Phasefield theory for fractional diffusion-reaction equations and applications

Cyril Imbert, Panagiotis E. Souganidis

Phasefield theory for fractional diffusion-reaction equations and applications

Cyril Imbert*and Panagiotis E. Souganidis†

July 31, 2009

Abstract. This paper is concerned with diffusion-reaction equations where the classical diffusion term, such as the Laplacian operator, is replaced with a singular integral term, such as the fractional Laplacian operator. As far as the reaction term is concerned, we consider bistable non-linearities. After properly rescaling (in time and space) these integro-differential evolution equations, we show that the limits of their solutions as the scaling parameter goes to zero exhibit interfaces moving by anisotropic mean curvature. The singularity and the unbounded support of the potential at stake are both the novelty and the challenging difficulty of this work.

Keywords: fractional diffusion-reaction equations, traveling wave, phasefield theory, anisotropic mean curvature motion, fractional Laplacian, Green-Kubo type formulae

Mathematics Subject Classification: 34E10, 45G05, 45K05, 47G20, 49L25, 53C44

1 Introduction

This paper is concerned with diffusion-reaction equations where the classical diffusion is replaced with a singular integral term. Our aim is somewhat classical: to show that the limit of their solutions after properly rescaling them in time and space exhibit a moving interface. However, we will deal with integral term whose potential is anisotropic, singular and with unbounded support.

*Université Paris-Dauphine, CEREMADE, UMR CNRS 7534, place de Lattre de Tassigny, 75775 Paris cedex 16, France, imbert@ceremade.dauphine.fr

†University of Chicago, Dept. of Mathematics, 5734 S. University Avenue Chicago, Illinois 60637, USA, souganidis@math.uchicago.edu

Fractional diffusion-reaction equations. We consider for $t > 0$ and $x \in \mathbb{R}^N$, $N \geq 2$,

$$\partial_t u^\varepsilon + \frac{1}{\varepsilon^2}\Big\{ - \mathcal{I}_\alpha^\varepsilon u^\varepsilon + f(u^\varepsilon)\Big\} = 0 \tag{1}$$

and

$$\partial_t u^\varepsilon + \frac{1}{\varepsilon^2|\ln\varepsilon|}\Big\{ - \mathcal{I}_1^\varepsilon u^\varepsilon + f(u^\varepsilon)\Big\} = 0 \tag{2}$$

where $\mathcal{I}_\alpha^\varepsilon$ is a singular integral operator depending on a parameter $\alpha \in (0,2)$ and f is a bistable non-linearity.

Example 1 (Standing example)**.** *The standing example for f and $\mathcal{I}_\alpha^\varepsilon$ are*

$$f(u) = u(u^2-1) \quad \textit{and} \quad \mathcal{I}_\alpha^\varepsilon u = \frac{1}{\varepsilon^\alpha}\Delta^{\frac{\alpha}{2}} u$$

where $\Delta^{\frac{\alpha}{2}}$ denotes the fractional Laplacian of the function u. We recall that

$$\Delta^{\frac{\alpha}{2}} u = \int (u(x+z) - u(x))\frac{dz}{|z|^{N+\alpha}}$$

where the singular integral must be undertood in the sense of Cauchy's principal value.

More generally, we will consider singular integral operators of the following form

$$\mathcal{I}_\alpha^\varepsilon u(x) = \int \Big(u(x+\varepsilon z) - u(x) - \varepsilon Du(x)\cdot z \mathbf{1}_B(\varepsilon z)\Big) J(z)dz \tag{3}$$

where B denotes the unit ball and where the function $J : \mathbb{R}^N \to \mathbb{R}$, which will be often referred to as the *potential*, can be of two types

$$\text{either} \qquad J(z) = g\left(\frac{z}{|z|}\right)\frac{1}{|z|^{N+\alpha}} \qquad \text{or} \qquad J \in L^1(\mathbb{R}^N)\cap C_c(\mathbb{R}^N) \tag{4}$$

with $\alpha \in [1,2)$ and $g : \mathbb{S}^{N-1} = \{z \in \mathbb{R}^N : |z| = 1\} \to (0,+\infty)$ continuous and where $C_c(\mathbb{R}^N)$ denotes the space of continuous functions with bounded support. The first potential will be referred to as the singular one while the second one will be referred to the regular one. As far as the standing example is concerned, J is singular with $g \equiv 1$.

Phasefield theory for diffusion-reaction equations. In [8], Chen proved rigourously that the solution of the Allen-Cahn equation [1] generates a front moving with mean curvature as long as the front is regular. Thanks to definition of fronts past singularities [18, 9], Evans, Soner and the second author [17] proved that this is still true after the appearence of singularities. Such results are generalized to a large class of bistable non-linearities by Barles, Soner and the second author [3] where a general phasefield theory for reaction-diffusion equations is introduced. In [4, 2], an abstract method is developed in order

to deal with more general reaction-diffusion equations and to handle boundary conditions. In particular, non-local reaction-diffusion equations are considered in [4] but integral operators are not singular. As the proofs of the present paper will show it, it is a challenging difficulty to be overcome.

Motivations. Recently, Caffarelli and the second author studied threshold dynamics-type algorithms corresponding to the fractional Laplace operator for $\alpha \in (0, 2)$. They proved that after properly rescaling them, they converge to an interface moving by mean curvature in the case $\alpha \geq 1$ and to a fractional mean curvature in the case $\alpha < 1$. Hypersurfaces with zero integral curvature are studied in [7]. See also [23] where the level-set approach [18, 9] is developed for such a geometric flow.

As far as applications are concerned, two main physical models motivate the present study. The first application we have in mind is dislocation dynamics. Dislocation theory aims at explaining the plastic behaviour of materials by the motion of linear defects in crystals. Peirls-Nabarro models [28] consist in approximating the geometric motion of these defects by non-local diffusion-reaction equations such as (2). In [19], such an approximation is also used and formal expansions are performed. In [20, 21], Garroni and Müller study a variational model for dislocations that can be viewed as the variational formulation of the stationary version of (2).

The second application we have in mind is statistical mechanics and more precisely stochastic Ising models. These models were introduced by Kač, Uhlenbeck and Hemmer [24] (see also [29]) to justify the validity of the Van der Waal's phase diagram. The interaction beween particles is described by the Kač potential. A lot of work has been done since then to understand the hydrodynamic limits of such interacting particle systems and it is beyond the scope of this paper to give a complete list of references. However, we can mention the papers by De Masi, Orlandi, Presutti and Triolo [11, 12, 13, 14, 15] and Katsoulakis and the second author [25, 26, 27]. The interested reader is also referred to the monograph of De Masi and Presutti [16] and the book of Spohn [30]. In the papers we mentioned before, hydrodynamic limit of stochastic Ising models with general dynamics are studied. In particular, a mean field equation is derived, see [12]. By many ways, these equations can be viewed as non-local reaction-diffusion equations. The next step is to show that for appropriate scalings the solution of the mean field equation approximates an anisotropic mean curvature motion; see for instance [11, 25]. Green-Kubo type formulae are provided for the mobility and the diffusion matrix in terms of a standing wave associated with the mean field equation.

On one hand, the Kač potential is assumed, in most papers, to be regular with a compact support. On the other hand, Lebowitz and Penrose [29, Eq.(1.20b),(1.21a),p.100] consider potentials J that are singular; more precisely, they assume that for small z, the singularity of J is of the form $|z|^{-N-\alpha}$ for $\alpha > 0$.

Description of the results. Our main result states that, as $\varepsilon \to 0$, the solutions u^ε of (1) and (2) can only have two limits: the stable equilibria of the bistable non-linearity f (see Section 2 for definitions). The resulting interface evolves by anisotropic mean curvature; moreover, Green-Kubo-type formulae are obtained: the mobility and the diffusion matrix of the geometric flow are expressed in terms of the standing wave associated with the bi-stable non-linearity; see Eq. (22), (23) and (25) below. Even if the proof follows the classical idea of constructing barriers by using traveling waves, the reader will see that classical arguments fail when extending the barrier away from the front; several new ideas are needed to handle the unboundedness of the support. In order to handle the anisotropy of the potential, we have to use ideas developed by Katsoulakis and the second author [26] and introduce correctors to cancel oscillating terms by averaging them. This implies in particular that anisotropic traveling waves must be considered. But because the integral term involves a singular potential, passing to the limit in averaged oscillating terms is challenging and this constitutes the core of the proof of the convergence theorem.

As the reader will see it when going through the preliminary section or in the statement of the convergence theorem, several assumptions on traveling waves and the linearized traveling wave equation are necessary (if not mandatory). Even if we do not construct such waves and correctors and assume that they exist, the reader can check that the assumptions we make are natural. For instance, the decay estimate (13) is expected since its corresponds to the one of the kernel of the fractional Laplacian in the one dimensional space. See also [6]. We plan to construct them in a compagnion paper.

As explained above, we will consider two kinds of potentials: singular and regular ones. As far as the singular case is concerned, we distinguish two subcases, depending how singular is the potential at the origin. Since potentials are positively homogeneous in the singular case, potentials in the subcase $\alpha = 1$ decay as $|z|^{-N-1}$ when $|z| \to +\infty$. This corresponds to the dislocation dynamics model. As the reader can see it, the scaling involves a logarithmic term; this factor is well-known in physics and the interested reader is referred to [5] for instance; see also [10, 6]. An additional comment about singular and regular potentials concern the Green-Kubo-type formulae. It turns out that these formulae are different in singular and regular cases. However, we give in appendix a formal argument to shed some light on the link between these two formulae.

Additional comments. As the reader can see it, we are not able to deal with the case $\alpha < 1$ even if, in view of the results of [6], we should observe an interface moving with fractional mean curvature (see Section 2 for a definition). In this case, the equation should be rescaled in time as follows

$$\partial_t u^\varepsilon + \frac{1}{\varepsilon^{1+\alpha}} \Big\{ - \mathcal{I}_\alpha^\varepsilon u^\varepsilon + f(u^\varepsilon) \Big\} = 0 \tag{5}$$

with

$$\mathcal{I}_\alpha^\varepsilon u(x) = \int \Big(u(x+\varepsilon z) - u(x) \Big) g\left(\frac{z}{|z|} \right) \frac{dz}{|z|^{N+\alpha}}$$

for some $\alpha \in (0,1)$. The reader can check that we are able to pass to the limit in (the average of) oscillating terms (see Lemma 10 below), which is usually the difficult part of the convergence proof. We are even able to construct a barrier close to the front. But because the diffusion-reaction is non-local, we are stuck with extending the solution away from it. In particular, the very slow decay of the potential at infinity does not permit us to use the new ideas we introduced in the singular case $\alpha \geq 1$. This difficulty is unexpected since in [6], this case is the easiest one. We hope to find a path toward this result in a future work.

In the one dimensional space, moving interfaces are points. Gonzalez and Monneau [22] considered such a case and proved a result analogous to our main one by taking advantage of the fact that the limit is a (system of) ordinary differential equation(s). In particular, the restriction on the strength of the singularity can be relaxed in this case.

Organization of the article. The first section is devoted to preliminaries. In particular, traveling waves are introduced as well as the linearized traveling wave equation which is the equation satisfied by correctors; see Subsections 2.3 and 2.4. We also introduce the geometric motion by mean curvature (Subsection 2.5) together with its equivalent definition in terms of generalized flows (Subsection 2.6). Our main result is stated in Section 3. In the remaining of this section, we explain how to reduce the proof of this convergence result to the construction of an appropriate barrier (see above). Section 4 is dedicated to this construction. The last section (Section 5) contains to core of the proof of the convergence result: the limit of the average of oscillating terms. Finally, we give in appendix a formal argument to explain the link between the two Green-Kubo formulae obtained in the convergence theorem.

Notation. The Euclidian norm of $x \in \mathbb{R}^N$ is denoted by $|x|$. The ball of center x and of radius r is denoted by $B_r(x)$. We simply write B_r for $B_r(0)$ and $B = B_1$ denotes the unit ball. The scalar product of x and y is denoted by $x \cdot y$. The unit sphere of $\mathbb{R}^N$ is denoted by $\mathbb{S}^{N-1}$. The set of symmetric $N \times N$ matrices is denoted by $\mathcal{S}_N$. The identity matrix (in any dimension) is denoted by I.

Given two real numbers $a, b \in \mathbb{R}$, $a \vee b$ denotes $\max(a,b)$ and $a \wedge b$ denotes $\min(a,b)$, a^+ denotes $a \vee 0$ and a^- denotes $-(a \wedge 0)$. In particular, $a^\pm \geq 0$ and $a = a^+ - a^-$.

Given a set A, $\mathbf{1}_A$ denotes its indicator function that equals 1 in A and 0 outside. The signed distance function $d_A(x)$ associated with A equals the distance function to $\mathbb{R}^N \setminus A$ if $x \in A$ and the opposite of the distance function to A if $x \notin A$.

The set of continuous functions $f : \mathbb{R}^N \to \mathbb{R}$ with compact support is denoted by C_c^0.

Given a family of locally bounded functions $f_\varepsilon : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ indexed by

$\varepsilon > 0$, the relaxed upper and lower limits are defined as follows

$$\liminf{}_* f_\varepsilon(x) = \liminf_{\varepsilon\to 0, y\to x} f_\varepsilon(y) \quad \text{and} \quad \limsup{}_* f_\varepsilon(x) = \limsup_{\varepsilon\to 0, y\to x} f_\varepsilon(y)\ .$$

If $f_\varepsilon = f$ for any $\varepsilon > 0$, these relaxed semi-limits coincide with the lower and upper semi-continuous of a locally bounded function f.

For traveling waves $q(r, e)$ and correctors $Q(r, e)$, $\dot{q}$ and $\dot{Q}$ denote derivatives with respect to r.

2 Preliminaries

This section is devoted to the presentation of the assumptions we make about non-linearities, traveling waves and linearized traveling wave equations. We also briefly describe the construction of fronts at stake after rescaling fractional diffusion-reaction equations.

2.1 Fractional diffusion-reaction equations

We can write (1), (2) and (5) as follows

$$\partial_t u^\varepsilon + \frac{1}{\varepsilon\eta}\Big\{ -\mathcal{I}_\alpha^\varepsilon u^\varepsilon + f(u^\varepsilon)\Big\} = 0 \tag{6}$$

with

$$\eta = \begin{cases} \varepsilon & \text{if } \alpha > 1\,, \\ \varepsilon|\ln\varepsilon| & \text{if } \alpha = 1\,, \\ \varepsilon^\alpha & \text{if } \alpha < 1 \end{cases} \tag{7}$$

(see Remark 4).

We will use later on that the potential J satisfies in the singular case the following properties

$$\begin{cases} J \text{ is smooth on } \mathbb{R}^N \setminus \{0\}, \text{ even and non-negative} \\ |J(z)| \le \frac{C_J}{|z|^{N+\alpha}} \\ J(z) \sim g\left(\frac{z}{|z|}\right)\frac{1}{|z|^{N+\alpha}} \text{ as } |z| \to +\infty \end{cases} \tag{8}$$

with $\alpha \in (0, 2)$. We also mention that if $\alpha < 1$, then

$$|z|^2 J_\varepsilon(z) + |\nabla(|z|^2 J_\varepsilon(z))| \le K(z) \in L^1(B) \tag{9}$$

where $J_\varepsilon(z) = \varepsilon^{-N-\alpha} J(\varepsilon^{-1} z)$.

2.2 Bistable non-linearity

We make the following assumptions.

Assumption 1 (Bistable non-linearity)**.** *The non-linearity $f : \mathbb{R} \to \mathbb{R}$ is C^1 and such that*

- *for all $h \in (0, H)$, they are constants $m_\pm(h)$ and $m_0(h)$ such that*

$$f(m_i(h)) = h, \quad i \in \{0, +, -\} \qquad \text{and} \qquad m_-(h) < m_0(h) < m_+(h). \tag{10}$$

- *$f > 0$ in $(\bar{m}_-, \bar{m}_0)$ and $f < 0$ in $(\bar{m}_0, \bar{m}_+)$.*
- *$f'(\bar{m}^\pm) > 0$ and $f'(\bar{m}_0) < 0$ where $\bar{m}_\pm = m_\pm(0)$ and $\bar{m}_0 = m_0(0)$.*

In particular, if $h > 0$, we have $\bar{m}_\pm \le m_\pm(h)$ and $\bar{m}_0 \ge m_0(h)$ and

$$|m_\pm(h) - \bar{m}_\pm| \le C_f h \, . \tag{11}$$

2.3 Anisotropic traveling wave

In this subsection, we describe the anisotropic traveling waves we will use in the construction of barriers in order to get the main convergence result. In particular, we make precise the decay we expect for such waves. This construction will be achieved in a future work.

Assumption 2 (Anisotropic traveling wave)**.** *For $h \in (0, H)$, there then exist two continuous functions $q : \mathbb{R} \times \mathbb{S}^{N-1} \to \mathbb{R}$ and $c : \mathbb{R} \times \mathbb{S}^{N-1} \to \mathbb{R}$ such that*

$$q(r, e, h) \to m_\pm(h) \text{ as } r \to \pm\infty$$

(the limit being uniform with respect to $e \in \mathbb{S}^{N-1}$) and

$$c\dot{q} - \mathcal{I}_e[q] + f(q) = h \tag{12}$$

where

$$\mathcal{I}_e[q](\xi) = \begin{cases} \displaystyle\int \Big(q(\xi + e \cdot z) - q(\xi) - \dot{q}(\xi) e \cdot z \mathbf{1}_B(z) \Big) J(z) dz & \text{if } \alpha \ge 1 \\ \displaystyle\int \Big(q(\xi + e \cdot z) - q(\xi) \Big) J(z) dz & \text{if } \alpha < 1 \end{cases}$$

for any $e \in \mathbb{S}^{N-1}$.

The traveling wave q is increasing in r and the following estimates hold true for any $(r, e) \in \mathbb{R} \times \mathbb{S}^{N-1}$

$$|q(r, e, h) - m_\pm(h)| = O\left(\frac{1}{|r|^{1+\alpha}} \right) \quad \text{as } r \to \pm\infty \tag{13}$$

$$\sup_{h>0} \{ \|D_e q\|_\infty + \|D^2_{e,e} q\|_\infty \} \le C_q \tag{14}$$

for some constant $C > 0$ and with a limit uniform in $e \in \mathbb{S}^{N-1}$. The function q also satisfies

$$\int \dot{q}^2(\xi) d\xi \to \int (\dot{q^0})^2(\xi) d\xi \tag{15}$$

as $h \to 0$ and the limit is uniform in e.

The speed c satisfies

$$hc(e,h) > 0 \quad \text{if} \quad h \neq 0 \tag{16}$$

and

$$\frac{c(e,h)}{h} \to \overline{c}(e) \quad \text{as} \quad h \to 0 \tag{17}$$

and the previous limit is uniform with respect to e. Moreover, the function $\overline{c}(e)$ is continuous on $\mathbb{S}^{N-1}$.

Standing wave. Notice that (17) implies in particular that $c(e,0) = 0$. Hence $q(e,0)$ is a standing wave. It is denoted by q^0 in the remaing of the paper.

Reduced integral operator. The operator $\mathcal{I}_e$ does not depend on e if $J(z)$ is radially symmetric, *i.e.* when $J(z) = j(|z|)$. We illustrate this fact in the next lemma where $\mathcal{I}_e$ is computed in the case where $J(z) = g(\hat{z})|z|^{-N-\alpha}$.

Lemma 1. *Assume that $J(z) = g(\frac{z}{|z|})\frac{1}{|z|^{N+\alpha}}$. Then*

$$\mathcal{I}_e[q](r) = a_{11}(e) \int_{\mathbb{R}} (q(\xi + r) - q(\xi)) \frac{d\xi}{|\xi|^{1+\alpha}}$$

where

$$a_{11}(e) = \int_{\mathbb{R}^{N-1}} g(1,u) \frac{du}{(1+|u|^2)^{(N+\alpha)/2}}$$

Remark 1. We recognize the fractional Laplacian of order α in the one dimensional space (up to a multiplicative constant).

2.4 Linearized traveling wave equation

In this subsection, the linearized traveling wave equation is considered. Loosely speaking, we need to know that the kernel of the linearized operator $\mathcal{L}$ reduces to $\mathbb{R}\dot{q}$ and, if f is regular enough, so is the solution Q of $\mathcal{L}Q(\xi) = Pf(\xi,t,x)$ where Pf is the projection of f on the space orthogonal to $\dot{q}$. We need in particular to be able to say that Q decays at infinity. Let us be more precise now.

The linearized operator $\mathcal{L}$ associated with (12) around a solution q is

$$\mathcal{L}Q = c\dot{Q} - \mathcal{I}_e[Q] + f'(q)\,Q. \tag{18}$$

Given a smooth function $d : [0,T] \times \mathbb{R}^N \to \mathbb{R}$, let us consider

$$a(r,e,t,x) = \frac{1}{h} \int \left\{ q(r + \frac{d(t,x+\varepsilon z) - d(t,x)}{\varepsilon}, e) - q(r + e \cdot z, e) \right\} J(z) dz$$

where $h = \varepsilon$ (resp. $\varepsilon|\ln\varepsilon|$, ε^α) if $\alpha > 1$ (resp. $\alpha = 1$, $\alpha < 1$).

Assumption 3 (The linearized TW equation)**.**

$$\text{Ker } \mathcal{L} = \text{Ker } (\mathcal{L})^* = \text{span } \dot{q}.$$

Moreover, if $d : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ *is a smooth function, there then exists a continuous solution* $Q : \mathbb{R} \times \mathbb{S}^{N-1} \times [0,T] \times \mathbb{R}^N$ *to*

$$\mathcal{L}Q = P_{\dot{q}} a(\cdot, e, t, x)$$

where $P_{\dot{q}}$ *stands for the projection on the space orthogonal to* span $\dot{q}$*. In particular, there exists* $C_Q > 0$ *such that for any* h, e, ξ*,*

$$|\dot{Q}| + |\partial_t Q| + |D_x Q| + |D_e Q| + |D^2_{e,e} Q| + |D^2_{x,x} Q| \leq C_Q \tag{19}$$

$$Q \to 0 \quad \text{as } r \to +\infty \tag{20}$$

where C_Q *does not depend on* h, e, t, x *and the limit is uniform in* h, e, t, x*.*

2.5 Geometric motions

In this subsection, we introduce the geometric motions of fronts at stake when rescaling the fractional diffusion-reaction equations.

It is well-known that singularities can appear on the front in finite time when considering, for instance, the mean curvature motion. We thus classically use the level-set approach to define a front for all times. We recall that this approach consists in looking for a front Γ_t under the form $\{x : u(t,x) = 0\}$ and to derive a PDE satisfied by u.

Anisotropic mean curvature motion. In the case of an anisotropic mean curvature motion, we obtain the following degenerate and singular parabolic equation,

$$\partial_t u = \mu\left(\widehat{Du}\right) \text{Tr}\left((I - \widehat{Du} \otimes \widehat{Du}) A\left(\widehat{Du}\right) D^2 u\right) \tag{21}$$

where $\mu : \mathbb{S}^{N-1} \to \mathbb{R}^+$ and $A : \mathbb{S}^{N-1} \to \mathcal{S}_N$ are continuous functions and I stands for the $N \times N$ identity matrix and $e = \frac{p}{|p|}$. We will see that the function μ (which will be referred to as the mobibility) is given by the following formula

$$\mu(e) = \left\{ \int (\dot{q^0})^2(\xi, e) d\xi \right\}^{-1}. \tag{22}$$

As far as the function A is concerned, we distinguish cases. In the singular case and if $\alpha > 1$, we have for all $e \in \mathbb{S}^{N-1}$

$$A(e) = \left\{ \int \int (q^0(\xi + z_1, e) - q^0(\xi, e))^2 \frac{dz_1}{|z_1|^{1+\alpha}} d\xi \right\} A_g(e) \tag{23}$$

with

$$A_g(e) = \alpha(\alpha - 1) \int_{\mathbb{R}^{N-1}} (1, u) \otimes (1, u) g(1, u) \frac{du}{(1 + |u|^2)^{(N+\alpha)/2}} \tag{24}$$

(where the space orthogonal to e is identified with $\mathbb{R}^{N-1}$). If $\alpha = 1$,

$$(\bar{m}_+ - \bar{m}_-)^2 \int_{\mathbb{S}^{N-2}=\{z\in\mathbb{S}^{N-1}:z\cdot e=0\}} \Big(\theta\otimes\theta J(\theta)\Big)\sigma(d\theta)\,. \tag{25}$$

In the regular case, we have for all $e \in \mathbb{S}^{N-1}$

$$A(e) = \int \dot{q^0}(\xi, e)\dot{q^0}(\xi + e\cdot z)z\otimes zJ(z)dz\,. \tag{26}$$

Remark 2. As a matter of fact, in the singular case with $\alpha > 1$, A's given by (23) and (26) are the same, at least formally. But it is not even clear that the integral defining $A(e)$ in (26) is well defined in the case $\alpha > 1$. A formal argument is given in Appendix.

Fractional mean curvature motion. A fractional version of this motion can be defined. More precisely, for any $\alpha < 1$, one can consider the following PDE,

$$\partial_t u = \mu\left(\widehat{Du}\right)\kappa[x, u(t,\cdot)]|Du| \tag{27}$$

where μ is defined by (22) and

$$\kappa[x,U] = \kappa^*[x,U] = \int \Big\{ \mathbf{1}_{\{U(x+z)\geq U(t,x), e\cdot z\leq 0\}} - \mathbf{1}_{\{U(t,x+z)<U(t,x), e\cdot z>0\}} \Big\} g\left(\frac{z}{|z|}\right)\frac{dz}{|z|^{N+\alpha}}\,.$$

This can also be written under the general form

$$\kappa^*[x,U] = \nu\{z\in\mathbb{R}^N : U(x+z)\geq U(t,x), e\cdot z\leq 0\} - \nu\{z\in\mathbb{R}^N : U(t,x+z) < U(t,x), e\cdot z > 0\}$$

for some non-negative Borel measure ν which is eventually singular. A general theory is developed in [23] to prove that the geometric flow is well defined. The definition of a viscosity solution for (27) implies the use of the following quantity

$$\kappa_*[x,U] = \nu\{z\in\mathbb{R}^N : U(x+z) > U(t,x), e\cdot z < 0\} - \nu\{z\in\mathbb{R}^N : U(t,x+z)\leq U(t,x), e\cdot z\geq 0\}\,.$$

Geometric non-linearities. In the following, we will use the notation: if $\alpha \geq 1$,

$$F(p,X) = -\mu(\hat{p})\mathrm{Tr}\left(\tilde{A}(\hat{p})X\right) - \overline{c}(0,\hat{p})|p| \tag{28}$$

and if $\alpha < 1$,

$$F(p,X,[\phi]) = -\mu(p)\kappa^*[x,\phi]|p| - \overline{c}(0,\hat{p})|p| \tag{29}$$

for $p \neq 0$ and $\hat{p} = p/|p|$. Since non-linearities F are discontinuous, it is necessary to use the lower and upper semi-continuous envelopes F_* and F^* of F in order to define viscosity solutions of (21). In the case $\alpha < 1$, we have

$$F^*(p, X, [\phi]) = -\mu_*(\hat{p})\kappa_*[x, \phi]|p| - \overline{c}_*(0, \hat{p})|p|$$
$$F_*(p, X, [\phi]) = -\mu^*(\hat{p})\kappa^*[x, \phi]|p| - \overline{c}^*(0, \hat{p})|p|$$

with the convention $\hat{0} = 0$.

2.6 Generalized flows

As we explained it above, the level-set approach is necessary in order to define the anisotropic mean curvature motion of a curvature after the onset of singularities. It is proved in [4] (see also [2]) that this notion of solution is intimately related with the notion of generalized flows of interfaces whose definition is recalled next.

Definition 1 (Generalized flow for (21))**.** *An family $\Omega = (\Omega_t)_{t>0}$ (resp. $U = (U_t)_{t>0}$) of open (resp. closed) sets of $\mathbb{R}^N$ is a* generalized super-flow *(resp.* generalized sub-flow*) of (21) if for all $(t_0, x_0) \in (0, +\infty) \times \mathbb{R}^N$, $h > 0$, and for all smooth function $\phi : (0; +\infty) \times \mathbb{R}^N \to \mathbb{R}$ such that*

(i) ***(Boundedness)*** *There exists $r > 0$ such that*

$$\{(t, x) \in [t_0, t_0 + h] \times \mathbb{R}^N : \phi(t, x) \geq 0\} \subset [t_0, t_0 + h] \times B(x_0, r),$$

(ii) ***(Speed)*** *There exists $\delta_\phi > 0$ such that*

$$\partial_t \phi + F^*(D\phi, D^2\phi) \leq -\delta_\phi \text{ in } [t_0, t_0 + h] \times \bar{B}(x_0, r)$$
$$\text{(resp. } \partial_t \phi + F_*(D\phi, D^2\phi) \geq \delta_\phi \text{ in } [t_0, t_0 + h] \times \bar{B}(x_0, r)\text{),}$$

(iii) ***(Non-degeneracy)***

$$D\phi \neq 0 \text{ in } \{(s, y) \in [t_0, t_0 + h] \times \bar{B}(x_0, r) : \phi(s, y) = 0\},$$

(iv) ***(Initial condition)***

$$\{y \in \bar{B}(x_0, r) : \phi(t_0, y) \geq 0\} \subset \Omega_{t_0},$$
$$\text{(resp. } \{y \in \bar{B}(x_0, r) : \phi(t_0, y) \leq 0\} \subset \mathbb{R}^N \setminus F_{t_0}\text{),}$$

then

$$\{y \in \bar{B}(x_0, r) : \phi(t_0 + h, y) > 0\} \subset \Omega_{t_0+h}$$
$$\text{(resp. } \{y \in \bar{B}(x_0, r) : \phi(t_0 + h, y) < 0\} \subset \mathbb{R}^N \setminus F_{t_0+h}\text{).}$$

Remark 3. Remark that this definition slightly differs from the one introduced in [4]. However, a quick look at the proof of the abstract method from [4] that will be used below should convince the reader that this definition is adequate too.

3 The convergence result

This section is devoted to statement of the main result of this paper. We also explain why its proof reduces to the construction of an appropriate "barrier" which will be constructed in the next section.

Loosely speaking, we will prove that solutions of the fractional diffusion-reaction equation (6) approximate, as ε go to 0, the motion of a front moving with a normal speed equal to its mean curvature. Moreover, the results state that the mean curvature motion is anisotropic and that mobilities and diffusion matrices are given by Green-Kubo formulae (see (22), (23) and (25)).

3.1 Statement of the main result

We now state our main result.

Theorem 1 (Convergence result when $\alpha \geq 1$)**.** *Let J be given by* (4) *with $\alpha \geq 1$ in the singular case and let f be a bistable non-linearity. We suppose that Assumptions 1, 2, 3 are satisfied by f. Let u^ε be the unique solution of* (1) *if $\alpha > 1$ and of* (2) *if $\alpha = 1$ associated with a continuous initial datum $u_0^\varepsilon : \mathbb{R}^N \to [\bar{m}_-, \bar{m}_+]$ defined by*

$$u_0^\varepsilon(x) = q^0\left(\frac{d_0(x)}{\varepsilon}, Dd_0(x)\right) \tag{30}$$

where q^0 is the standing wave associated with the diffusion-reaction equation and d_0 is the signed distance function to the boundary of a smooth set Ω_0.

Let u be the unique solution of the geometric equation (21) *supplemented with the initial condition $u(0,x) = d_0(x)$, where μ is given by* (22) *and A is defined in* (26) *in the regular case and* (23), (25) *in the singular case.*

Then the function u^ε satisfies

$$\begin{cases} u^\varepsilon \to \bar{m}_+ & \text{in } \{u > 0\} \\ u^\varepsilon \to \bar{m}_- & \text{in } \{u < 0\} \end{cases} \qquad \text{as} \quad \varepsilon \to 0$$

where $\bar{m}_\pm$ denote the stable zeros of f; moreover both limits are local uniform.

Proof. In order to prove this theorem, we use the abstract method developed in [4] and [2]. Consider two open sets defined as

$$\begin{aligned} \Omega^1 &= \text{Int}\{(t,x) \in (0,+\infty) \times \mathbb{R}^N : \liminf_* \frac{u^\varepsilon - \bar{m}_+}{\eta} \geq 0\} \subset (0,+\infty) \times \mathbb{R}^N \\ \Omega^2 &= \text{Int}\{(t,x) \in (0,+\infty) \times \mathbb{R}^N : \limsup_* \frac{u^\varepsilon - \bar{m}_-}{\eta} \leq 0\} \subset (0,+\infty) \times \mathbb{R}^N \end{aligned}$$

where interior is considered with respect to $(0,+\infty) \times \mathbb{R}^N$.

We next define their traces at initial time by considering the lower semi-continuous function $\chi = \mathbf{1}_{\Omega^1} - \mathbf{1}_{(\Omega^1)^c}$ and the upper semi-continuous function $\bar{\chi} = \mathbf{1}_{(\Omega^2)^c} - \mathbf{1}_{\Omega^2}$ defined on $(0,+\infty) \times \mathbb{R}^N$. They can be extended at $t = 0$

by setting $\chi(0,x) = \liminf_{t\to 0, y\to x} \chi(t,y)$ and $\bar\chi(0,x) = \limsup_{t\to 0, y\to x} \bar\chi(t,y)$. We now define

$$\Omega_0^1 = \{x \in \mathbb{R}^N : \chi(0,x) = 1\} \quad \text{and} \quad \Omega_0^2 = \{x \in \mathbb{R}^N : \bar\chi(0,x) = -1\}\,.$$

The method developed in [4] consists in proving the following propositions.

Proposition 1 (Initial time)**.** *The set $\{x \in \mathbb{R}^N : d_0(x) > 0\}$ is contained in Ω_0^1. Similarly, the set $\{x \in \mathbb{R}^N : d_0(x) < 0\}$ is contained in Ω_0^2.*

Proposition 2 (Propagation)**.** *The set Ω^1 (resp. $\overline{\Omega^2}$) defined above is a generalized super-flow (resp. sub-flow) of* (21)*.*

The proofs of both propositions are postponed. Applying next [2, Corollary 2.1] (see also [4, Corollary 3.1]), we conclude the proof of Theorem 1. □

It remains to prove Propositions 1 and 2. Both rely on the construction of barriers for smooth fronts; in our case, the term "barrier" refers to a sub- or super-solution of the fractional diffusion-reaction equation.

3.2 Proofs of Propositions 1 and 2

As we shall see it, proofs of Propositions 1 and 2 reduce to the proof of the following one.

Proposition 3 (Construction of a barrier)**.** *Given $t_0 > 0$ and $x_0 \in \mathbb{R}^N$, consider a smooth function $\phi : (0,+\infty)\times\mathbb{R}^N \to \mathbb{R}$ such that (i), (ii), (iii) from Definition 1 are satisfied. For all $\beta > 0$, there exists a sub-solution $U^{\varepsilon,\beta}$ of* (1) *if $\alpha > 1$ and* (2) *if $\alpha = 1$ such that*

$$U^{\varepsilon,\beta}(t_0,x) \le (m_+(-\beta\eta) - \beta\eta)\mathbf{1}_{\{d(t_0,\cdot)\ge\beta\}} + m_-(-\beta\eta)\mathbf{1}_{\{d(t_0,\cdot)<\beta\}} \text{ for } x \in \mathbb{R}^N \tag{31}$$

where $d(t,x)$ denotes the signed distance to the set $\{y : \phi(s,y) = 0\}$ which has the same signs as ϕ and $m_\pm(-\beta\eta)$ are the stable equilibria of $f+\beta\eta$. Moreover, if $d(t,x) > -2\beta$, then

$$\liminf{}_*\, \frac{U^{\varepsilon,\beta} - \bar m_+}{\eta}(t,x) \ge -(C_f+2)\beta \tag{32}$$

where C_f appears in (11)*.*

We first derive Proposition 2 from the construction of the barrier.

Proof of Proposition 2. Let us explain why Proposition 3 together with the comparison principle for (6) yield the desired result.

We consider a smooth function ϕ such that (i)-(iv) hold true. Let d denote the signed distance function to $\{\phi = 0\}$. We derive from (iv) that

$$\{d(t_0,\cdot) \ge 0\} \subset \{\phi(t_0,\cdot) \ge 0\} \subset \Omega_{t_0}^1 = \{\liminf{}_*\, \frac{u^\varepsilon - \bar m_+}{\eta} \ge 0\}\,.$$

For the sake of clarity, $m_\pm$ denotes $m_\pm(-\beta\eta)$. And we will do so in the remaining of the paper. Hence

$$\{d(t_0,\cdot)\geq\beta\}\subset \mathrm{int}\{\liminf{}_*\frac{u^\varepsilon-\bar m_+}{\eta}\geq 0\}\,.$$

Hence by using (i), we conclude that u^ε satisfies on one hand

$$u^\varepsilon\geq \bar m_+-\beta\eta\geq m_+-\beta\eta\quad\text{ in }\{d(t_0,\cdot)\geq\beta\}\,.$$

On the other hand, since $\bar m_-$ is a trivial solution of the diffusion-reaction equation, we have $u^\varepsilon\geq\bar m_-\geq m_-$. We thus conclude that u^ε satisfies

$$u^\varepsilon(t_0,x)\geq (m_+-\beta\eta)\mathbf{1}_{\{d(t_0,\cdot)\geq\beta\}}+m_-\mathbf{1}_{\{d(t_0,\cdot)<\beta\}}\text{ for }x\in\mathbb{R}^N\,,\tag{33}$$

We now use Proposition 3 in order to get a sub-solution U^ε with the desired properties. Combining (31) and (33) yields that $U^\varepsilon\leq u^\varepsilon$ at $t=t_0$. We thus conclude by using the comparison principle for (6) that $U^\varepsilon\leq u^\varepsilon$ on $[t_0,t_0+h]\times\mathbb{R}^N$ and (32) implies that

$$\liminf{}_*\frac{u^\varepsilon(t_0+h,x)-\bar m_+}{\eta}\geq -(C_f+2)\beta$$

as soon as $d(t_0+h,x)>2\beta$. Since β is arbitrary, the proof is complete. □

We now prove Proposition 1.

Proof of Proposition 1. We only prove the result for Ω_0^1 since the proof for Ω_0^2 is similar.

Let x_0 be such that $d_0(x_0)=:2\delta>0$. We have to prove that $x_0\in\Omega_0^1$. In other words, for all (t,x) in a neighbourhood of $(0,x_0)$, we would like to prove

$$\liminf{}_*\frac{u^\varepsilon-\bar m_+}{\eta}(t,x)\geq 0\,.$$

In order to get such a result, we construct for any small $\beta>0$ a subsolution $U^{\varepsilon,\beta}$ of (6) such that

$$U^{\varepsilon,\beta}(0,x)\leq u^\varepsilon(0,x)$$

and satisfying (32) for some function $d(t,x)$ such that $\{d>2\beta\}$ contains a neighbourhood of $(0,x_0)$.

There exists $r>0$ such that for any $x\in\bar B(x_0,r)$, $d_0(x)\geq\delta>0$. Consider next the smooth function

$$\phi(t,x)=(r-Ct)_+^2-|x-x_0|^2\,.$$

The associated distance function is given by the following formula

$$d(t,x)=r-Ct-|x-x_0|\,.$$

Remark that $\{d>2\beta\}=\cup_{t\geq0}\{t\}\times B(x_0,r-Ct-2\beta)$.

We claim that (33) holds true with $t_0 = 0$. Indeed, when $d(0,x) \geq \beta$, we know that $d_0(x) \geq \delta$ and this implies that $u^\varepsilon(0,x) \geq m_+ - \beta\eta$ for ε small enough as showed now

$$\begin{aligned} u^\varepsilon(0,x) &\geq q^0\left(\frac{d_0(x)}{\varepsilon}, Dd_0(x)\right) \\ &\geq q^0\left(\frac{\delta}{\varepsilon}, Dd_0(x)\right) \\ &\geq \bar{m}_+ + o(\varepsilon^{1+\alpha}) \\ &\geq m_+ + o(\varepsilon^{1+\alpha}) \\ &\geq m_+ - \beta\eta\,. \end{aligned}$$

Notice that (i) and (iii) are satisfied. As far as (ii) is concerned, it is only used in the construction of the barrier in order to get (34) below. We thus have to prove that we can choose $C > 0$ such that (34) also holds true. The constant C is chosen as follows

$$C \geq \sup_{e\in\mathbb{S}^{N-1}} (-\mu(e)\mathrm{Tr}(A(e)) - \bar{c}(e)) + \frac{\delta_\phi}{2}$$

and (34) holds true for γ and h small enough. □

4 Construction of the barrier

This section is devoted to the proof of Proposition 3.

Proof of Proposition 3. The proof proceeds in several steps. We first construct a sub-solution U of the diffusion-reaction equation close to the smooth front and we then extend it to the whole space.

A barrier close to the front. Using (i), (ii) and (iii), we know that there exists $\gamma > 0$ such that d is smooth on the set

$$Q_\gamma = \{|d| < \gamma\}$$

and $D\phi(s,x) \neq 0$ on Q_γ and

$$\partial_t d \leq \mu(Dd)\mathrm{Tr}(A(Dd)D^2 d) - \frac{\delta_\phi}{2} \quad \text{in } Q_\gamma\,. \tag{34}$$

We used the fact that $|Dd| = 1$ in Q_γ which also implies that $D^2 dDd = 0$ in Q_γ. For $\beta \leq \gamma/2$, we next define a "barrier" as

$$\begin{aligned} U(t,x) = q\left(\frac{d(t,x)-2\beta}{\varepsilon}, \mathcal{D}(t,x), -\beta h\right) \\ + hQ\left(\frac{d(t,x)-2\beta}{\varepsilon}, \mathcal{D}(t,x), t, x, -\beta h\right) - 2\beta h \end{aligned}$$

where $h > 0$ will be chosen later, q denotes the traveling wave given by Assumption 2; the function $\mathcal{D}$ is assumed to be smooth, to coincide with Dd in Q_γ and to be such that $\frac{1}{2} \leq |\mathcal{D}| \leq \frac{3}{2}$. Let us point out that we would like to choose $\mathcal{D} = Dd$ but this function is not well defined everywhere away from the front and even if we prove that U is a subsolution close to the front, U has to be defined everywhere since the diffusion-reaction equation is not local. As far as the function Q is concerned, it will be chosen later.

Plugging the barrier into the diffusion-reaction equation. In order to prove that the barrier we introduced in the previous step is a sub-solution of the diffusion-reaction equation close to the front, we first plug it into the equation.

Lemma 2. *If $h = \eta$ and $\beta \leq \bar{\beta}$ (depending only on δ_ϕ), then the function U satisfies the following inequality in Q_γ*

$$\partial_t U + \frac{1}{\varepsilon\eta}\{-\mathcal{I}_\alpha^\varepsilon U + f(U)\} \leq \frac{\dot{q}(r,e)}{\varepsilon}\left[\mu(e)\mathrm{Tr}(A(e)B) - \mu_\varepsilon \bar{a}_\varepsilon - \frac{\delta_\phi}{4}\right] - \frac{\beta}{\varepsilon} + \frac{1}{\varepsilon}\left[\mathcal{L}Q(r,e,t,x) - (a_\varepsilon(r,e,t,x) - \dot{q}(r,e)\mu_\varepsilon\bar{a}_\varepsilon)\right] + (\mathrm{err}) \quad (35)$$

where $r = \frac{d(t,x)-2\beta}{\varepsilon}$, $e = Dd(t,x)$, $B = D^2 d(t,x)$, $\mu_\varepsilon, \bar{a}_\varepsilon \in \mathbb{R}$ are two real numbers to be chosen later and

$$a_\varepsilon(r,e,t,x) = \frac{1}{\eta}\int\left\{q(r + \frac{d(t,x+\varepsilon z) - d(t,x)}{\varepsilon}, e) - q(r + e\cdot z, e)\right\}J(z)dz\,. \quad (36)$$

As far as error terms are concerned, we have

$$(\mathrm{err}) = \frac{1}{\varepsilon\eta}R[T_q] + \frac{1}{\varepsilon}R[T_Q] + \frac{1}{\varepsilon}\left(-\dot{q}\frac{\delta_\phi}{8} - 2f'(q)\beta\right) + o(\varepsilon^{-1}) \quad (37)$$

and

$$R[T_q] = \int\left\{q\left(\frac{d(t,x+\varepsilon z) - 2\beta}{\varepsilon}, e\right) - q\left(\frac{d(t,x+\varepsilon z) - 2\beta}{\varepsilon}, \mathcal{D}(t,x+\varepsilon z)\right) + BD_e q(r,e)\cdot\varepsilon z \mathbf{1}_B(\varepsilon z)\right\}J(z)dz, \quad (38)$$

$$R[T_Q] = \int\Big\{Q(r + e\cdot z, e, t, x) - Q\left(\frac{d(t,x+\varepsilon z) - 2\beta}{\varepsilon}, \mathcal{D}(t,x+\varepsilon z), t, x+\varepsilon z\right) + (BD_e Q + D_x Q)\cdot\varepsilon z\mathbf{1}_B(\varepsilon z)\Big\}J(z)dz\,. \quad (39)$$

Proof. We compute the quantity $\partial_t U + (\varepsilon\eta)^{-1}\{-\mathcal{I}_\alpha^\varepsilon U + f(U)\}$.

The fact that q is a traveling wave (see (12)) together with a uniform bound on Q with respect to all its variables (we will choose Q below so that it satisfies such a condition) permits to get

$$\begin{aligned} f(U) &= f(q) + hf'(q)Q - 2f'(q)\beta h + O(h^2) \\ &= -\beta h + \mathcal{I}_e[q] - c(e,-\beta h)\dot{q} + hf'(q)Q - 2f'(q)\beta h + O(h^2)\,. \end{aligned}$$

Rearranging terms, we thus obtain, for $(t,x) \in Q_\gamma$,

$$\begin{aligned} \frac{1}{\varepsilon\eta}\{-\mathcal{I}_\alpha^\varepsilon U + f(U)\} \leq \frac{\dot{q}}{\varepsilon}\left[-\frac{c(e,-\beta h)}{\eta}(0,Dd)\right] - \frac{\beta h}{\varepsilon\eta} - 2f'(q)\beta\frac{h}{\varepsilon\eta} \\ + \frac{1}{\varepsilon\eta}(\mathcal{I}_e[q] + hf'(q)Q - \mathcal{I}_\alpha^\varepsilon U) + O(\frac{h^2}{\varepsilon\eta}) \quad (40) \end{aligned}$$

We immediately see from this computation and in view of (17) that h must be chosen as follows

$$h = \eta\,.$$

We next write

$$\mathcal{I}_e[q](r,e) + \eta f'(q)Q - \mathcal{I}_\alpha^\varepsilon U(t,x) = \eta T_Q + T_q \quad (41)$$

where

$$T_Q = f'(q)\,Q - \mathcal{I}_\alpha^\varepsilon(Q(\varepsilon^{-1}(d-2\beta), Dd, t, \cdot)) = \mathcal{L}Q - c\dot{Q} + R[T_Q] \quad (42)$$

and

$$\begin{aligned} T_q &= \mathcal{I}_e[q](r) - \mathcal{I}_\alpha^\varepsilon[q(\varepsilon^{-1}(d-2\beta), \mathcal{D})](x) \\ &= \int\left\{q(r+e\cdot z, e) - q(r,e) - \dot{q}(r,e)e\cdot z\mathbf{1}_B(z)\right\}J(z)dz \\ &\quad - \int\left\{q\left(\frac{d(t,x+\varepsilon z)-2\beta}{\varepsilon}, \mathcal{D}(t,x+\varepsilon z)\right) - q(r,e)\right. \\ &\qquad \left. - \left(\dot{q}(r,e)\frac{e}{\varepsilon} + BD_e q(r,e)\right)\cdot\varepsilon z\mathbf{1}_B(\varepsilon z)\right\}J(z)dz\,. \end{aligned}$$

Hence, T_q can be written as follows

$$T_q = -\eta a_\varepsilon + R[T_q]\,. \quad (43)$$

We now compute the time derivative of the barrier U. We use (34) in order to get

$$\begin{aligned} &\partial_t U = \frac{\dot{q}}{\varepsilon}[\partial_t d] + (D_e q + \eta D_e Q)\cdot D(\partial_t d) + \eta\varepsilon^{-1}\dot{Q}\partial_t d + \eta\partial_t Q \\ &\leq \frac{\dot{q}}{\varepsilon}\left[\mu(e)\mathrm{Tr}(A(e)B) - \frac{\delta_\phi}{2}\right] + (D_e q + \eta D_e Q)\cdot D(\partial_t d) + \eta\varepsilon^{-1}\dot{Q}\partial_t d + \eta\partial_t Q. \end{aligned} \quad (44)$$

We next combine (40), (41), (42), (43) and (44) to get

$$\partial_t U + \frac{1}{\varepsilon\eta}\{-\mathcal{I}_\alpha^\varepsilon U + f(U)\} \leq \frac{\dot q}{\varepsilon}\left[\mu(e)\mathrm{Tr}(A(e)B) - \frac{\delta_\phi}{4} - \frac{c(e,-\beta\eta)}{\eta} - \frac{\delta_\phi}{8}\right] - \frac{\beta}{\varepsilon} + \frac{1}{\varepsilon}\left[-a_\varepsilon + \mathcal{L}Q\right] + (\mathrm{err})$$

with

$$(\mathrm{err}) = \frac{1}{\varepsilon\eta}R[T_q] + \frac{1}{\varepsilon}R[T_Q] - \frac{c}{\eta}\dot Q + \frac{1}{\varepsilon}\left(-\dot q\frac{\delta_\phi}{8} - 2f'(q)\beta\right) + (D_e q + \eta D_e Q)\cdot D(\partial_t d) + \eta\varepsilon^{-1}\dot Q\partial_t d + \eta\partial_t Q + o(\varepsilon^{-1})\,.$$

Using (17), (14) and (19), we finally get (35) with the associated error term. □

Estimating error terms. In this paragraph, we prove that the right hand side of (35) is non-positive. We first construct a corrector Q in order to handle oscillating terms.

Lemma 3 (Choice of the corrector Q)**.** *There exist $\mu_\varepsilon, \bar a_\varepsilon \in \mathbb{R}$ such that there exists Q satisfying*

$$\mathcal{L}Q = a_\varepsilon - \dot q\mu_\varepsilon\bar a_\varepsilon\,. \tag{45}$$

Proof. In view of Proposition 3, it is enough to choose μ_ε and $\bar a_\varepsilon$ such that

$$\int (a_\varepsilon(\xi) - \mu_\varepsilon\bar a_\varepsilon\dot q(\xi))\dot q(\xi)d\xi = 0.$$

The following choices permit to ensure such a condition

$$\mu_\varepsilon(e) = \left(\int \dot q^2(\xi,e)d\xi\right)^{-1}$$

and

$$\bar a_\varepsilon(e,t,x) = \int \dot q(\xi,e)a_\varepsilon(\xi,e,t,x)d\xi = \frac{1}{\eta}\iint \dot q(\xi,e)\Big\{q(\xi + e\cdot z + \varepsilon W(t,x,z),e) - q(\xi + e\cdot z,e)\Big\}J(z)d\xi dz$$

with

$$W(t,x,z) = \frac{1}{\varepsilon^2}[d(t,x+\varepsilon z) - d(t,x) - \varepsilon Dd(t,x)\cdot z]\,.$$

□

Remark 4. The choice of h when rescaling fractional diffusion-reaction equations (6) is made such that $\bar a_\varepsilon$ has a limit as $\varepsilon \to 0$.

The following lemma is the core of the proof of Theorem 1 and its proof is rather involved. This is the reason why we postpone it until Section 5.

Lemma 4 (Uniform convergence of approximate coefficients (I)). *As $\varepsilon \to 0$,*

$$\bar{a}_\varepsilon(e,t,x) \to \text{tr}\,(A(e)D^2 d(t,x))$$

and the limit is uniform with respect to $(e,t,x) \in \mathbb{S}^{N-1} \times Q_\gamma$.

We next treat error terms appearing in (err).

Lemma 5 (Error terms (err)). *We have*

$$R[T_q] = o(\varepsilon^\alpha) = o(\eta) \tag{46}$$

and

$$R[T_Q] = o(1) \tag{47}$$

uniformly in $(e,t,x) \in \mathbb{S}^{N-1} \times Q_\gamma$ and for all $r \in \mathbb{R}$ and $\beta \le \bar{\beta} = \bar{\beta}(\delta_\phi)$

$$-\dot{q}(r)\frac{\delta_\phi}{8} - 2f'(q(r))\beta \le 0\,. \tag{48}$$

Proof. We first prove (46). Through a change of variables, we get

$$\begin{aligned} R[T_q] = \varepsilon^\alpha \int \Big\{ & q\left(\frac{d(t,x+\bar{z})-2\beta}{\varepsilon}, \mathcal{D}(t,x)\right) \\ & - q\left(\frac{d(t,x+\bar{z})-2\beta}{\varepsilon}, \mathcal{D}(t,x+\bar{z})\right) - BD_e q(r,e)\cdot \bar{z}\mathbf{1}_B(\bar{z})\Big\} J_\varepsilon(\bar{z})d\bar{z} \end{aligned}$$

where $J_\varepsilon(\bar{z}) = \varepsilon^{-(N+\alpha)}J(\varepsilon^{-1}\bar{z})$. By using (8) and (14), dominated convergence theorem permits to conclude.

We next turn to the proof of (47). To prove it, we first write

$$R[T_Q] = R^1[T_Q] - R^2[T_Q] \tag{49}$$

with

$$\begin{aligned} R^1[T_Q] &= \int \left\{ Q(r+e\cdot z, e,t,x) - Q\left(\frac{d(t,x+\varepsilon z)-2\beta}{\varepsilon}, e,t,x\right)\right\} J(z)dz\,, \\ R^2[T_Q] &= \int \Big\{ Q\left(r+e\cdot z, \mathcal{D}(t,x+\varepsilon z), t, x+\varepsilon z\right) \\ &\qquad -Q(r+e\cdot z,e,t,x) - (BD_eQ + D_xQ)\cdot \varepsilon z \mathbf{1}_B(\varepsilon z)\Big\} J(z)dz\,. \end{aligned}$$

As far as $R^1[T_Q]$ is concerned, we can write for any $R>0$

$$\begin{aligned} R^1[T_Q] &\le \|\dot{Q}\|_\infty \frac{1}{2}\|D^2 d\|_{L^\infty(B(x,\varepsilon R))}\varepsilon \int_{|z|\le R} |z|^2 J(z)dz \\ &\quad +2\|Q\|_\infty \int_{|z|\ge R} J(z)dz \\ &\le C\|\dot{Q}\|_\infty \frac{1}{2}\|D^2 d\|_{L^\infty(B(x,\varepsilon R))}\varepsilon R^{2-\alpha} + 2C\|Q\|_\infty R^{-\alpha} \end{aligned}$$

where we used (8) to get the second inequality. Choose now R such that $\varepsilon R \leq 1$, $R \to +\infty$ and $\varepsilon R^{2-\alpha} \to 0$; for instance $R = \varepsilon^{-1/2}$ permits to conclude in this case.

As far as $R^2[T_Q]$ is concerned, we use once again (8) in order to write

$$|R^2[T_Q]| \leq \varepsilon^\alpha \int \Big| Q\left(\dots, \mathcal{D}(t, x+\bar{z}), t, x+\bar{z}\right) - Q(\dots, \mathcal{D}(t,x), t, x) - (BD_eQ + D_xQ)\cdot \bar{z}\mathbf{1}_B(\bar{z}) \Big| C_J \frac{d\bar{z}}{|\bar{z}|^{N+\alpha}} \leq C\varepsilon^\alpha$$

where C depends on $\sup_h\{\|Q\|_\infty + \|D^2_{e,e}Q\|_\infty + \|D^2_{x,x}Q\|_\infty\}$ that is bounded by assumption (see Estimate (19)).

It remains to prove (48). It is enough to prove that there exists a constant C_{tw} which does not depend on h and such that for all $r \in \mathbb{R}$

$$\dot{q}(r) + C_{tw} f'(q(r)) \geq 0\,. \tag{50}$$

This inequality is trivial when $f'(q(r)) \geq 0$. Hence, we consider r such that $f'(q(r)) \leq 0$, that is to say

$$\bar{m}_- < \bar{q}_- \leq q(r) \leq \bar{q}_+ < \bar{m}_+$$

for some constants $\bar{q}_\pm$ which do not depend on h. If r satisfies the previous inequality, we deduce from (13) that

$$|r| \leq R$$

for some constant R which does not depend either on h. Now (50) is clear. It is enough to find an estimate from below for $\dot{q}$ on $[-R, R]$ which does not depend on h. The proof of the lemma is now complete. □

Using Lemmata 3, 4 and 5 we derive from (35) the following inequality

$$\partial_t U + \frac{1}{\varepsilon\eta}\{-\mathcal{I}^\varepsilon_\alpha U + f(U)\} \leq \frac{-\beta}{\varepsilon} + o(\frac{1}{\varepsilon}) \leq \frac{-\beta}{2\varepsilon}\,. \tag{51}$$

Extension of the barrier away from the front. The remaining of the construction of the barrier consists in extending the subsolution U we constructed before in order that it is a subsolution in $[t_0, t_0 + h] \times \bar{B}(x_0, r)$ (in particular, far from the front). More precisely, we modify U far from the front. Following [4, 2], we proceed in two steps. We first extend it on $\{d \leq \gamma\}$ by m_- and then extend it on $\{d \geq \gamma\}$ by $m_+ - \beta\eta$ The difficulty is to keep it a subsolution. We do this by truncating properly U. Truncating it from below by m_- is easy but truncating it from above by $m_+ - \beta\eta$ is more delicate.

UPPER ESTIMATES FOR U. We start by estimating from above the "barrier" function U we constructed before. We claim that the following inequalities hold true

$$\begin{aligned} U(t,x) &\leq m_- \quad \text{in } \{d \leq \beta\}\,, & (52)\\ U(t,x) &\leq m_+ - \beta\eta\,. & (53) \end{aligned}$$

We first justify (52). In view of the definition of U, we use (13) and (20), in order to get

$$U(t,x) \leq m_- + O(\varepsilon^{1+\alpha}) - 2\beta\eta \leq m_-\,.$$

We next justify (53) by adapting an argument from [4]. First, (20) implies that there exists $\bar{c} > 0$ which does not depend on h such that we have for $|r| \geq \bar{c}$

$$|Q(r,e,t,x)| \leq \beta\,.$$

Next, we claim that there exists $\nu(\bar{c}) > 0$ such that we have for $|r| \leq \bar{c}$

$$q(r) \leq m_+ - \nu(\bar{c})\,. \tag{54}$$

Now if $|d(t,x) - 2\beta| \geq \varepsilon\bar{c}$, then

$$U(t,x) \leq m_+ + \beta\eta - 2\beta\eta \leq m_+ - \beta\eta\,.$$

In the other case, $|d(t,x) - 2\beta| \leq \varepsilon\bar{c}$, then

$$U(t,x) \leq m_+ - \nu(\bar{c}) + \eta\|Q\|_\infty - 2\beta\eta \leq m_+ - \beta\eta$$

as soon as $\eta\|Q\|_\infty \leq \nu(\bar{c})$.

DEFINITION OF $\bar{U}$. We define for $(t,x) \in [t_0, t_0+h] \times \mathbb{R}^N$

$$\bar{U}(t,x) = \max(U(t,x), m_-)\,.$$

From (52) and (53), we get

$$\begin{aligned} \bar{U} &= m_- \quad \text{in } \{d \leq \beta\}\,, & (55)\\ m_- &\leq \bar{U} \leq m_+ - \beta\eta\,. & (56) \end{aligned}$$

On one hand, a classical argument implies that $\bar{U}$ is a subsolution of (6) on $Q_\gamma = \{-\gamma \leq d \leq \gamma\}$ since it is the maximum of two subsolutions. On the other hand, (52) implies that $\bar{U}(t,x) = m_-$ on $\{d \leq -\gamma/3\}$. Thus $\bar{U}$ is a subsolution on $\{d \leq \gamma\}$.

We also shed light on the fact that $\bar{U}$ satisfies (51) at points of Q_γ where $\bar{U} = U$. This will be used later on. We will also need the following lemma.

Lemma 6 (Gradient estimate for the barrier)**.** *There exists $\bar{C} > 0$ such that for all $(t,x) \in [t_0, t_0+h] \times \mathbb{R}^N$ and all $\varepsilon >$ small enough,*

$$|\varepsilon D\bar{U}(t,x)| \leq \bar{C}\,.$$

Proof. This is a simple consequence of the definition of $\bar{U}$ and of Estimates (14) and (19). □

DEFINITION OF $U^{\varepsilon,\beta}$. We finally define

$$U^{\varepsilon,\beta} = \begin{cases} \psi(d)\bar{U} + (1-\psi(d))(m_+ - \beta\eta) & \text{if } d < \gamma\,, \\ m_+ - \beta\eta & \text{if } d \geq \gamma \end{cases}$$

where $\psi : \mathbb{R} \to [0,1]$ is a smooth function such that $\psi(r) = 1$ if $r \leq \gamma/2$, $\psi(r) = 0$ if $r \geq 3\gamma/4$. We will see below that it is convenient to assume additionally that $\psi(5\gamma/8 + r) = 1 - \psi(5\gamma/8 - r)$. We deduce from properties of ψ and (55) and (56) that

$$U^{\varepsilon,\beta} = m_- \qquad \text{in } \{d \leq \beta\} \tag{57}$$

$$U^{\varepsilon,\beta} = m_+ - \beta\eta \qquad \text{in } \{d \geq 3\gamma/4\} \tag{58}$$

$$m_- \leq U^{\varepsilon,\beta} \leq m_+ - \beta\eta\,. \tag{59}$$

In particular, (31) is clearly satisfied.

We also deduce from (59) and the definitions of $\bar{U}$ and $U^{\varepsilon,\beta}$ that

$$U \leq \bar{U} \leq U^{\varepsilon,\beta}\,. \tag{60}$$

In particular, if $d(t,x) > 2\beta$, then $d > 2\beta$ in a neighbourhood $\mathcal{V}$ of (t,x). In particular,

$$U^{\varepsilon,\beta} \geq U = m_+ + o(\eta) - 2\beta\eta \quad \text{ in } \mathcal{V}\,.$$

Using now (11), we deduce that (32) also holds true.

The barrier $U^{\varepsilon,\beta}$ is a subsolution of (6) on $[t_0, t_0+h]\times\mathbb{R}^N$. We distinguish three cases.

Consider first a point (t,x) such that $d(t,x) < \gamma/2$. In this case, $U^{\varepsilon,\beta}(s,y) = \bar{U}(s,y)$ in a neighbourhood of (t,x) and this implies $\partial_t U^{\varepsilon,\beta}(t,x) = \partial_t \bar{U}(t,x)$ (in the viscosity sense). In order to prove that $U^{\varepsilon,\beta}$ is a subsolution of (6) at (t,x) it is enough to prove that $\mathcal{I}_\alpha^\varepsilon \bar{U}(t,x) \leq \mathcal{I}_\alpha^\varepsilon U^{\varepsilon,\beta}(t,x)$ since $\bar{U}$ is a subsolution. Such an inequality is a consequence of (60) and the fact that $U^{\varepsilon,\beta}(t,x) = \bar{U}(t,x)$.

Consider next a point (t,x) such that $d(t,x) > 3\gamma/4$. In this case, there exists $r_0 > 0$ such that $d(s,y) > 3\gamma/4$ for $y \in B((t,x), r_0)$. Consequently, $U^{\varepsilon,\beta}(s,y) = m_+ - \beta\eta$ for $(s,y) \in B((t,x), r_0)$. This yields that $\partial_t U^{\varepsilon,\beta}(t,x) = 0$ (in the viscosity sense) so we have to prove that

$$f(U^{\varepsilon,\beta}(t,x)) \leq \mathcal{I}_\alpha^\varepsilon(U^{\varepsilon,\beta})(t,x)\,.$$

To get the previous inequality, on one hand, we have

$$\begin{aligned} f(U^{\varepsilon,\beta}(t,x)) &= f(m_+ - \beta\eta) \\ &= f(m_+(-\beta\eta)) - f'(m_+ - \theta\beta\eta)\beta\eta \\ &\leq -\beta\eta \end{aligned}$$

and on the other hand,

$$\begin{aligned}\mathcal{I}_\alpha^\varepsilon(U^{\varepsilon,\beta})(t,x) &= \int [U^{\varepsilon,\beta}(t,x+\varepsilon z)-(m_+-\beta\eta)]J(z)dz \\ &= \int_{|z|\geq r_0/\varepsilon} [U^{\varepsilon,\beta}(t,x+\varepsilon z)-(m_+-\beta\eta)]J(z)dz \\ &\geq -C\int_{|z|\geq r_0/\varepsilon} J(z)dz = -C\varepsilon^\alpha \geq -\beta\eta\end{aligned}$$

in view of the definition of η. Notice that this argument fails in the case $\alpha < 1$.

Finally, we consider (t,x) such that $\gamma/2 \leq d(t,x) \leq 3\gamma/4$.

It is convenient to introduce $\psi_d(x) = \psi(d(x))$. Remark that $\bar{U} = U$ in a neighbourhood of (t,x). Hence, we mentioned above that $\bar{U}$ satisfies (51) at (t,x)

$$\partial_t \bar{U} + \frac{1}{\varepsilon\eta}\{-\mathcal{I}_\alpha^\varepsilon \bar{U} + f(\bar{U})\} \leq \frac{-\beta}{2\varepsilon}\,. \tag{61}$$

We use (61) and compute (in the viscosity sense)

$$\begin{aligned}&\partial_t U^{\varepsilon,\beta} + \frac{1}{\varepsilon\eta}\left(f(U^{\varepsilon,\beta}) - \mathcal{I}_\alpha^\varepsilon U^{\varepsilon,\beta}\right) \\ &\quad = \psi' \times (\partial_t d) \times (\bar{U} - m_+ + \beta\eta) + \psi_d \partial_t \bar{U} + \frac{1}{\varepsilon\eta} f(U^{\varepsilon,\beta}) - \frac{1}{\varepsilon\eta} I_\alpha^\varepsilon(U^{\varepsilon,\beta}) \\ &\qquad \leq C(\psi)\left|\bar{U} - (m^+(-\beta\eta)-\beta\eta)\right| + \frac{1}{\varepsilon\eta}\left[f(U^{\varepsilon,\beta}) - \psi_d f(\bar{U})\right] \\ &\qquad\qquad + \frac{1}{\varepsilon\eta}\left[\psi_d \mathcal{I}_\alpha^\varepsilon \bar{U} - \mathcal{I}_\alpha^\varepsilon U^{\varepsilon,\beta}\right] - \psi_d \frac{\beta}{2\varepsilon}\end{aligned} \tag{62}$$

where $C(\psi)$ only depends on ψ and γ. We now estimate each term of the right hand side of (62).

First, we derive directly from the equality $\bar{U} = U$ and the very definition of U the following lemma

Lemma 7. *We have $\bar{U} = m_+ - 2\beta\eta + o(\beta\eta)$. In particular,*

$$\left|\bar{U} - (m^+ - \beta\eta)\right| \leq 2\beta\eta\,. \tag{63}$$

We now estimate the second term of the right hand side of (62).

Lemma 8.

$$f(U^{\varepsilon,\beta}) - \psi_d f(\bar{U}) \leq -\frac{1}{2}(1-\psi_d)\beta\eta\,. \tag{64}$$

Proof. From Lemma 7, we have

$$\begin{aligned}U^{\varepsilon,\beta} &= \psi_d(m_+ - 2\beta\eta + o(\beta\eta)) + (1-\psi_d)(m_+ - \beta\eta) \\ &= m_+ - \beta\eta - \psi_d(1+o(1))\beta\eta\,.\end{aligned}$$

In particular, $U^{\varepsilon,\beta} < m_+$. Hence,

$$\begin{aligned} f(U^{\varepsilon,\beta}) &= f(U^{\varepsilon,\beta}) - f(m_+) + f(m_+) \\ &\leq -(f'(\bar{m}_+) + o(1))(1 + \psi_d + o(1))\beta\eta - \beta\eta \\ &\leq -(f'(\bar{m}_+) + 1 + o(1))\beta\eta \end{aligned}$$

Lemma 7 also implies

$$0 \leq U^{\varepsilon,\beta} - \bar{U} = (1 - \psi_d)(1 + o(1))\beta\eta\,.$$

Hence, we obtain

$$\begin{aligned} f(U^{\varepsilon,\beta}) - \psi_d f(\bar{U}) &= (1 - \psi_d) f(U^{\varepsilon,\beta}) + \psi_d (f(U^{\varepsilon,\beta}) - f(\bar{U})) \\ &\leq -(1 - \psi_d)(f'(\bar{m}_+) + 1 + o(1))\beta\eta \\ &\quad + \psi_d (f'(\bar{m}_+) + o(1))(1 - \psi_d)(1 + o(1))\beta\eta \\ &\leq (-1 + o(1))(1 - \psi_d)\beta\eta \leq -\frac{1}{2}(1 - \psi_d)\beta\eta\,. \end{aligned}$$

□

We now turn to the third term of the right hand side of (62) whose estimate is more delicate. It is given by the following technical lemma.

Lemma 9. *For any $\gamma_0 > 0$, there exists a function $O(\varepsilon^\alpha)$ such that*

$$\psi_d \mathcal{I}_\alpha^\varepsilon \bar{U} - \mathcal{I}_\alpha^\varepsilon U^{\varepsilon,\beta} \leq \gamma_0 \eta + O(\varepsilon^\alpha)\,. \tag{65}$$

Proof. We would like first to point out that we can forget the time variable in this proof since it plays no role.

We first remark that for $r_0 = \gamma/4$,

$$|z| \leq \frac{r_0}{\varepsilon} \Rightarrow U^{\varepsilon,\beta}(x + \varepsilon z) = \psi_d(x + \varepsilon z)(\bar{U}(x + \varepsilon z) - m_+ + \beta\eta) + (m_+ - \beta\eta)\,.$$

Indeed, $|d(x + \varepsilon z) - d(x)| \leq r_0 = \gamma/4$ and this implies $d(x + \varepsilon z) \in (\gamma/4, \gamma)$. In particular $d(x + \varepsilon z) \leq \gamma$ and (63) holds true.

We next approximate the quantity we are estimating by truncating large z's. Precisely, using the previous remark and the fact that the mass of J outside $B_{r_0/\varepsilon}$ is $O(\varepsilon^\alpha)$, we write

$$\begin{aligned} \psi_d \mathcal{I}_\alpha^\varepsilon \bar{U} - \mathcal{I}_\alpha^\varepsilon U^{\varepsilon,\beta} &= \psi_d \mathcal{J}^\varepsilon \bar{U} - \mathcal{J}^\varepsilon[\psi_d(\bar{U} - m_+ + \beta\eta) + (m_+ - \beta\eta)] + O(\varepsilon^\alpha) \\ &= \psi_d \mathcal{J}^\varepsilon(\bar{U} - m_+ + \beta\eta) - \mathcal{J}^\varepsilon[\psi_d(\bar{U} - m_+ + \beta\eta)] + O(\varepsilon^\alpha) \end{aligned}$$

where the operator $\mathcal{J}^\varepsilon$ is defined as follows

$$\mathcal{J}^\varepsilon \varphi(x) = \int_{\varepsilon|z| \leq r_0} [\varphi(x + \varepsilon z) - \varphi(x) - D\varphi(x) \cdot \varepsilon z \mathbf{1}_B(z)] J(z) dz$$

and where $O(\varepsilon^\alpha)$ only depends on $\|Q\|_\infty$ and $\bar{m}_\pm$ (for ε small enough).

We next use the following equality

$$\mathcal{J}^\varepsilon(\varphi\varphi')(x) - \varphi(x)\mathcal{J}^\varepsilon\varphi'(x) - \varphi'(x)\mathcal{J}^\varepsilon\varphi(x) \\ = \int_{\varepsilon|z|\le r_0} (\varphi(x+\varepsilon z)-\varphi(x))(\varphi'(x+\varepsilon z)-\varphi'(x))J(z)dz$$

with $\varphi = \psi_d$ and $\varphi' = \bar{U} - m_+ + \beta\eta$. We obtain

$$\psi_d\mathcal{I}^\varepsilon_\alpha\bar{U} - \mathcal{I}^\varepsilon_\alpha U^{\varepsilon,\beta} = -(\bar{U}-m_+ +\beta\eta)\mathcal{J}^\varepsilon\psi_d \\ - \int_{\varepsilon|z|\le r_0} (\psi_d(x+\varepsilon z)-\psi_d(x))(\bar{U}(x+\varepsilon z)-\bar{U}(x))J(z)dz + O(\varepsilon^\alpha)\,.$$

Recalling that (63) holds true for $|z| \le \varepsilon^{-1}r_0$, we write

$$\psi_d\mathcal{I}^\varepsilon_\alpha\bar{U} - \mathcal{I}^\varepsilon_\alpha U^{\varepsilon,\beta} \le 2\beta\eta|\mathcal{J}^\varepsilon\psi_d(x)| \\ + \varepsilon\|D\psi_d\|_{L^\infty(B(x,r_0))}\left(\varepsilon\|D\bar{U}\|_{L^\infty(B(x,r_0))}\right)\int_{|z|\le\varepsilon}|z|^2J(z)dz \\ + \varepsilon\|D\psi_d\|_{L^\infty(B(x,r_0))}4\beta\eta\int_{\varepsilon\le|z|\le r_0\varepsilon^{-1}}|z|J(z)dz + O(\varepsilon^\alpha)\,.$$

We next estimate each term as follows.

$$\begin{aligned} |\mathcal{J}^\varepsilon\psi_d(x)| &\le C(\psi_d)\varepsilon^\alpha\,, \\ \int_{|z|\le\varepsilon}|z|^2J(z)dz &\le C\varepsilon^{2-\alpha}\,, \\ \int_{\varepsilon\le|z|\le r_0\varepsilon^{-1}}|z|J(z)dz &\le \begin{cases} C\varepsilon^{1-\alpha} & \text{if } \alpha>1 \\ C|\ln\varepsilon| & \text{if } \alpha=1 \end{cases} \le \frac{1}{\varepsilon}\,. \end{aligned}$$

The first estimate is easily obtained by adapting the arguments used above to estimate $R^1[T_Q]$ and $R^2[T_Q]$. Moreover, the constant $C(\psi_d)$ only depends on $\|\psi_d\|_\infty = 1$ and $\|D^2\psi_d\|_{L^\infty(B(x,r_0))}$. This last quantity only depends on γ and $\|Dd\|_{L^\infty(B(x,r_0))}$, $\|D^2d\|_{L^\infty(B(x,r_0))}$. Hence, by using Lemma 6, we have

$$\psi_d\mathcal{I}^\varepsilon_\alpha\bar{U} - \mathcal{I}^\varepsilon_\alpha U^{\varepsilon,\beta} \le C(\eta\varepsilon^\alpha + \varepsilon^{3-\alpha} + \eta) + O(\varepsilon^\alpha)$$

(we used that $\beta \le 1$ for instance). We achieve the proof by choosing ε small enough so that

$$C(\eta\varepsilon^\alpha + \varepsilon^{3-\alpha} + \eta) \le \gamma_0\,.$$

□

We now combine (62), (64) and (65) to get

$$\partial_t U^{\varepsilon,\beta} + \frac{1}{\varepsilon\eta}\left(f(U^{\varepsilon,\beta}) - \mathcal{I}^\varepsilon_\alpha U^{\varepsilon,\beta}\right) \le C\beta\eta - (1-\psi_d)\frac{\beta}{2\varepsilon} - \psi_d\frac{\beta}{2\varepsilon} + \frac{\gamma_0}{\varepsilon} + o(\frac{1}{\varepsilon})\,.$$

This is where it is convenient to choose ψ such that $\psi(5\gamma/8+r) = 1-\psi(5\gamma/8-r)$ since in this case, $\max(\psi_d, 1-\psi_d) \geq 1/2$ and we obtain

$$\partial_t U^{\varepsilon,\beta} + \frac{1}{\varepsilon\eta}\left(f(U^{\varepsilon,\beta}) - \mathcal{I}_\alpha^\varepsilon U^{\varepsilon,\beta}\right) \leq C\beta\eta - \frac{\beta}{4\varepsilon} + \frac{\gamma_0}{\varepsilon} + o\left(\frac{1}{\varepsilon}\right) .$$

Choosing now γ_0 small enough, we finally get

$$\partial_t U^{\varepsilon,\beta} + \frac{1}{\varepsilon\eta}\left(f(U^{\varepsilon,\beta}) - \mathcal{I}_\alpha^\varepsilon U^{\varepsilon,\beta}\right) \leq C(\psi)\beta\eta - \frac{\beta}{8\varepsilon} .$$

It is now clear that for ε small enough, $U^{\varepsilon,\beta}$ is a subsolution of (6) in $[t_0, t_0 + h] \times \mathbb{R}^N$. □

5 Proof of Lemma 4

This section is devoted to the study of the average of oscillating terms. Their behaviour as $\varepsilon \to 0$ was given by Lemma 4 whose proof was postponed. We first deal with the singular case with $\alpha > 1$. We next prove the result in the regular case. We then state the equivalent lemma for the case $\alpha < 1$ since ideas will be used in the case $\alpha = 1$. We finally prove Lemma 4 in the case $\alpha = 1$.

Proof of Lemma 4 in the singular case for $\alpha > 1$. We first recall the definition of $\bar{a}_\varepsilon$ and W. For the sake of clarity, we do not write e and h variables of q since they play no role in the present argument.

$$\begin{aligned}\bar{a}_\varepsilon(e,t,x) &= \int \dot{q}(\xi) a_\varepsilon(\xi,e,t,x) d\xi \\ &= \frac{1}{\varepsilon}\iint \dot{q}(\xi)\Big\{q(\xi + e\cdot z + \varepsilon W(t,x,z)) - q(\xi + e\cdot z)\Big\} J(z) d\xi dz\end{aligned}$$

with

$$W(t,x,z) = \frac{1}{\varepsilon^2}[d(t,x+\varepsilon z) - d(t,x) - \varepsilon Dd(t,x)\cdot z] .$$

We proceed in several steps.

Step 1: reduction to the study of the singular integral around the origin for quadratic W's. Let us choose r_ε such that we also have (see (8))

$$\frac{1}{\varepsilon}\int_{|z|\geq r_\varepsilon} J(z) dz \to 0 \quad \text{as } \varepsilon \to 0 .$$

For instance we consider $r_\varepsilon = \varepsilon^{-\beta}$ with $\beta > 1/\alpha$. In view of Condition (8), we thus can assume from now on that

$$J(z) = g(\hat{z})\frac{1}{|z|^{N+\alpha}} .$$

Since q is bounded, it is therefore enough to study the convergence of

$$b_\varepsilon(e,t,x) = \int\int_{|z|\le r_\varepsilon} \dot q(\xi)\frac{1}{\varepsilon}\Big\{q(\xi + e\cdot z + \varepsilon W(t,x,z),e) - q(\xi + e\cdot z)\Big\}J(z)d\xi dz\ .$$

For $|z|\le r_\varepsilon$,

$$W(t,x,z) \to \frac{1}{2}D^2 d(t,x)z\cdot z \quad \text{ as } \varepsilon\to 0$$

as soon as one chooses r_ε such that $\varepsilon r_\varepsilon \to 0$. Hence we take $\beta\in(\alpha^{-1},1)$. If B denotes $\frac{1}{2}D^2d(t,x)$, we have for $|z|\le r_\delta$,

$$|W(t,x,z) - Bz\cdot z| \le \delta|z|^2\ .$$

By using the monotonicity of q, we thus can reduce the study of b_ε to the study of

$$c_\varepsilon = \int\int_{|z|\le r_\varepsilon} \dot q(\xi)\frac{1}{\varepsilon}\Big\{q(\xi + e\cdot z + \varepsilon Cz\cdot z) - q(\xi + e\cdot z)\Big\}J(z)d\xi dz\ .$$

Step 2: integrating by parts. By using a system of coordinates where $z_1 = e\cdot z$ and $z=(z_1,z')$, we can decompose the matrix C as follows

$$C = \left[\begin{array}{cc} c_1 & v^* \\ v & C'\end{array}\right]$$

Hence, we can write

$$\begin{aligned} c_\varepsilon &= \int\int_{|z|\le r_\varepsilon}\int_0^1 \dot q(\xi)\dot q(\xi + z_1 + \varepsilon\tau Cz\cdot z)(Cz\cdot z)J(z_1,z')d\xi dz_1 dz' d\tau \\ &= \int\int_{|z|\le r_\varepsilon}\int_0^1 \dot q(\xi)\dot q(\xi + z_1 + \varepsilon\tau Cz\cdot z)(Cz\cdot z)J(z_1,z')d\xi dz_1 dz' d\tau \\ &= \int\int_{|z|\le r_\varepsilon}\int_0^1 \dot q(\xi)\partial_{z_1}\Big\{q(\xi + z_1 + \varepsilon\tau Cz\cdot z) - q(\xi)\Big\} \\ &\quad \times\frac{Cz\cdot z}{1+2\varepsilon\tau(c_1z_1 + v\cdot z')}J(z_1,z')d\xi dz_1 dz' d\tau \end{aligned}$$

We now integrate by parts with respect to z_1.

$$\begin{aligned} c_\varepsilon &= -\int\int_{|z|\le r_\varepsilon}\int_0^1 \dot q(\xi)\Big\{q(\xi + z_1 + \varepsilon\tau Cz\cdot z)\Big\} \\ &\quad \times\partial_{z_1}\left\{\frac{Cz\cdot z}{1+2\varepsilon\tau(c_1z_1 + v\cdot z')}J(z_1,z')\right\}d\xi dz_1 dz' d\tau + (BT)^1_+ - (BT)^1_- \end{aligned}$$

with

$$(BT)^1_\pm = \int\int_{|z'|\le r_\varepsilon}\int_0^1 \dot q(\xi)\Big\{q(\xi \pm \sqrt{r_\varepsilon^2-|z'|^2} + \varepsilon\tau C(\pm\sqrt{r_\varepsilon^2-|z'|^2},z')\cdot(\pm\sqrt{r_\varepsilon^2-|z'|^2},z'))\Big\} \times\left\{\frac{C(\pm\sqrt{r_\varepsilon^2-|z'|^2},z')\cdot(\pm\sqrt{r_\varepsilon^2-|z'|^2},z')}{1+2\varepsilon\tau(\pm c_1\sqrt{r_\varepsilon^2-|z'|^2}+v\cdot z')}J(\pm\sqrt{r_\varepsilon^2-|z'|^2},z')\right\}d\xi dz'd\tau\,.$$

We next integrate by parts with respect to ξ.

$$\begin{aligned} c_\varepsilon &= \int\int_{|z|\le r_\varepsilon}\int_0^1 q(\xi)\Big\{\dot q(\xi+z_1+\varepsilon\tau Cz\cdot z)\Big\} \\ &\quad\times\partial_{z_1}\left\{\frac{Cz\cdot z}{1+2\varepsilon\tau(c_1z_1+v\cdot z')}J(z_1,z')\right\}d\xi dz_1dz'd\tau \\ &\quad+(BT)^1_+-(BT)^1_- \\ &= \int\int_{|z|\le r_\varepsilon}\int_0^1 q(\xi)\partial_{z_1}\Big\{q(\xi+z_1+\varepsilon\tau Cz\cdot z)-q(\xi)\Big\} \\ &\quad\times\frac{1}{1+2\varepsilon\tau(c_1z_1+v\cdot z')}\partial_{z_1}\left\{\frac{Cz\cdot z}{1+2\varepsilon\tau(c_1z_1+v\cdot z')}J(z_1,z')\right\}d\xi dz_1dz'd\tau \\ &\quad+(BT)^1_+-(BT)^1_- \end{aligned}$$

We finally integrate by parts in z_1 and we get

$$c_\varepsilon = d_\varepsilon + (BT)^1_+-(BT)^1_-+(BT)^2_+-(BT)^2_- \tag{66}$$

with

$$d_\varepsilon = -\int\int_{|z|\le r_\varepsilon}\int_0^1 q(\xi)\Big\{q(\xi+z_1+\varepsilon\tau Cz\cdot z)-q(\xi)\Big\} \times\partial_{z_1}\left\{\frac{1}{1+2\varepsilon\tau(c_1z_1+v\cdot z')}\partial_{z_1}\left\{\frac{Cz\cdot z}{1+2\varepsilon\tau(c_1z_1+v\cdot z')}J(z_1,z')\right\}\right\}d\xi dz_1dz'd\tau$$

$$\begin{aligned}(BT)^2_\pm &= \int\int_{|z'|\le r_\varepsilon}\int_0^1 q(\xi)\Big\{q(\xi\pm\sqrt{r_\varepsilon^2-|z'|^2} \\ &\quad+\varepsilon\tau(\pm\sqrt{r_\varepsilon^2-|z'|^2},z')\cdot(\pm\sqrt{r_\varepsilon^2-|z'|^2},z'))-q(\xi)\Big\} \\ &\quad\times\frac{1}{1+2\varepsilon\tau(\pm c_1\sqrt{r_\varepsilon^2-|z'|^2}+v\cdot z')} \\ &\quad\partial_{z_1}\left\{\frac{Cz\cdot z}{1+2\varepsilon\tau(c_1(\cdot)+v\cdot z')}J(\cdot,z')\right\}(\pm\sqrt{r_\varepsilon^2-|z'|^2})d\xi dz'd\tau\,.\end{aligned}$$

We now study the limits of all terms in (66).

Step 3: study of boundary terms. We start with $(BT)^1_\pm$.

$$\begin{aligned} |(BT)^1_\pm| &\leq 2(m_+(h)-m_-(h))\|q\|_\infty\|g\|_\infty \int_{|z'|\leq r_\varepsilon} 2r_\varepsilon^2 \frac{dz'}{r_\varepsilon^{N+\alpha}} \\ &\leq C r_\varepsilon^{1-\alpha} \end{aligned}$$

and this goes to 0 as $\varepsilon \to 0$.

We now turn to $(BT)^2_\pm$. It is convenient to introduce the function

$$\Gamma(\tau, z_1) = (1 + 2\varepsilon\tau(c_1 z_1 + v\cdot z'))^{-1}\,.$$

Since $\varepsilon r_\varepsilon \to 0$ as $\varepsilon \to 0$, we deduce that for ε small enough, we have

$$\begin{aligned} |\Gamma(\tau,z_1)| &\leq 2 \\ |\partial_{z_1}\Gamma(\tau,z_1)| &\leq 8|c_1|\varepsilon \\ |\partial_{z_1}(\Gamma^2)(\tau,z_1)| &\leq 32|c_1|\varepsilon \\ |\partial^2_{z_1,z_1}\Gamma(\tau,z_1)| &\leq 64|c_1|^2\varepsilon^2\,. \end{aligned}$$

We next compute

$$\begin{aligned} &\partial_{z_1}((Cz\cdot z)\Gamma(z_1)J(z_1,z')) \\ &\qquad = 2c_1 z_1 \Gamma J + (c_1 z_1^2 + C'z'\cdot z')(\partial_{z_1}\Gamma)J + (c_1 z_1^2 + C'z'\cdot z')\Gamma\partial_{z_1}J\,. \end{aligned}$$

Now, since $J(z) = g(\hat{z})|z|^{-N-\alpha}$, we deduce that

$$|\partial_{z_1}J(z)| \leq \frac{C}{|z|^{N+\alpha+1}}\,.$$

We thus conclude that for $|z'| \leq r_\varepsilon$ and z_1 such that $|z| = r_\varepsilon$, we have

$$|\partial_{z_1}((Cz\cdot z)\Gamma J)| \leq C\frac{r_\varepsilon}{r_\varepsilon^{N+\alpha}} + C\frac{\varepsilon r_\varepsilon^2}{r_\varepsilon^{N+\alpha}} + C\frac{r_\varepsilon^2}{r_\varepsilon^{N+\alpha+1}} \leq C r_\varepsilon^{-N-\alpha}$$

and we get

$$\int_{|z'|\leq r_\varepsilon, z_1^2+|z'|^2=r_\varepsilon^2} |\partial_{z_1}((Cz\cdot z)\Gamma(\tau,z_1)J(z_1,z'))|dz' \leq C r_\varepsilon^{-\alpha}\,.$$

With this inequality in hand, we now derive

$$\begin{aligned} G(-Cr_\varepsilon) &= -Cr_\varepsilon^{-\alpha}\int (q(\xi) - q(\xi - Cr_\varepsilon))d\xi \\ &\leq (BT)^2_\pm \leq Cr_\varepsilon^{-\alpha}\int (q(\xi+Cr_\varepsilon) - q(\xi))d\xi = G(Cr_\varepsilon) \end{aligned}$$

where

$$G(r) = \int (q(\xi + r) - q(\xi))d\xi\,.$$

It is clear that G is Lipschitz continuous and equals 0 at 0. Hence

$$|(BT)^2_\pm| \leq C r_\varepsilon^{1-\alpha}\,.$$

It thus goes to 0 as $\varepsilon \to 0$.

Step 4: study of d_ε. In order to study the main term d_ε, we first write it as follows

$$d_\varepsilon = e_\varepsilon + R_\varepsilon$$

with

$$\begin{aligned} e_\varepsilon &= -\int\int_{|z|\leq r_\varepsilon}\int_0^1 q(\xi)\Big\{q(\xi+z_1+\varepsilon\tau Cz\cdot z)-q(\xi)\Big\} \\ &\quad \times\Gamma^2(\tau,z_1)\partial^2_{z_1,z_1}\Big\{(Cz\cdot z)J(z_1,z')\Big\}d\xi dz_1 dz' d\tau \\ R_\varepsilon &= -\int\int_{|z|\leq r_\varepsilon}\int_0^1 q(\xi)\Big\{q(\xi+z_1+\varepsilon\tau Cz\cdot z)-q(\xi)\Big\} \\ &\quad \times\Big[\Big\{(\partial_{z_1}\Gamma)^2+\Gamma\partial^2_{z_1,z_1}\Gamma\Big\}\Big\{(Cz\cdot z)J(z_1,z')\Big\} \\ &\quad +\partial_{z_1}(\Gamma^2)\partial_{z_1}\Big\{(Cz\cdot z)J(z_1,z')\Big\}\Big]d\xi dz_1 dz' d\tau\,. \end{aligned}$$

Let us now prove that R_ε goes to 0 as $\varepsilon \to 0$. We proceed as we did with $(BT)^2_\pm$. We first estimate the quantity $[\dots]$ in the definition of R_ε. We use the estimates on Γ and its derivatives, together with the estimate of $\partial_{z_1}J$. We obtain for $|z|\leq r_\varepsilon$,

$$|[\dots]| \leq C\frac{\varepsilon^2}{|z|^{N+\alpha-2}} + C\frac{\varepsilon}{|z|^{N+\alpha-1}} \leq \frac{C\varepsilon}{|z|^{N+\alpha-1}}$$

since $\varepsilon|z| \leq \varepsilon r_\varepsilon \to 0$. By arguing as for $(BT)^2_\pm$, we conclude that

$$|R_\varepsilon| \leq C\varepsilon G(Cr_\varepsilon)\int_{|z|\leq r_\varepsilon}|z|^{-N-\alpha+1}dz \leq C(\varepsilon r_\varepsilon)r_\varepsilon^{1-\alpha}$$

and the right hand side of the previous inequality goes to 0 as $\varepsilon \to 0$.

It remains to study the limit of e_ε. By dominated convergence theorem, we

obtain that it converges towards

$$\begin{aligned} e_0 &= -\int_{\mathbb{R}_\xi} q(\xi) \int_{\mathbb{R}_{z_1}} (q(\xi+z_1)-q(\xi)) \\ &\quad \times \partial^2_{z_1,z_1} \left\{ \int_{z'} (Cz\cdot z) g(\hat{z}) \frac{dz'}{(|z_1^2+|z'|^2)^{(N+\alpha)/2}} \right\} d\xi dz_1 \\ &= -\int_{\mathbb{R}_\xi} q(\xi) \int_{\mathbb{R}_{z_1}} (q(\xi+z_1)-q(\xi)) \partial^2_{z_1,z_1} \left\{ \frac{|z_1|^{1-\alpha}}{\alpha(\alpha-1)} \mathrm{Tr}(A_g(e)C) \right\} d\xi dz_1 \\ &= -\int_{\mathbb{R}_\xi} q(\xi) \int_{\mathbb{R}_{z_1}} (q(\xi+z_1)-q(\xi)) \frac{dz_1}{|z_1|^\alpha} d\xi \Big(\mathrm{Tr}(A_g(e)C) \Big) \\ &= \iint (q(\xi+z_1)-q(\xi))^2 \frac{dz_1 d\xi}{|z_1|^{1+\alpha}} \Big(\mathrm{Tr}(A_g(e)C) \Big) \end{aligned}$$

where $A_g(e)$ is defined in (24). The proof is now complete. □

Proof of Lemma 4 in the regular case. The proof of the lemma in this case is divided into two steps.

Step 1: reduction to the study of the singular integral around the origin for quadratic W's. As explained in the proof of Lemma 4 for $J(z) = g(\hat{z})|z|^{-N-\alpha}$ and $\alpha > 1$, it is enough to study the convergence of

$$\begin{aligned} b_\varepsilon(e,t,x) &= \int\int_{|z|\le r_\varepsilon} \dot{q}(\xi) a_\varepsilon(\xi,e,t,x) d\xi dz \\ &= \int\int_{|z|\le r_\varepsilon} \dot{q}(\xi) \frac{1}{\varepsilon} \Big\{ q(\xi + e\cdot z + \varepsilon W(t,x,z), e) - q(\xi + e\cdot z) \Big\} J(z) d\xi dz \end{aligned}$$

where we recall that $r_\varepsilon = \varepsilon^{-\beta}$ with $\beta > 1/\alpha$.

Remark that there exists $C_R > 0$ such that for any $(t,x) \in B_R$ and $z \in B$,

$$|W(t,x,z)| \le C_W |z|^2 \tag{67}$$

and that, for $|z| \le r_\varepsilon$,

$$W(t,x,z) \to \frac{1}{2} D^2 d(t,x) z\cdot z \quad \text{ as } \varepsilon \to 0$$

as soon as one chooses r_ε such that $\varepsilon r_\varepsilon \to 0$. We conclude that the integrand of b_ε converges towards

$$\frac{1}{2} \dot{q^0}(\xi) \dot{q^0}(\xi + e\cdot z)(D^2 d(t,x) z\cdot z)\, J(z)\,.$$

This explains why we expect the limit of $\bar{a}_\varepsilon$ to be given by (26).

We can apply dominated convergence theorem outside the unit ball B. Hence, we reduce the study of the limit of b_ε to the one of

$$\begin{aligned} &c_\varepsilon(e,t,x) \\ &\quad = \int\int_{|z|\le 1} \dot{q}(\xi) \frac{1}{\varepsilon} \Big\{ q(\xi + e\cdot z + \varepsilon W(t,x,z), e) - q(\xi + e\cdot z) \Big\} J(z) d\xi dz\,. \end{aligned}$$

In order to do so, we introduce

$$c_\varepsilon^\pm \;=\; \int\!\!\int_{|z|\le 1} \dot q(\xi)\frac{1}{\varepsilon}\Big\{q(\xi+e\cdot z\pm\varepsilon C_W|z|^2,e)-q(\xi+e\cdot z)\Big\}J(z)d\xi dz\;.$$

We know from (67) and the monotonicity property of q that

$$c_\varepsilon^- \le c_\varepsilon \le c_\varepsilon^+ \tag{68}$$

and it is thus enough to prove that integrals $c_\varepsilon^\pm$ have limits that are uniform with respect to e,t,x to conclude.

Step 2: integrating by parts. Recall that $|e|=1$ and let z_1 denote $e\cdot z$ and $z=(z_1,z')$. We now write

$$\begin{aligned}
&c_\varepsilon^+(e)\\
&=\int\!\!\int_{|z|\le 1}\dot q(\xi)\frac{1}{\varepsilon}\Big\{q(\xi+z_1+\varepsilon C_W z_1^2+\varepsilon C_W|z'|^2,e)-q(\xi+z_1)\Big\}J(z)d\xi dz\\
&=C_W\int\!\!\int_{|z|\le 1}\int_0^1\dot q(\xi)\dot q(\xi+z_1+\varepsilon\tau C_W z_1^2+\varepsilon\tau C_W|z'|^2,e)\;|z|^2J(z)\;d\xi dz d\tau\\
&\qquad=C_W\int\!\!\int_0^1\int_{|z|\le 1}\dot q(\xi)\partial_{z_1}\Big(q(\xi+z_1+\varepsilon\tau C_W z_1^2+\varepsilon\tau C_W|z'|^2,e)\Big)\\
&\qquad\qquad\qquad\qquad\times\frac{|z|^2J(z)}{1+2\varepsilon\tau C_W z_1}\;d\xi d\tau dz' dz_1\;.
\end{aligned}$$

We next integrate by parts and get

$$c_\varepsilon^+ = d_\varepsilon^+ + (BT)_+ + (BT)_-\;.$$

with

$$\begin{aligned}
d_\varepsilon^+ \;&=\; -C_W\int\!\!\int_0^1\int_{|z|\le 1}\dot q(\xi)q(\xi+z_1+\varepsilon\tau C_W z_1+\varepsilon\tau C_W|z'|^2,e)\\
&\quad\times K(\tau,z)\;d\xi d\tau dz' dz_1\\
(BT)_\pm \;&=\; \pm C_W\int\!\!\int_0^1\int_{|z|\le 1}\dot q(\xi)q(\xi\pm\sqrt{1-|z'|^2}+\varepsilon\tau C_W,e)\\
&\quad\times\frac{J(\pm\sqrt{1-|z'|^2},z')}{1\pm 2\varepsilon\tau C_W\sqrt{1-|z'|^2}}
\end{aligned}$$

where

$$\begin{aligned}
K(\tau,z)\;&=\;\partial_{z_1}\left(\frac{|z|^2J(z)}{1+2\varepsilon\tau C_W z_1}\right)\\
&=\;\frac{\partial_{z_1}(|z|^2J(z))}{1+2\varepsilon\tau C_W z_1}-2\varepsilon\tau C_W\frac{|z|^2J(z)}{(1+2\varepsilon\tau C_W z_1)^2}\;.
\end{aligned}$$

After remarking that for $J \in L^1 \cap C^0_c$, we have

$$|K(\tau, z)| \leq C(|z|^2 J(z) + |\nabla(|z|^2 J(z))|) \in L^1(B)\,,$$

it is clear that we can apply dominated convergence theorem in each integral. The proof is now complete. □

In the case $\alpha < 1$, the ansatz used to treat the case $\alpha \geq 1$ yield oscillating terms with the following form.

$$a_\varepsilon(\zeta, s, y) \quad = \quad \frac{1}{\varepsilon^\alpha} \int \left(q(\zeta + \frac{\phi(s, y + \varepsilon z) - \phi(s, y)}{\varepsilon}) - q(\zeta + e \cdot z) \right) \frac{dz}{|z|^{N+\alpha}}\,.$$

Their average is thus defined as follows

$$\bar{a}_\varepsilon(s, y) = \int a_\varepsilon(r, s, y) \dot{q}(r) dr\,.$$

Even if we are not able (yet!) to treat the case $\alpha < 1$, we think this can be of interest to explain what is the limit of the average as $\varepsilon \to 0$ in order to justify our conjecture about the limit we expect in the case $\alpha < 1$. Another reason for including such a result is that its proof shares ideas with the one corresponding one for the case $\alpha = 1$.

Lemma 10 (Uniform convergence of approximate coefficients (II))**.** *Consider a smooth function $d : (0, +\infty) \times \mathbb{R}^N \to \mathbb{R}$ such that $|Dd(t, x)| = 1$. Then, as $\varepsilon \to 0$,*

$$\bar{a}_\varepsilon(e, t, x) \to \kappa[x, d(t, \cdot)]$$

and the limit is uniform with respect to $(e, t, x) \in \mathbb{S}^{N-1} \times Q_\gamma$.

Proof of Lemma 10. In the case $\alpha < 1$, we first make a change of variables as follows

$$\begin{aligned}
\bar{a}_\varepsilon(e, t, x) &= \int \dot{q}(\xi) a_\varepsilon(\xi, e, t, x) d\xi \\
&= \frac{1}{\varepsilon^\alpha} \iint \dot{q}(\xi) \left\{ q(\xi + \frac{d(t, x + \varepsilon z) - d(t, x)}{\varepsilon}, e) - q(\xi + e \cdot z) \right\} J(z) d\xi dz \\
&= \iint \dot{q}(\xi) \left\{ q(\xi + \frac{d(t, x + \bar{z}) - d(t, x)}{\varepsilon}, e) - q(\xi + \frac{e \cdot \bar{z}}{\varepsilon}) \right\} J_\varepsilon(\bar{z}) d\xi d\bar{z}
\end{aligned}$$

with

$$J_\varepsilon(\bar{z}) = \frac{1}{\varepsilon^{N+\alpha}} J\left(\frac{\bar{z}}{\varepsilon}\right)\,.$$

Step 1: reduction to the study of the singular integral around the origin for quadratic W's. Remark next that it is easy to pass to the limit in the integrand; indeed,

$$\begin{aligned}
q(\xi + \frac{d(t, x + \bar{z}) - d(t, x)}{\varepsilon}, e) - q(\xi + \frac{e \cdot \bar{z}}{\varepsilon}) \\
\to (\bar{m}_+ - \bar{m}_-) \left(\mathbf{1}_{(d(t,x+\cdot) > d(t,x), e\cdot(\cdot) < 0)} - \mathbf{1}_{(d(t,x+\cdot) < d(t,x), e\cdot(\cdot) > 0)} \right)
\end{aligned}$$

and

$$J_\varepsilon(z) \to g(\frac{z}{|z|})\frac{1}{|z|^{N+\alpha}}\,.$$

The difficulty is to deal with the singular measure. Hence, it is enough to study, as in the case $\alpha = 1$,

$$\begin{aligned}
c_\varepsilon^+ &= \int_{\mathbb{R}_\xi}\int_{|\bar z|\le\delta} \dot q(\xi)\Big\{q(\xi+\frac{z_1+C_W z_1^2+C_W|z'|^2}{\varepsilon},e)-q(\xi+\frac{z_1}{\varepsilon})\Big\}\\
&\quad\times J_\varepsilon(z_1,z')d\xi dz'dz_1\\
&= C_W\int_{\mathbb{R}_\xi}\int_{|\bar z|\le\delta}\int_0^1 \dot q(\xi)\dot q(\xi+\frac{z_1+C_W\tau z_1^2+C_W\tau|z'|^2}{\varepsilon},e)\\
&\quad\times\frac{|z|^2}{\varepsilon}J_\varepsilon(z_1,z')d\xi dz d\tau\\
&= C_W\int_{\mathbb{R}_\xi}\int_{|\bar z|\le\delta}\int_0^1 \dot q(\xi)\partial_{z_1}\left(q(\xi+\frac{z_1+C_W\tau z_1^2+C_W\tau|z'|^2}{\varepsilon},e)\right)\\
&\quad\times\frac{|z|^2}{1+2C_W\tau z_1}J_\varepsilon(z)d\xi dz d\tau
\end{aligned}$$

where $\bar z = z_1 e + z'$.

Step 2: integrating by parts. By integrating by parts, we obtain

$$c_\varepsilon^+ = d_\varepsilon^+ + (BT)_+ + (BT)_-$$

where

$$\begin{aligned}
d_\varepsilon^+ &= -C_W\int_{\mathbb{R}_\xi}\int_{|\bar z|\le\delta}\int_0^1 \dot q(\xi)q(\xi+\frac{z_1+C_W\tau z_1^2+C_W\tau|z'|^2}{\varepsilon},e)\\
&\quad\times L_\varepsilon(\tau,z)d\xi dz d\tau\\
(BT)_\pm &= \pm C_W\int_{\mathbb{R}_\xi}\int_{|z'|\le\delta}\int_0^1 \dot q(\xi)q(\xi+\frac{\pm\sqrt{\delta^2-|z'|^2}+C_W\tau\delta^2}{\varepsilon},e)\\
&\quad\times\delta^2 J_\varepsilon(\pm\sqrt{\delta^2-|z'|^2},z')dz'd\xi d\tau
\end{aligned}$$

with

$$\begin{aligned}
L_\varepsilon(\tau,z) &= \partial_{z_1}\left(\frac{|z|^2J_\varepsilon(z)}{1+2C_W\tau z_1}\right)\\
&= \frac{\partial_{z_1}(|z|^2J_\varepsilon(z))}{1+2C_W\tau z_1}-2C_W\tau\frac{|z|^2J_\varepsilon(z)}{(1+2C_W\tau z_1)^2}\,.
\end{aligned}$$

Condition (9) ensures that

$$|L_\varepsilon(\tau,z)|\le CK(z)\in L^1(B)$$

and dominated convergence can be used to prove the convergence of d_ε^+. As far as boundary terms are concerned, we use (8) to get a constant $C > 0$ such that

$$J_\varepsilon(z) \le \frac{C}{|z|^{N+\alpha}}$$

and this implies

$$J_\varepsilon(\pm\sqrt{\delta^2 - |z'|^2}, z') \le \frac{C}{\delta^{N+\alpha}}\,.$$

Hence, dominated convergence can be applied to boundary terms too. The proof is now complete. □

Proof of Lemma 4 in the singular case for $\alpha = 1$. The proof is divided in several steps.

Step 1: reduction to the study of the singular integral around the origin for quadratic W's. Let us fix $\delta > 0$. There exists $r_\delta > 0$ such that for any $\bar z \in B_{r_\delta}$,

$$|d(t, x + \bar z) - d(t, x) - Dd(t, x) \cdot \bar z - \frac{1}{2} D^2 d(t, x)\bar z \cdot \bar z| \le \delta |\bar z|^2\,.$$

Consequently, for z such that $|z| \le r_\delta \varepsilon^{-1} =: R_\varepsilon$, we have

$$|W(t, x, z) - \frac{1}{2} Bz \cdot z| \le \delta |z|^2\,. \tag{69}$$

We remark next that

$$\begin{aligned} \int_{|z|\le 1} |z|^2 J(z)dz &= O(1) = o(|\ln \varepsilon|)\,, \\ \int_{|z|\ge R_\varepsilon} J(z)dz &= O(\varepsilon) = o(\varepsilon|\ln \varepsilon|)\,. \end{aligned}$$

It is therefore enough to study the convergence of

$$\begin{aligned} &b_\varepsilon \\ &= \frac{1}{\varepsilon|\ln \varepsilon|} \iint_{1\le|z|\le R_\varepsilon} \dot q(\xi)[q(\xi + e \cdot z + \varepsilon W(t, x, z)) - q(\xi + e \cdot z)]J(z)dzd\xi\,. \end{aligned}$$

By using (69) together with the monotonicity of q, it is even enough to study the convergence of

$$\begin{aligned} &c_\varepsilon = \\ &\frac{1}{\varepsilon|\ln \varepsilon|} \iint_{1\le|z|\le R_\varepsilon} \dot q(\xi)[q(\xi + z_1 + \varepsilon c_1 z_1^2 + \varepsilon C' z' \cdot z') - q(\xi + e \cdot z)]J(z)dzd\xi \end{aligned}$$

for any $(N-1)\times(N-1)$ symmetric matrix C and any constant $c_1 \in \mathbb{R}$. Precisely, we next prove that, as $\varepsilon \to 0$,

$$c_\varepsilon \to 2(\bar m_+ - \bar m_-)^2 \int_{\mathbb{S}^{N-2}} g(\theta)\; C'\theta \cdot \theta\; \sigma(d\theta)$$

where g appears in (8).

Step 2: change of variables and domain decomposition. We now introduce the function F_q defined as follows

$$F_q(a) = \int \dot{q}(\xi) q(\xi + a) d\xi \,.$$

We remark that F_q is bounded, non-decreasing, Lipschitz continuous and sastisfies

$$F_q(a) \to \begin{cases} \bar{m}_+(\bar{m}_+ - \bar{m}_-) & \text{as } a \to +\infty \\ \bar{m}_-(\bar{m}_+ - \bar{m}_-) & \text{as } a \to -\infty \end{cases}$$

and

$$\|F_q'\|_\infty \leq \left(\int \dot{q}^2(\xi) d\xi \right)^{1/2} .$$

We now rewrite c_ε with this new function

$$c_\varepsilon = \frac{1}{\varepsilon |\ln \varepsilon|} \int_{1 \leq |z| \leq R_\varepsilon} \Big\{ F_q(z_1 + \varepsilon c_1 z_1^2 + \varepsilon C' z' \cdot z') - F_q(z_1) \Big\} J(z) dz$$

Through the change of variables $z' = \varepsilon^{-1} r\theta$ and $z_1 = \varepsilon^{-1} r t_1$, we get

$$c_\varepsilon = \frac{1}{|\ln \varepsilon|} \int_{\theta \in \mathbb{S}^{N-2}} d\sigma(\theta) \int_0^{r_\delta} \frac{dr}{r^2} \int_{\varepsilon^2 r^{-2} \leq 1 + t_1^2 \leq r_\delta^2 r^{-2}} dt_1$$
$$\times \left\{ F_q\left(\frac{r}{\varepsilon} (t_1 + r c_1 t_1^2 + r C' \theta \cdot \theta) \right) - F_q\left(\frac{r}{\varepsilon} t_1 \right) \right\} g\left(\frac{(t_1, \theta)}{\sqrt{t_1^2 + 1}} \right) \frac{1}{(1 + t_1^2)^{(N+1)/2}} \,.$$

Without loss of generality, we assumed here for simplicity that

$$J(z) = g(z/|z|) |z|^{-N-1}$$

(see (9)). We next cut it into two pieces as follows

$$c_\varepsilon = d_\varepsilon + R_\varepsilon^1$$

with

$$\begin{aligned} d_\varepsilon &= \frac{1}{|\ln \varepsilon|} \int_{\theta \in \mathbb{S}^{N-2}} d\sigma(\theta) \int_{\Theta_\varepsilon}^{r_\delta} \frac{dr}{r^2} \int_{\varepsilon^2 r^{-2} \leq 1 + t_1^2 \leq r_\delta^2 r^{-2}} dt_1 \\ &\quad \times \Big\{ F_q(\dots) - F_q(\dots) \Big\} g(\dots) \frac{1}{(1 + t_1^2)^{(N+1)/2}} \,. \\ R_\varepsilon^1 &= \frac{1}{|\ln \varepsilon|} \int_{\theta \in \mathbb{S}^{N-2}} d\sigma(\theta) \int_0^{\Theta\varepsilon} \frac{dr}{r^2} \int_{\varepsilon^2 r^{-2} \leq 1 + t_1^2 \leq r_\delta^2 r^{-2}} dt_1 \\ &\quad \times \Big\{ F_q(\dots) - F_q(\dots) \Big\} g(\dots) \frac{1}{(1 + t_1^2)^{(N+1)/2}} \end{aligned}$$

for some $\Theta > 0$ to be fixed later.

Step 3: study of R^1_ε. We now use the fact that F_q is Lispchitz continuous in order to get, for Θ large enough,

$$\begin{aligned} |R^1_\varepsilon| &\leq \|g\|_\infty \|F_q'\|_\infty (|c_1| + |C'|) |\mathbb{S}^{N-2}| \frac{1}{|\ln \varepsilon|} \int_0^{\Theta\varepsilon} dr \\ &\quad \times \int_{\varepsilon^2 r^{-2} \leq 1 + t_1^2 \leq r_\delta^2 r^{-2}} dt_1 \frac{1}{\varepsilon} \frac{1}{(1+t_1^2)^{(N-1)/2}} \\ &\leq C \frac{1}{\varepsilon |\ln \varepsilon|} \int_0^{\Theta\varepsilon} dr \int_{1 \leq |t_1| \leq r_\delta r^{-1}} dt_1 \, t_1^{-N+1} \\ &\leq C_\delta \frac{1}{\varepsilon |\ln \varepsilon|} (\varepsilon - \varepsilon^{N-1}) \leq \frac{C_\delta}{|\ln \varepsilon|} \, . \end{aligned}$$

Hence

$$R^1_\varepsilon \to 0 \text{ as } \varepsilon \to 0 \, .$$

Step 4: study of d_ε. First, we rewrite d_ε (for $\Theta \geq 1$ large enough) as follows

$$\begin{aligned} d_\varepsilon = \frac{1}{|\ln \varepsilon|} \int_{\theta \in \mathbb{S}^{N-2}} d\sigma(\theta) \int_{\Theta\varepsilon}^{r_\delta} \frac{dr}{r^2} \\ \times \int_{|t_1| \leq \sqrt{r_\delta^2 r^{-2} - 1}} \Big\{ F_q(\dots) - F_q(\dots) \Big\} g(\dots) \frac{1}{(1+t_1^2)^{(N+1)/2}} \, . \end{aligned}$$

We now study the limit of the integrand of d_ε. In view of the limits of $F_q(a)$ when $a \to \pm\infty$, this limit is not 0 if t_1 satisfies

$$\left\{ \begin{array}{l} t_1 \geq 0 \\ t_1 + c_1 r t_1^2 + r c_\theta \leq 0 \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} t_1 \leq 0 \\ t_1 + c_1 r t_1^2 + r c_\theta \geq 0 \end{array} \right.$$

where $c_\theta = C'\theta \cdot \theta$. This is equivalent to

$$0 \leq t_1 \leq \mathcal{T}_1(r) \quad \text{or} \quad \left\{ \begin{array}{l} t_1 \leq -\frac{1}{2c_1 r}(1 + \sqrt{1 - 4c_1 c_\theta r^2}) \simeq -\frac{1}{c_1 r} \\ \text{or } \ -c_\theta r \leq t_1 \leq 0 \end{array} \right.$$

with

$$\mathcal{T}_1(r) = \frac{1}{2c_1 r}(\sqrt{1 - 4c_1 c_\theta r^2} - 1) \simeq -c_\theta r \, .$$

We use now the fact that $|t_1| \leq \sqrt{r_\delta^2 r^{-2} - 1}$ to get

$$0 \leq t_1 \leq \mathcal{T}_1(r) \quad \text{or} \quad \mathcal{T}_1(r) \leq t_1 \leq 0 \, .$$

We next cut d_ε into pieces as follows

$$d_\varepsilon = e^+_\varepsilon + e^-_\varepsilon + R^2_\varepsilon$$

with

$$\begin{aligned} e_\varepsilon^\pm &= \frac{1}{|\ln\varepsilon|}\int_{\theta\in\mathbb{S}^{N-2}} d\sigma(\theta)\int_{\Theta_\varepsilon}^{r_\delta}\frac{dr}{r^2}\int_{0\le\pm t_1\le|\mathcal{T}_1(r)|} \\ &\quad\times\Big\{F_q(\dots)-F_q(\dots)\Big\}g(\dots)\frac{1}{(1+t_1^2)^{(N+1)/2}} \\ R_\varepsilon^2 &= \frac{1}{|\ln\varepsilon|}\int_{\theta\in\mathbb{S}^{N-2}} d\sigma(\theta)\int_{\Theta_\varepsilon}^{r_\delta}\frac{dr}{r^2}\int_{|\mathcal{T}_1(r)|\le|t_1|\le\sqrt{r_\delta^2 r^{-2}-1}} \\ &\quad\times\Big\{F_q(\dots)-F_q(\dots)\Big\}g(\dots)\frac{1}{(1+t_1^2)^{(N+1)/2}}\,. \end{aligned}$$

Step 5: study of $e_\varepsilon^\pm$. Since $r\le r_\delta$ and r_δ very small, we can replace $e_\varepsilon^\pm$ with

$$\begin{aligned} f_\varepsilon^\pm &= \frac{1}{|\ln\varepsilon|}\int_{\theta\in\mathbb{S}^{N-2}} d\sigma(\theta)\int_{\Theta_\varepsilon}^{r_\delta}\frac{dr}{r^2}\int_{0\le\pm t_1\le|\mathcal{T}_1(r)|} dz_1 \\ &\quad\times\Big\{F_q\Big(\frac{r}{\varepsilon}(t_1+rc_1t_1^2+rC'\theta\cdot\theta)\Big)-F_q\Big(\frac{r}{\varepsilon}t_1\Big)\Big\}g(0,\theta) \\ &= \frac{1}{|\ln\varepsilon|}\int_{\theta\in\mathbb{S}^{N-2}} d\sigma(\theta)\int_{\Theta_\varepsilon}^{r_\delta}\frac{dr}{r}\int_{0\le\pm z_1\le r^{-1}|\mathcal{T}_1(r)|} dz_1 \\ &\quad\times\Big\{F_q\Big(\frac{1}{\varepsilon}(z_1+rc_1z_1^2+r^2C'\theta\cdot\theta)\Big)-F_q\Big(\frac{z_1}{\varepsilon}\Big)\Big\}g(0,\theta) \end{aligned}$$

Now it is easy to conclude that

$$\begin{aligned} f_\varepsilon^\pm &\to \int_{\theta\in\mathbb{S}^{N-2}}((c_\theta)^+ + (c_\theta)^-)g(0,\theta)d\sigma(\theta)(F_{q^0}(+\infty)-F_{q^0}(-\infty)) \\ &= (\bar m_+ - \bar m_-)^2\int_{\theta\in\mathbb{S}^{N-2}}(C\theta\cdot\theta)g(0,\theta)d\sigma(\theta)\,. \end{aligned}$$

It remains to prove that R_ε^2 converges towards 0.

Step 6: study of R_ε^2. By the study we made in Step 4, we conclude that the integrand of R_ε^2 converges towards 0. To dominate convergence, we simply write

$$\begin{aligned} |R_\varepsilon^2| &\le \frac{1}{|\ln\varepsilon|}|\mathbb{S}^{N-2}|\int_{\Theta_\varepsilon}^{r_\delta}\frac{dr}{r}\int_{|z_1|\ge|c_\theta|/2} 2\|F_q\|_\infty\frac{1}{(1+t_1^2)^{(N+1)/2}} \\ &\le |\mathbb{S}^{N-2}|\frac{\ln(r_\delta)+|\ln\varepsilon|}{|\ln\varepsilon|}\int_{|z_1|\ge|c_\theta|/2} 2\|F_q\|_\infty\frac{1}{(1+t_1^2)^{(N+1)/2}}\,. \end{aligned}$$

The proof of the lemma is now complete. □

A Link between (23) and (26)

In this section, we explain the link between the two first formulae in (23) and (26) by giving a formal argument.

If z_1 denotes $e \cdot z$ and $z' = z - (e \cdot z)e \in \mathbb{R}^{N-1}$, then this equation can be written as follows

$$A(e) = \int_{\mathbb{R}_\xi} d\xi \dot{q^0}(\xi) \int_{\mathbb{R}_{z_1}} dz_1 \dot{q^0}(\xi + z_1) \int_{z' \in \mathbb{R}^{N-1}} dz' \begin{bmatrix} z_1^2 & z_1 z' \\ z_1 (z')^* & z' \otimes z' \end{bmatrix} J(z_1, z')$$

We remark next that

$$\int_{z' \in \mathbb{R}^{N-1}} \begin{bmatrix} z_1^2 & z_1 z' \\ z_1 (z')^* & z' \otimes z' \end{bmatrix} J(z_1, z') dz' = \frac{1}{\alpha(\alpha-1)} A_g(e) |z_1|^{1-\alpha} .$$

Hence, $A(e) = K A_g(e)$ with

$$K = \frac{1}{\alpha(\alpha-1)} \int_{\mathbb{R}_\xi} d\xi \dot{q^0}(\xi) \int_{\mathbb{R}_{z_1}} dz_1 \dot{q^0}(\xi + z_1) |z_1|^{1-\alpha} . \tag{70}$$

By integrating by parts in z_1 and ξ and z_1 successively, we obtain

$$\begin{aligned} J_i &= \frac{1}{\alpha} \int_{\mathbb{R}_\xi} d\xi \dot{q^0}(\xi) \int_{\mathbb{R}_{z_1}} dz_1 (q^0(\xi + z_1) - q^0(\xi)) |z_1|^{-(1+\alpha)} z_1 \\ &= -\frac{1}{\alpha} \int_{\mathbb{R}_\xi} d\xi q^0(\xi) \int_{\mathbb{R}_{z_1}} dz_1 (\dot{q^0}(\xi + z_1) - \dot{q^0}(\xi)) |z_1|^{-(1+\alpha)} z_1 \\ &= -\int_{\mathbb{R}_\xi} d\xi q^0(\xi) \int_{\mathbb{R}_{z_1}} dz_1 (q^0(\xi + z_1) - q^0(\xi) - \dot{q^0}(\xi) z_1) |z_1|^{-\alpha} \\ &= \int q^0(\xi) (-\Delta)^{(\alpha-1)/2} [q^0](\xi) d\xi \\ &= \int \int (q^0(\xi + z_1) - q^0(\xi))^2 \frac{dz_1}{|z_1|^\alpha} d\xi . \end{aligned}$$

Acknowledgments. The first author was partially supported by ANR project "MICA" (ANR-06-BLAN-0082) and "EVOL" ().

References

[1] S. Allen and J. Cahn, *A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening*, Act. Metall., 27 (1979), pp. 1089–1095.

[2] G. Barles and F. Da Lio, *A geometrical approach to front propagation problems in bounded domains with Neumann-type boundary conditions*, Interfaces Free Bound., 5 (2003), pp. 239–274.

[3] G. Barles, H. M. Soner, and P. E. Souganidis, *Front propagation and phase field theory*, SIAM J. Control Optim., 31 (1993), pp. 439–469.

[4] G. Barles and P. E. Souganidis, *A new approach to front propagation problems: theory and applications*, Arch. Rational Mech. Anal., 141 (1998), pp. 237–296.

[5] L. Brown, *The self-stress of dislocations and the shape of extended nodes*, Phil. Mag., 10 (1964), p. 441.

[6] X. Cabré and J. Solà-Morales, *Layer solutions in a half-space for boundary reactions*, Comm. Pure Appl. Math., 58 (2005), pp. 1678–1732.

[7] L. Caffarelli, J.-M. Roquejoffre, and O. Savin. Personal communication.

[8] X. Chen, *Generation and propagation of interfaces for reaction-diffusion equations*, J. Differential Equations, 96 (1992), pp. 116–141.

[9] Y. G. Chen, Y. Giga, and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, Proc. Japan Acad. Ser. A Math. Sci., 65 (1989), pp. 207–210.

[10] F. Da Lio, N. Forcadel, and R. Monneau, *Convergence of a nonlocal eikonal equation to anisotropic mean curvature motion. Application to dislocation dynamics*, J. Eur. Math. Soc. (JEMS), 10 (2008), pp. 1061–1104.

[11] A. De Masi, E. Orlandi, E. Presutti, and L. Triolo, *Motion by curvature by scaling nonlocal evolution equations*, J. Statist. Phys., 73 (1993), pp. 543–570.

[12] ———, *Glauber evolution with the Kac potentials. I. Mesoscopic and macroscopic limits, interface dynamics*, Nonlinearity, 7 (1994), pp. 633–696.

[13] ———, *Stability of the interface in a model of phase separation*, Proc. Roy. Soc. Edinburgh Sect. A, 124 (1994), pp. 1013–1022.

[14] ———, *Glauber evolution with Kac potentials. II. Fluctuations*, Nonlinearity, 9 (1996), pp. 27–51.

[15] ———, *Glauber evolution with Kac potentials. III. Spinodal decomposition*, Nonlinearity, 9 (1996), pp. 53–114.

[16] A. De Masi and E. Presutti, *Mathematical methods for hydrodynamic limits*, vol. 1501 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1991.

[17] L. C. Evans, H. M. Soner, and P. E. Souganidis, *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math., 45 (1992), pp. 1097–1123.

[18] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature. I*, J. Differential Geom., 33 (1991), pp. 635–681.

[19] N. Forcadel and R. Monneau, *Existence of solutions for a model describing the dynamics of junctions between dislocations*, SIAM J. Math. Anal., 40 (2009), pp. 2517–2535.

[20] A. Garroni and S. Müller, *Γ-limit of a phase-field model of dislocations*, SIAM J. Math. Anal., 36 (2005), pp. 1943–1964 (electronic).

[21] A. Garroni and S. Müller, *A variational model for dislocations in the line tension limit*, Arch. Ration. Mech. Anal., 181 (2006), pp. 535–578.

[22] M. Gonzalez and R. Monneau. Personal communication.

[23] C. Imbert, *Level set approach for fractional mean curvature flows*, Interfaces Free Bound., 11 (2009), pp. 153–176.

[24] M. Kac, G. E. Uhlenbeck, and P. C. Hemmer, *On the van der Waals theory of the vapor-liquid equilibrium. I. Discussion of a one-dimensional model*, J. Mathematical Phys., 4 (1963), pp. 216–228.

[25] M. A. Katsoulakis and P. E. Souganidis, *Interacting particle systems and generalized evolution of fronts*, Arch. Rational Mech. Anal., 127 (1994), pp. 133–157.

[26] ———, *Generalized motion by mean curvature as a macroscopic limit of stochastic Ising models with long range interactions and Glauber dynamics*, Comm. Math. Phys., 169 (1995), pp. 61–97.

[27] M. A. Katsoulakis and P. E. Souganidis, *Stochastic Ising models and anisotropic front propagation*, J. Statist. Phys., 87 (1997), pp. 63–89.

[28] M. Koslowski, A. M. Cuitiño, and M. Ortiz, *A phasefield theory of dislocation dynamics, strain hardening and hysteresis in ductile single crystal*, J. Mech. Phys. Solids, 50 (2002), pp. 2597–2635.

[29] J. L. Lebowitz and O. Penrose, *Rigorous treatment of the van der Waals-Maxwell theory of the liquid-vapor transition*, J. Mathematical Phys., 7 (1966), pp. 98–113.

[30] H. Spohn, *Large scale dynamics of interacting particles*, Springer-Verlag, New York, 1991.