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# MULTIPLICATIVE BIAS CORRECTED NONPARAMETRIC SMOOTHERS WITH APPLICATION TO NUCLEAR ENERGY SPECTRUM ESTIMATION

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## Abstract

The paper presents a multiplicative bias reduction estimator for nonparametric regression. The approach consists to apply a multiplicative bias correction to an over-smooth pilot estimator. We study the asymptotic properties of the resulting estimate and prove that this estimate has zero asymptotic bias and the same asymptotic variance as the local linear estimate. Simulations show that our asymptotic results are available for modest sample sizes. We also illustrate the benefit of this new method on nuclear energy spectrum estimation.

**Index terms:** Nonparametric regression, bias reduction, local linear estimate.

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# 1 Introduction

The decay of radioactive isotopes often generates gamma particles whose energy can be measured using specialized detectors. Typically, these detectors count the number of particle in various energy bins over short time intervals such as one to ten minutes. This enables estimation of the energy distribution of the emitted particles, which is called the energy spectrum. For low or medium resolution detectors, the spectrum is typically composed of multiple broad peaks whose location and area characterize the radio-isotope.

Because the actual bin counts are noisy, and the energy spectrum is fairly smooth, it has been proposed to estimate the energy spectrum using non-parametric smoothing techniques (Sullivan et al. [2006], Gang et al. [2004]). However, it is known that many classical smoothers, such as kernel-based regression smoothers,  $k$ -nearest neighbors, and smoothing splines, typically under-estimate in the peaks and over-estimate in the valleys of the regression function. See for example Simonoff [1996], Fan and Gijbels [1996], Wand and Jones [1995], Scott [1992].

This bias degrades isotope identification performance for any algorithm that includes peak area or ratios of areas (Casson et al. [2006]) and motivates studying methods to reduce bias at peaks and valleys. There are many approaches to reducing the bias, but most of them do so at the cost of an increase in the variance of the estimator. For example, one may chose to under-smooth the energy spectrum. Under-smoothing will reduce the bias but will have a tendency of generating spurious peaks. One can also use higher order smoother, such as local polynomial smoother with a polynomial of order larger than one. While again this will lead to a smaller bias, the smoother will have a larger variance. Another approach is to start with a pilot

smoother and to estimate its bias by smoothing the residuals (Cornillon et al. [2009], Di Marzio and Taylor [2008]). Subtracting the estimated bias from the smoother produces a regression smoother with smaller bias and larger variance. In the context of estimating an energy spectrum, the additive bias correction and the higher order smoothers have the unfortunate side effect of possibly generating a non-positive estimate.

An attractive alternative to the linear bias correction is the multiplicative bias correction pioneered by Linton and Nielsen [1994]. Because the multiplicative correction does not alter the sign of the regression function, this type of correction is particularly well suited for adjusting non-negative regression functions. Jones et al. [1995] showed that if the true regression function has four continuous derivatives, then the multiplicative bias reduction is operationally equivalent to using an order four kernel. And while this does remove the bias, it also increases the variance.

Although the bias-variance tradeoff for nonparametric smoothers is always present in finite samples, it is possible to construct smoothers whose *asymptotic bias* converges to zero while keeping the same asymptotic variance. Hengartner and Matzner-Løber [2009] has exhibited a nonparametric density estimator based on multiplicative bias correction with that property, and have shown in simulations that their estimator also enjoyed good finite sample properties. In this paper, we present such an estimator for nonparametric regression. We emphasize that a major difference between our work and that of Jones et al. [1995] is that we do not assume that the regression function has four continuous derivatives.

This paper is organized as follows. Section 2 introduces the notation and defines the estimator. Section 3 gives the asymptotic behavior of the proposed

estimate. A brief simulation study on finite sample comparison is presented in section 4. Finally, in section 5, the procedure is applied to estimate the energy spectrum. The interested reader is referred to the Appendix where we have gathered the technical proofs.

## 2 Preliminaries

### 2.1 Notation.

Let  $(X_1, Y_1), \dots, (X_n, Y_n)$  be  $n$  independent copies of the pair of random variables  $(X, Y)$ . We suppose that the explanatory variable  $X \in \mathbb{R}$  has probability density  $f$  and model the dependence of the univariate response variable  $Y$  to the explanatory variable  $X$  through the nonparametric regression model

$$Y = m(X) + \varepsilon. \quad (1)$$

We assume that the regression function  $m(\cdot)$  is smooth and that the disturbance  $\varepsilon$  is a mean zero random variable with finite variance  $\sigma^2$  that is independent of the covariate  $X$ . Consider the linear smoothers for the regression function  $m(x)$  which we can write as

$$\hat{m}(x) = \sum_{j=1}^n \omega_j(x; h) Y_j,$$

where the weight function  $\omega_j(x; h)$  depends on a tuning parameter  $h$ , that we think of as the bandwidth.

If the weight functions are such that  $\sum_{j=1}^n \omega_j(x; h) = 1$  and  $\sum_{j=1}^n \omega_j(x; h)^2 = (nh)^{-1} \tau^2$ , and if the disturbances satisfy the Lindberg-Feller condition, then the linear smoother obeys the central limit theorem

$$\sqrt{nh} \left( \hat{m}(x) - \sum_{j=1}^n \omega_j(x; h) m(X_j) \right) \longrightarrow \mathcal{N}(0, \tau^2). \quad (2)$$

We can use (2) to construct asymptotic pointwise confidence intervals for the unknown regression function  $m(x)$ . But unless the limit of the scaled bias

$$b(x) = \lim_{n \rightarrow \infty} \sqrt{nh} \left( \sum_{j=1}^n w_j(x; h) m(X_j) - m(x) \right),$$

which we call the asymptotic bias, is zero, the confidence interval

$$\left[ \hat{m}(x) - Z_{1-\alpha/2} \sqrt{nh} \tau, \hat{m}(x) + Z_{1-\alpha/2} \sqrt{nh} \tau \right]$$

will not cover asymptotically the true regression function  $m(x)$  at the nominal  $1 - \alpha$  level. The construction of valid pointwise  $1 - \alpha$  confidence intervals for regression smoothers is the another motivation for developing estimators with zero asymptotic bias.

## 2.2 Multiplicative bias reduction

Here we present a framework for multiplicative bias reduction. Given a pilot smoother

$$\tilde{m}_n(x) = \sum_{j=1}^n \omega_j(x; h_0) Y_j,$$

the ratio

$$V_j = \frac{Y_j}{\tilde{m}_n(X_j)}$$

is a noisy estimate of  $m(X_j)/\tilde{m}_n(X_j)$ , the inverse relative estimation error of the smoother  $\tilde{m}_n$  at each of the observations. Smoothing  $V_j$  by

$$\hat{\alpha}_n(x) = \sum_{j=1}^n \omega_j(x; h_1) V_j$$

yields an estimate for the inverse of the relative estimation error which can be used as a multiplicative correction of the pilot smoother. This leads to the (nonlinear) smoother

$$\hat{m}_n(x) = \hat{\alpha}_n(x) \tilde{m}_n(x). \tag{3}$$

The estimator (3) was studied for fixed design by Linton and Nielsen [1994] and further studied by Jones et al. [1995]. In both cases, they assumed that the regression function had four continuous derivatives, and show an improvement in the convergence rate of the corrected estimator. Glad [1998a,b] proposes to use a parametrically guided local linear smoother and Nadaraya-Watson smoother by starting with a parametric pilot. She shows that the resulting estimates improves on the local polynomial estimate as soon as the pilot captures some of the features of the regression function.

### 3 Theoretical Analysis of Multiplicative Bias Reduction

In this section, we will show that the multiplicative smoother has smaller bias with essentially no cost to the variance, assuming only two derivatives of the regression function. While the derivation of our results are for local linear smoothers, the technique used in the proofs can be easily adapted for other linear smoothers, and the conclusions remain essentially unchanged.

#### 3.1 Assumptions

We make the following assumptions:

1. The regression function is bounded and strictly positive, that is,  $b \geq m(x) \geq a > 0$  for all  $x$ .
2. The regression function is twice continuously differentiable everywhere.
3. The density of the covariate is strictly positive on the interior of its support in the sense that  $f(x) \geq b(\mathcal{K}) > 0$  over every compact  $\mathcal{K}$

contained in the support of  $f$ .

4.  $\varepsilon$  has finite fourth moments and has a symmetric distribution around zero.
5. Given a symmetric probability density  $K(\cdot)$ , consider the weights  $\omega_j(x; h)$  associated to the local linear smoother. That is, denote by  $K_h(\cdot) = K(\cdot/h)/h$  the scaled kernel by the bandwidth  $h$  and define for  $k = 0, 1, 2, 3$  the sums

$$S_k(x) \equiv S_k(x; h) = \sum_{j=1}^n (X_j - x)^k K_h(X_j - x).$$

Then

$$\omega_j(x; h) = \frac{S_2(x; h) - (X_j - x)S_1(x; h)}{S_2(x; h)S_0(x; h) - S_1^2(x; h)} K_h(X_j - x).$$

We set

$$\omega_{0j}(x) = \omega_j(x; h_0) \quad \text{and} \quad \omega_{1j}(x) = \omega_j(x; h_1).$$

6. The bandwidths  $h_0$  and  $h_1$  are such that

$$h_0 \longrightarrow 0, \quad h_1 \longrightarrow 0 \quad nh_0 \longrightarrow \infty, \quad nh_1^3 \longrightarrow \infty \quad \text{and} \quad \frac{h_1}{h_0} \longrightarrow 0.$$

## 3.2 A technical aside

The proof of Theorems (3.1) and (3.2) rests on establishing a stochastic approximation of estimator (3) in which each term can be directly analyzed.

**Proposition 3.1.** *We have*

$$\widehat{m}_n(x) = \mu_n(x) + \sum_{j=1}^n \omega_{1j}(x)A_j(x) + \sum_{j=1}^n \omega_{1j}(x)B_j(x) + \sum_{j=1}^n \omega_{1j}(x)\xi_j,$$



where  $\mu_n(x)$ , conditionally on  $X_1, \dots, X_n$  is a deterministic function,  $A_j, B_j$  and  $\xi_j$  are random variables. Under condition  $nh_0 \rightarrow \infty$ , the remainder  $\xi_j$  converges to 0 in probability and we have

$$\widehat{m}_n(x) = \mu_n(x) + \sum_{j=1}^n \omega_{1j}(x)A_j(x) + \sum_{j=1}^n \omega_{1j}(x)B_j(x) + O_P\left(\frac{1}{nh_0}\right).$$

**Remark:** A technical difficulty arises because even though  $\xi_j$  may be small in probability, its expectation may not be small. We resolve this problem by showing that only needs to modify  $\xi_j$  on a set of vanishingly small probability to guarantee that its expectation is also small.

DEFINITION Given a sequence of real numbers  $a_n$ , say that a sequence of random variables  $\xi_n = o_p(a_n)$  if for all fixed  $t > 0$ ,

$$\limsup_{n \rightarrow \infty} \mathbb{P}[|\xi_n| > ta_n] = 0.$$

We will need the following Lemma.

**Lemma 3.1.** *If  $\xi_n = o_p(a_n)$ , then there exists a sequence of random variables  $\xi_n^*$  such that*

$$\limsup_{n \rightarrow \infty} \mathbb{P}[\xi_n^* \neq \xi_n] = 0 \quad \text{and} \quad \mathbb{E}[\xi_n^*] = o(a_n).$$

We shall use the following notation

$$\mathbb{E}_\star[\xi_n] = \mathbb{E}[\xi_n^*].$$

### 3.3 Main results

We deduce from Proposition 3.1 and Lemma 3.1 the following Theorem.

**Theorem 3.1.** *Under the assumptions (1)-(6), the estimator  $\widehat{m}_n$  satisfies:*

$$\mathbb{E}_\star(\widehat{m}_n(x)|X_1, \dots, X_n) = \mu_n(x) + O_p\left(\frac{1}{n\sqrt{h_0h_1}}\right) + O_p\left(\frac{1}{nh_0}\right)$$

and

$$\mathbb{V}_\star(\widehat{m}_n(x)|X_1, \dots, X_n) = \sigma^2 \sum_{j=1}^n w_{1j}^2(x) + O_p\left(\frac{1}{nh_0}\right) + o_p\left(\frac{1}{nh_1}\right).$$

If the bandwidth  $h_0$  of the pilot estimator converges to zero much slower than  $h_1$ , then  $\widehat{m}_n$  has the same asymptotic variance as the local linear smoother of the original data with bandwidth  $h_1$ . However, for finite samples, the two step local linear smoother can have a slightly larger variance depending on the choice of  $h_0$ . A limited Taylor expansion of  $\mu_n(x)$  leads to the following result.

**Theorem 3.2.** *Under the assumptions (1)-(6), the estimator  $\widehat{m}_n$  satisfies:*

$$\mathbb{E}_\star(\widehat{m}_n(x)|X_1, \dots, X_n) = m(x) + o_p(h_1^2).$$

Combining Theorem 3.1 and Theorem 3.2, we conclude that the multiplicative adjustment performs a bias reduction on the pilot estimator without increasing the asymptotic variance. The asymptotic behavior of the bandwidths  $h_0$  and  $h_1$  is constrained by assumption 6. However, it is easily seen that this assumption is satisfied for a large set of values of  $h_0$  and  $h_1$ . For example, the choice  $h_1 = c_1 n^{-1/5}$  and  $h_0 = c_0 n^{-\alpha}$  for  $0 < \alpha < 1/5$  leads to

$$\mathbb{E}_\star(\widehat{m}_n(x)|X_1, \dots, X_n) - m(x) = o_p(n^{-2/5})$$

and

$$\mathbb{V}_\star(\widehat{m}_n(x)|X_1, \dots, X_n) = O_p(n^{-4/5}).$$

## 4 Numerical examples

While the amount of the bias reduction depends on the curvature of the regression function, a decrease is expected (asymptotically) everywhere, and this, at no cost to the variance. The simulation study in this section shows that this asymptotic behavior emerges already at modest sample sizes.

### 4.1 Local study

To illustrate numerically the possible reduction in the bias and associate increase of the variance achieved by the multiplicative bias correction, consider estimating the regression function

$$m(x) = 3 + 3|x|^{5/2} + x^2 + 4 \cos(10x)$$

at  $x = 0$  (see Figure 1).

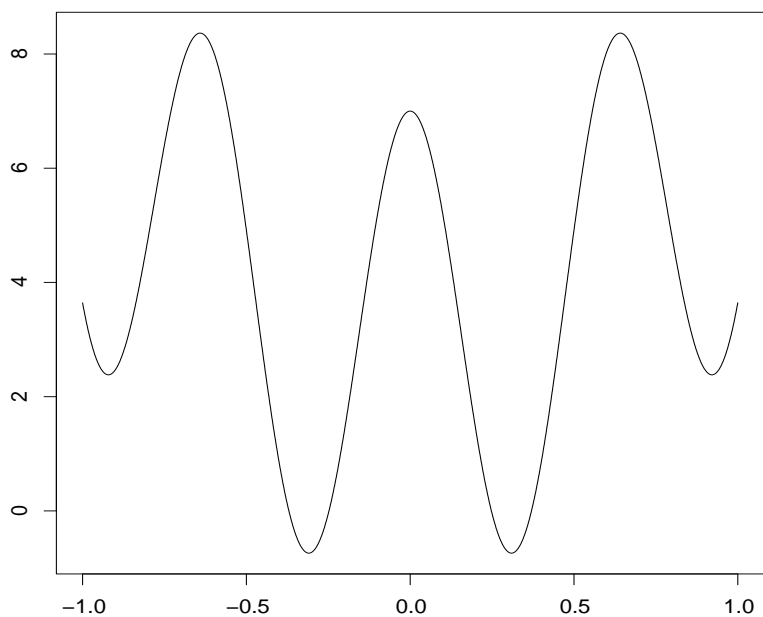


Figure 1: The regression function to be estimated.

The local linear smoother tends to under-estimate the regression function at their maximum, and hence, this example will provide a good example. Furthermore, because the second derivative of this regression function is continuous but not differentiable at the origin, the results previously obtained by Linton and Nielsen [1994] do not apply.

The data are simulated according to the model

$$Y_i = m(X_i) + \varepsilon_i, \quad i = 1, \dots, 100,$$

where  $\varepsilon_i$  are independent  $\mathcal{N}(0, 0.1^2)$  variables. We first consider the local linear estimate with a Gaussian kernel function and we study its performances over a grid of bandwidths  $\mathcal{H} = [0.005, 0.040]$ . For the new estimate, the theory recommends to start with an over-smooth pilot estimate. In this regard, we take  $h_0 = 0.03$  and study the performance of the multiplicative bias corrected estimate for  $h_1 \in \mathcal{H}_1 = [0.005, 0.060]$ . In order to explore the sensitivity of our two stages estimator on  $h_0$ , we also consider the choice  $h_0 = 0.008$ . For such a choice, the pilot estimate clearly under-smoothes the regression function.

Bias and the variance of each estimate are calculated at  $x = 0$ . To do this, we compute the value of each estimate at  $x = 0$  for 200 samples  $(X_i, Y_i), i = 1, \dots, 100$ . The same design  $X_i, i = 1, \dots, 100$  is used for each sample. It is generated according to a uniform distribution over  $[-1, 1]$ . The bias at point  $x = 0$  is estimated by subtracting  $m(0)$  at the mean value of the estimate at  $x = 0$  (the mean value is computed over the 200 replications). Similarly we estimate the variance at  $x = 0$  by the variance of the values of the estimate at this point. Figure 2 represents squared bias, variance and mean square error of each estimate for different values of bandwidth  $h$  for the local linear smoother and  $h_1$  for our estimate.

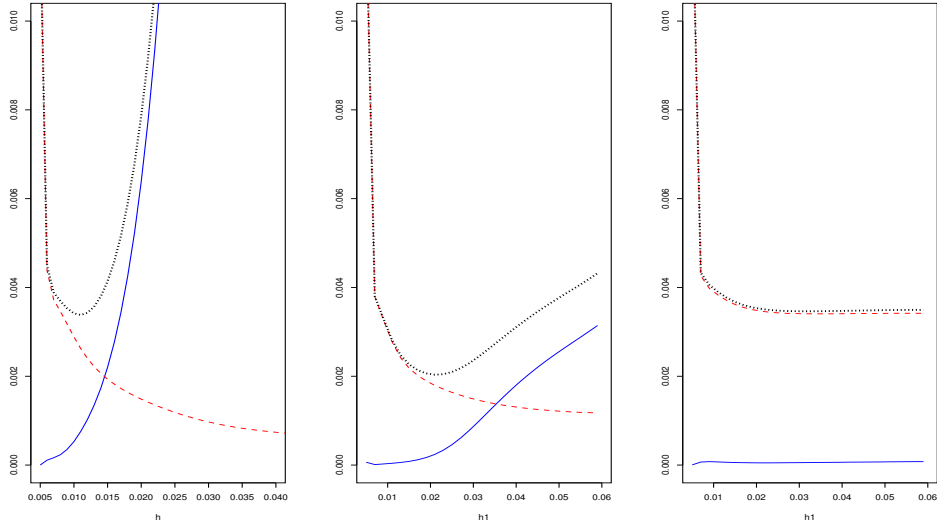


Figure 2: Mean square error (dotted line), squared bias (solid line) and variance (dashed line) of the local linear estimate (left) and multiplicative bias corrected estimate with  $h_0 = 0.03$  (center) and  $h_0 = 0.008$  (right) at point  $x = 0$ .

The first conclusion is that the corrected estimate has smaller bias than the local linear estimate provided the pilot estimate over-smoothes the regression function. Small values of  $h_0$  clearly under-smooth the regression function, whatever the choice of  $h_1$ . Moreover, it is worth pointing out that our procedure does not significantly increase the variance. Even if Theorem 3.1 and Theorem 3.2 provide asymptotic results, our simulations show that the asymptotic behavior of our estimate emerges already at modest sample size. Finally, due to the bias reduction, we note that our procedure also reduces the optimal mean square error (see Table 1).

	MSE	Bias <sup>2</sup>	Variance
LLE	3.38 10 <sup>-3</sup>	0.75 10 <sup>-3</sup>	2.63 10 <sup>-3</sup>
MBCE	2.04 10 <sup>-3</sup>	0.24 10 <sup>-3</sup>	1.80 10 <sup>-3</sup>

Table 1: Optimal mean square error (MSE) for the local linear estimate (LLE) and the multiplicative bias corrected estimate (MBCE) with  $h_0 = 0.03$  at point  $x = 0$ .

## 4.2 Global study

This paper does not conduct any theory to select the two bandwidths  $h_0$  and  $h_1$  in an optimal way. If automatic procedures are needed, they can be obtained by adjusting traditional automatic selection procedures for the classical nonparametric estimators (see Burr et al. [2009]). In this part, we propose to use leave-one-out cross validation to choose both  $h_0$  and  $h_1$ . We then compare the performance of the selected estimate with the local polynomial estimate in term of integrated square error.

Hurvich et al. [1998] report a comprehensive numerical study that compares standard smoothing methods on various test functions. Here, we take the same setting to compare the local linear estimate with its multiplicative bias corrected smoother. In each of the examples, we take the Gaussian kernel  $K(x) = \exp(-x^2/2)/\sqrt{2\pi}$ . We consider the following regression functions (see Figure 3):

- (1)  $m_1(x) = \sin(5\pi x)$
- (2)  $m_2(x) = \sin(15\pi x)$
- (3)  $m_3(x) = 1 - 48x + 218x^2 - 315x^3 + 145x^4$
- (4)  $m_4(x) = 0.3 \exp[-64(x - .25)^2] + 0.7 \exp[-256(x - .75)^2]$

and we take a Gaussian error distribution with standard deviation  $\sigma = 0.3$  for  $m_1, m_2, m_3$  and  $\sigma = 0.05$  for  $m_4$ .

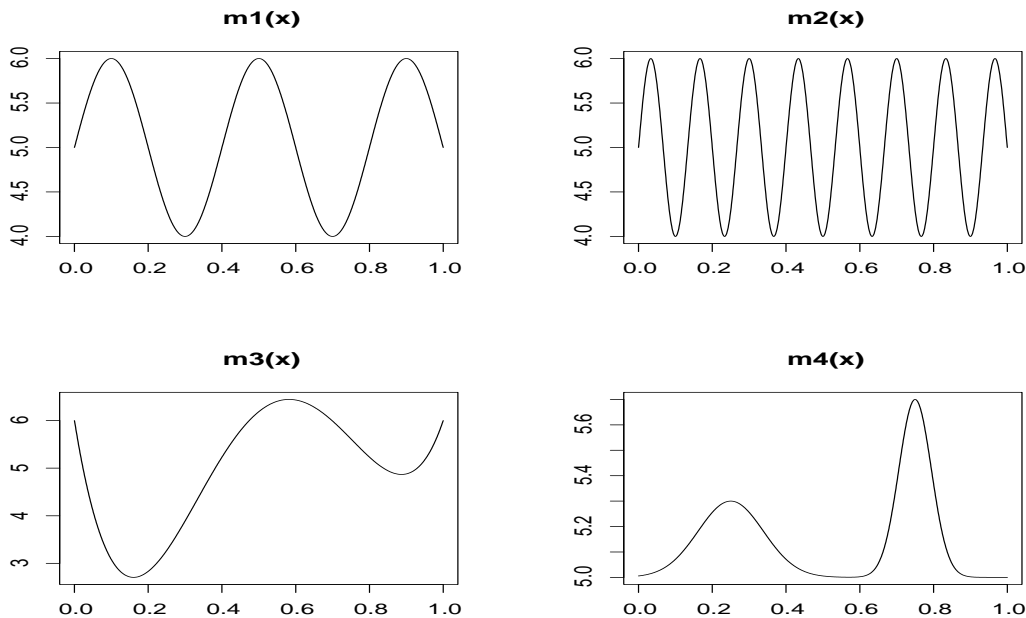


Figure 3: Regression functions to be estimated.

We use a cross validation device to select both  $h_0$  and  $h_1$ . This selection procedure involves solving minimization problem that necessitate a search over a finite grid  $\mathcal{H}$  of bandwidths  $h_0$  and  $h_1$ . Formally, given  $\mathcal{H}$ , we choose  $\hat{h}_0$  and  $\hat{h}_1$  such as

$$(\hat{h}_0, \hat{h}_1) = \underset{(h_0, h_1) \in \mathcal{H} \times \mathcal{H}}{\operatorname{argmin}} \frac{1}{n} \sum_{i=1}^n (Y_i - \hat{m}_n^i(X_i))^2.$$

Here  $\hat{m}_n^i$  stands for the corrected local polynomial estimate after deleted the  $i$ th observation. To assess the quality of the selected estimate, we compare its performances with the local polynomial estimate for which the bandwidth is again selected by leave-one-out cross validation. The performance of an

estimator  $\widehat{m}$  is measured by the integrated square error

$$ISE(\widehat{m}) = \int_0^1 (m(x) - \widehat{m}(x))^2 dx,$$

and to avoid the boundary effects, the design  $X_1, \dots, X_n$  is generated according to a uniform distribution over  $[-0.2, 1.2]$ .

Table 2 presents the median over 100 replications of

- the selected bandwidths;
- the integrated square error;
- the integrated square error of the local linear estimate divided by the integrated square error of the corrected estimate ( $R_{ISE}$ ).

	LLE		MBCE			
	$h$	ISE	$h_0$	$h_1$	ISE	$R_{ISE}$
$m_1$	0.029	0.021	0.053	0.041	0.017	1.226
$m_2$	0.014	0.092	0.026	0.015	0.078	1.156
$m_3$	0.029	0.021	0.070	0.056	0.012	1.600
$m_4$	0.019	0.0010	0.033	0.024	0.0009	1.135

Table 2: Median over 100 replications of the selected bandwidths and of the integrated square error of the selected estimates. LLE and MBCE stands for local linear estimate and multiplicative bias corrected estimate.

We obtain significant ISE reduction. As predicted by Theorem 3.1, the data-driven procedure chooses  $h_0$  bigger than  $h$ : the pilot estimate is over-smoothing the true regression function. Of course, selecting both  $h_0$  and  $h_1$



is time consuming and can appear as the price to be paid to improve the local linear smoother.

The following picture presents, for the regression function  $m_1$  with  $n = 100$  and 100 iterations, different estimators on a grid of points. In lines is the true regression function which is unknown. For every point on a fixed grid, we plot, side by side, the mean over 100 replications of our estimator at that point (left side) and on the right side of that point the mean over 100 replications of the local polynomial estimator. Leave-one-out cross validation is applied to select the bandwidths  $h_0$  and  $h_1$  for our estimator and the bandwidth  $h$  for the local polynomial estimator. We add also the interquartile interval in order to see the fluctuations of the different estimators. On this

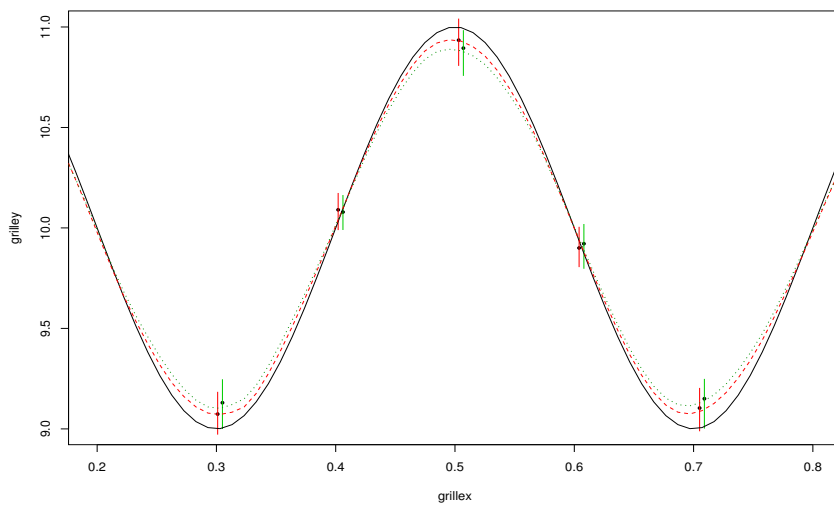


Figure 4: The solid curve represents the true regression function, our estimator is in dashed line and local linear smoother is dotted.

example, our estimator reduces the bias by increasing the peak and decreasing

the valley and the interquartile intervals look similar for both estimator, as predicted by the theory.

## 5 Example: Estimation of an energy spectrum

The energy spectrum of Ba133 is measured at Los Alamos National Laboratory using a 1024-energy channel Sodium Iodide detector for a one-minute count time. The calibration of channel to energy is not important in this context so we consider the one-minute counts versus bin number.

Figure 5 shows the raw counts versus bin number for the first 250 bins, the smoother histogram using a local linear smoother (dotted line) and the multiplicative adjusted energy spectrum (dashed line). Observe that the multiplicative adjusted smoother does indeed fit better the peaks and the valleys of the data without introducing undue variability on the rest of the curve. This suggests that the multiplicative adjustment prior to peak height and area estimation will improve isotope identification performance. Isotope identification algorithms (see Casson et al. [2006]) in the broadest context must consider multiple unknown source isotopes, unknown form (gas, liquid, solid), with unknown shielding between the source and detector, which modifies spectral shape. Many algorithms rely on peak location, height, and/or area so the multiplicative adjustment is an appealing data processing step. Sullivan et al. [2006] report success using wavelet smoothing to locate peaks but do not consider the impact of smoothing on estimated peak height or area.

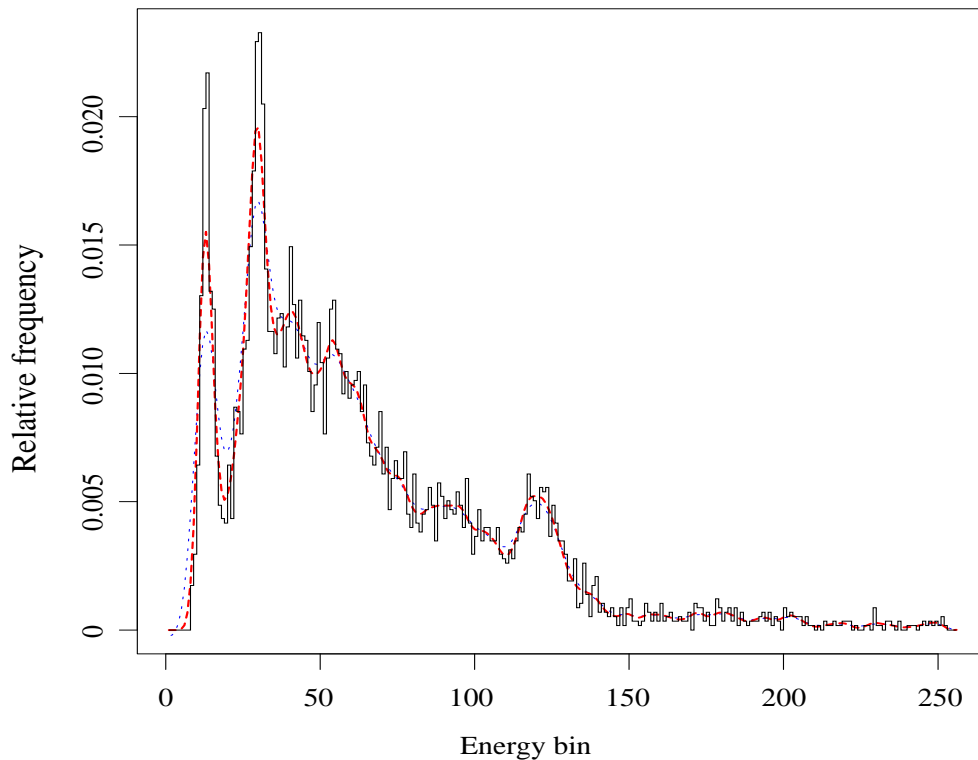


Figure 5: See text.

## 6 Proofs

This section is devoted the technical proofs.

### 6.1 Proof of Proposition 3.1

Write the bias corrected estimator

$$\hat{m}_n(x) = \sum_{j=1}^n \omega_{1j}(x) \frac{\tilde{m}_n(x)}{\tilde{m}_n(X_j)} Y_j = \sum_{j=1}^n \omega_{1j}(x) R_j(x) Y_j,$$

and let us approximate the quantity  $R_j(x)$ . Define

$$\bar{m}_n(x) = \sum_{j=1}^n \omega_{0j}(x) m(X_j) = \mathbb{E}(\tilde{m}_n(x) | X_1, \dots, X_n),$$

and observe that

$$\begin{aligned}
R_j(x) &= \frac{\tilde{m}_n(x)}{\tilde{m}_n(X_j)} \\
&= \frac{\bar{m}_n(x)}{\bar{m}_n(X_j)} \times \left(1 + \frac{\tilde{m}_n(x) - \bar{m}_n(x)}{\bar{m}_n(x)}\right) \times \left(1 + \frac{\tilde{m}_n(X_j) - \bar{m}_n(X_j)}{\bar{m}_n(X_j)}\right)^{-1} \\
&= \frac{\bar{m}_n(x)}{\bar{m}_n(X_j)} \times [1 + \Delta_n(x)] \times \frac{1}{1 + \Delta_n(X_j)},
\end{aligned}$$

where

$$\Delta_n(x) = \frac{\tilde{m}_n(x) - \bar{m}_n(x)}{\bar{m}_n(x)} = \frac{\sum_{l \leq n} \omega_{0l}(x) \varepsilon_l}{\sum_{l \leq n} \omega_{0l}(x) m(X_l)}.$$

Write now  $R_j(x)$  as

$$R_j(x) = \frac{\bar{m}_n(x)}{\bar{m}_n(X_j)} [1 + \Delta_n(x) - \Delta_n(X_j) + r_j(x, X_j)]$$

where  $r_j(x, X_j)$  is a random variable converging to 0 to be define latter on.

Given the last expression and model (1), estimator (3) could be written as

$$\begin{aligned}
\hat{m}_n(x) &= \sum_{j=1}^n \omega_{1j}(x) R_j(x) Y_j \\
&= \sum_{j=1}^n \omega_{1j}(x) \frac{\bar{m}_n(x)}{\bar{m}_n(X_j)} m(X_j) \\
&\quad + \sum_{j=1}^n \omega_{1j}(x) \frac{\bar{m}_n(x)}{\bar{m}_n(X_j)} [\varepsilon_j + m(X_j) (\Delta_n(x) - \Delta_n(X_j))] \\
&\quad + \sum_{j=1}^n \omega_{1j}(x) \frac{\bar{m}_n(x)}{\bar{m}_n(X_j)} (\Delta_n(x) - \Delta_n(X_j)) \varepsilon_j \\
&\quad + \sum_{j=1}^n \omega_{1j}(x) \frac{\bar{m}_n(x)}{\bar{m}_n(X_j)} r_j(x, X_j) Y_j \\
&= \mu_n(x) + \sum_{j=1}^n \omega_{1j}(x) A_j(x) + \sum_{j=1}^n \omega_{1j}(x) B_j(x) + \sum_{j=1}^n \omega_{1j}(x) \xi_j.
\end{aligned}$$

which is the first part of the lemma. Under assumption set forth in Section 3.1, the pilot smoother  $\tilde{m}_n$  converges to the true regression function  $m(x)$ . Bickel and Rosenblatt [1973] show that this convergence is uniform over compact sets  $\mathcal{K}$  contained in the support of the density of the covariate  $X$ . As a result

$$\sup_{x \in \mathcal{K}} |\tilde{m}_n(x) - \bar{m}_n(x)| \leq \frac{1}{2}.$$

So a limited expansion of  $(1 + u)^{-1}$  yields for  $x \in \mathcal{K}$

$$R_j(x) = \frac{\bar{m}_n(x)}{\bar{m}_n(X_j)} \left[ 1 + \Delta_n(x) - \Delta_n(X_j) + O_p(|\Delta_n(x)\Delta_n(X_j)| + \Delta_n^2(X_j)) \right],$$

thus

$$\xi_j = O_p(|\Delta_n(x)\Delta_n(X_j)| + \Delta_n^2(X_j)).$$

Under the stated regularity assumptions, we deduce that

$$\xi_j = O_p\left(\frac{1}{nh_0}\right).$$

leading to the announced result. Theorem (3.1) is proved.

## 6.2 Proof of lemma (3.1)

By definition

$$\limsup_{n \rightarrow \infty} \mathbb{P}[|\xi_n| > ta_n] = 0$$

for all  $t > 0$ , so that a triangular array argument shows that there exists an increasing sequence  $m = m(k)$  such that

$$\mathbb{P}\left[|\xi_n| > \frac{a_n}{k}\right] \leq \frac{1}{k} \quad \text{for all } n \geq m(k).$$

For  $m(k) \leq n \leq m(k+1) - 1$ , define

$$\xi_n^* = \begin{cases} \xi_n & \text{if } |\xi_n| < k^{-1}a_n \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the construction of  $\xi_n^*$  that for  $n \in (m(k), m(k+1) - 1)$ ,

$$\mathbb{P}[\xi_n \neq \xi_n^*] = \mathbb{P}[|\xi_n| > k^{-1}a_n] \leq \frac{1}{k},$$

which converges to zero as  $n$  goes to infinity. Finally set  $k(n) = \sup\{k : m(k) \leq n\}$ , we obtain

$$\mathbb{E}[|\xi_n^*|] \leq \frac{a_n}{k(n)} = o(a_n).$$

### 6.3 Proof of Theorem (3.1)

Recall that

$$\widehat{m}_n(x) = \mu_n(x) + \sum_{j=1}^n \omega_{1j}(x)A_j(x) + \sum_{j=1}^n \omega_{1j}(x)B_j(x) + O_P\left(\frac{1}{nh_0}\right).$$

Focus on the conditional bias, we get

$$\begin{aligned} \mathbb{E}(\mu_n(x)|X_1, \dots, X_n) &= \mu_n(x) \\ \mathbb{E}(A_j(x)|X_1, \dots, X_n) &= 0 \\ \mathbb{E}(B_j(x)|X_1, \dots, X_n) &= \frac{\bar{m}_n(x)}{\bar{m}_n(X_j)} \sigma^2 \left( \frac{\omega_{0j}(x)}{\bar{m}_n(x)} - \frac{\omega_{0j}(X_j)}{\bar{m}_n(X_j)} \right). \end{aligned}$$

Since

$$\sum_{j=1}^n \omega_{1j}(x)\omega_{0j}(x) \leq \sqrt{\sum_{j=1}^n \omega_{1j}(x)^2} \sqrt{\sum_{j=1}^n \omega_{0j}(x)^2} = O_p\left(\frac{1}{n\sqrt{h_0h_1}}\right),$$

we deduce that

$$\mathbb{E}\left(\sum_{j=1}^n \omega_{1j}(x)B_j(x) \middle| X_1, \dots, X_n\right) = O_p\left(\frac{1}{n\sqrt{h_0h_1}}\right).$$

This proves the first part of the Theorem.

For the conditional variance, we use the following expansion of the two stages estimator

$$\widehat{m}_n(x) = \sum_{j=1}^n \omega_{1j}(x) \frac{\bar{m}_n(x)}{\bar{m}_n(X_j)} Y_j (1 + [\Delta_n(x) - \Delta_n(X_j)]) + O_p\left(\frac{1}{nh_0}\right).$$

Using the fact that the residuals have four finite moments and have a symmetric distribution around 0, a moment's thought shows that

$$\mathbb{V}(Y_j [\Delta_n(x) - \Delta_n(X_j)] | X_1, \dots, X_n) = O_p\left(\frac{1}{nh_0}\right)$$

and

$$\text{Cov}(Y_j, Y_j [\Delta_n(x) - \Delta_n(X_j)] | X_1, \dots, X_n) = O_p\left(\frac{1}{nh_0}\right).$$

Hence

$$\begin{aligned} \mathbb{V}_*(\widehat{m}_n(x) | X_1, \dots, X_n) &= \mathbb{V}\left(\sum_{j=1}^n \omega_{1j}(x) \frac{\bar{m}_n(x)}{\bar{m}_n(X_j)} Y_j \middle| X_1, \dots, X_n\right) \\ &\quad + O_p\left(\frac{1}{nh_0}\right). \end{aligned}$$

Observe that the first term on the right hand side of this equality can be seen as the variance of the two stages estimator with a deterministic pilot estimator. It follows from Glad [1998a] that

$$\mathbb{V}\left(\sum_{j=1}^n \omega_{1j}(x) \frac{\bar{m}_n(x)}{\bar{m}_n(X_j)} Y_j \middle| X_1, \dots, X_n\right) = \sigma^2 \sum_{j=1}^n \omega_{1j}^2(x) + o_p\left(\frac{1}{nh_1}\right),$$

which proves the theorem.

## 6.4 Proof of theorem (3.2)

Recall that

$$\mu_n(x) = \sum_{j \leq n} \omega_{1j}(x) \frac{\bar{m}_n(x)}{\bar{m}_n(X_j)} m(X_j).$$

We consider the limited Taylor expansion of the ratio

$$\frac{m(X_j)}{\bar{m}_n(X_j)} = \frac{m(x)}{\bar{m}_n(x)} + (X_j - x) \left( \frac{m(x)}{\bar{m}_n(x)} \right)' + \frac{1}{2} (X_j - x)^2 \left( \frac{m(x)}{\bar{m}_n(x)} \right)'' (1 + o_p(1)),$$

then

$$\begin{aligned} \mu_n(x) = & \bar{m}_n(x) \left\{ \frac{m(x)}{\bar{m}_n(x)} \sum_{j=1}^n \omega_{1j}(x) + \left( \frac{m(x)}{\bar{m}_n(x)} \right)' \sum_{j=1}^n (X_j - x) \omega_{1j}(x) \right. \\ & \left. + \frac{1}{2} \left( \frac{m(x)}{\bar{m}_n(x)} \right)'' \sum_{j=1}^n (X_j - x)^2 \omega_{1j}(x) (1 + o_p(1)) \right\}. \end{aligned}$$

It is easy to verify that

$$\begin{aligned} \Sigma_0(x; h_1) &= \sum_{j=1}^n \omega_{1j}(x) = 1, \\ \Sigma_1(x; h_1) &= \sum_{j=1}^n (X_j - x) \omega_{1j}(x) = 0 \\ \Sigma_2(x; h_1) &= \sum_{j=1}^n (X_j - x)^2 \omega_{1j}(x) = \frac{S_2^2(x; h_1) - S_3(x; h_1)S_1(x; h_1)}{S_2(x; h_1)S_0(x; h_1) - S_1^2(x; h_1)}. \end{aligned}$$

For random designs, we can further approximate (see, e.g., Wand and Jones [1995])

$$S_k(x, h_1) = \begin{cases} h^k \sigma_K^k f(x) + o_p(h^k) & \text{for } k \text{ even} \\ h^{k+1} \sigma_K^{k+1} f'(x) + o_p(h^{k+1}) & \text{for } k \text{ odd.} \end{cases},$$

where  $\sigma_K^k = \int u^k K(u) du$ . Therefore

$$\begin{aligned} \Sigma_2(x; h_1) &= h_1^2 \int u^2 K(u) du + o_p(h_1^2) \\ &\equiv \sigma_K^2 h_1^2 + o_p(h_1^2), \end{aligned}$$

so that we can write  $\mu_n(x)$  as

$$\begin{aligned} \mu_n(x) &= \bar{m}_n(x) \left\{ \frac{m(x)}{\bar{m}_n(x)} + \frac{\sigma_K^2 h_1^2}{2} \left( \frac{m(x)}{\bar{m}_n(x)} \right)'' + o_p(h_1^2) \right\} \\ &= m(x) + \frac{\sigma_K^2 h_1^2}{2} \bar{m}_n(x) \left( \frac{m(x)}{\bar{m}_n(x)} \right)'' + o_p(h_1^2). \end{aligned}$$



Expression

$$\begin{aligned} \left( \frac{m(x)}{\bar{m}_n(x)} \right)'' &= \frac{\bar{m}_n^2(x)m''(x)}{\bar{m}_n^3(x)} - 2\frac{\bar{m}_n(x)\bar{m}_n'(x)m'(x)}{\bar{m}_n^3(x)} \\ &\quad - \frac{m(x)\bar{m}_n(x)\bar{m}_n''(x)}{\bar{m}_n^3(x)} + 2\frac{m(x)(\bar{m}_n'(x))^2}{\bar{m}_n^3(x)} \end{aligned}$$

and applying the usual approximations, we conclude that

$$\left( \frac{m(x)}{\bar{m}_n(x)} \right)'' = o_p(1).$$

Putting all pieces together, we obtain

$$\mathbb{E}_*(\hat{m}_n(x)|X_1, \dots, X_n) - m(x) = o_p(h_1^2) + O_p\left(\frac{1}{n\sqrt{h_0h_1}}\right) + O_p\left(\frac{1}{nh_0}\right).$$

Since

$$nh_1^3 \longrightarrow \infty \quad \text{and} \quad \frac{h_1}{h_0} \longrightarrow 0,$$

we conclude that the bias is of order  $o_p(h_1^2)$ .

## References

- P. Bickel and M. Rosenblatt. On some global measures of the deviations of density function estimates. *The Annals of Statistics*, 1:1071–1095, 1973.
- T. Burr, N. Hengartner, E. Matzner-Løber, E. Myers, and L. Rouvière. Smoothing low resolution spectral data. *Submitted*, 2009.
- W. Casson, C. Sullivan, J. Blackadar, R. Paternoster, J. Matzke, M. Rawool-Sullivan, and L. Atencio. *Nuclear Reachback Reference Manual*, chapter 6. LosAlamos National Laboratory, 2006.
- P.-A. Cornillon, N. Hengartner, and E. Matzner-Løber. Recursive bias estimation for multivariate regression smoothers. *Submitted*, 2009.

- M. Di Marzio and C. Taylor. On boosting kernel regression. *Journal of Statistical Planning and Inference*, 138:2483–2498, 2008.
- J. Fan and I. Gijbels. *Local Polynomial Modeling and Its Application, Theory and Methodologies*. Chapman et Hall, New York, 1996.
- X. Gang, D. Li, Z. Benai, and Z. Jianshi. A nonlinear wavelet method for data smoothing of low-level gamma-ray spectra. *J. Nuclear Science and Technology*, 41:73–76, 2004.
- I. Glad. Parametrically guided non-parametric regression. *Scandinavian Journal of Statistics*, 25:649–668, 1998a.
- I. Glad. A note on unconditional properties of parametrically guided Nadaraya-Watson estimator. *Statistics and Probability Letters*, 37:101–108, 1998b.
- N. Hengartner and E. Matzner-Løber. Asymptotic unbiased density estimators. *ESAIM*, 13:1–14, 2009.
- C. Hurvich, G. Simonoff, and C. L. Tsai. Smoothing parameter selection in nonparametric regression using and improved akaike information criterion. *Journal of the Royal Statistical Society*, 60:271–294, 1998.
- M. Jones, O. Linton, and J. Nielsen. A simple and effective bias reduction method for kernel density estimation. *Biometrika*, 82:327–338, 1995.
- O. Linton and J. P. Nielsen. A multiplicative bias reduction method for nonparametric regression. *Statistics and Probability Letters*, 19(181–187), 1994.

- D. Scott. *Multivariate Density Estimation: Theory, Practice, and Visualization*. Wiley, New-York, 1992.
- J. Simonoff. *Smoothing Methods in Statistics*. Springer, New York, 1996.
- C. Sullivan, M. Martinez, and S. Garner. Wavelet analysis of sodium iodide spectra. *IEEE transaction on nuclear science*, 53:2916–2922, 2006.
- M. Wand and M. Jones. *Kernel Smoothing*. Chapman and Hall, London, 1995.