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# The Veldkamp Space of the Smallest Slim Dense Near Hexagon

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## Abstract

We give a detailed description of the Veldkamp space of the smallest slim dense near hexagon. This space is isomorphic to  $PG(7, 2)$  and its  $2^8 - 1 = 255$  Veldkamp points (that is, geometric hyperplanes of the near hexagon) fall into five distinct classes, each of which is uniquely characterized by the number of points/lines as well as by a sequence of the cardinalities of points of given orders and/or that of (grid-)quads of given types. For each type we also give its weight, stabilizer group within the full automorphism group of the near hexagon and the total number of copies. The totality of  $(255 \text{ choose } 2)/3 = 10795$  Veldkamp lines split into 41 different types. We give a complete classification of them in terms of the properties of their cores (i. e., subconfigurations of points and lines common to all the three hyperplanes comprising a given Veldkamp line) and the types of the hyperplanes they are composed of. These findings may lend themselves into important physical applications, especially in view of recent emergence of a variety of closely related finite geometrical concepts linking quantum information with black holes.

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**Keywords:** Near Hexagons – Veldkamp Spaces – Quantum Information – Stringy Black Holes

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## 1 Introduction

It has only relatively recently been recognized that the properties of certain finite groups and the structure of certain finite geometries/point-line incidence structures are tied very closely to each other. There exists, in particular, a large family of groups relevant for physics where the (non)commutativity of two distinct elements can be expressed in the language of finite symplectic polar spaces (the corresponding points (not) being joined by an isotropic line; see, e. g., [1]–[14]) and/or finite projective ring lines (the corresponding unimodular vectors (not) lying on the same free cyclic submodule; see, e. g., [15]–[26]). Most recently, invoking also some finite generalized polygons, this link has been employed in [27, 28] to shed a novel light on and get deeper insights into the so-called black hole analogy — a still puzzling formal relation between the entropy of certain stringy black holes and the entanglement properties of some small-level quantum systems (for a recent comprehensive review, see [29]). A concept that played a crucial role in the latter discovery turned out to be that of the Veldkamp space of a point-line incidence structure [30, 31].

The point-line geometry currently central to understanding some particular aspects of the black hole analogy is the unique generalized quadrangle of order  $(2, 4)$ ,  $\text{GQ}(2, 4)$  [28]. Its Veldkamp space is isomorphic to  $\text{PG}(5, 2)$  [30], which is the natural embedding space not only for  $\text{GQ}(2, 4)$  itself, but also for the split Cayley hexagon of order two and the Klein quadric — other two prominent finite geometries linking quantum information with black holes [27]. The geometry  $\text{GQ}(2, 4)$  contains several notable subgeometries, one of them being the unique generalized quadrangle of order two,  $\text{GQ}(2, 2)$ . This quadrangle and its associated Veldkamp space ( $\simeq \text{PG}(4, 2)$ ) were found to underlie the commutation relations between the elements of two-qubit Pauli group and to also clarify some conceptual issues of quantum mechanics [20, 24, 25]. Based on these observations, Vrana and Lévy [31] have even discovered a whole family of Veldkamp spaces, which are associated with the Pauli groups of qubits of arbitrary multiplicity.

To know the structure of a Veldkamp space is vital not only for its own sake, but also because this space fully encodes information about the point-line incidence geometry it is associated with (or generated by). In this paper, with our interest mainly stirred by the above-described remarkable physical applications, we shall focus on the Veldkamp space of the point-line incidence structure that is known under several names: the smallest slim dense near hexagon [32, 33], the  $27_3$  Gray configuration [34], or, simply, a  $(3 \times 3 \times 3)$ -grid (and in what follows denoted as  $L_3^{\times 3}$ ). Although  $L_3^{\times 3}$  may seem to be a rather trivial geometry, its Veldkamp space is endowed with a rich and complex structure well worth deserving a closer look at.

## 2 Near Polygons, Quads, Geometric Hyperplanes and Veldkamp Spaces

In this section we gather all the basic notions and well-established theoretical results that will be needed in the sequel.

A *near polygon* (see, e.g., [33] and references therein) is a connected partial linear space  $S = (P, L, I)$ ,  $I \subset P \times L$ , with the property that given a point  $x$  and a line  $L$ , there always exists a unique point on  $L$  nearest to  $x$ . (Here distances are measured in the point graph, or collinearity graph of the geometry.) If the maximal distance between two points of  $S$  is equal to  $d$ , then the near polygon is called a near  $2d$ -gon. A near 0-gon is a point and a near 2-gon is a line; the class of near quadrangles coincides with the class of generalized quadrangles.

A nonempty set  $X$  of points in a near polygon  $S = (P, L, I)$  is called a subspace if every line meeting  $X$  in at least two points is completely contained in  $X$ . A subspace  $X$  is called geodetically closed if every point on a shortest path between two points of  $X$  is contained in  $X$ . Given a subspace  $X$ , one can define a sub-geometry  $S_X$  of  $S$  by considering only those points and lines of  $S$  which are completely contained in  $X$ . If  $X$  is geodetically closed, then  $S_X$  clearly is a sub-near-polygon of  $S$ . If a geodetically closed sub-near-polygon  $S_X$  is a non-degenerate generalized quadrangle, then  $X$  (and often also  $S_X$ ) is called a *quad*.

A near polygon is said to have order  $(s, t)$  if every line is incident with precisely  $s + 1$  points and if every point is on precisely  $t + 1$  lines. If  $s = t$ , then the near polygon is said to have order  $s$ . A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance two have at least two common neighbours. A near polygon is called *slim* if every line is incident with precisely three points. It is well known (see, e.g., [35]) that there are, up to isomorphism, three slim non-degenerate generalized quadrangles. The  $(3 \times 3)$ -grid is the unique generalized quadrangle of order  $(2, 1)$ ,  $\text{GQ}(2, 1)$ . The unique generalized quadrangle of order 2,  $\text{GQ}(2, 2)$ , is the generalized quadrangle of the points and lines of  $\text{PG}(3, 2)$  which are totally isotropic with respect to a given symplectic polarity. The points and lines lying on a given nonsingular elliptic quadric of  $\text{PG}(5, 2)$  define the unique generalized quadrangle of order  $(2, 4)$ ,  $\text{GQ}(2, 4)$ . Any *slim dense* near polygon contains quads, which are necessarily isomorphic to either  $\text{GQ}(2, 1)$ ,  $\text{GQ}(2, 2)$  or  $\text{GQ}(2, 4)$ .

Next, a *geometric hyperplane* of a partial linear space is a proper subspace meeting each line (necessarily in a unique point or the whole line). The set of points at non-maximal distance from a given point  $x$  of a dense near polygon  $S$  is a hyperplane of  $S$ , usually called the *singular* hyperplane with *deepest* point  $x$ . Given a hyperplane  $H$  of  $S$ , one defines the *order* of any of

its points as the number of lines through the point which are fully contained in  $H$ ; a point of a hyperplane/sub-configuration is called *deep* if all the lines passing through it are fully contained in the hyperplane/sub-configuration. If  $H$  is a hyperplane of a dense near polygon  $S$  and if  $Q$  is a quad of  $S$ , then precisely one of the following possibilities occurs: (1)  $Q \subseteq H$ ; (2)  $Q \cap H = x^\perp \cap Q$  for some point  $x$  of  $Q$ ; (3)  $Q \cap H$  is a sub-quadrangle of  $Q$ ; and (4)  $Q \cap H$  is an ovoid of  $Q$ . If case (1), case (2), case (3), or case (4) occurs, then  $Q$  is called, respectively, *deep*, *singular*, *sub-quadrangular*, or *ovoidal* with respect to  $H$ . If  $S$  is slim and  $H_1$  and  $H_2$  are its two distinct hyperplanes, then the complement of symmetric difference of  $H_1$  and  $H_2$ ,  $\overline{H_1 \Delta H_2}$ , is again a hyperplane; this means that the totality of hyperplanes of a slim near polygon form a vector space over the Galois field with two elements,  $\text{GF}(2)$ . In what follows, we shall put  $\overline{H_1 \Delta H_2} \equiv H_1 \oplus H_2$  and call it the (Veldkamp) sum of the two hyperplanes.

Finally, we shall introduce the notion of the *Veldkamp space* of a point-line incidence geometry  $\Gamma(P, L)$ ,  $\mathcal{V}(\Gamma)$  [36]. Here,  $\mathcal{V}(\Gamma)$  is the space in which (i) a point is a geometric hyperplane of  $\Gamma$  and (ii) a line is the collection  $H'H''$  of all geometric hyperplanes  $H$  of  $\Gamma$  such that  $H' \cap H'' = H' \cap H = H'' \cap H$  or  $H = H', H''$ , where  $H'$  and  $H''$  are distinct points of  $\mathcal{V}(\Gamma)$ . Following [20, 30], we adopt also here the definition of Veldkamp space given by Buekenhout and Cohen [36] instead of that of Shult [37], as the latter is much too restrictive by requiring any three distinct hyperplanes  $H', H''$  and  $H'''$  of  $\Gamma$  to satisfy the following condition:  $H' \cap H'' \subseteq H'''$  implies  $H' \subset H'''$  or  $H' \cap H'' = H' \cap H'''$ .

### 3 Veldkamp Space of $L_3^{\times 3}$

#### 3.1 Geometric Hyperplanes

Our point-line geometry  $L_3^{\times 3}$  consists of 27 points and the same number of lines, with three points on a line and, dually, three lines through a point. Its full group of automorphisms  $G$  is isomorphic to  $S_3 \wr S_3$ , of order 1296 [32]. As already mentioned, apart from being the smallest slim dense near hexagon, it is also one of a pair of dual to each other  $27_3$  configurations whose incidence graph is the *Gray* graph [34] — the smallest cubic graph which is edge-transitive and regular, but not vertex-transitive [38]. Being a slim geometry, its geometric hyperplanes are readily found by employing the above-mentioned property that the complement of symmetric difference of any two of its hyperplanes is again a hyperplane.

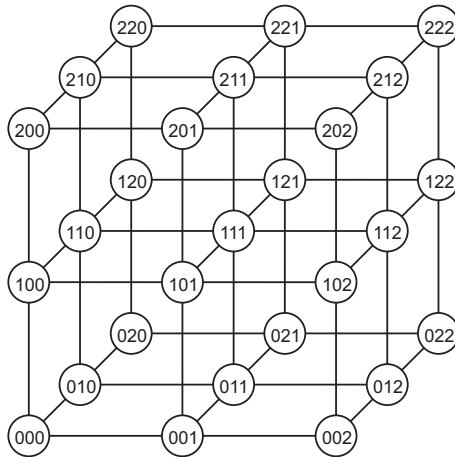


Figure 1: A diagrammatic illustration of the structure of  $L_3^{\times 3}$ . The points are represented by circles and the lines by straight segments joining triples of points. Also shown is a ternary digits labelling of the points, employed in the sequel.

To this end, we shall use a diagrammatical representation of our geometry as a  $(3 \times 3 \times 3)$ -grid, Figure 1. From this representation we find that all its quads are  $\text{GQ}(2, 1)$ s, and there are altogether nine of them. Its singular hyperplane,  $H_1$ , is endowed with 19 points and 15 lines,

with 12 points of order two and seven of order three, and three deep and six singular quads — as easily discerned from Figure 2-1. To find the stabilizer group of this type of geometric hyperplane, one thinks of  $L_3^{\times 3}$  as a collection of triples of ternary digits (Figure 1). We can act on the first, second and third digit independently by permuting by  $S_3$ . Finally, we can permute the digits themselves by another  $S_3$ . The stabilizer of an  $H_1$  is the same as the stabilizer of its deepest point, i. e.  $Z_2 \wr S_3$ , of order 48; taking this point to be 000, the  $Z_2$  decides whether or not to exchange 1 and 2 in each position and the  $S_3$  permutes the positions. There are altogether  $|G|/|Z_2 \wr S_3| = 27$  copies of  $H_1$  in  $L_3^{\times 3}$ .

Next, pick up two distinct copies  $H_1(x)$  and  $H_1(y)$  of  $H_1$ , where  $x$  and  $y$  are the corresponding deepest points. If  $x$  and  $y$  are at distance two from each other, then  $H_1(x) \oplus H_1(y) \equiv H_2$  is a new type of hyperplane, not isomorphic to  $H_1$ . As depicted in Figure 2-2, an  $H_2$  comprises 15 points and nine lines, with six points of order one and two, and three deep points; it features one deep, six singular and two ovoidal quads. Its stabilizer is  $Z_2 \times S_3 \times Z_2$ , of order 24. The first  $Z_2$  switches the two planes/quads parallel to the included one (0ab, say). We also need to permute the points 000, 011 and 022 among themselves; this means that we can do one permutation ( $S_3$ ) affecting the second and third coordinates simultaneously, and also swap the second and third coordinates (that is the second  $Z_2$ ). We find altogether 54 copies of  $H_2$  in  $L_3^{\times 3}$ .

If the two deepest points are at maximum distance from each other, then  $H_1(x) \oplus H_1(y) \equiv H_3$  is again a new kind of hyperplane, not isomorphic to the previous two — see Figure 2-3. An  $H_3$  comprises 13 points and six lines, with one point of order zero (henceforth called the nucleus), six points of orders one and two, and no deep points. There are no deep quads there, only six singular and three ovoidal ones. The stabilizer is  $D_{12}$ . For if one looks at the copy of  $H_3$  shown in Figure 2-3 along the appropriate axis, it looks like a regular hexagon with a dot (= the nucleus) in the middle; from this perspective, the result is obvious. Altogether 108 copies of  $H_3$  sit inside  $L_3^{\times 3}$ .

There exist two more types of geometric hyperplanes. A hyperplane of type  $H_4$ , Figure 2-4, features 11 points and three lines, four points of order zero, six of order one and one deep point, and, like the previous type, it exhibits only singular (three) and ovoidal (six) quads. It is stabilized by  $S_4$ . This can be seen from Figure 2-4, as the four points of order zero form the vertices of a regular tetrahedron and the symmetry group of our  $H_4$  is that of the tetrahedron (including reflections). One can get a copy of  $H_4$  in the following manner. Take a copy of  $H_2$ ; we already know that any  $H_2$  is the complement of the symmetric difference of two  $H_1$ s whose deeps are at distance two from each other. An  $H_4$  is then the complement of the symmetric difference of the  $H_2$  and the copy of  $H_1$  whose deep is at distance three from each of the previous two. Hence, one gets an  $H_4$  as the sum of *three*  $H_1$ s two of which deeps are at distance two from each other and the third one being at the maximum distance from the two. The cardinality of  $H_4$  is 54.

The final type,  $H_5$ , is an ovoid of  $L_3^{\times 3}$ , that is a set of nine pairwise non-collinear points (Figure 2-5). Hence, in this case all the points are of zeroth order and all the quads are ovoidal. The stabilizer group is  $E \rtimes S_3$ , where  $E \simeq (Z_3 \times Z_3) \rtimes Z_2$ , and the  $Z_2$  acts as inversion. There is a nice description of this using the ternary digits method. Take all coordinates whose digits sum to zero mod 3. The  $S_3$  just permutes the coordinates. The group  $E$  has order 18 and is generated by an  $S_3$  and an element  $g$  of order 3. The element  $g$  acts by increasing (respectively, decreasing) the second (respectively, third) digit by 1 mod 3. The  $S_3$  acts as the same permutation (value permutation, not place permutation) on each digit simultaneously. To get a copy of  $H_5$ , one selects an  $H_3$ , any of which is the complement of the symmetric difference of two  $H_1$ s whose deeps are at maximum distance from each other. A copy of  $H_5$  is then found as the complement of the symmetric difference of the  $H_3$  and the copy of  $H_1$  whose deep is at maximum distance from either of the two. Hence, one gets an  $H_5$  as the sum of *three*  $H_1$ s whose deeps are pairwise at maximum distance from each other (compare with the preceding case). There are only 12 copies of  $H_5$  inside  $L_3^{\times 3}$ .

The attentive reader might have noticed that  $H_1$  plays a special role amongst the hyperplanes, because each of the remaining types can be obtained as the sum of several copies of  $H_1$ ; the smallest such number is called the *weight* of a hyperplane. From what we said above it follows that  $H_2$  and  $H_3$  are of weight two, whereas  $H_4$  and  $H_5$  have weight three. In this respect,  $L_3^{\times 3}$

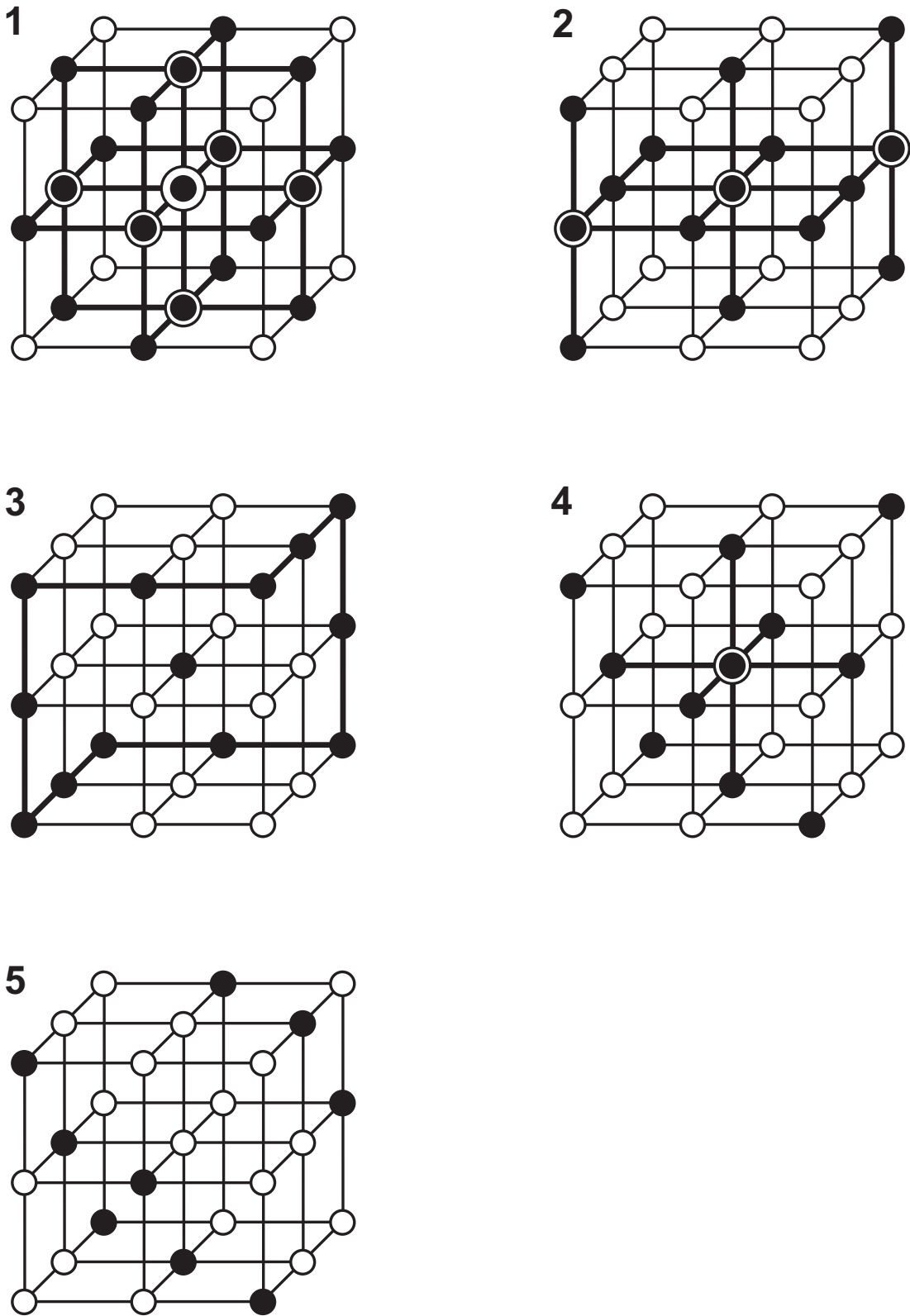


Figure 2: A diagrammatic illustration of the five distinct types of geometric hyperplanes of the smallest slim dense near hexagon. The points/lines belonging to a hyperplane are boldfaced; double circles stand for deep points.

Table 1: The types of geometric hyperplanes of the smallest slim dense near-hexagon.

Hp	Pts	Lns	# of Points of Order				# of Quads of Type			StGr	Wgt	Crd
			0	1	2	3	deep	sng	ovd			
$H_1$	19	15	0	0	12	7	3	6	0	$Z_2 \wr S_3$	1	27
$H_2$	15	9	0	6	6	3	1	6	2	$Z_2^{\times 2} \times S_3$	2	54
$H_3$	13	6	1	6	6	0	0	6	3	$D_{12}$	2	108
$H_4$	11	3	4	6	0	1	0	3	6	$S_4$	3	54
$H_5$	9	0	9	0	0	0	0	0	9	$E \rtimes S_3$	3	12

resembles the *dual* of the split Cayley hexagon of order two [39], save for the fact that the latter features many more types of geometric hyperplanes.

All our findings are summarized in Table 1. We see that the total number of hyperplanes is  $27 + 54 + 108 + 54 + 12 = 255$ . As we already know that hyperplanes from a  $\text{GF}(2)$ -vector space and  $255 = 2^8 - 1 = |\text{PG}(7,2)|$ , one is immediately tempted to conjecture that  $\mathcal{V}(L_3^{\times 3})$  is isomorphic to  $\text{PG}(7,2)$ . In the next section we shall show that this is indeed so.

### 3.2 Veldkamp Lines

As each Veldkamp line is obviously of the form  $\{H', H'', H' \oplus H''\}$ , and since  $L_3^{\times 3}$  is rather small, diagrams of the form given in Figure 2 were easy to use to find out “by hand” all 41 different types of Veldkamp lines, whose basic properties are listed in Table 2. A rather coarse classification in terms of the composition is refined by that of the properties of the cores. To classify each type unambiguously, we have to further refine, in the relevant cases, the core line cardinality entries by stating whether the lines of the core are concurrent (“c”) or parallel/mutually skew (“p”) and specify the core point cardinality entries even in a greater variety of ways, as follows:

- $4_{(3;1)}$  means that one of the points is at maximum distance from the other three, in contrast to  $4_{(2;2)}$  where two points are such that either of them is at maximum distance from the remaining three (to distinguish type 35 from type 36);
- $7_{(2/3)}$  or  $9_{(2/3)}$  stand for the fact that the two isolated (= zeroth order) points of the core are at distance  $2/3$  from each other (to distinguish type 16 from type 17);
- $X_{[n]}$  illustrates the fact that there exist exactly  $n$  quads each of which cuts the core in an ovoid, i. e. a triple of pairwise non-collinear points (to distinguish types 25 to 28 from one another); and, finally,
- $5_{(n)}$  tells us that out of five points there are  $n$  such that each is coplanar (that is, shares a quad) with any of the remaining four.

Table 2 reveals a number of interesting properties of the Veldkamp lines. First, we notice that there is only one type (41) whose core is an empty set; all the three Veldkamp points of this line are of the same kind, namely  $H_5$ . Next, there are nine different types of Veldkamp lines each of which consists of the hyperplanes of the same type; these are types 1 ( $H_1$ ), 6, 10, 11 ( $H_2$ ), 23, 24, 28 ( $H_3$ ), 37 ( $H_4$ ) and 41 ( $H_5$ ). On the other hand, we find nine (out of theoretically possible  $10 = \binom{3}{3}$ ) types featuring three different kinds of hyperplanes; these are types 5, 8, 13, 20, 22, 25, 27, 31 and 39. Interestingly, the only nonexistent “heterogeneous” combination is the  $H_1$ - $H_2$ - $H_5$  one. The most “abundant” type of Veldkamp points is  $H_3$ , which occurs in 23 types of lines, followed by  $H_2$  (20),  $H_4$  (17),  $H_1$  (13) and finally by  $H_5$  (9). Remarkably,  $H_3$  also prevails in multiplicity  $\geq 2$  (12 types), whereas in “singles” the primacy belongs to  $H_2$  (13 types). It is also worth mentioning that there are as many as four different types of the form  $H_2$ - $H_3$ - $H_3$  (15, 16, 17 and 19). All these observations are gathered in Table 3.

Table 2: The types of Veldkamp lines of the Veldkamp space of the smallest slim dense near-hexagon. The details of the symbols/notation are explained in the text.

Type	Core		Composition					Crđ
	Pts	Lns	$H_1$	$H_2$	$H_3$	$H_4$	$H_5$	
1	15	11	3	–	–	–	–	27
2	13	8	2	1	–	–	–	162
3	12	6	2	–	1	–	–	108
4	11	7	1	2	–	–	–	81
5	10	4	1	1	1	–	–	648
6	9	6	–	3	–	–	–	18
7	9	4	1	–	2	–	–	324
8	9 <sub>(2)</sub>	3c	1	1	–	1	–	324
9	9	3	1	–	2	–	–	324
10	9	3p	–	3	–	–	–	18
11	9 <sub>(3)</sub>	3c	–	3	–	–	–	108
12	8	3	–	2	1	–	–	648
13	8	2	1	–	1	1	–	648
14	7	3	1	–	–	2	–	27
15	7	2p	–	1	2	–	–	162
16	7 <sub>(2)</sub>	2c	–	1	2	–	–	324
17	7 <sub>(3)</sub>	2c	–	1	2	–	–	324
18	7 <sub>[2]</sub>	1	–	2	–	1	–	162
19	7 <sub>[1]</sub>	1	–	1	2	–	–	324
20	7	0	1	–	1	–	1	108
21	7	0	1	–	–	2	–	108
22	6	2c	–	1	1	1	–	648
23	6	2p	–	–	3	–	–	108
24	6	1	–	–	3	–	–	648
25	6 <sub>[3]</sub>	0	1	–	–	1	1	216
26	6 <sub>[2]</sub>	0	–	2	–	–	1	108
27	6 <sub>[1]</sub>	0	–	1	1	1	–	648
28	6 <sub>[0]</sub>	0	–	–	3	–	–	36
29	5 <sub>[1]</sub>	1	–	1	–	2	–	162
30	5 <sub>[0]</sub>	1	–	–	2	1	–	648
31	5 <sub>(2)</sub>	0	–	1	1	–	1	324
32	5 <sub>(1)</sub>	0	–	1	–	2	–	324
33	5 <sub>(0)</sub>	0	–	–	2	1	–	648
34	4	0	–	–	2	–	1	324
35	4 <sub>(3:1)</sub>	0	–	–	1	2	–	216
36	4 <sub>(2:2)</sub>	0	–	–	1	2	–	324
37	3	1	–	–	–	3	–	54
38	3 <sub>[1]</sub>	0	–	1	–	–	2	54
39	3 <sub>[0]</sub>	0	–	–	1	1	1	216
40	2	0	–	–	–	2	1	108
41	0	0	–	–	–	–	3	4

Table 3: A succinct overview of the composition of the Veldkamp lines; here, for example, number 6 in the row  $H_2$  and the column  $H_4$  means that there are 6 different types of lines featuring both  $H_2$  and  $H_4$ .

	$H_1$	$H_2$	$H_3$	$H_4$	$H_5$
$H_1$	3	4	6	5	2
$H_2$		7	9	6	3
$H_3$			12	8	4
$H_4$				8	3
$H_5$					2

Table 4: The set of representatives, after discarding the relation of equality, of double cosets of the group  $G$  of automorphisms of the near hexagon with respect to the stabilizer groups of all five different kinds of hyperplanes. By abuse of notation, the stabilizer group of a hyperplane is denoted by the same symbol as the hyperplane itself. Thus, for example, the entry 6 in the row  $H_2$  and the column  $H_4$  should read that there are 6 distinct double cosets  $H_2aH_4$ ,  $a \in G$ .

	$H_1$	$H_2$	$H_3$	$H_4$	$H_5$
$H_1$	3	4	6	5	2
$H_2$		7	9	6	3
$H_3$			15	9	4
$H_4$				9	3
$H_5$					2

The structure of Table 3 can easily be demystified if one looks at a decomposition of the group of automorphisms into double cosets with respect to stabilizer (sub-)groups of hyperplanes. Let us recall (see, e. g., [40]) that given a group  $G$  and its two (not necessarily distinct) subgroups  $K_1$  and  $K_2$ , the set

$$K_1aK_2 \equiv \{k_1ak_2 \in G | a \in G, k_1 \in K_1, k_2 \in K_2\}$$

is called a double coset with respect to  $K_1$  and  $K_2$ . So, for  $G \simeq S_3 \wr S_3$  and with  $K_i$ ,  $i = 1, 2$ , identified with the stabilizers of five different kinds of hyperplanes, one finds — using, for example, a GAP code [41] — the number of representatives of double cosets for different combinations of hyperplanes as given in Table 4. Comparing the last two tables one sees that they are *almost* identical; the only discrepancies occur in the  $H_3$ - $H_3$ ,  $H_3$ - $H_4$  and  $H_4$ - $H_4$  entries, where we always find more double coset representatives than line types. This means that there are particular Veldkamp line types that correspond to more than one double coset. And, indeed, after some work one finds that types 33 and 35 correspond each to two double coset representatives, whereas type 24 corresponds to as many as three. To illustrate the origin of this puzzling “multi-valuedness” property, we shall consider the last mentioned case.

We first introduce some more terminology. Given a copy of  $H_3$  and disregarding its nucleus, the 12 remaining points are such that there exist exactly two distinct points coplanar with all of them. Let us call this pair of points the “axis” of the  $H_3$ . Next, let us select three distinct copies  $A$ ,  $B$ , and  $C$  of  $H_3$  as follows (employing again our ternary digits representation of the points of the near hexagon):  $A$  is the  $H_3$  with nucleus 111 and axis  $\{000, 222\}$ ,  $B$  is the  $H_3$  with nucleus 020 and axis  $\{102, 211\}$ , and  $C$  is the  $H_3$  with nucleus 122 and axis  $\{001, 210\}$ . It is not difficult to verify that the triple  $\{A, B, C\}$ , with its core comprising the points 012, 020, 111, 200, 201 and 202, is indeed a Veldkamp line of type 24. Now, consider the six *ordered* pairs  $\{(A, B), (A, C), (B, A), (B, C), (C, A), (C, B)\}$ . The orbits of  $G$  acting on this set are  $\{(A, B), (B, A)\}$ ,  $\{(A, C), (B, C)\}$ , and  $\{(C, A), (C, B)\}$  because  $(A, C)$  is not  $G$ -conjugate to  $(C, A)$  even though  $A$  is  $G$ -conjugate to  $C$ ! This just means that the *single* type of a Veldkamp

line, that is to say the triple  $\{A, B, C\}$ , corresponds to *three* distinct double cosets; one of them is symmetric, the other two are transposes of each other.

The remaining issue is to find the cardinality for each type of Veldkamp lines and verify that their sum indeed amounts to 10 795, that is to the number of lines of  $\text{PG}(7, 2)$ . These calculations were mostly done by computer, again using a particular GAP code [41], and their results are given in the last column of Table 2. It must be stressed that types 6, 23, 37 and 41 behave in a rather peculiar way as in these cases it is impossible to fully reconstruct/recover the hyperplanes of the line from the configuration of the core and, therefore, one has to be more careful in counting here. It is also worth noting that there are only four distinct Veldkamp lines with the empty core, that cardinalities 4, 36 and 81 occur just for a single line type, and that there exist as many as 10 different types of lines of cardinality 324; the next most frequent cardinality is 648, occurring in 8 distinct types.

We shall conclude this section by the following important observation. Suppose we have a Veldkamp space over  $\text{GF}(2)$  whose associated point-line incidence geometry has an odd number of points and also all its geometric hyperplanes have an odd cardinality. Then, given any two distinct hyperplanes  $H'$  and  $H''$ , there exists on this Veldkamp space a  $G$ -invariant  $\text{GF}(2)$ -bilinear form  $B(H', H'')$  defined as follows:  $B(H', H'') = 0$  if  $|H' \cap H''|$  is odd and  $B(H', H'') = 1$  if  $|H' \cap H''|$  is even. The  $G$ -invariance of the form is immediate, and it turns out that this is the only nontrivial  $G$ -invariant bilinear form over  $\text{GF}(2)$  on the Veldkamp space. The  $\text{GF}(2)$ -bilinearity stems from the facts that: a) if  $H', H''$  and  $H'''$  form a Veldkamp line then their pairwise intersection is the same as their triple intersection and b) the total number of points is odd and so is the number of points in each hyperplane. In our case, the corresponding form is the unique symplectic form of  $\text{PG}(7, 2)$  with respect to which the totality of 10 795 Veldkamp lines split into two disjoint sets: 5 355 isotropic lines (odd core point cardinality) and 5 440 non-isotropic ones (even core point cardinality).

## 4 Summary and Conclusion

A comprehensive description of the Veldkamp space of the smallest slim dense near hexagon (*alias* the Gray configuration, or a  $(3 \times 3 \times 3)$ -grid) has been presented. Being isomorphic to  $\text{PG}(7, 2)$ , its  $2^8 - 1 = 255$  Veldkamp points fall into five distinct classes. Each class is uniquely characterized by the number of points/lines as well as by a sequence of the cardinalities of points of given orders and/or that of (grid-)quads of given types; in addition, we also provide its weight, stabilizer group within the full automorphism group of the near hexagon and the total number of copies. Similarly, the totality of  $\binom{255}{2}/3 = 10\,795$  Veldkamp lines are shown to form 41 different types. A complete classification of them, partly based on the properties of double cosets of the group of automorphisms with respect to the stabilizer groups of hyperplanes, is given in terms of the properties of their cores and the types of the hyperplanes they are composed of. Given a specific symplectic polarity in  $\text{PG}(7, 2)$ , we also show that isotropic/non-isotropic Veldkamp lines are of those types whose core point cardinality is odd/even.

As already stressed, we expect this space to have interesting physical applications. We just give an outline of a couple of arguments giving support to such expectations.

It is a well-known fact that the group  $\text{Spin}(8)$ , the double-cover of  $\text{SO}(8)$ , has exactly three irreducible real representations of degree eight and, accordingly, three representation spaces: one vectorial and two spinorial. Remarkably, the three spaces are on equal footing and there exists an extra automorphism, known as *triality*, that permutes them. This feature has already been found to play a distinguished role in certain supersymmetry and supergravity theories (see, e. g., [42]). Strikingly, it is also our Veldkamp space that is endowed with a *triality*, albeit a geometric one. To be more specific (see, e. g., [43]), let us consider a non-singular hyperbolic quadric  $Q^+(7, 2)$  in  $\text{PG}(7, 2)$ . The generators of this particular quadric are three-dimensional subspaces. The set of generators can be divided into two families  $P_1$  and  $P_2$  such that two distinct generators belong to the same family if and only if they intersect in a subspace of odd dimension (i. e., in the empty set, a line or a three-dimensional subspace (in which case they coincide)). Let  $P_0$  be the set of points of  $Q^+(7, 2)$  and call the elements of  $P_i$  the  $i$ -points,  $i \in \{0, 1, 2\}$ ; let  $L$  denote the set of lines on  $Q^+(7, 2)$ . A simple counting argument yields

$|P_0| = |P_1| = |P_2|$ . A *triatlity* of  $Q^+(7, 2)$  is a bijection  $\mu$  that maps:  $P_0 \mapsto P_1$ ,  $P_1 \mapsto P_2$ ,  $P_2 \mapsto P_0$ ,  $L \mapsto L$ , is incidence preserving and satisfies  $\mu^3 = 1$ . An absolute  $i$ -point is an  $i$ -point that is incident with its image under the triatlity  $\mu$ ,  $i \in \{0, 1, 2\}$ ; an absolute line is a line that is fixed by  $\mu$ . For certain triatlities the incidence structure formed by the absolute  $i$ -points and the absolute lines, with given incidence, is a generalized hexagon. And for a *specific* triatlity we get the *split Cayley hexagon of order two* — an important representative of finite geometry found to underlie the properties of entropy formulas of a distinguished class of stringy black holes [27]. We surmise that the geometric triatlity inside our Veldkamp space of  $L_3^{\times 3}$ ,  $\mathcal{V}(L_3^{\times 3})$ , and the algebraic one of  $Spin(8)$  are intimately connected, being simply two sides of *the same* coin.

The second argument is related to the concept of superqubits as proposed in a very recent paper by Borsten *et al.* [44]. Here, the structure of  $L_3^{\times 3}$  seems to be *directly* related to the Hilbert space of three-superqubits. This super Hilbert space is 27-dimensional and associating bijectively its dimensions with the 27 points of  $L_3^{\times 3}$  one immediately finds that its 13 fermionic dimensions should correspond to a geometric hyperplane of type  $H_3$  and its 14 bosonic ones to the complement of the hyperplane in question. Moreover, any truncation to two-superqubits would correspond to a GQ(2, 1)-quad in  $L_3^{\times 3}$ , and the associated splitting of the 9-dimensional super Hilbert sub-space into 5 boson and 4 fermion dimensions would simply correspond to a perp-set (of the GQ(2, 1)) and its complement, respectively.

Further explorations along these lines are clearly vital as they may, among other things, shed further light on the relation between quantum information theory and physics of stringy black holes outlined in the introduction. In view of these facts, proper understanding of the structure of  $\mathcal{V}(L_3^{\times 3})$  — as expounded in preceding sections — is of great importance.

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