|^2 r \, dr d\tilde{\theta}$$

In order to recover the global weighted norm of  $q$  in  $W_\alpha^{0,2}(\Pi)$ , we multiply the inequality by  $2^{2\alpha(n+1)}$  on both sides; we use that for  $r \in [2^{n-1}, 2^n]$ ,  $\rho := (1 + r^2)^{\frac{1}{2}}$  can be estimated as  $2^{2\alpha(n-1)} \leq \rho^{2\alpha} \leq 2^{2\alpha(n+1)}$  giving finally

$$\int_{C_n} |\nabla \mathbf{v}_n|^2 \rho^{2\alpha} dy \leq \int_{C_n} |\nabla \mathbf{v}_n|^2 2^{2\alpha(n+1)} dy \\ \leq k(C_1, 0) 2^{2\alpha(n+1)} \int_{C_n} q_n^2 dy \leq 2^{4\alpha} k(C_1, 0) \int_{C_n} q_n^2 \rho^{2\alpha} dy$$

One defines  $\mathbf{v} := \sum_n \mathbf{v}_n \mathbb{1}_{C_n}$ , because of the boundary conditions imposed on each of the  $C_n$ ,  $\mathbf{v}$  is continuous on  $\Pi$  and thus belongs to  $\mathbf{W}_\alpha^{1,2}(\Pi)$ . This gives the result.  $\square$

We lift problem (21) by subtracting to  $\mathbf{w}$  a function  $\mathcal{R}(\mathbf{w})$  satisfying:

$$\mathcal{R}(\mathbf{w}) \in \mathbf{W}_\alpha^{1,2}(\Pi), \quad \mathcal{R}(\mathbf{w}) = \mathbf{f} \text{ on } D \cup N \text{ and } \mathcal{R}(\mathbf{w}) = 0 \text{ on } B$$

Such a lift exists (cf p. 249 [16] for an explicit form of  $\mathcal{R}(\mathbf{w})$ ). We correct the divergence of  $\mathcal{R}(\mathbf{w})$  by setting :

$$\mathcal{S}(\mathbf{w}) \in \mathbf{W}_0^{1,2}(\Pi) \text{ s.t. } \operatorname{div} \mathcal{S}(\mathbf{w}) = -\operatorname{div}(\mathcal{R}(\mathbf{w})) \text{ and } \|\mathcal{S}(\mathbf{w})\|_{\mathbf{W}_0^{1,2}(\Pi)} \leq k \|\mathcal{R}(\mathbf{w})\|_{\mathbf{W}_\alpha^{1,2}(\Pi)}$$

which is possible thanks to Proposition 8. The new variables  $(\tilde{\mathbf{w}} := \mathbf{w} - \mathcal{R}(\mathbf{w}) - \mathcal{S}(\mathbf{w}), \theta)$  solve the homogeneous problem:

$$\begin{cases} -\Delta \tilde{\mathbf{w}} + \nabla \theta = \Delta \mathcal{R}(\mathbf{w}) + \Delta \mathcal{S}(\mathbf{w}) & \text{in } \Pi, \\ \operatorname{div} \tilde{\mathbf{w}} = 0 & \text{in } \Pi, \\ \tilde{\mathbf{w}} = 0 & \text{on } D, \\ \tilde{\mathbf{w}} \cdot \boldsymbol{\tau} = 0, \text{ and } \theta = \tilde{h} & \text{on } N, \\ \tilde{\mathbf{w}} = 0 & \text{on } B, \end{cases} \quad (22)$$

where  $\tilde{h} := h + (\partial_{\mathbf{n}}(\mathcal{R}(\mathbf{w}) + \mathcal{S}(\mathbf{w})), \mathbf{n})$ . The onto mapping between  $\mathbf{W}_{\alpha}^{1,2}(\Pi) \times W_{\alpha}^{0,2}(\Pi)$  and  $\mathbf{W}_0^{1,2}(\Pi) \times W_0^{0,2}(\Pi)$  (cf Theorem I.3 p. 243 in [16]) allows us to claim that  $(\mathbb{F}, \mathbb{G}) := (\rho^{\alpha} \tilde{\mathbf{w}}, \rho^{\alpha} \theta)$  solve in an equivalent way the problem: find  $(\mathbb{F}, \mathbb{G})$  in  $\mathbf{W}_0^{1,2}(\Pi) \times W_0^{0,2}(\Pi)$  s.t.

$$\begin{cases} \mathcal{A}_{\alpha} \mathbb{F} + \mathcal{B}_{\alpha}^T \mathbb{G} = \rho^{\alpha} (\Delta \mathcal{R}(\mathbf{w}) + \Delta \mathcal{S}(\mathbf{w})) & \text{in } \Pi, \\ \mathcal{B}_{\alpha} \mathbb{F} = 0 & \text{in } \Pi, \\ \mathbb{F} = 0 & \text{on } D \cup B \\ \mathbb{F} \cdot \boldsymbol{\tau} = 0 & \text{on } N \\ \mathbb{G} = \rho^{\alpha} \tilde{h} & \text{on } N \end{cases} \quad (23)$$

where

$$\mathcal{A}_{\alpha} \mathbb{F} := -\Delta \mathbb{F} - 2\rho^{\alpha} \nabla \mathbb{F} \cdot \nabla \frac{1}{\rho^{\alpha}} - \rho^{\alpha} \Delta \frac{1}{\rho^{\alpha}} \mathbb{F}, \quad \text{and} \quad \mathcal{B}_{\alpha} \mathbb{F} := \operatorname{div} \mathbb{F} + \rho^{\alpha} \nabla \left( \frac{1}{\rho^{\alpha}} \right) \cdot \mathbb{F}. \quad (24)$$

Note that the rhs in (23) belongs to  $\mathbf{W}_0^{-1,2}(\Pi)$  and the boundary data to  $W_0^{-\frac{1}{2},2}(N)$ . The well-posedness of problem (23) is equivalent to two conditions (Theorem A.56 p. 474 [12]):

- (i)  $\tilde{P} \mathcal{A}_{\alpha} : \ker(\mathcal{B}_{\alpha}) \rightarrow \ker(\mathcal{B}_{\alpha})'$  is an isomorphism
- (ii)  $\mathcal{B}_{\alpha} : \dot{\mathbf{W}}_0^{1,2}(\Pi) \rightarrow W_0^{0,2}(\Pi)$  is surjective

where  $\tilde{P}$  is the restriction of  $\mathcal{A}_{\alpha}$  to the kernel of  $\mathcal{B}_{\alpha}$  and  $\dot{\mathbf{W}}_0^{1,2}(\Pi) \equiv X_0$ . Here we prove that these conditions are actually fulfilled.

**Proposition 3.1.** *If  $\alpha < 1$  then  $\mathcal{A}_{\alpha}, \mathcal{B}_{\alpha}$  satisfy conditions (i) and (ii) above*

*Proof.* We prove at first that condition (i) is satisfied. Expanding (24) one has:

$$\mathcal{A}_{\alpha} \mathbb{F} = -\Delta \mathbb{F} + \frac{2\alpha}{\rho^2} \nabla \mathbb{F} \cdot \nabla y + \alpha \left( \frac{2}{\rho^2} - \frac{(\alpha+2)|y|^2}{\rho^4} \right) \mathbb{F}$$

If  $\mathbb{F} \in \dot{\mathbf{W}}_0^{1,2}(\Pi) \cap \ker(\mathcal{B}_{\alpha})$  then on the boundary  $N$ , one has necessarily that

$$\left. \begin{array}{l} \mathbb{F}_2 = 0 \implies \partial_{y_2} \mathbb{F}_2 = 0 \text{ on } N \\ \mathcal{B}_{\alpha} \mathbb{F} = \operatorname{div} \mathbb{F} - \frac{\nabla \rho^{\alpha}}{\rho^{\alpha}} \cdot \mathbb{F} = 0 \end{array} \right\} \implies \partial_{y_1} \mathbb{F} = \frac{\nabla \rho^{\alpha}}{\rho^{\alpha}} \begin{pmatrix} \mathbb{F}_1 \\ 0 \end{pmatrix} = \frac{y}{\rho^{\alpha}} \cdot \begin{pmatrix} \mathbb{F}_1 \\ 0 \end{pmatrix} = 0$$

thus  $\partial_{\mathbf{n}}\mathbb{F} = 0$  on  $N$ , whereas  $\mathbb{F} = 0$  on  $D$ . Thanks to that, for every vector  $\mathbb{F}$  in  $\ker(\mathcal{B}_\alpha)$ , one has

$$\begin{aligned} (\mathcal{A}_\alpha\mathbb{F}, \mathbb{F})_\Pi &= \left( -\partial_{\mathbf{n}}\mathbb{F} + \alpha \frac{(y \cdot \mathbf{n})}{\rho^2} \mathbb{F}, \mathbb{F} \right)_{\partial\Pi} + |\nabla\mathbb{F}|_{L^2(\Pi)}^2 \\ &\quad - \alpha \left( \operatorname{div} \left( \frac{y}{\rho^2} \right), |\mathbb{F}|^2 \right)_\Pi + \alpha \left( \left( \frac{2}{\rho^2} - \frac{(\alpha+2)|y|^2}{\rho^4} \right) \mathbb{F}^2 \right)_\Pi \\ &= |\nabla\mathbb{F}|_{L^2(\Pi)}^2 - \alpha \left( \left( \frac{2}{\rho^2} - \frac{2|y|^2}{\rho^4} \right) \mathbb{F}^2 \right)_\Pi + \alpha \left( \left( \frac{2}{\rho^2} - \frac{(\alpha+2)|y|^2}{\rho^4} \right) \mathbb{F}^2 \right)_\Pi \\ &= |\nabla\mathbb{F}|_{L^2(\Pi)}^2 - \alpha^2 \left( \frac{|y|^2}{\rho^4} \mathbb{F}^2 \right)_\Pi \end{aligned}$$

note that the boundary term on the first line above vanishes also because  $(y \cdot \mathbf{n}) = 0$  on  $N$ , although  $\mathbb{F}_1 \neq 0$ . We use optimal Poincaré-Wirtinger estimates already presented in the proof of Theorem 5.3 p. 20 in [24]:

$$\int_\Pi \left| \frac{\mathbb{F}|y|}{\rho^2} \right|^2 dy \leq \left| \frac{\mathbb{F}}{\rho} \right|_{L^2(\Pi)}^2 \leq |\nabla\mathbb{F}|_{L^2(\Pi)}^2$$

Finally one has

$$(\mathcal{A}_\alpha\mathbb{F}, \mathbb{F})_\Pi \geq (1 - \alpha^2) |\nabla\mathbb{F}|_{L^2(\Pi)}^2$$

which implies coercivity of the operator if  $|\alpha| < 1$ . Note that this result (also valid in the scalar case) improves Lemma 4.3 in [5]. This is essentially due to the integration by parts performed on the term  $(\nabla\mathbb{F}y/\rho^2, \mathbb{F})_\Pi$  which avoids estimating this term in a direct way.

We focus on the condition (ii). For all  $q \in W_0^{0,2}(\Pi)$  we look for  $\mathbb{F} \in X_0$  s.t.

$$\mathcal{B}_\alpha\mathbb{F} = q, \text{ and } \|\mathbb{F}\|_{X_0} \leq k \|q\|_{W_0^{0,2}(\Pi)}$$

but this is equivalent to solve

$$\operatorname{div} \left( \frac{\mathbb{F}}{\rho^\alpha} \right) = \frac{q}{\rho^\alpha}$$

Then if  $q \in W_0^{0,2}(\Pi)$  then  $q/\rho^\alpha \in W_\alpha^{0,2}(\Pi)$  and by Proposition 8 there exists  $\mathbf{v} \in X_\alpha$  such that

$$\operatorname{div} \mathbf{v} = \frac{q}{\rho^\alpha} \text{ and } |\mathbf{v}|_{\mathbf{W}_\alpha^{1,2}(\Pi)} \leq k(C_1, \alpha) \left| \frac{q}{\rho^\alpha} \right|_{W_\alpha^{0,2}(\Pi)}$$

Set  $\mathbb{F} := \rho^\alpha \mathbf{v}$  thanks to the isomorphism between  $\mathbf{W}_\alpha^{1,2}(\Pi)$  and  $\mathbf{W}_0^{1,2}(\Pi)$  there exists a constant s.t.

$$|\mathbb{F}|_{\mathbf{W}_0^{1,2}(\Pi)} \leq k_1 |\mathbf{v}|_{\mathbf{W}_\alpha^{1,2}(\Pi)} \leq k_1 k(C_1, \alpha) \left| \frac{q}{\rho^\alpha} \right|_{W_\alpha^{0,2}(\Pi)} = k' \|q\|_{W_0^{0,2}(\Pi)}$$

□

*Proof of Theorem 3.1.* Thanks to the equivalence between well-posedness and conditions (i) and (ii) one concludes the existence and uniqueness of a pair  $(\mathbb{F}, \mathbb{G})$  solving problem (23). Moreover one has the *a priori* estimates :

$$\|\mathbb{F}\|_{\mathbf{W}_0^{1,2}(\Pi)} + \|\mathbb{G}\|_{W_0^{0,2}(\Pi)} \leq k' (\|\rho^\alpha \Delta(\mathcal{R}(\mathbf{w})) + \mathcal{S}(\mathbf{w})\|_{\mathbf{W}_0^{-1,2}(\Pi)} + \|\rho^\alpha \tilde{h}\|_{W_0^{-\frac{1}{2},2}(N)})$$

they are obtained similarly to those of Theorem 2.34 p. 100 in [12]. The isomorphism between weighted spaces mentioned above and the equivalence of problems (23) and (22) gives existence and uniqueness of  $(\tilde{\mathbf{w}}, \theta)$  solving problem (22) and *a priori* estimates

$$\|\tilde{\mathbf{w}}\|_{\mathbf{W}_\alpha^{1,2}(\Pi)} + \|\theta\|_{W_\alpha^{0,2}(\Pi)} \leq k'' \left\{ \|\Delta(\mathcal{R}(\mathbf{w})) + \mathcal{S}(\mathbf{w})\|_{\mathbf{W}_\alpha^{-1,2}(\Pi)} + \|\tilde{h}\|_{W_\alpha^{-\frac{1}{2},2}(N)} \right\}$$

where the negative Sobolev spaces are defined as usually in the weighted Sobolev context [16] :  $\mathbf{W}_\alpha^{-1,2}(\Pi) = (\mathbf{W}_{-\alpha}^{1,2}(\Pi))'$ . This gives existence and uniqueness of  $(\mathbf{w}, \theta)$  and due to the continuity of the lifts  $\mathcal{R}(\mathbf{w})$  and  $\mathcal{S}(\mathbf{w})$  with respect to the data, one easily proves that

$$\|\mathbf{w}\|_{\mathbf{W}_\alpha^{1,2}(\Pi)} + \|\theta\|_{W_\alpha^{0,2}(\Pi)} \leq k''' \left\{ \|\mathbf{f}\|_{\mathbf{W}_\alpha^{\frac{1}{2},2}(D \cup N)} + \|h\|_{W_{-\alpha}^{-\frac{1}{2},2}(N)} \right\}$$

which ends the proof  $\square$

*Proof of Theorem 2.2.* For every pair  $(\mathbf{w}_i, \theta)$  the boundary data is decreasing exponentially with respect to  $|y_2|$  on  $D \cup N$ . The data can be discontinuous near the origin, but this occurs only in the horizontal component which does not need to be lifted up to  $N$ , so that one could smoothly extend it in  $N$  keeping only the first component in  $D$  without changing the problem. This implies that these data belong to  $W_\alpha^{\frac{1}{2},2}(D \cup N)$  for the Dirichlet part and  $W_{-\alpha}^{-\frac{1}{2},2}(N)$  for the pressure part, for every  $\alpha \in \mathbb{R}$ . Thus we fulfill hypotheses of Theorem 3.1 and the result follows.  $\square$

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