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Singular ordinary differential equations homogeneous of degree 0 near a codimension 2 set

Didier Bresch [1], Benoît Desjardins [2], Emmanuel Grenier [3]

Abstract

This note deals with an example of class of ordinary differential equations which are singular near a codimension 2 set, with an homogeneous singularity, of degree 0. Under some structural assumptions, we prove that for almost all initial data there exists a unique global solution and study the evolution of the Lebesgue measure of a transported set of initial data.

Keywords : Singular ODE's, co-dimension 2 singularity, global existence and uniqueness

AMS subject classification : 47305, 37N10, 35A05, 74H35.

1 Introduction

This note deals with a simple example of a class of ordinary differential equations which are singular on a manifold of codimension 2, the singularity being homogeneous of degree 0 near this singularity.

Let H be an Hilbert space. Let Π denotes the orthogonal projection on a given plane P. Let $x_h = \Pi x$ and let $x_v = x - \Pi x$. Note that $x = x_h + x_v$. By a slight abuse of notation we shall say that $x = (x_h, x_v)$, x_h and x_v referring to the "horizontal" and "vertical" components of x. We then identify the plane P with $\mathbb{C}$ by choosing arbitrarily a basis on P. This allows to use polar coordinates on P and thus to define the modulus r and argument θ of x_h, and to use the notation $x_h = r\exp(i\theta)$.

Let ϕ be a smooth function defined on $H \times \mathbb{S}^1$ ($\mathbb{S}^1$ being the unit circle). Then

$$\dot{x} = \phi\Big(x, \frac{x_h}{|x_h|}\Big) \tag{1}$$

is a dynamical system, singular on $P^\perp$, orthogonal of P, which is of codimension 2. Moreover, this system is homogeneous of degree 0 in any direction orthogonal to $P^\perp$.

Under some assumptions on ϕ, we show global existence and uniqueness of solution for almost every initial data. Note that we only take care of the behavior of $t \mapsto x(t)$ near $P^\perp$ and by a slight abuse of language we shall say that a solution is global if it does not reach the singularity $P^\perp$ in finite time.

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2 Motivation

The toy model (1) mimics phenomena occurring in the low Mach number limit problem for non-isentropic flows in a periodic box for instance. Such flows are described through a velocity field u, a density ρ implicitly given by a state law depending on the entropy S and the pressure p, see [4]). After some change of variables, namely introducing q defined by $p = \bar{p}e^{\varepsilon q}$ with $\overline{p}$ a prescribed constant and ε the low Mach number, the adimensionnalized system reads

$$a(\partial_t q + u \cdot \nabla q) + \frac{1}{\varepsilon}\mathrm{div} u = 0, \tag{2}$$

$$r(\partial_t u + u \cdot \nabla u) + \frac{1}{\varepsilon}\nabla p = 0, \tag{3}$$

$$\partial_t S + u \cdot \nabla S = 0, \tag{4}$$

where a and r are two smooth functions of S and $\varepsilon\, q$. The singular limit consists in letting the Mach number ε go to 0. In the ill prepared-data case, namely when the initial data do not satisfy the condition

$$\mathrm{div}\, u^0 = 0, \qquad \nabla p^0 = 0. \tag{5}$$

an oscillatory limit with changing eigenvalues occurs. Indeed the eigenvalues and the spectrum of the singular operator $\mathcal{A}$ where

$$\mathcal{A} = \begin{pmatrix} 0 & a^{-1}\mathrm{div} \\ r^{-1}\nabla & 0 \end{pmatrix}. \tag{6}$$

will depend on the solution itself. This leads to complex problem, since eigenvalues may cross. The wave equation, related to the operator $\mathcal{A}$, may also be written in this case

$$\varepsilon\partial_{tt}\psi - \mathrm{div}\,(S^{-1}\nabla\psi) = 0 \tag{7}$$

where S is the entropy quantity, see [4] for more details.

In the very special case of only one spatial dimension, the limit can be both calculated completely and justified, see [4]. In the multi-dimensional case the formal calculation of the extra term in the limit, which once again involves the spectral decomposition of the fast operator, assumes that the spectrum of that fast operator is simple and non-resonant, see [3] for the viscous case and [4] for the inviscid one. For certain finite-dimension al truncations of the equations those assumptions can be shown to be generic and to ensure convergence to the limit equations. This has been done in the paper [4].

The main difficulty in the general study is to prove, after defining appropriate infinite dimension measures, that for almost all initial data, the limit flow does not meet double eigenvalues and crosses the resonance set transversally. Difficulties occur since the flow is singular across the resonant set which may be shown to define a codimension 2 set. This explain the necessity to study ODEs of the form (1). Our result will be used to treat oscillatory limits with changing eigenvalues and particularly the low Mach number limit in a forthcoming paper [2]. Note that another type of singular ordinary differential equation occuring in fluid mechanics has been studied recently, see for instance [1].

3 Setup and main result

3.1 Notations.

Let us first introduce some notations. We define $\psi(x_h, x_v, \theta)$ as the argument of $\Pi\phi\Big((x_h, x_v), \exp(i\theta)\Big)$, and $\tilde{\psi}(x_h, x_v, \theta)$ its modulus in such a way that

$$\Pi\phi\Big((x_h, x_v), \exp(i\theta)\Big) = \tilde{\psi}(x_h, x_v, \theta) \exp\Big(i\psi(x_h, x_v, \theta)\Big).$$

Structural properties. We will assume that there exists an integer $N_0 > 0$ and N_0 smooth functions of x_v, $\Theta_1(x_h, x_v)$, ..., $\Theta_{N_0}(x_h, x_v)$, satisfying the following properties :

(H1) For all $x = (x_h, x_v) \in H$, the equation in θ

$$\psi(x_h, x_v, \theta) \in \theta + \pi\mathbb{Z},$$

has exactly N_0 solutions $\Theta_1(x_h, x_v) \dots \Theta_{N_0}(x_h, x_v)$.

(H2) For every j, the following sign condition holds for all $x = (x_h, x_v) \in H$

$$\partial_\theta \psi(x_h, x_v, \Theta_j(x_h, x_v)) < 1.$$

Note that this implies that the solutions Θ_j are all simple.

(H3) $\tilde{\psi}$ does not vanish.

Note that (H1) implies that $\Big\{(\tau \exp(i\Theta_j(0, x_v)), x_v), \quad \tau > 0\Big\}$ is a trajectory for

$$\dot{x}_h = \Pi\phi\Big((0, x_v), \frac{x_h}{\|x_h\|}\Big). \tag{8}$$

This trajectory goes to the singularity or leaves it, depending on its orientation. In particular for some initial data we reach the singularity in finite time. The flow is not defined everywhere and we can only get almost everywhere results.

3.2 Main result and consequences

The main result of this paper is :

Theorem 3.1 Stable and unstable manifolds.
Let us assume that (H1), (H2) and (H3) *hold true. Let* $x_0 \in P^\perp$ *and let* $\rho > 0$. *There exists a finite number of manifolds* V_k, *of codimension* 1, *with boundary* Σ, *such that :*

- *For any initial data* x_1 *in one of the manifolds* V_k, *the corresponding solution of* (9) *reaches* Σ *in finite time (in the past or in the future).*
- *For any initial data* x_1 *in* $B(x_0, \rho)$ *outside all these manifold* V_k, *the corresponding solution of* (9) *reaches the boundary of* $B(x_0, \rho)$ *before* Σ.

With this result one can define the flow $\Psi(t)$ of this equation, flow which is defined everywhere except on the manifolds V_k. Let A be some set of initial data. We now want to bound the measure of $\Psi(t)A$ in terms of the measure of A. This of course depends on the measure of H, which is not canonical. In order to fix the ideas we will restrict ourselves to a finite dimensional space H, with the Lebesgue measure μ, in order to prove

Theorem 3.2 Control of the divergence.
Let $x_0 \in P^\perp$. Let $\rho > 0$. There exists a constant $C_0(t)$ such that for any borelian set A of H,

$$\mu\Big(\Psi(t)(A \cap B(x_0, \rho))\Big) \leq C_0(t)\mu(A \cap B(x_0, \rho)),$$

for any $t > 0$.

The outline of the paper is as follows. First we study the simplified equation

$$\dot{x} = \phi\Big(\frac{x_h}{|x_h|}\Big) \tag{9}$$

and then extend it by perturbation arguments to equations of the general form (1). More precisely, in Section 4, we will study trajectories near the singular set. The last section concerns measure type estimate of a transported set by the flow which is defined everywhere except on the manifolds V_k of codimension one.

4 Study of systems of the form (9)

We first prove Theorem 2.1 in the particular case of systems of the form (9). Note that (9) is a two dimensional dynamical system, which takes places in $x_h + P$. Changing the notations, (9) is of the form

$$\dot{y} = \phi\Big(\frac{y}{\|y\|}\Big)$$

where $y \in \mathbb{C}$. First we turn to polar coordinates, define $y = r\exp(i\theta)$ and make the change of time defined by $d\tau/dt = 1/r$ to get

$$\frac{dr}{d\tau} = r\tilde{\psi}(\theta)\cos(\psi(\theta) - \theta) \tag{10}$$

$$\frac{d\theta}{d\tau} = \tilde{\psi}(\theta)\sin(\psi(\theta) - \theta). \tag{11}$$

Note that (11) does not involve r, which greatly simplifies the analysis of (9). Note also that $\tilde{\psi}$ is always positive. Therefore the dynamics of (11) is given by Assumption (H1). There exists N_0 fixed points $\Theta_j(x_v)$ which are stable provided

$$(\psi'(\Theta_j) - 1)\cos(\psi(\Theta_j) - \Theta_j) < 0$$

and unstable if $(\psi'(\Theta_j) - 1)\cos(\psi(\Theta_j) - \Theta_j) > 0$ (the null case being ruled out by (H2)).

Moreover the Θ_j are the only fixed points of (11) and the dynamics of solutions $t \mapsto \theta(t)$ of (11) is very simple : $t \mapsto \theta(t)$ goes in a monotonic way from some $\Theta_j(x_v)$ (unstable equilibrium, limit value as τ goes to $-\infty$) to $\Theta_{j-1}(x_v)$ or $\Theta_{j+1}(x_v)$ as τ goes to $+\infty$ (stable equilibria). All the solutions of (11) are global in the τ variable and go from an unstable Θ_j to a close stable one.

It then remains to solve (10). The behavior depends on the sign of

$$\tilde{\psi}(\theta)\cos(\psi(\theta) - \theta).$$

As $\tilde{\psi}(\theta) > 0$ and as $\psi'(\Theta_j) < 1$, $\tilde{\psi}(\theta)\cos(\psi(\theta) - \theta)$ is positive if Θ_j is stable and negative if Θ_j is unstable.

Therefore solutions of (9) are global (except if θ constantly equals some of the Θ_j where the solution goes to the singularity in finite time in the future or in the past), and are asymptotic in $+\infty$ to some stable Θ_j and in $-\infty$ to some instable Θ_j.

The phase portrait can be described as follows :

- There exists N_0 particular solutions which are straight lines, going to or coming from the origin in finite positive or negative times.
- All the other trajectories are global in time and are asymptotic to two of the particular solutions as time goes to $+\infty$ or $-\infty$.

Theorem 2.1 is then straightforward. □

Note that Hypothesis (H2) is crucial. If we assume $\psi'(\Theta') > 1$, then the conclusion is completely changed : all the trajectories come from the singularity and go back to the singularity in finite time, except for $\theta = \Theta_j$. In this case, almost all the trajectories blow up in finite time.

5 Trajectories near the singular set

In this section we will describe the behavior of solutions near the singular set. Let $x_0 = (0, x_v^0)$ be a point of $P^\perp$. Locally the geometry of the flow is described by the angles $\Theta_j(x_v^0)$ which split the space into angular sectors

$$\Omega_j = \{\Theta_j(x_v^0) < \theta < \Theta_{j+1}(x_v^0)\}.$$

If there were no x_h dependence of the flow, the angles Ω_j would be invariant under the flow as in the previous section. This is not the case here and we have to be more precise in the spatial description.

5.1 Domain decomposition.

Let $\alpha > 0$ and $\eta > 0$ be small. Then by continuity there exists angles θ_j^+ and θ_j^- such that

$$\left|\tilde{\psi}(x_h, x_v, \theta)\sin(\psi(x_h, x_v, \theta) - \theta)\right| \geq \alpha \tag{12}$$

when $|x_h| + |x_v - x_v^0| < 2\eta$ and $\Theta_j(x_v^0) < \theta_j^+ < \theta < \theta_{j+1}^- < \Theta_{j+1}(x_v^0)$.

Let us now introduce the following sets, for ε chosen later on with $\varepsilon < \eta$:

$$\Omega_j(\varepsilon,\eta) = \Big\{(x_h, x_v, \theta), \quad | \quad |x_h| < \varepsilon, \quad |x_v - x_v^0| < \eta, \quad \theta_j^- < \theta < \theta_j^+\Big\},$$

$$\Sigma_j(\varepsilon,\eta) = \Big\{(x_h, x_v, \theta), \quad | \quad |x_h| < \varepsilon, \quad |x_v - x_v^0| < \eta, \quad \theta_j^+ < \theta < \theta_{j+1}^-\Big\},$$

and

$$\Omega_v(\varepsilon,\eta) = \Big\{(x_h, x_v, \theta), \quad | \quad |x_h| < \varepsilon, \quad |x_v - x_v^0| < \eta\Big\}.$$

Note that $\Omega_v(\varepsilon,\eta)$ is the union of the $\Omega_j(\varepsilon,\eta)$ and $\Sigma_j(\varepsilon,\eta)$ and of their boundaries. The $\Omega_j(\varepsilon,\eta)$ will be the neighborhood of the stable and unstable manifolds.

5.2 Study of the trajectory.

Let us assume to fix the ideas that Θ_j is unstable and Θ_{j+1} is stable (the discussion is similar if the stabilities are interchanged). Let $x(t)$ be a trajectory with $x(0) \in \Sigma_j(\varepsilon, \eta/2)$. We claim

Claim : If ε is small enough, the trajectory $x(t)$ is contained in $\Omega_j(\eta,\eta)$ for $t < t_-$ and is contained in $\Omega_{j+1}(\eta,\eta)$ for $t > t_+$. Moreover, for $|t|$ large enough, it exits $\Omega_v(\eta)$.

Proof of the claim : Let $x(t)$ be a solution of (1), such that $x(0) = (x_h(0), x_v(0))$ with $x_h(0) = r(0)\exp(i\theta(0))$. Using polar coordinates and introducing again the change of time variable, we get

$$\dot{x}_v = r(1-\Pi)\phi\Big((x_h, x_v), \frac{x_h}{\|x_h\|}\Big), \tag{13}$$

with

$$\dot{r} = r\tilde{\psi}(x_h, x_v, \theta)\cos(\psi(x_h, x_v, \theta) - \theta) \tag{14}$$

$$\dot{\theta} = \tilde{\psi}(x_h, x_v, \theta)\sin(\psi(x_h, x_v, \theta) - \theta). \tag{15}$$

Let $]a, b[$ be the maximal time interval containing 0 such that $x(t) \in \Omega_v(\eta)$ for $t \in]a, b[$. By definition of θ_j and θ_{j+1}, $\theta(t)$ is increasing as long as it belongs to $\Omega_v(\eta,\eta) - \Omega_j(\eta,\eta) - \Omega_{j+1}(\eta,\eta)$. Let $]a_1, b_1[$ the maximum time interval containing 0 such that $\theta_j < \theta(t) < \theta_{j+1}$. Of course $a \le a_1 < b_1 \le b$.

If $a < a_1$ then for $t = a_1$, $x(t) \in \Omega_j \cap \Omega_v$ and therefore $x(t) \in \Omega_j \cap \Omega_v$ for any $a < t < a_1$. Similarly, if $b > b_1$ then for any $b_1 < t < b$, $x(t) \in \Omega_{j+1} \cap \Omega_v$.

Note that, using (15) and (12), we can bound $b_1 - a_1$, by

$$b_1 - a_1 \le \frac{\theta_3 - \theta_2}{\alpha}. \tag{16}$$

This implies that b_1 and $-a_1$ are less than $(\theta_3 - \theta_2)/\alpha$ Moreover

$$\frac{d}{d\tau}\log r = \tilde{\psi}(x_h, x_v, \theta)\cos(\psi(x_h, x_v, \theta) - \theta) \tag{17}$$

Note that on Ω_v, $\tilde{\psi}$ is bounded, by some constant C_0. Hence for $a_1 < \tau < b_1$, integrating (17) with respect to τ and using (16), we get

$$\exp(-C_0\alpha^{-1}(\theta_3 - \theta_2)) \leq \frac{r(\tau)}{R_0(x)} \leq \exp(C_0\alpha^{-1}(\theta_3 - \theta_2)). \tag{18}$$

Therefore if

$$\varepsilon < \frac{\eta}{1 + 2C_0} \exp(-2C_0\alpha^{-1}(\theta_3 - \theta_2)), \tag{19}$$

$r(\tau)$ remains smaller than η for $a_1 < \tau < b_1$. Hence the solution can not leave Ω_v at $\tau = a_1$ or $\tau = b_1$ at the boundary $|x_h| = \eta$.

Now using (13), we see that as, using (18)–(19), $r(\tau)$ remains smaller than $\eta/2C_0$, the trajectory can not leave Ω_v at its horizontal boundary. Therefore the trajectory goes from $\Omega_j(\eta)$ to $\Omega_{j+1}(\eta)$, which ends the proof of the claim.

Bounds on $r(t)$: Let us go back to the genuine time t. On $\Omega_j(\eta)$ and $\Omega_{j+1}(\eta)$, provided α is small enough, $|\cos(\psi(x_h, x_v, \theta) - \theta)|$ is larger than $1/2$. Therefore

$$\frac{d}{dt}|x_h(t)|^2 = 2r(t)\dot{r}(t)$$

is bounded away from 0 by $r(t) \min \tilde{\psi}/2$. Therefore, as long as $r(t)$ remains in Ω_v, for $t > b_1$ we have

$$r(b_1) + \gamma_1(t - b_1) \leq r(t) \leq r(b_1) + \gamma_2(t - b_1)$$

for some non negative constants γ_1, γ_2. A similar result is true for $t < a_1$.

Dynamics on Ω_j : As $\dot{r} > 0$ we change time to get

$$\dot{x}_v = \frac{r}{\cos(\psi(x_h, x_v, \theta) - \theta)}(1 - \Pi)\phi\Big((x_h, x_v), \frac{x_h}{\|x_h\|}\Big), \tag{20}$$

$$\dot{r} = r, \tag{21}$$

$$\dot{\theta} = \tan(\psi(x_h, x_v, \theta) - \theta). \tag{22}$$

Note that (20,21,22) is a standard dynamical system, studied near the equilibrium set $\{(x_v, 0, \Theta_j) | x_v \in P^\perp\}$ which appears to be unstable. It therefore admits an unstable manifold, which is precisely the manifold V_j.

6 Measure of a transported set

This section is divided in three parts. The first one concerns a control of the integral of the divergence along a trajectory $x(t)$ passing through a point x_0 in terms of its minimum distance $R_0(x)$ to the singularity. The second part concerns the local intergrability of $\log(R_0^{-1})$ that will be used in the third part to prove Theorem 2.2 through the bound established in the first part.

Control of the divergence. Let us first bound the divergence of our vector field. This divergence D equals

$$D = \frac{1}{r}\Big(\cos(\psi(x_h, x_v, \theta) - \theta)(\partial_\theta \psi - 1)\tilde{\psi} + \partial_\theta \tilde{\psi} \sin((\psi(x_h, x_v, \theta) - \theta)\Big) + D_1$$

where D_1 is a sum of terms which are bounded as r goes to 0. If α and η are small enough (see the domain decomposition part for their definitions), the sine is nearly 0 and much smaller than the cosine, hence D has the sign of $\cos(\Theta_j - \theta)$ and, as $\partial_\theta \psi \neq 1$ and $\tilde{\psi}$ is bounded away from 0, D is bounded in modulus by

$$|D| \leq \frac{c_0}{2r} \tag{23}$$

provided α is small enough.

Let us now evaluate the integral of the divergence along a trajectory. Let $x(t)$ with $t_1 < t < t_2$ be a part of trajectory included in Ω_{j+1}. Note that $r(t)$ is increasing and

$$r(t) \geq r(t_1) + C_0(t - t_1)$$

for some constant C_0 depending only on $\tilde{\psi}$. Therefore,

$$\int_{t_1}^{t_2} \frac{1}{r(t)} dt \leq \int_{t_1}^{t_2} \frac{1}{r(t_1) + C_0(t - t_1)} dt = \frac{1}{C_0} \log\Big(\frac{r(t_1) + C_0(t_2 - t_1)}{r(t_1)}\Big) \tag{24}$$

$$\leq \frac{1}{C_0} \log\Big(\frac{\eta}{R_0(x)}\Big)$$

since $r(t_2) + C_0(t_2 - t_1) \leq \eta$. A similar result is true for parts of trajectories included in Ω_j. Note that the integral behaves like $\log(1/R_0(x))$.

Local integrability and minimum distance to the singularity set. It remains to provide a local integrability property related to $\log(R_0^{-1})$, where R_0^{-1} denotes the minimum distance between the singularity and a trajectory $x(t)$ passing trough x_0. Let us look at the formulation (13,14,15), and let us study a trajectory $x(t)$. Up to a shift in time, we may assume that $x(t)$ reaches R_0 at $t = 0$. Let us study $x(t)$ when t becomes negative and large. First $x(t)$ enters $\Omega_j(\varepsilon, \eta)$ at some time $t_1 < 0$ with $|t_1| = O(1)$ as we consider sequences of trajectories with corresponding R_0 which go to 0.

For $t < t_1$, when $d(x(t), P^\perp) = \delta$ (δ being fixed) then, using (14) and integrating from 0 to t, t is of order $C_1 \log(\delta/R_0)$. Since the trajectory $x(t)$ approaches V_j with an exponential speed, we get that $d(x(t), V_j)$ is of order $C_2(R_0/\delta)^\gamma$ for some positive constants C_2 and γ.

Therefore the measure of

$$\{x_0 \in \Omega_j(\varepsilon, \eta) | d(x(t), P^\perp) = \delta, \text{ the trajectory } x(t) \text{ passes at a distance smaller than } r_0\}$$

is of order $C_3(r_0/\delta)^\gamma$ for some positive constants C_3 and γ. Thus

$$\mu\Big(x \,:\, R_0(x) \leq r_0 \text{ and } d(x, P^\perp) = \delta\Big) \leq C(r_0/\delta)^\gamma.$$

Hence $\log(R_0^{-1})$ is locally integrable.

Bound of the measure of a transported set. Let $F(t_0, t_1)$ denotes the resolvent of the flow, which is defined outside the manifolds V_j, and thus defined almost everywhere. Let $x_0 \in P^\perp$. Let $\rho > 0$, then for any measurable set A,

$$\mu\Big(F(t_0,t_1)(A \cap B(x_0,\rho)\Big) = \mu(A) \exp\Big(\int_{t_0}^{t_1} \int_{F(t_0,\tau)(A\cap B(x_0,\rho))} D(\tau,x) d\tau dx\Big)$$

$$= \mu(A \cap B(x_0,\rho)) \exp\Big(\int_{A\cap B(x_0,\rho)} \int_{t_0}^{t_1} D(\tau, F(t_0,\tau)x) d\tau dx\Big)$$

$$\leq \mu(A \cap B(x_0,\rho)) \exp\Big(C_0^{-1} \int_{A\cap B(x_0,\rho)} \log(\eta R_0^{-1}(x)) dx\Big) \leq C\mu(A \cap B(x_0,\rho))$$

for some constant C since $\log(R_0^{-1})$ is locally integrable. This ends the proof of Theorem 2.2. □

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