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# Finite Larmor radius effects : The two-fluid approach

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**ABSTRACT.** The linear properties (polarization, dispersion, damping) of the three low frequency ion modes : fast and slow compressional (or magnetosonic), and shear Alfvén, are studied for frequencies lower than the proton gyrofrequency. It is shown that a two-fluid theory is sufficient for obtaining most of the results, including first order finite Larmor radius effects. The key point is the closure of the system of fluid equations : the adiabatic laws currently used are critically discussed ; in particular it is shown that the concept of polytropic index ( $\gamma$ ) has to be ruled out. In order to bridge the gap between two-fluid and kinetic theories we point out where kinetic effects have to be introduced to complete the two-fluid calculations. The results are compared with fully kinetic ones, obtained from a numerical computation. The methods presented here are relevant for a wide range of phenomena in collisionless plasmas, since many structures can be constructed from the studied three linear modes, wave-like ones as well as stationary ones (e.g. weak discontinuities).

**Key words :** finite Larmor radius, kinetic Alfvén wave, MHD, two-fluid theory, polytropic indexes, ULF waves.

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## 1. INTRODUCTION

In many key regions of the ionized terrestrial environment and other collisionless space plasmas, an intense electromagnetic activity can be observed in the frequency range from zero up to ion gyrofrequency and above. A large amount of literature has been devoted to these observations of ULF fluctuations as well in the magnetosphere (Southwood, 1981) as in the magnetosheath (Moustaizis *et al.*, 1986) or the solar wind (Barnes, 1979) or near other planets (Dessler, 1983). When the amplitudes of these oscillations are not too large, the first work of the experimenter generally consists in trying to fit this experimental spectrum with the set of linear modes that can theoretically propagate in this medium.

From elementary cold plasma theory (Quémada, 1968), it is known that three low frequency modes can propagate. They carry three different kinds of information : parallel compressions for the slow (magnetosonic) mode, perpendicular velocity shears for the shear (Alfvén) mode and perpendicular compressions for the fast (magnetosonic) mode. Nevertheless these elementary theoretical solutions cannot be expected to precisely fit the data when the frequency is not negligible compared to the ion gyrofrequency, or when the wavelength is not infinitely larger than the ion Larmor radius and electron skin depth. The comparison must be done in such cases with more refined theories. The most difficult effects are those

related to temperature and most of the analytical works in this field are based upon some classical expansions in finite Larmor radii (Chew *et al.*, 1956 ; Mac Mahon, 1965 ; Rutherford and Frieman, 1968). A great amount of work has been done about this topic, in particular concerning the modified shear Alfvén wave (re-named « kinetic » Alfvén wave in these conditions) by Hasegawa (see Hasegawa and Uberoi, 1982, for an overview), Goertz (1983 and 1984) and other authors. The kinetic methods generally used in these works are very powerful, but rather cumbersome, so that the authors, to get analytical calculations and tractable results, have been led to limit themselves to particular cases : for instance, the low beta case, with quasi-perpendicular propagation ( $\cos^2 \theta \ll \beta \ll 1$ ) is frequently met. Out of such particular cases, the calculation appears to be very difficult, while in fluid methods a general calculation is possible. The experimental results can also be directly compared with complete kinetic ones obtained by numerical computation ; this can be done for instance thanks to program WHAMP, kindly provided to us by K. Rönnmark (Rönnmark, 1982 and 1983). This program (Waves in Homogeneous Anisotropic Multicomponent Plasma) solves the full dispersion relation (for real wave number and complex frequency) and calculates the polarization of linear electromagnetic waves in a magnetized plasma. The dielectric tensor is derived using the kinetic theory of homogeneous plasmas with various distribution func-

tions (for the present study we used only maxwellian distributions). Nevertheless as this numerical method does not allow us to understand how each parameter and each physical phenomenon play a role in the final results, analytical calculation remains necessary. Thus both methods are complementary, since the numerical one enables to check the analytical one and to qualify the approximations made.

As the departures of the low frequency modes from the MHD results are often weak, it can be suspected that it is no use changing drastically the method. We show that these departures can be correctly accounted for by using a complete two-fluid method, i.e. by removing the approximations that reduce it to MHD. In this way, we follow the example of Southwood and Hughes (1983), who spoke of the so-called kinetic Alfvén wave in fluid terms in a review untitled « Theory of Hydromagnetic waves in the Magnetosphere ». The present two-fluid theory is based on a system of two basic equations (section 2.2) which happens not to be closed because of the pressure terms. So a third equation is needed, giving the pressures as functions of velocities. This closure equation in the MHD frame and the concept of polytropic index are critically discussed in section 2.3, and afterwards two different closure equations are proposed. First, (section 2.4) the electrons are assumed isothermal and the ions adiabatic (ion heat flux divergence free). Second, (section 2.5) electron and ion heat fluxes are deduced from Vlasov equation through approximations of the same kind as Mac Mahon's expansion (1965). We would like to emphasize that we use here a two-fluid theory although it is sometimes believed that this procedure only holds for collisional plasmas. Indeed these two-fluid equations are derived from Vlasov equation through BBGKY hierarchy : neither these equations nor the following approximations depend on the collisions. In section 2.5, we precise the link between this theory and the kinetic one, by pointing out where the kinetic effects must be introduced to complete the two-fluid method. It allows us to show where the imaginary part of the frequency comes in, and thus to obtain the damping, which is of course not possible from the pure two-fluid theory. From this general method it is possible to infer many particular results needed by the experimenter : dispersion, damping and polarization (without the usual *a priori* assumption of the existence of a perpendicular electrostatic potential). In section 3, we give some such examples concerning the so-called « kinetic » Alfvén wave and the two compressional modes, for which our results are compared to those of the literature and to numerical computations. Finally, some other topics in which the explained theory might be relevant are also suggested.

## 2. DERIVATION OF THE PROPAGATION EQUATION

### 1) Notations

As we will have to write  $3 \times 3$  matrices whose elements are rather long expressions, we will use the following condensed notations :

$$F = \omega / \omega_{ci}$$

$$K = kV_A / \omega_{ci}$$

$$\theta = (\bar{B}, \bar{k})$$

$$s = \sin \theta$$

$$c = \cos \theta$$

$$\bar{l} = \bar{k}/k$$

$$\beta_{i,e} = \kappa T_{i,e} / M_i V_A^2$$

$$V_{thi,e} = \sqrt{\kappa T_{i,e} / M_{i,e}}$$

$$\mu = M_e / M_i$$

$$\bar{u} \bar{x}$$

frequency normalized to the ion gyrofrequency.

wave vector modulus normalized to the ratio of the ion gyrofrequency to the Alfvén velocity.

wave vector angle with reference to the static magnetic field.

longitudinal unit vector.

ion or electron kinetic pressure normalized to magnetic pressure.

ion or electron thermal velocity.

mass ratio.

tensor defined by  $\bar{u} \bar{x} \cdot \bar{v} = \bar{u} \times \bar{v}$ .

The coordinate system is defined by the unit vectors :  $\bar{e}_z$  along the static magnetic field,  $\bar{e}_x$  along the perpendicular component of the wave vector  $\bar{k}$ , and  $\bar{e}_y$  as third coordinate. In this system, the components of the longitudinal unit vector are :  $\bar{l} = (s, 0, c)$ .

In the present linear theory perturbed quantities are indicated by superscripts 1 (for instance  $\bar{v}_i^1$  is the perturbation of ion velocity). Unperturbed quantities are without any index (for instance  $\bar{B}$  is the static magnetic field).

### 2) Basic two-fluid equations

We will use two basic equations, the classical hydrodynamic equation and the Maxwell-Faraday law in which the electric field is expressed in terms of ion velocity and pressure, through the ion momentum equation :

$$\begin{aligned} \rho \partial \bar{v}_i^1 / \partial t &= (\bar{\nabla} \times \bar{B}^1 / \mu_0) \times \bar{B} - \bar{\nabla} \cdot (\bar{p}_e^1 + \bar{p}_i^1) \\ \bar{\nabla} \times [-\bar{v}_i^1 \times \bar{B} + m_i / e \partial \bar{v}_i^1 / \partial t + \bar{\nabla} \cdot \bar{p}_i^1 / ne] &= \\ &= -\partial \bar{B}^1 / \partial t. \end{aligned}$$

The first one can be rewritten, after Fourier transform, and in dimensionless notations :

$$\bar{v}_i^1 / V_A = K / F [\bar{D}_1 \cdot \bar{B}^1 / B + \beta_e \bar{l} \cdot \bar{p}_e^1 / p_e + \beta_i \bar{l} \cdot \bar{p}_i^1 / p_i] \quad (1)$$

where

$$\bar{D}_1 = \bar{l} \bar{e}_z - \bar{l} \cdot \bar{e}_z \bar{l} = \begin{bmatrix} -c & 0 & s \\ 0 & -c & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$\bar{l}$  being the unit tensor.

The unperturbed pressures  $\bar{p}_e$  and  $\bar{p}_i$  are supposed to be isotropic and of the form  $\bar{p}_{e,i} = n \kappa T_{e,i} \bar{l}$ . We have also substituted  $\bar{\nabla} \times \bar{B}^1 / \mu_0$  for the current density  $\bar{j}^1$ , which means that the displacement current is neglected. This assumption is equivalent to the quasi neutrality hypothesis ( $n_e = n_i$ ) which is always used for the Alfvén wave (see Hasegawa and Uberoi,

1982) ; it proves to be valid as long as corrective terms of the order of  $V_A^2/c^2$  can be neglected.

We have made another assumption in (1) by replacing  $\bar{v}^1$  (fluid velocity) by  $\bar{v}_i^1$  (ion velocity), which means we have neglected the electron inertia terms. They can be obtained if needed by using :

$$\bar{v}^1 = \bar{v}_i^1 - \mu/ne \bar{\nabla} \times \bar{B}^1/\mu_0.$$

The only consecutive change in (1) is the modification of  $\bar{D}_1$  into :

$$\bar{D}_1^* = \bar{D}_1 + iF\mu\bar{l}\bar{x} = \begin{bmatrix} -c & -iF\mu c & s \\ iF\mu c & -c & -iF\mu s \\ 0 & iF\mu s & 0 \end{bmatrix}.$$

The insertion of these new terms, of order  $iF\mu$ , does not introduce any difficulty in its principle, and they are known to lead to corrective terms in  $K^2\mu$  ( $=k^2c^2/\omega_{pe}^2$ ) in the dispersion relation. As we want to focus on the more difficult problem of the temperature effects, and for the sake of simplicity, we shall not include them in the present paper. It means that our results will be valid only when  $K^2\mu \ll 1$ , and since we will keep terms in  $K^2\beta_{i,e}$  and  $K^2c^2$ , we will implicitly assume  $c^2, \beta_{i,e} \gg \mu$ .

The second basic equation (Maxwell-Faraday law) rewrites, after the same handling as for (1) :

$$\bar{B}^1/B = K/F[\bar{D}_2 \cdot \bar{v}_i^1/V_A + iK\beta_i \bar{l} \times (\bar{l} \cdot \bar{p}_i^1/p_i)] \quad (2)$$

where

$$\bar{D}_2 = \bar{\epsilon}_z \bar{l} - \bar{\epsilon}_z \cdot \bar{l} \bar{l} - iF\bar{l}\bar{x} = \begin{bmatrix} -c & iFc & 0 \\ -iFc & -c & iFs \\ s & -iFs & 0 \end{bmatrix}.$$

It must be noticed that the term  $\bar{l} \times (\bar{l} \cdot \bar{p}_i^1)$  is not generally equal to zero as long as the pressure perturbation  $\bar{p}_i^1$  cannot be assumed scalar.

For beginning to solve the system of equations (1), (2),  $\bar{B}^1$  can be eliminated, which gives :

$$\bar{D}_c \cdot \bar{v}_i^1/V_A = F/K[\beta_e \bar{l} \cdot \bar{p}_e^1/p_e + \beta_i \bar{U}_i \cdot (\bar{l} \cdot \bar{p}_i^1/p_i)] \quad (3)$$

where

$$\begin{aligned} \bar{D}_c &= F^2/K^2 \bar{l} - \bar{D}_1 \cdot \bar{D}_2 = \\ &= \begin{bmatrix} F^2/K^2 - 1 & iF & 0 \\ -iFc^2 & F^2/K^2 - c^2 & iFsc \\ 0 & 0 & F^2/K^2 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \bar{U}_i &= \bar{l} + iK^2/F \bar{D}_1 \cdot \bar{l}\bar{x} = \\ &= \begin{bmatrix} 1 & iK^2/F & 0 \\ -iK^2c^2/F & 1 & iK^2sc/F \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

The remaining work is to close the system by finding the expression of  $\bar{p}_e^1$  and  $\bar{p}_i^1$ . This will be done in the next sections with different assumptions. In all cases the results will be written in the same general form :

$$\begin{aligned} \bar{l} \cdot \bar{p}_e^1/p_e &= K/F \bar{P}_e \cdot \bar{v}_e^1/V_A \\ \bar{l} \cdot \bar{p}_i^1/p_i &= K/F \bar{P}_i \cdot \bar{v}_i^1/V_A. \end{aligned} \quad (4)$$

This is always possible, thanks to linearity and it is convenient since we only need the pressure gradients. The three next sections will thus be devoted to different estimations of the two new tensors  $\bar{P}_e$  and  $\bar{P}_i$ . The system being so closed, the final solution will take the form of the following propagation equation for ion velocity :

$$\bar{D} \cdot \bar{v}_i^1/V_A = 0 \quad (5)$$

where

$$\bar{D} = \bar{D}_c - \beta_e \bar{P}_e \cdot \bar{U}_e - \beta_i \bar{U}_i \cdot \bar{P}_i.$$

The tensor  $\bar{U}_e$  comes from the expression of  $\bar{v}_e^1$  versus  $\bar{v}_i^1$ :  $\bar{v}_e^1 = \bar{v}_i^1 - \bar{j}^1/ne$ , which, after some handling can be written in the form  $\bar{v}_e^1 = \bar{U}_e \cdot \bar{v}_i^1$ , where :

$$\begin{aligned} \bar{U}_e \cdot \bar{l} &= \bar{l} - iK^2/F(\bar{l} - \bar{l}\bar{l}) \times \\ &\quad \times [\bar{\epsilon}_z \bar{x} - i(K^2/F)\beta_e \bar{P}_e]. \end{aligned}$$

It should be noticed that  $K^2/F$  being first order in  $K$  and  $F$ , we will have  $\bar{U}_e = \bar{U}_i = \bar{l}$  in the zero frequency limit.

The dispersion equation is thus obtained as  $\text{Det}(\bar{D}) = 0$ , and the polarizations of  $\bar{v}_i^1$  will be the corresponding non trivial solutions of (5).

### 3) Connexion with classical MHD treatment

Two kinds of approximations have to be made to reduce the preceding equations to the classical MHD treatment. We first have to take the zero frequency limit ( $\omega \ll \omega_{ci}$ ) of the basic equations (1) and (2), and second to use the classical MHD closure equation. The first assumption reduces the « cold part » of the propagation matrix  $\bar{D}_c$  to :

$$\bar{D}_{c0} = \begin{bmatrix} F^2/K^2 - 1 & 0 & 0 \\ 0 & F^2/K^2 - c^2 & 0 \\ 0 & 0 & F^2/K^2 \end{bmatrix}.$$

Results obtained later will be compared to this simple cold plasma result for which each solution corresponds to one term of the diagonal :  $F = K$  (fast mode) has a velocity polarization along  $x$ ,  $F = Kc$  (shear mode) along  $y$ , and  $F = 0$  (the slow mode does not propagate in cold plasma) along  $z$ .

The second assumption consists in using the following closure equation for each species :

$$\bar{p}_{i,e}^1/p = \gamma_{i,e}(n_{i,e}^1/n)\bar{l} \quad (6)$$

which yields, thanks to the equation of continuity :

$$\bar{P}_{i,e} = \gamma_{i,e} \bar{l}\bar{l} = \gamma_{i,e} \begin{bmatrix} s^2 & 0 & sc \\ 0 & 0 & 0 \\ sc & 0 & c^2 \end{bmatrix}$$

where  $\gamma_{i,e}$  are called the polytropic indexes.

So that the final MHD propagation matrix writes :

$$\overline{\overline{D}} = \begin{bmatrix} F^2/K^2 - 1 - \beta s^2 & 0 & -\beta sc \\ 0 & F^2/K^2 - c^2 & 0 \\ -\beta sc & 0 & F^2/K^2 - \beta c^2 \end{bmatrix}$$

where  $\beta = \gamma_i \beta_i + \gamma_e \beta_e$ .

It can be seen on this result that the fast and slow modes become coupled through  $\beta$ , while the shear mode remains unchanged in these approximations. However, it must be noticed that the name « adiabatic laws » for equations (6) is not correct in a magnetoplasma. As we will see in next section, a true adiabatic hypothesis ( $\overline{\nabla} \cdot \overline{\overline{Q}}^1 = 0$ , where  $\overline{\overline{Q}}$  is the heat flux) would not give this kind of result, except if  $\overline{\overline{B}}$  were zero, i.e. if the medium were isotropic (and then  $\gamma = 5/3$ ).

#### 4) Closure equations of fluid type

In this section, instead of using the closure equations (6), we consider more physical assumptions: we assume the electrons are isothermal and the ions adiabatic. It will be shown, in the next section, that the very major part of the results about the low frequency modes can be accurately found *via* this correct two-fluid theory and that the corresponding assumptions for phase velocity and frequency are :

(i) the parallel phase velocity is slow compared to the electron thermal velocity, and fast compared to the ion thermal velocity :

$$V_{thi} \ll \omega/k_1 \ll V_{the} \quad (7)$$

$$\overline{\overline{P}}_i = \begin{bmatrix} 2s^2 - F^2(d_2 s^2/4 + d_1 c^2) & -iF(d_2 s^2/2 + d_1 c^2) & (1 - F^2 d_1) sc \\ iF(d_2 s^2/2 + d_1 c^2) & -F^2(d_2 s^2/4 + d_1 c^2) & iF d_1 sc \\ (1 - F^2 d_1) sc & -iF d_1 sc & 3c^2 - F^2 s^2 d_1 \end{bmatrix}. \quad (10)$$

The quantities

$$d_1 = 1/(1 - F^2) \quad \text{and} \quad d_2 = 1/(1 - F^2/4)$$

show that poles at first and second gyroharmonics appear, as long as they can be approached without violating condition (8). The zero frequency limit of this result is to be noticed :

$$\overline{\overline{P}}_{i0} = \begin{bmatrix} 2s^2 & 0 & sc \\ 0 & 0 & 0 \\ sc & 0 & 3c^2 \end{bmatrix}.$$

This limit, which can be directly obtained from (9) in a very simple way, can also be reached by other methods, in particular by writing  $(p_{\perp}/nB)^1 = 0$ , as suggested by Krall and Trivelpiece (1973) from consideration of invariant conservation. But the most important point we would like to emphasize is that this limit cannot be obtained from any method involving polytropic indexes since even the more sophisticated « double adiabatic » law (see for instance

(ii) the frequency is not too close to the ion gyroharmonics  $\omega = N\omega_{ci}$  ( $N$  integer) :

$$|N - F| \gg k_1 V_{thi}/\omega_{ci}. \quad (8)$$

These hypotheses correspond to ions free from both Cerenkov and cyclotron resonances.

Under these conditions,  $\overline{\overline{P}}_e$  has actually the form (6), with  $\gamma_e = 1$ , but  $\overline{\overline{P}}_i$  is still to be determined. To find out the correct expression of the adiabatic law, giving  $\overline{\overline{p}}_i^1$  versus  $\overline{\overline{v}}_i^1$  components, we have to write one more equation of the BBGKY hierarchy (Buneman, 1961) :

$$\partial(\overline{\overline{p}}_i/n)^1/\partial t + p_i/n \{ \overline{\overline{\nabla}}_i^1 \}^{sym} + \{ \overline{\overline{\omega}}_{ci} \times (\overline{\overline{p}}_i/n)^1 \}^{sym} + \overline{\overline{\nabla}} \cdot \overline{\overline{Q}}^1/n = 0$$

which writes :

$$\overline{\overline{p}}_i^1/p_i - 1/iF \{ \overline{\overline{e}}_z \times \overline{\overline{p}}_i^1/p_i \}^{sym} = K/F \{ \overline{\overline{T}}_i^1/V_A \}^{sym} + \overline{\overline{T}} \cdot \overline{\overline{v}}_i^1/V_A \overline{\overline{T}} + \overline{\overline{T}} \cdot \overline{\overline{Q}}^1/(p_i V_A). \quad (9)$$

In these equations,  $\{ \overline{\overline{T}} \}^{sym}$  stands for the sum of tensor  $\overline{\overline{T}}$  and its transposed, and the unperturbed pressure  $\overline{\overline{p}}_i$  is assumed to be scalar.

In this section  $\overline{\overline{\nabla}} \cdot \overline{\overline{Q}}^1$  is taken equal to zero (adiabatic hypothesis) : equation (9) thus directly provides the law we were looking for, without any assumption about the form of the result, and without any *ad hoc*  $\gamma_i$  coefficient, contrary to the MHD procedure. The result is found through a six equations system (since  $\overline{\overline{p}}^1$  is symmetric), and it writes :

Buneman, 1961 ; Krall and Trivelpiece, 1973) provides in this case :

$$\overline{\overline{P}}_i = \begin{bmatrix} \gamma_{\perp} s^2 & 0 & \gamma_{\perp} sc \\ 0 & 0 & 0 \\ \gamma_{\parallel} sc & 0 & \gamma_{\parallel} c^2 \end{bmatrix}$$

which cannot fit our result, whatever  $\gamma_{\perp}$  and  $\gamma_{\parallel}$  are. Moreover, all terms of order  $F$  (and  $K$ ) or more, which do exist in equation (10) are never taken into account in usual fluid theories, although they modify the modes, in particular by coupling the shear mode with the compressional ones. In our opinion, the usual disregard of these terms is the reason why such theories are generally considered as unable of correctly accounting for finite Larmor radii corrections.

#### 5) Closure equations of kinetic type

Let us consider now a last kind of closure of the two-fluid system, which is a generalization of the preceding

one : the closure equation is solved with an estimation of  $\bar{\nabla} \cdot \bar{Q}^1$  found with a kinetic method, instead of  $\bar{\nabla} \cdot \bar{Q}^1 = 0$  in the above section. It will allow us to get the departures from the preceding results when the hypotheses (7) and (8) are no longer verified. Furthermore, it will also enable us to obtain the imaginary parts of the results (damping, for a maxwellian distribution), and to specify where they get in, by comparing the obtained matrices with the above fluid ones. We start from the linearized Vlasov equation, whose general solution can be found in Krall and Trivelpiece (1973). To get tractable results, we first expand this solution at zeroth order in  $k_{\perp} \rho$  (the validity of this expansion will have to be questioned for each particular result, and if other expansions are needed for further simplifications, we will have to ensure that they are consistent with this one). This expanded solution writes :

$$f^1/V_A = -\partial f/\partial V_1 \left[ \frac{F v_z^1}{F^* V_A} - \frac{K V_{th}^2}{F^* V_A^2} \bar{I} \cdot \bar{p}^1 \cdot \bar{e}_z \right] - \partial f/\partial V_{\perp} \left[ \left( \cos \phi + \frac{k_{\perp} V_{\perp}/\omega_c}{2 F^*} \right) \frac{v_x^1}{V_A} + \sin \phi \frac{v_y^1}{V_A} \right] \quad (11)$$

where  $f$  (resp.  $f^1$ ) is the distribution function of variable  $\bar{V}$  at zeroth order (resp. first order),  $\bar{v}$  is the fluid speed,  $F = \omega/\omega_c$ ,  $F^* = (\omega - k_{\parallel} V_1)/\omega_c$  and  $\phi$  is the azimuthal angle of  $\bar{V}_{\perp}$  around  $\bar{B}$ . The electric field which appears in Vlasov equation has been expressed in term of velocity and pressure components, through the momentum equation. In this section, the calculation being valid as well for ions and electrons,  $F$  and  $K$  will be normalized to  $\omega_{ci}$  or  $\omega_{ce}$  depending on the considered species. The heat flux perturbation

$$\bar{Q}^1 = m \int [(\bar{V} - \bar{v})(\bar{V} - \bar{v})(\bar{V} - \bar{v}) f(\bar{V})]^1 d^3V$$

can thus be obtained through integrations of  $f^1$  upon  $V_1$ ,  $V_{\perp}$  and  $\phi$ . When these integrations are performed, 20 of the 27 components of  $\bar{Q}^1$  are found to be zero, and the remaining 7 components have only two distinct values (all components whose indices differ only by a permutation are equal) :

$$\begin{aligned} Q_{xz}^1/pV_A &= (\alpha' - 1) v_z^1/V_A - (\alpha - 1) \times \\ &\quad \times (F/Kc^2) \bar{I} \cdot \bar{p}^1/p \cdot \bar{e}_z + (\alpha - 1)(2s/c) v_z^1/V_A \\ Q_{yz}^1/pV_A &= (\alpha'' - 3) v_z^1/V_A - (\alpha' - 1) \times \\ &\quad \times (F/Kc^2) \bar{I} \cdot \bar{p}^1/p \cdot \bar{e}_z + (\alpha' - 1)(s/c) v_z^1/V_A \\ Q_{xy}^1 &= Q_{yx}^1 \end{aligned} \quad (12)$$

where the coefficients  $\alpha$ ,  $\alpha'$ ,  $\alpha''$  contain the Fried and Conte function (Fried and Conte, 1961) when the unperturbed distribution function is specified to be maxwellian :

$$\begin{aligned} \alpha &= -\xi Z(\xi) \\ \alpha' &= 2\xi^2(\alpha - 1) \quad \xi = \omega/(\sqrt{2} k_{\parallel} V_{th}) \\ \alpha'' &= 2\xi^2(\alpha' - 1) \end{aligned}$$

Let us notice that so many components of  $\bar{Q}^1$  are zero because, at zeroth order in  $k_{\perp} \rho$ , the only pole appearing in equation (11) is the Cerenkov one (no cyclotron resonance). It can be checked here that for  $\xi_i \gg 1$  and  $\xi_e \ll 1$  (as in the above section) ions are adiabatic ( $\bar{Q}^1 = 0$ ) and electrons are not :

$$\begin{aligned} \alpha_i &\sim \alpha'_i \sim 1, \quad \alpha''_i \sim 3 \quad \text{for ions} \\ \alpha_e &\sim \alpha'_e \sim \alpha''_e \sim 2\xi_e^2 \sim 0 \quad \text{for electrons.} \end{aligned}$$

This point underlines that the adiabatic hypothesis depends on the existence of resonant particles and not of a thermodynamical equilibrium that would assume collisions.

In the general case, using result (12),  $\bar{\nabla} \cdot \bar{Q}^1$  can be easily determined, from which the solution of (9) can be inferred. This is again done through a 6-equation system, but much more cumbersome. Nevertheless, the procedure is straightforward, and the result can be found. The general expression of the  $\bar{P}$  components are given in appendix I. This matrix is not *a priori* easily tractable, but thanks to this general result, we will be able to particularize it in any needed range of parameters and thus simplify it with well controlled approximations. Examples of such treatments are given in next section.

### 3. ION WAVES IN LOW FREQUENCY LIMIT

We will now show how the general results can be particularized for the low frequency modes ( $F \ll 1$ ). In the case of the shear Alfvén wave we shall emphasize that the pressure gradients obtained in section 2.4 are sufficient to get the first order finite Larmor radius results. We shall then check that the kinetic closure of section 2.5 gives the same real part of the frequency (in the same conditions) and also the imaginary part.

#### 1) Expansion of the general result

We will make the expansion with respect to  $K$ , assuming  $F$  is the same order of magnitude as  $K$  (i.e.  $\omega/k = O(V_A)$ ). Taking also into account the form of the propagation matrix :

$$\bar{D} = \bar{D}_{ci} + \begin{bmatrix} d_{xx} & iFd_{xy} & d_{xz} \\ iFd_{yx} & -F^2 d_{yy} & iFd_{yz} \\ d_{zx} & iFd_{zy} & d_{zz} \end{bmatrix}$$

we will expand all corrective terms  $d_{ij}$  to zeroth order, and it will be sufficient to get corrections to second order for the shear Alfvén mode. Meanwhile, the compressional modes will only be analysed at zeroth order. With these expansions, the different matrices appearing in (5) greatly simplify (main lines of the calculation are reported to appendix II), and the resulting propagation matrix is :

$$\bar{D} = \begin{bmatrix} F^2/K^2 - 1 - s^2[\beta_e(1 + \alpha_e) + \beta_i(1 + \alpha_i)] & & & & \\ & iF \left[ 1 + \beta_e s^2 \left( \frac{K^2 c^2}{F^2} \alpha_e + \frac{\mu}{2 \xi_e^2} \right) + \frac{\beta_i}{2} \left( 1 + c^2 - \frac{s^2}{2 \xi_i^2} \right) \right] & & & -sc(\beta_e + \beta_i) \\ & iF \left[ -c^2 + \frac{\beta_i}{2} \left( 1 - 3c^2 - 2s^2 \alpha_i \left( 1 - \frac{K^2 c^2}{F^2} \right) \right) \right] & & & \\ & & F^2/K^2 - c^2 - \frac{\beta_i F^2}{4} \left[ 1 - 9c^2 - 4s^2 \left( 1 - \frac{1}{\alpha_i'} \right) - \frac{s^2}{\xi_i^2} \left( 2 - \frac{K^2 c^2}{F^2} \right) \right] & & \\ & & & & iFsc \left[ 1 - \beta_i \left[ 1 + \frac{K^2 c^2}{F^2} (G_i - 1) \right] \right] \\ -sc(\beta_e + \beta_i) & iFsc \left[ \beta_e \left( \frac{K^2 c^2}{F^2} (1 - G_e) + \frac{\mu}{\alpha_e'} \right) + \frac{\beta_i}{\alpha_i'} \right] & & & F^2/K^2 - c^2(\beta_e G_e + \beta_i G_i) \end{bmatrix} \quad (13)$$

where  $G = \alpha''/\alpha'$ .

The simpler form of (13) corresponding to the usual case of intermediate  $\beta$  ( $\xi_e \ll 1$ ,  $\xi_i \gg 1$ ) must also be retained :

$$\bar{D}_{int} = \begin{bmatrix} F^2/K^2 - 1 - s^2(\beta_e + 2\beta_i) & iF \left[ 1 + \frac{\beta_i}{2} (1 + c^2) \right] & & & -sc(\beta_e + \beta_i) \\ & iF \left[ -c^2 + \frac{\beta_i}{2} \left( 1 - 3c^2 - 2s^2 \left( 1 - \frac{K^2 c^2}{F^2} \right) \right) \right] & & & \\ & & F^2/K^2 - c^2 - \beta_i \frac{F^2}{4} (1 - 9c^2) & iFsc \left[ 1 - \beta_i \left( 1 + 2 \frac{K^2 c^2}{F^2} \right) \right] & \\ & & & & \\ -sc(\beta_e + \beta_i) & & iFsc\beta_i & & F^2/K^2 - c^2(\beta_e + 3\beta_i) \end{bmatrix} \quad (14)$$

This result can be obtained by setting  $\alpha_e = \mu/\xi_e^2 = \mu/\alpha_e' = 0$ ,  $G_e = 1$ ,  $\alpha_i = \alpha_i' = 1$ ,  $1/\xi_i^2 = 0$ ,  $G_i = 3$  in the preceding result, or by deriving it directly from the two-fluid treatment of section 2.4. The very major part of the results are contained in this matrix (14) in case of intermediate  $\beta$ , except the estimation of the damping, since the imaginary parts stand in coefficients  $\alpha$ ,  $\alpha'$  and  $G$  of (13).

## 2) Shear Alfvén wave for intermediate $\beta$

The case  $\beta_i \ll 1$  and  $\beta_e \gg M_e/M_i$  is one of the most frequently met in the literature (Hasegawa and Chen, 1975 ; Hasegawa and Uberoi, 1982 ; Goertz, 1983, 1984 ; Southwood and Hughes, 1983). In this case matrix (14) which is directly obtainable from two-fluid theory, is sufficient to derive the dispersion and polarization of the shear Alfvén mode. To get these results at order  $K^2$ ,  $F/Kc$  can be replaced by its value at zeroth order in all coefficients  $d_{ij}$ . This value is 1, since when  $K = 0$  the shear Alfvén mode is strictly decoupled of compressional modes and its dispersion relation is  $F = Kc$ . If we further simplify by expanding also to first order in  $\beta_i$  and  $\beta_e$ , which is consistent

with the reduction of (13) into (14), equating the determinant of this matrix to zero provides :

$$F^2 = K^2 c^2 (1 + A_0 K^2) \quad (15)$$

where  $A_0 = (\beta_e - c^2)/s^2 + 3/4 \beta_i s^2$ .

This dispersion equation calls for comments :

(i) Its validity is limited : the above procedure, consisting in an expansion around the zero frequency mode  $F = Kc$ , is only valid when  $|A_0| K^2 \ll 1$  (i.e.  $K^2 \ll s^2/c^2$ ), excluding quasi parallel propagation.

(ii) The curvature of the dispersion curve can change, as  $A_0$  can change its sign. The curvature is upward when the propagation is perpendicular, but it becomes downward as soon as  $c^2/s^2 \gg 3/4 \beta_i s^2 + \beta_e$ . This can be checked on figure 1, where the shear Alfvén dispersion curve is displayed for various  $\theta$  angles. The agreement between the above simple analytical formula, which predicts an inversion of curvature for  $\theta \sim 79^\circ$ , and the numerical result ( $\theta = 81^\circ$ ) is rather good. Let us notice that the knowledge of the curvature of the linear dispersion relation happens to be important for nonlinear calculations (wave coupling, existence of solitons...).

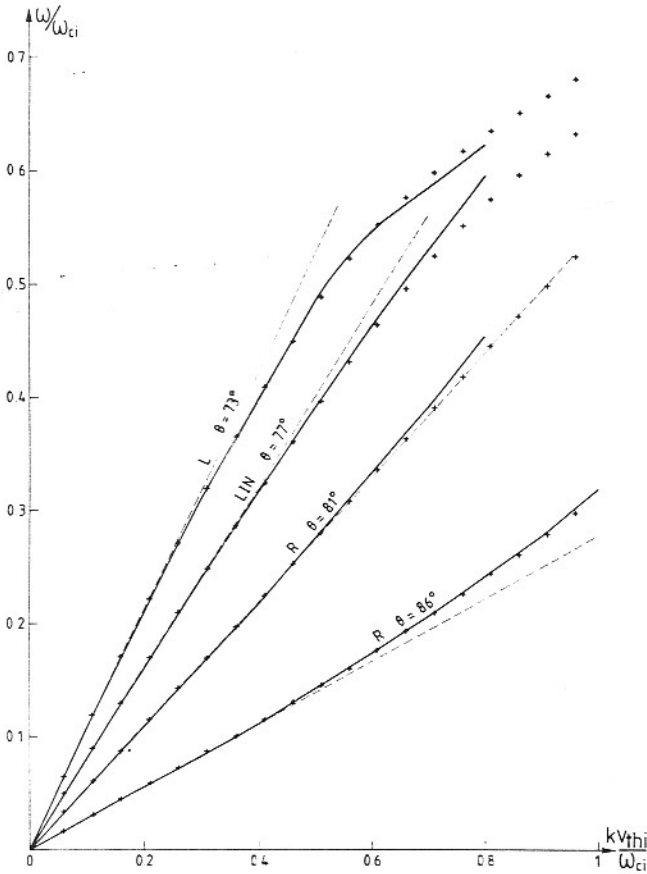


Figure 1  
Shear Alfvén dispersion relation for various propagation angles  $\theta$ . The solid lines are analytical results, compared to numerical ones (crosses) and to the cold plasma and low frequency approximation (dashed line), in the following conditions:  $\beta_e = 7.7 \times 10^{-3}$  and  $\beta_i = 3.9 \times 10^{-2}$ . Changes in the curvature and in the polarization sense are to be noticed: R stands for right-handed, L for left-handed and LIN for linear polarization.

(iii) It is interesting to notice that it is not necessary to solve the fully kinetic problem to find out the « kinetic » correction, due to the coupling of the shear Alfvén mode with the compressional ones.

We can compare our result to the one given by Hasegawa and Uberoi (1982) for the case of quasi-perpendicular propagation ( $c^2 \ll \beta \ll 1$ ). In this case, our result writes :

$$F^2/K^2 c^2 = 1 + (\beta_e + 3/4 \beta_i) K^2$$

which is exactly the same as theirs, since  $K_{\perp} = K$  in this limit.

We can also compare our result with a result from Bilichenko *et al.* (1984) who show a change in the curvature of the shear Alfvén mode for  $\beta_e = c^2$  in the case of cold ions ( $\beta_i = 0$ ); this property also results obviously from equation (15).

As we have equated the determinant of matrix (14) to zero, we can look for the corresponding  $\bar{v}_i^1$  solutions, that is for the polarization of the ion velocity perturbation in the shear Alfvén mode. Knowing this velocity, many interesting points can be deduced: the electric field and the polarization of the wave, the components of the fields which are parallel to the equilibrium magnetic field ( $z$ -components) and the density fluctu-

Table 1

Description of the shear Alfvén wave

	$\frac{E_y}{E_x}$	$\frac{E_z}{E_x}$	$\frac{B_z}{B_y}$	$\frac{n}{n_0}$
(1)	$-i 4.9 \times 10^{-3}$	$-1.4 \times 10^{-3}$	$-i 2.4 \times 10^{-2}$	
(2)	$-i 5.6 \times 10^{-3}$	$-1.5 \times 10^{-3}$	$-i 2.7 \times 10^{-2}$	$-11 \frac{B_z}{B_0}$

(1) Numerical results.

(2) Analytical results in the following conditions:  $\beta_e = 0.028$ ,  $\beta_i = 0.085$  and  $(\bar{B}, \bar{k}) = 79^\circ$ , for  $F = 0.059$  which implies  $kV_{thi}/\omega_{ci} = 0.07$ .

ation associated to the wave. We will compare our analytical results to those of program WHAMP, except for the density which is not numerically calculated. The comparison is made in table 1, at one point of the dispersion relation.

After a little handling and taking into account the fact that  $F \sim Kc$ , we find :

$$v_{ix}^1/v_{iy}^1 = iKc/s^2 \left[ 1 - \beta_e - \beta_i \frac{3-c^2}{2} \right] \quad (16)$$

$$v_{iz}^1/v_{iy}^1 = iK/s [\beta_e + \beta_i c^2].$$

Of course, we find again that  $\bar{v}_i$  is nearly polarized along  $y$ , that is perpendicular to both the static magnetic field and the wave vector; but we also see that, as soon as  $K$  is not zero, a parallel component appears together with a left-handed perpendicular component (since  $\beta_e, \beta_i \ll 1$ ).

A more surprising result occurs when the polarization is calculated for the electromagnetic fields: deducing the components of the electric field from (16), it arises :

$$\begin{aligned} E_y^1/E_x^1 &= -\frac{v_{ix}^1}{v_{iy}^1} (1 - iKc v_y^1/v_x^1) = \\ &= -\frac{iKc}{s^2} \left[ c^2 - \beta_e - \beta_i \frac{3-c^2}{2} \right]. \end{aligned} \quad (17)$$

This result shows that the electric field perturbation is left-handed when  $c^2 > \beta_e + \beta_i \frac{3-c^2}{2}$ , as it is well-

known in cold plasma theory, but is right-handed in the opposite case, in particular in quasi perpendicular propagation. It is worth noticing that in this last case, the perturbations of electromagnetic fields and velocity are rotating in opposite senses. Bilichenko *et al.* (1984) have already found such a change of polarization in the case when  $\beta_i = 0$ . In this situation the change of curvature of the dispersion relation and the change of polarization happen for the same  $(\bar{B}, \bar{k})$  angle defined by  $c^2 = \beta_e$ . From our result it appears that when the ions are hot these changes happen respectively for  $c^2 \sim \beta_e + 3/4 \beta_i$  and  $c^2 \sim \beta_e + 3/2 \beta_i$ . We have compared these results to the numerical kinetic ones (see fig. 1): the change of polarization is expected for  $\theta \sim 75^\circ$  and it happens for  $\theta \sim 77^\circ$ , which is again fairly good (in fact the agreement becomes less good for smaller  $\beta$  values).



For the parallel component we find :

$$E_z^1/E_x^1 = -K^2 \frac{c}{s} \beta_e \quad (18)$$

which means that this parallel component is only due to electron temperature. This result is identical to those set down by Hasegawa (1976) and Southwood and Hughes (1983).

At last we will give two results concerning the compressibility of the shear Alfvén mode, which can be derived in the same way :

$$B_z^1/B = iK/s \left( c^2 - \beta_e - \beta_i \frac{3-c^2}{2} \right) B_y^1/B \quad (19)$$

$$n^1/n = iK/s (1 - 3/2 \beta_i s^2) B_y^1/B.$$

The first terms in these compressional effects are due to ion inertia, so that they exist even in a cold plasma ; this reminds us that this mode is never purely torsional when  $K \neq 0$ . Nevertheless, the other terms, due to temperature, happen to have an unlike sign, and thus might diminish the compressional effects.

### 3) Shear Alfvén damping

The preceding calculation was made from pure two-fluid theory (matrix (14)). Let us generalize it by using matrix (13) obtained from the kinetic closure. It allows one to see the corrections which must be made in the real part of the dispersion relation when the hypothesis « intermediate beta » is not fulfilled, but we will not present here the corresponding results since they can take different forms depending on which of the conditions  $\beta_i < 1$  or  $\beta_e > M_e/M_i$  is violated. So we will use equation (13) only to determine the damping rate of the wave in the same « intermediate beta » case as above. In these conditions, the dispersion equation (15) keeps the same form but with a slightly different constant  $A_0$  :

$$A_0 = 1/s^2 [\beta_e (1 - c^2 \alpha_e - s^4 (1 - G_e + \mu/\alpha_e')) - c^2] + \beta_i [3/4 s^2 - (1 - \alpha_i) c^2/s^2]. \quad (20)$$

Taking into account the exponential decrease of the imaginary part of  $\alpha_i$  with  $\xi_i$ , the damping of the shear Alfvén wave is proved to be almost entirely due to electrons, through coefficients  $\alpha_e$  and  $G_e$ . (The imaginary part of  $\mu/\alpha_e'$  can also be neglected since it is smaller by a factor  $\beta_e$  than these ones). It is interesting to note that both coefficients arise in the calculation of  $\bar{U}_e$ , which introduces the difference of velocity between electrons and ions. It corresponds to the well-known idea that Landau damping originates in the motion of electrons in the parallel electric field.

Referring to definitions of  $\alpha_e$  and  $G_e$ , we find that

$$\text{Im } \alpha_e = \text{Im } G_e = -i \sqrt{\pi \mu / (2 \beta_e)}$$

so that the damping has the simple expression :

$$\text{Im } F / \text{Re } F = -K^2 \frac{s^4 + c^4}{2s^2} \sqrt{\pi \mu \beta_e / 2}. \quad (21)$$

These results agree with numerical ones (fig. 3).

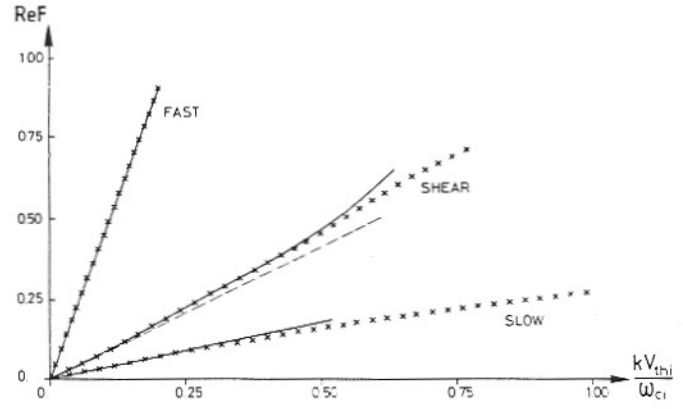


Figure 2

Real part of the frequency versus wave number, for the three modes. Analytical results (solid line) are compared to numerical ones (crosses) in the following conditions :  $\beta_e = 0.028$ ,  $\beta_i = 0.085$  and  $(\bar{B}, \bar{k}) = 79^\circ$ .

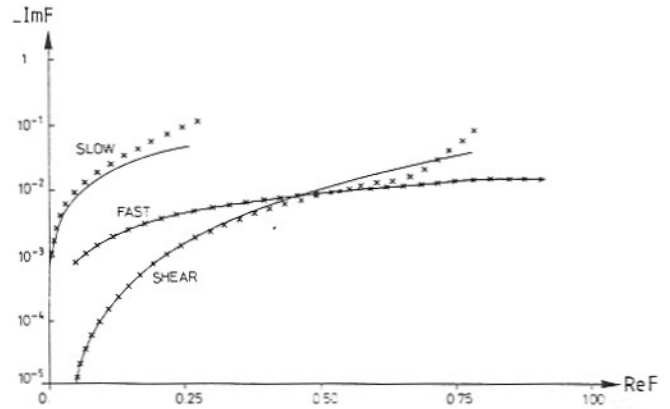


Figure 3

Imaginary part of the frequency versus real part, in the same conditions as in figure 2. Analytical results (solid line) are compared to numerical ones (crosses).

### 4) Compressional modes

We present now a last example of what can be done in the frame of the same theory. At zeroth order in  $K^2$ , the dispersion equation of fast and slow modes are easily obtained :

$$F^2/K^2 = c^2 (G_i \beta_i + G_e \beta_e) \quad \text{Slow} \quad (22)$$

$$F^2/K^2 = 1 + s^2 [(1 + \alpha_i) \beta_i + (1 + \alpha_e) \beta_e] \quad \text{Fast}.$$

These implicit equations ( $\alpha_{e,i}$  and  $G_{e,i}$  depend on the unknown  $F/K$ ) provide both real and imaginary parts of the solution. In the fluid case, which corresponds to situations where  $\xi_i \gg 1$  and  $\xi_e \ll 1$  (but notice that these conditions are not verified for the same values of  $\beta_{i,e}$  for each mode), these results write :

$$F^2/K^2 = c^2 (3 \beta_i + \beta_e) \quad \text{Slow}$$

$$F^2/K^2 = 1 + s^2 (2 \beta_i + \beta_e) \quad \text{Fast}.$$

These results can be considered as identical to MHD ones, provided that the  $\gamma$  coefficients of this theory are allowed to be different depending on the mode  $\gamma_e = 1$  in both cases, but  $\gamma_i = 3$  for slow mode and

$\gamma_i = 2$  for fast mode. However, if we consider the same range of parameters as for the shear Alfvén mode (intermediate  $\beta$  case), the preceding fluid inequalities are not always verified for the compressional modes. Indeed, we have to consider the more realistic assumptions :

(i) *Slow mode* :  $\xi_e \ll 1$  but  $\xi_i \sim 1$ .

The dispersion equation thus becomes :

$$F^2/K^2 = c^2(G_i \beta_i + \beta_e). \quad (23)$$

It can be checked that this equation can be put in the very convenient form :  $Z'(\xi_i) = 2 T_e/T_i$  where  $Z'$  is the derivative of the Fried and Conte function (Fried and Conte, 1961). From the solution  $\xi_i$  of equation (23), the result can be drawn :

$$\text{Re } F = Kc[\beta_i \text{Re } G_i + \beta_e]^{1/2}$$

$$\text{Im } F/\text{Re } F = 1/2 \text{Im } G_i \beta_i / (\text{Re } G_i \beta_i + \beta_e).$$

These solutions are plotted in figures 2 and 3, and the comparison with numerical results is shown. The agreement can be considered as good for results at zeroth order in  $K^2$ . A point which is worth noticing is that in this example, where we have chosen  $T_e = 3 T_i$ , the slow compressional mode is drastically damped by ions. The study of equation (23) confirms that if we lessen  $T_e/T_i$  the damping increases, reaching 40 % if  $T_e = T_i$ . These results, about the absorption of the slow mode, although quite well-known, remind us that a MHD dispersion relation containing the slow mode solution but not its damping, must not be used for plasmas where  $T_e \lesssim T_i$ . We thus provide an alternative simple dispersion equation which could be used in these conditions. We think that this would be of interest, for instance, for the study of the magnetopause Kelvin-Helmoltz instability where such modes are supposed to exist (Southwood, 1968 ; Pu and Kivelson, 1983).

(ii) *Fast mode* :  $\xi_i \gg 1$ , but  $\xi_e \sim 1$ .

In these conditions equation (19) leads to :

$$\text{Re } F = K \left[ \frac{1 + s^2(2\beta_i + \beta_e)}{1 - \mu s^2/c^2} \right]^{1/2} \approx \sqrt{\frac{2\beta_e}{\mu}} Kc\xi_e$$

$$\text{Im } F/\text{Re } F = -\frac{\sqrt{\pi}}{2} \frac{\beta_e s^2 \xi_e^2 \exp - \xi_e^2}{1 + s^2(2\beta_i + \beta_e(1 + 2\xi_e^2))} \quad (24)$$

The damping is of course linked to electrons in this case. The agreement with numerical results (figs. 2 and 3) is quite good in spite of the poor expansion made.

#### 4. CONCLUSION

The purpose of our study was to obtain theoretical results for the low frequency ion modes that could be easily compared with experimental data. That implied

we could not make *a priori* assumptions concerning the wave itself such as its polarization or its propagation direction, since these assumptions may be difficult to verify in experimental situations. We have achieved this program thanks to a theoretical procedure of a two-fluid type : it allowed us to get the first order Larmor radius effects without the usual kinetic calculation, and also to obtain the damping of the modes when completed by a kinetic closure. The main results presented here concern the « kinetic » Alfvén wave, in particular : (i) the curvature of the dispersion relation may change and thus the parallel phase velocity may be sub-Alfvénic or super-Alfvénic, depending on the plasma temperature and propagation angle ; (ii) although the wave has a linear polarization for very low frequencies, when  $\omega \neq 0$  it has a circular component which is not always left-handed (as in cold plasma). Some results have also been inferred from the same procedure for the dispersion and the damping of the compressional modes.

Beyond the results summarized above, we expect that this paper might also be considered as a tool and allow to infer many other results that could be needed for interpreting experimental observations. For that purpose we have tried to present the theoretical procedure as clearly and systematically as possible. For instance, the study of the compressional mode properties could be carried on further just by changing the expansion orders in the general matrices ; a temperature anisotropy could also be introduced in the same frame, to investigate magnetosheath waves (Rodgers *et al.*, 1984) and possible mirror modes (Hasegawa, 1975), etc. Let us outline that a good knowledge of the linear ion modes is also interesting for other studies, non linear ones (knowledge of the dispersion relation curvature) or weak discontinuities theory : the fast and slow compressional modes can give rise to fast and slow shocks, and the shear Alfvén mode to rotational discontinuities (the coupling between slow and shear modes in the degenerate case of purely perpendicular propagation giving rise to tangential discontinuities). A two-fluid theory of the type presented above could thus be useful for studying the role of the « gyroviscous stress » in the magnetopause equilibrium as suggested by Staciewicz (1986).

#### Acknowledgements

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#### APPENDIX I

##### COMPLETE EXPRESSION OF TENSOR $\overline{\overline{P}}$

$\overline{\overline{P}}$  is defined by :

$$\overline{\overline{P}} \cdot \overline{\overline{P}}^1/p = K/F \overline{\overline{P}} \cdot \overline{\overline{v}}^1/V_A$$

and calculated when  $\overline{\overline{Q}}^1$  is taken at zeroth order in  $k_\perp \rho$ .

We use the condensed notations :

$$\begin{aligned} s &= \sin \theta, \quad c = \cos \theta, \\ F &= \omega / \omega_{ci}, \quad K = kV_A / \omega_{ci}, \\ \xi &= \omega / (\sqrt{2} k_1 V_{th}), \quad \alpha = -\xi Z(\xi), \\ \alpha' &= 2\xi^2(\alpha - 1), \quad \alpha'' = 2\xi^2(\alpha' - 1), \\ d_1^* &= 1 / \left[ 1 - F^2 \left( 1 + \frac{s^2}{2\xi^2 c^2} \right) \right], \\ d_2 &= 1 / (1 - F^2/4). \end{aligned}$$

Then :

$$\begin{aligned} P_{xx} &= (1 + \alpha) s^2 - \\ &- F^2 \left[ s^2 \frac{d_2}{4} + c^2 d_1^* \left( 1 + \frac{s^2}{c^2} (\alpha - 1) \right) \left( 1 + \frac{1}{2\xi^2 c^2} \right) \right] \end{aligned}$$

$$\begin{aligned} P_{xy} &= -iF \left[ s^2 \left( \frac{d_2}{2} - \frac{d_1^*}{2\xi^2} \right) + c^2 d_1^* \right] \\ P_{xz} &= sc \left[ 1 - F^2 d_1^* \left( 1 - \frac{s^2}{2\xi^2 c^2} \right) \right] \\ P_{yx} &= iF \left[ s^2 \frac{d_2}{2} + c^2 d_1^* \left( 1 + \frac{s^2}{c^2} (\alpha - 1) \right) \right] \\ P_{yy} &= -F^2 \left[ s^2 \frac{d_2}{4} + c^2 d_1^* \left( 1 + \frac{s^2}{2\xi^2 c^2} \right) \right] \\ P_{yz} &= iFsc d_1^* \\ P_{zx} &= sc \left[ 1 - F^2 \frac{d_1^*}{\alpha'} \left( 1 + \frac{s^2}{c^2} (\alpha - 1) \right) \right] \\ P_{zy} &= -iFsc d_1^* / \alpha' \\ P_{zz} &= \alpha'' c^2 / \alpha' - F^2 s^2 d_1^* / \alpha' \end{aligned}$$

## APPENDIX II

### LOW FREQUENCY EXPANSION

Here are some of the intermediate steps of the calculation of the propagation matrix in the case of the expansion defined in section 3.1. ( $\alpha$ ,  $\alpha'$ ,  $\alpha''$  are defined in section 2.5 and  $G = \alpha''/\alpha'$ ).

$$\begin{aligned} \bar{\bar{P}}_e &= \begin{bmatrix} s^2(1 + \alpha_e) & iFs^2\mu/(2\xi_e^2) & sc \\ 0 & 0 & 0 \\ sc & -iFsc\mu/\alpha'_e & G_e c^2 \end{bmatrix} \\ \bar{\bar{P}}_e \cdot \bar{\bar{U}}_e &= \begin{bmatrix} s^2(1 + \alpha_e) & -iFs^2 \left( \frac{K^2 c^2}{F^2} \alpha_e + \frac{\mu}{2\xi_e^2} \right) & sc \\ 0 & 0 & 0 \\ sc & -iFsc \left[ \frac{K^2 c^2}{F^2} (1 - G_e) + \mu/\alpha'_e \right] & G_e c^2 \end{bmatrix} \\ \bar{\bar{P}}_i &= \begin{bmatrix} s^2(1 + \alpha_i) & -\frac{iF}{2} \left( 1 + c^2 - \frac{s^2}{\xi_i^2} \right) & sc \\ \frac{iF}{2} [1 + c^2 + 2s^2(\alpha_i - 1)] & -\frac{F^2}{4} \left( 1 + 3c^2 + \frac{2s^2}{\xi_i^2} \right) & iFsc \\ sc & -iFsc/\alpha'_i & G_i c^2 \end{bmatrix} \\ \bar{\bar{U}}_i \cdot \bar{\bar{P}}_i &= \begin{bmatrix} s^2(1 + \alpha_i) & -\frac{iF}{2} \left( 1 + c^2 - \frac{s^2}{2\xi_i^2} \right) & sc \\ -\frac{iF}{2} \left[ 1 - 3c^2 - 2s^2\alpha_i \left( 1 - \frac{K^2 c^2}{F^2} \right) \right] & A & B \\ sc & -iFsc/\alpha'_i & G_i c^2 \end{bmatrix} \end{aligned}$$

where

$$A = -\frac{F^2}{4} \left[ 1 + 3c^2 + \frac{2s^2}{\xi_i^2} - \frac{2K^2 c^2}{F^2} \left( \frac{2s^2}{\alpha'_i} - 1 - c^2 + \frac{s^2}{2\xi_i^2} \right) \right]$$

and

$$B = iFsc \left[ 1 + \frac{K^2 c^2}{F^2} (G_i - 1) \right].$$

## REFERENCES

- Barnes A., 1979. Hydromagnetic waves and turbulence in the solar wind. In: *Solar System Plasma Physics*, I. Eds. Parker E. N., Kennel C. F., Lanzerotti L. J., North-Holland Publ. Co., Amsterdam, New York, Oxford.
- Bilichenko S. V., Kostin V. M., Chmyrev V. M., 1984. Properties of the polarization of electromagnetic bursts in ionospheric and magnetospheric plasmas. Preprint 57 (531), Moscow, IZMIRAN.
- Buneman O., 1961. Gas law and conductivity of a collision-free plasma. *Phys. Fluids*, **4**, 669-680.
- Chew G. F., Goldberger M. L., Low F. E., 1956. The Boltzmann equation and the one fluid hydromagnetic equations in the absence of particle collisions. *Proc. Roy. Soc. (London)* **A236**, 112-118.
- Dessler A. J., 1983. *Physics of the Jovian magnetosphere*. Cambridge Univ. Press.
- Fried B. O., Conte S. D., 1961. *The plasma dispersion function*. Academic Press.
- Goertz C. K., 1983. Non-stationary coupling between the magnetosphere and ionosphere. *ESA/SP-183*, 221-228.
- Goertz C. K., 1984. Kinetic Alfvén waves on auroral field lines. *Planet. Space Sci.*, **32**, 1387-1392.
- Hasegawa A., Chen L., 1975. Kinetic process of plasma heating due to Alfvén wave excitation. *Phys. Rev. Lett.*, **35**, 370-372.
- Hasegawa A., 1975. *Plasma instabilities and nonlinear effects*. Springer-Verlag.
- Hasegawa A., 1976. Kinetic theory of MHD instabilities in a non-uniform plasma. *Solar Phys.*, **47**, 325-330.
- Hasegawa A., Uberoi C., 1982. *The Alfvén wave*. U.S. Dept. of Energy.
- Krall N. A., Trivelpiece A. W., 1973. *Principles of plasma physics*. Mac Graw Hill.
- Mac Mahon A., 1965. Finite gyro-radius corrections to the hydromagnetic equations for a Vlasov plasma. *Phys. Fluids*, **8**, 1840-1845.
- Moustazis S., Hubert D., Mangeney A., Harvey C. C., Perche C., Russell C. T., 1986. Magnetohydrodynamic turbulence in the magnetosheath. *Ann. Geophysicae*, **4A**, 355-362.
- Pu Z. Y., Kivelson M. G., 1983. Kelvin-Helmoltz instability at the magnetopause: solution for compressible plasmas. *J. Geophys. Res.*, **88**, 841-852.
- Quémada D., 1968. *Ondes dans les plasmas*. Hermann, Paris.
- Rodgers D. J., Wrenn G. L., Cowley S. W. H., 1984. Evidence for the heating of thermal electrons at the magnetopause boundary layer. *Planet. Space Sci.*, **32**, 657-666.
- Rönnmark K., 1982. WHAMP-waves in homogeneous, anisotropic multicomponent plasmas. Kiruna Geophysical Institute Report n° 179.
- Rönnmark K., 1983. Computation of the dielectric tensor of a maxwellian plasma. *Plasma Phys.*, **25**, 699-701.
- Rutherford P. H., Frieman E. A., 1968. Drift instabilities in general magnetic field configurations. *Phys. Fluids*, **11**, 569-585.
- Southwood D. J., 1968. The hydromagnetic stability of the magnetospheric boundary. *Planet. Space Sci.*, **16**, 587-605.
- Southwood D. J., 1981. *ULF pulsations in the magnetosphere*. Center for Academic Publications, Tokyo and D. Reidel Publ. Co., Dordrecht.
- Southwood D. J., Hughes W. J., 1983. Theory of hydromagnetic waves in the magnetosphere. *Space Sci. Rev.*, **35**, 301-366.
- Staciewicz K., 1986. Formation of the magnetospheric boundary layer by gyro-viscous interactions with the solar wind. Sixth International Symposium on Solar Terrestrial Physics, Toulouse, France.

