# Clifford bundles: a unifying framework for images (videos), vector fields and orthonormal frame fields regularization.

Thomas Batard

# CLIFFORD BUNDLES: A UNIFYING FRAMEWORK FOR IMAGES(VIDEOS), VECTOR FIELDS AND ORTHONORMAL FRAME FIELDS REGULARIZATION *

THOMAS BATARD †

**Abstract.** The aim of this paper is to present a new framework for regularization by diffusion. The methods we develop in the sequel can be used to smooth nD images, nD videos, vector fields and orthonormal frame fields in any dimension.[1]
From a mathematical viewpoint, we deal with vector bundles over Riemannian manifolds and so-called generalized Laplacians. Sections are regularized from heat equations associated to generalized Laplacians, the solution being given as convolution by generalized heat kernels. The anisotropy of the diffusion is controlled by the metric of the base manifold and by the connection of the vector bundle. It finds applications to images and videos anisotropic regularization. The main topic of this paper is to show that this approach can be extended to other fields such as vector fields and orthonormal frame fields by considering the context of Clifford algebras. We introduce a Clifford-Hodge operator (and the corresponding Clifford-Hodge flow) as a generalized Laplacian on the Clifford bundle of a Riemannian manifold. Laplace-Beltrami diffusion appears as a particular case of Clifford-Hodge diffusion, namely diffusion for degree 0 sections (functions). Considering base manifolds of dimension 2 and 3, applications to multispectral images, multispectral videos, 2D and 3D vector fields and orthonormal frame fields regularization may be envisaged. Some of them are presented in this paper.

**Key words.** regularization, heat equations, Clifford algebras, vector bundles, differential geometry

**AMS subject classifications.** 68U10-53CXX-15A66-58J35

## 1. Introduction.

Most multivalued image smoothing process are based on PDE's of the form

$$\frac{\partial I^i}{\partial t} = \sum_{j,k=1}^{2} f_{jk} \frac{\partial^2 I^i}{\partial j \, \partial k} + \text{first-order part}$$

of initial condition $I\colon (x,y) \longmapsto (I^1(0,x,y), \cdots, I^n(0,x,y))$ a nD image, where $f_{jk}$ are real functions. We refer to [20] for an overview on related works. From a geometric viewpoint, the set of right terms, for $i = 1 \cdots n$, may be viewed as a second-order differential operator acting on sections of a vector bundle over a Riemannian manifold called a **generalized Laplacian** $H$ [4]. As a consequence, it ensures existence and unicity of a kernel $K_t(x,y,H)$, called the **heat kernel of** $H$, generating the solution of the heat equation

$$\frac{\partial I}{\partial t} + HI = 0 \tag{1.1}$$

from a 'convolution' with the initial condition. For arbitrary Riemannian manifold and vector bundle, it is usually impossible to find a closed form for this kernel, and consequently a closed form of the solution. However, for the problem we consider in this paper, the use of an approximate solution appears to be sufficient. A generalized Laplacian $H$ on a vector bundle $E$ over a Riemannian manifold $X$ is determined by

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†Laboratory Mathematics Image and Applications, University of La Rochelle, 17042 La Rochelle Cedex, France (tbatar01@univ-lr.fr).

[1]Preliminary form of the paper in [3].

three pieces of data: the metric $g$ of the base manifold $X$ that determines the second order part, a connection on $E$ that determines the first order part, and a section $F \in \Gamma(\mathrm{End}(E))$ that determines the zero order part. It follows that the anisotropy of a regularization is determined by these data. For example, the so-called oriented Laplacians may be viewed as generalized Laplacians [1]. Similarly, on the vector bundle $C^\infty(X)$ of smooth functions on $X$, there is a canonical generalized Laplacian, the scalar Laplacian, which corresponds to the Laplace-Beltrami operator up to a sign. Considering each component $Ik$ of a nD image as a function over a well-chosen Riemannian manifold, we obtain the Beltrami flow of Sochen et al. in the context of image regularization [14],[15]. The aim of this paper is to show that this Beltrami flow, acting on $C^\infty(X)$, can be extended in a natural way to tangent vector fields and oriented orthonormal frame fields on $(X, g)$ by considering the Clifford bundle of $(X, g)$. The former is devoted to regularization of vector fields on the charts, the latter to regularization of orthonormal frame fields on the charts with respect to the metric $g$. Both isotropic and anisotropic regularizations of fields differing from nD images(videos) were widely investigated. Concerning vector fields, we may refer to [20],[8], where the smoothing is made through the smoothing of a corresponding image and where the anisotropy is controlled at each point by the orientation of the vector field. More precisely, an oriented Laplacian is applied on each component of the image. Concerning $\mathrm{SO}(n)$-valued fields, we may refer to [16] in the framework of principal bundles, where the anisotropy is determined by the flow itself too. In this paper, the context is slightly different since the anisotropy of the regularization is entirely determined by the geometry of the manifold where fields are defined. Additionally, let us refer to [19] for $S^n$-valued fieds regularization and [22] for constrained matrix-valued fields regularization.

Generalizing the Laplace-Beltrami operator acting on functions to tangent vector fields and oriented orthonormal frame fields is not straightforward. This was introduced in [3] for $X$ of dimension $m = 2$. First, considering the Clifford bundle $Cl(X, g)$ of $X$ [11], both functions and tangent vector fields may be viewed as sections of $Cl(X, g)$. This is not the case for oriented orthonormal frame fields but, by the trivializations of the fiber bundle, oriented orthonormal frame fields are locally mappings from $X$ to $\mathrm{SO}(m)$, and sections of $Cl(X, g)$ with values in the bivector part are locally mappings from $X$ to $\mathfrak{so}(m)$. By the canonical isomorphism between the bundle of differential forms of a Riemannian manifold and its Clifford bundle, the Hodge Laplacian $\Delta$ [7] may be viewed as a generalized Laplacian on $Cl(X, g)$ and its restriction to functions is minus the Laplace-Beltrami operator. As a consequence, it makes sense to consider the flow generated by the solution of the heat equation $\partial I/\partial t + \Delta I = 0$ on $Cl(X, g)$, that provides anisotropic smoothing of the initial condition. By construction of the Clifford bundle, the anisotropy is determined by the metric of the base manifold. From now on, we call **Clifford-Hodge Laplacian** the Hodge Laplacian on $Cl(X, g)$. In particular, the Clifford-Hodge Laplacian preserves the structures of functions, tangent vector fields and bivector-valued fields. Whereas smoothing of functions and tangent vector fields is done in a natural way, the regularization of oriented orthonormal frame fields needs the use of the exponential map of $\mathrm{SO}(m)$.

Clifford algebras framework [6],[9] finds a wide range of applications in computer science [17]. Application of Clifford bundles to image processing was introduced in [2] where the Di Zenzo's gradient, devoted to nD image segmentation, is generalized using covariant derivatives instead of usual derivatives. Most properties of Clifford

algebras we use in this paper are presented in [2].
This paper is organized as follows. Section 2 is devoted to introduce heat equations on vector bundles associated to generalized Laplacians. In particular, we discuss approximations of the solutions. It requires to determine geodesic curves and compute geodesic distances on the base manifold, and to compute the parallel transport map on the vector bundle. We treat the particular case of the scalar Laplacian, and relate it with the Laplace-Beltrami operator. In Section 3, we introduce the Clifford bundle $Cl(X,g)$. We determine the connection on $Cl(X,g)$ and the section $F \in \Gamma(\mathrm{End}(Cl(X,g)))$ that make the Clifford-Hodge Laplacian be a generalized Laplacian on $Cl(X,g)$. Then, we detail the method to regularize oriented orthonormal frame fields. In section 4, we treat the case $m=2$. We first compute explicit formulae of $\Delta$ and transport parallel map on $Cl(X,g)$ for $X$ of dimension 2 . Then, we are concerned in the particular case of images. We construct the Riemannian manifold being the base manifold of the Clifford bundle we consider, and a global frame field of the tangent bundle, generating a global frame field of the Clifford bundle. At last, we present applications to regularization of a color image, and regularizations of a vector field and a orientation field related to the image. In section 5, we treat the case $m=3$. We first compute explicit formulae of $\Delta$ and transport parallel map on $Cl(X,g)$, for $X$ of dimension 3. Then, we are concerned in the particular case of videos. We construct the Riemannian manifold being the base manifold of the Clifford bundle we consider. At last, we present an application to regularization of a color video.

**2. Heat equations on vector bundles.** For details on this part, see [4].

**2.1. Generalized Laplacians on vector bundles.** We refer to [18],[10] for an introduction to differential geometry. For a smooth vector bundle $E$ over a manifold $X$, the symbol $\Gamma(E)$ denotes the space of smooth sections of $E$. For $x \in X$, $E_x$ denotes the fiber over $x$.

DEFINITION 2.1. *Let $E$ be a vector bundle over a Riemannian manifold $(X,g)$. A **generalized Laplacian** on $E$ is a second-order differential operator $H : \Gamma(E) \longrightarrow \Gamma(E)$, that may be written*

$$H = -\sum_{ij} g^{ij}(x)\partial_i\,\partial_j \; + \textit{ first-order part}$$

*in any local coordinates system, where $(g^{ij}(x))$ is the inverse of the matrix $g(x) = (g_{ij}(x))$.*

The heat kernel of a generalized Laplacian on a vector bundle $E$ is related with a connection on $E$.

DEFINITION 2.2. *Let $E$ be a vector bundle over a Riemannian manifold $(X,g)$, endowed with a connection $\nabla^E$. Since $(X,g)$ is Riemannian, it possesses a canonical connection, the Levi-Cevita connection $\nabla$. To any pair of tangent vector fields $V$ and $W$ on $X$, we associate an invariant second derivative $\nabla^2_{V,W} : \Gamma(E) \longrightarrow \Gamma(E)$ by setting*

$$\nabla^2_{V,W}\varphi \equiv \nabla^E_V \nabla^E_W \varphi - \nabla^E_{\nabla_V W}\varphi \tag{2.1}$$

*Then, the **connection Laplacian** $\Delta^E : \Gamma(E) \longrightarrow \Gamma(E)$ is defined by*

$$\Delta^E\varphi = -trace(\nabla^2_{\cdot,\cdot}\,\varphi)$$

*where trace denotes the contraction with the metric $g$.*

*In particular, if $e_i$ is a local orthonormal frame of $TX$, the operator $\Delta^E$ is given by*

$$\Delta^E = -\sum_i \left( \nabla^E_{e_i} \nabla^E_{e_i} - \nabla^E_{\nabla_{e_i} e_i} \right)$$

*With respect to the frame $\partial_i := \partial/\partial x_i$ defined by a coordinates system around a point in $X$, the operator $\Delta^E$ equals*

$$\Delta^E = -\sum_{ij} g^{ij} \left( \nabla^E_{\partial_i} \nabla^E_{\partial_j} - \sum_k \Gamma^k_{ij} \nabla^E_{\partial_k} \right) \tag{2.2}$$

*where $\Gamma^k_{ij}$'s are defined by $\nabla_{\partial_i} \partial_j = \sum_k \Gamma^k_{ij} \partial_k$.*

We have the following result:
**Any generalized Laplacian is of the form $\Delta^E + F$, where $\Delta^E$ is the connection Laplacian associated to some connection $\nabla^E$, and $F$ is a section of the bundle End($E$). In particular, any connection Laplacian $\Delta^E$ is a generalized Laplacian.**

**2.2. The heat kernel of a generalized Laplacian.** To any generalized Laplacian $H$, one may associate an operator $e^{-tH} \colon \Gamma(E) \longrightarrow \Gamma(E)$, for $t > 0$, with the property that $I(t,x) = e^{-tH} I(x)$ satisfies the heat equation $\partial I/\partial t + HI = 0$.
We shall define $e^{-tH}$ as an integral operator of the form

$$(e^{-tH} I)(x) = \int_X K_t(x,y,H) I(y) dy \tag{2.3}$$

where $K_t(x,y,H) \colon E_y \longrightarrow E_x$ is a linear map depending smoothly on $x, y$ and $t$. This kernel $K$ is called the **heat kernel of** $H$.

In the following theorem, we give some results on approximations of the heat kernel and solutions of the heat equation.

THEOREM 2.3. *Let $x \in X$ and a normal coordinates system around $x$, we denote by $\mathbf{y^i}$ the normal coordinates of a point $y$ in the injectivity radius of $X$ at $x$, $\partial_i$ the corresponding partial derivatives, and by $g_{ij}(\mathbf{y})$ the scalar product of $\partial_i$ and $\partial_j$ at $\mathbf{y}$. Moreover, we define*

$$J(x,y) = det(g_{ij}(\mathbf{y}))^{1/2} \qquad for\ y = exp_x(\mathbf{y})$$

*Let $\epsilon$ chosen smaller than the injectivy radius of $X$. Let $\Psi \colon \mathbb{R}_+ \longrightarrow [0,1]$ be a smooth function such that $\Psi(s) = 1$ if $s < \epsilon^2/4$ and $\Psi(s) = 0$ if $s > \epsilon^2$.*
*Let $\tau(x,y) \colon E_y \longrightarrow E_x$ be the parallel transport along the unique geodesic curve joining $x$ and $y$, and $d(x,y)$ its length.*
*Let $K^N_t(x,y,H)$ be the kernel defined by*

$$\left(\frac{1}{4\pi t}\right)^{\frac{m}{2}} e^{-d(x,y)^2/4t}\ \Psi(d(x,y)^2) \sum_{i=0}^{N} t^i \Phi_i(x,y,H) J(x,y)^{-\frac{1}{2}}$$

*where the sections $\Phi_i$ are given by $\Phi_0(x,y,H)=\tau(x,y)$ and*

$$\tau(x,y)^{-1}\Phi_i(x,y,H) = -\int_0^1 s^{i-1}\tau(x_s,y)^{-1}(B_x.\Phi_{i-1})(x_s,y,H)ds$$

*$B_x$ is the operator $J^{1/2}\circ H_x\circ J^{-1/2}$ where $H_x$ is the operator $H$ with respect to the first variable.*

*1. For every $N > m/2$, the kernel $K_t^N(x,y,H)$ is asymptotic to $K_t(x,y,H)$:*

$$\left\| \partial_t^k [\, K_t(x,y,H) - K_t^N(x,y,H)\,] \right\|_l = O(t^{N-m/2-l/2-k})$$

*where $\|\ \|_l$ is a norm on $C^l$ sections.*

*2. Let us denote by $k_t^N$ the operator defined by*

$$(k_t^N I)(x) = \int_X K_t^N(x,y,H)I(y)dy \tag{2.4}$$

*Then for every $N$, $\lim_{t\to 0}\|k_t^N I - I\|_l = 0$.*

*3. We have the following estimate:*

$$\left\| e^{-tH}I - \sum_{i=0}^{k}\frac{(-tH)^i}{i!}I \right\|_j = O(t^{k+1})$$

*which justifies the notation $e^{-tH}$.*

In the sequel, we make use the following property:

(P1) $J(x,y) = 1 + O(\|\mathbf{y}\|^2)$.

**2.3. Discrete approximations of heat equations solutions.** As mentionned above, for arbitrary base manifold and vector bundle, it is not possible to compute the solution of the equation (1.1), but for the problem we consider in this paper, the computation of an approximation of (2.3) appears to be sufficient. A natural choice is to consider the operator $k_t^N$ (2.4), for some $N$. Additionally, there is a freedom in the choice of the function $\Psi$ of Theorem 2.3, that determines the neighborhoods where the convolutions are computed. For the problem we consider in this paper, we need to compute discretizations of such approximations. As a consequence, the function $\Psi$ determines the size of the masks involved in the discrete convolutions. From (P1), we approximate $J(x,y)$ by 1.

Let us first consider the particular case of the operator $k_t^0$, defined by

$$\begin{aligned}(k_t^0 I)(x) &= \int_X K_t^0(x,y,H)I(y)dy \\ &= \Big(\frac{1}{4\pi t}\Big)^{\frac{m}{2}}\int_X e^{-d(x,y)^2/4t}\ \Psi(d(x,y)^2)\,\tau(x,y)I(y)\,J(x,y)^{-\frac{1}{2}}\,dy\end{aligned}$$

We discretize it as the discrete convolution of $\tau(x,y)I(y)$ with a mask whose inputs are geodesic distances from the current point to $x$ (up to the normalization of the mask).

*Remark:* the kernel $K_t^0$ entirely depends on the metric of the base manifold and the parallel transport map of the vector bundle. The first one determines geodesics and their lengths on the base manifold, the second one the map $\tau$. As a consequence, any generalized Laplacian on a vector bundle equipped with a connection generates the same operator $k_t^0$, and the subsequent diffusion process.

As last, let us mention that the Euler scheme may be obtained from this point of view. We have $\tau(x,x) = Id$ and $\Phi_1(x,x,H) = -J^{1/2} \circ H \circ J^{-1/2}(x,x)$. Then, considering the operator $k_t^1$ of (2.4), defined by

$$\begin{aligned}(k_t^1 I)(x) &= \int_X K_t^1(x,y,H)I(y)dy \\ &= \Big(\frac{1}{4\pi t}\Big)^{\frac{m}{2}} \int_X e^{-d(x,y)^2/4t}\, \Psi(d(x,y)^2)\, (\tau(x,y) + t\Phi_1(x,y,H))I(y)\, J(x,y)^{-\frac{1}{2}}\, dy\end{aligned}$$

and choosing $\Psi$ generating sufficiently small neighborhoods, we obtain that the discretization of $k_t^1 I$ at a point $x$ results from the convolution of $(\tau(x,y)+t\Phi_1(x,y,H))I(y)$ with a normalized 1x1 mask, i.e. with 1. In other words, we have

$$I(x) - tHI(x) \tag{2.5}$$

This is the Euler scheme of (1.1).

**2.4. An example: the scalar/Beltrami Laplacian.** The scalar Laplacian on a Riemannian manifold $(X,g)$ of dimension $m$ is the connection Laplacian (2.2) on the vector bundle $E = C^\infty(X)$. In other words, it is the connection Laplacian on a smooth vector bundle $E$ of rank 1 with connection $\nabla^E$ defined by the symbols

$$\Upsilon_1 = \cdots = \Upsilon_m = 0$$

i.e. $\nabla_X^E(f) = d_X f$. It follows the transport parallel map on $C^\infty(X)$

Proposition 2.4. *Let $(X,g)$, $E$ and $\nabla^E$ as defined above, $\gamma$ be a $C^1$ curve in $X$ such that $\gamma(0) = y$. Then, the parallel transport $Y$ along $\gamma$ of the scalar $Y_0 = Y_0^1 1(y)$ is $Y(t) = Y_0^1 1(\gamma(t))$.*

In local coordinates, the scalar Laplacian is defined by

$$\Delta(f) = -\sum_{ij} g^{ij}\Big(\partial_i\partial_j - \sum_k \Gamma_{ij}^k \partial_k\Big) f \tag{2.6}$$

The Laplace-Beltrami operator $\Delta_g$ is the differential operator of order 2 on $C^\infty(X)$ defined by

$$\Delta_g(f) = \frac{1}{\sqrt{g}} \sum_{jk} \partial_j(\sqrt{g} g^{jk} \partial_k f)$$

It is well-known that if a Riemannian manifold $(X,g)$ is equipped with the Levi-Cevita connection, then $\Delta_g$ is minus the scalar Laplacian (2.6). This makes the PDE's $\partial I/\partial t = \Delta_g I$ and $\partial I/\partial t + \Delta I = 0$ be equivalent.

## 3. The Clifford-Hodge Laplacian: an extension of the scalar Laplacian for multivector fields smoothing.

**3.1. The Clifford-Hodge Laplacian: a generalized Laplacian on the Clifford bundle of a Riemannian manifold.** The Hodge Laplacian $\Delta$ is a generalized Laplacian acting on differential forms of a Riemannian manifold. It is defined by

$$\Delta = d\delta + \delta d$$

where $d$ is the exterior derivative operator and $\delta$ its formal adjoint [7].
In particular, when applied to 0-forms, i.e. functions, $\Delta$ corresponds to the scalar Laplacian (2.6).

One of the main ideas of this paper is the following: the Hodge Laplacian can also be applied to tangent vector fields and generators of orthonormal frame fields by considering the Clifford bundle of $X$.

DEFINITION 3.1 (Clifford bundle). *Let $(X, g)$ be a Riemannian manifold, and let us denote by $TX$ its tangent bundle. The Clifford bundle $Cl(X, g)$ of $(X, g)$ is the quotient bundle*

$$Cl(X, g) = \mathcal{T}(TX) \Big/ I(TX)$$

*where $\mathcal{T}(TX)$ is the bundle whose fiber at $x \in X$ is the tensor algebra of $T_xX$ and $I(TX)$ is the bundle whose fiber at $x \in X$ is the two-sided ideal $I(T_xX)$ in $\mathcal{T}(T_xX)$, generated by elements $v \otimes v + \|v\|^2$ for $v \in T_xX$.*

We obtain a bundle of Clifford algebras over $X$, and the fiberwise multiplication in $Cl(X, g)$ gives an algebra structure to the space $\Gamma(Cl(X, g))$ of sections of $Cl(X, g)$.

More precisely, let $(e_1, \cdots, e_m)$ be a local oriented orthonormal frame field on $\Omega \subset X$. Then, any section $s$ of $Cl(X, g)$ takes the form

$$s = s_1\, 1 + s_2\, e_1 + s_{m+1}\, e_m + \cdots + s_{2^m}\, e_1 \cdots e_m$$

on $\Omega$, for some functions $s_{i_1 \cdots i_k}$ defined on $\Omega$ where $i_1 < \cdots < i_k$ and $k \in \{1 \cdots m\}$.

Given a finite-dimensional vector space $V$ equipped with a quadratic form $Q$, there is a vector space isomorphism between the exterior algebra $\bigwedge V$ of $V$ and $Cl(V, Q)$. In particular, if $Q \equiv 0$, they are isomorphic as algebras. Let $(X, g)$ be a Riemannian manifold, we denote by $\bigwedge T^*X$ the bundle of differential forms of $(X, g)$. By the constructions of $\bigwedge T^*X$ and $Cl(X, g)$ as associated bundles to the principal bundle $\mathrm{PSO}(X)$ of oriented orthonormal frame fields of $(X, g)$, it follows a canonical vector bundle isomorphism between $\bigwedge T^*X$ and $Cl(X, g)$. As a consequence, we may identify $\Gamma(\bigwedge T^*X)$ and $\Gamma(Cl(X, g))$, as follows

$$\text{k-form}: \sum_{\substack{i_1 < \cdots < i_k \\ 1 \le i_k \le m}} \omega_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k} \longleftrightarrow \sum_{\substack{i_1 < \cdots < i_k \\ 1 \le i_k \le m}} \omega_{i_1 \cdots i_k}\, e_{i_1} \cdots e_{i_k}$$

$$\text{0-form}: \qquad f \longleftrightarrow f1$$

Under this identification, the Hodge Laplacian may be applied to $\Gamma(Cl(X,g))$ (see [11] for details). In the Clifford algebras context, it is the square of a first order differential operator, namely the **Dirac operator**. Let us first introduce the connection on $Cl(X,g)$ induced by the Levi-Cevita connection on $TX$.

PROPOSITION 3.2. *The Levi-Cevita connection of a Riemannian manifold $(X,g)$ induces an algebra connection on its Clifford bundle, i.e. a connection $\nabla^C$ satisfying*

$$\nabla^C(\varphi\psi) = (\nabla^C\varphi)\psi + \varphi(\nabla^C\psi)$$

*for any $\varphi, \psi \in \Gamma(Cl(X,g))$.*

*Proof.* [5] The Levi-Cevita connection may be extended in a unique way to a connection $\nabla$ on $\mathcal{T}(TX)$ by linearity and postulation of the Leibniz rule.
For $U, V \in \Gamma(TX)$ we have

$$\begin{aligned}\nabla_U(V\otimes V - g(V,V)) &= \frac{1}{2}\Big[(\nabla_U V + V)\otimes(\nabla_U V + V) - g(\nabla_U V + V, \nabla_U V + V)\Big] \\ &\quad -\frac{1}{2}\Big[(\nabla_U V - V)\otimes(\nabla_U V - V) - g(\nabla_U V - V, \nabla_U V - V)\Big]\end{aligned} \tag{3.1}$$

We prove (3.1) by developping both right and left terms of the equality.

For the first one we obtain

$$\nabla_U V\otimes V + V\otimes\nabla_U V - 2g(V,\nabla_U V)$$

Indeed, by property of the Levi-Cevita connection, for any $U, W, Z \in \Gamma(TX)$

$$d_U g(W,Z) = g(\nabla_U W, Z) + g(W, \nabla_U Z)$$

Then, taking $W = Z = V$, it gives

$$d_U g(V,V) = 2g(\nabla_U V, V)$$

Developping the left term we have

$$\nabla_U(V\otimes V - g(V,V)) = \nabla_U V\otimes V + V\otimes\nabla_U V - d_U g(V,V)$$

and the equality (3.1) is proved.

As the right term of (3.1) belongs to $I(TX)$, it proves that $\nabla$ preserves the ideal $I(TX)$.

We deduce that $\nabla$ induces a connection $\nabla^C$ on $Cl(X,g)$. Indeed, for $a, b \in \Gamma(\mathcal{T}(TX))$ in the same equivalence class (denoted by $\dot{a} = \dot{b}$), $\nabla(b) = \nabla(a + I_1)$ for some $I_1 \in \Gamma(I(TX))$. Hence $\nabla(b) = \nabla(a) + I_2$ for some $I_2 \in \Gamma(I(TX))$. Therefore $\widehat{\nabla(a)} = \widehat{\nabla(b)}$, and $\nabla^C$ is well-defined.

The rest of the proof is devoted to show that $\nabla^C$ is an algebra connection.

Let $\varphi, \psi \in \Gamma(Cl(X,g))$. By definition, there exist $a, b \in \Gamma(\mathcal{T}(TX))$ such that $\varphi = \dot{a}$ and $\psi = \dot{b}$. Then,

$$\begin{aligned}
\nabla^C(\varphi\,\psi) &= \widehat{\nabla(a \otimes b)}^{\cdot} \\
&= \widehat{\nabla(a) \otimes b + a \otimes \nabla(b)}^{\cdot} \\
&= \widehat{\nabla(a) \otimes b}^{\cdot} + \widehat{a \otimes \nabla(b)}^{\cdot} \\
&= \widehat{\nabla(a)}^{\cdot}\,\dot{b} + \dot{a}\,\widehat{\nabla(b)}^{\cdot} \\
&= \nabla^C(\varphi)\psi + \varphi\nabla^C(\psi)
\end{aligned}$$

which ends the proof. □

Definition 3.3. *Let $Cl(X,g)$ be the Clifford bundle of a Riemannian manifold $(X,g)$ equipped with the connection $\nabla^C$. Let $(e_1, \cdots, e_m)$ be a local orthormal frame field of $TX$. The Dirac operator is the first-order differential operator $D\colon \Gamma(Cl(X,g)) \longrightarrow \Gamma(Cl(X,g))$ defined locally by*

$$D\sigma = \sum_{i=1}^{m} e_i\, \nabla^C_{e_i}\sigma \tag{3.2}$$

It can be shown that this definition is independant of the choice of the local orthonormal frame field.

Proposition 3.4. *Under the isomorphism between $Cl(X,g)$ and $\bigwedge T^*X$, the Clifford-Hodge operator $D^2$ may be identified with the Hodge Laplacian $\Delta$.*

*Proof.* [11] From the fact that $d^2 = \delta^2 = 0$, we have

$$\Delta = (d+\delta)^2$$

on $\bigwedge T^*X$. Hence it suffices to prove that $D$ may be identified with $d+\delta$.
Endowing Clifford algebras with the outer $\wedge$ and the inner $\cdot$ products, $D$ may be decomposed as follows

$$D = \sum_{j=1}^{m} e_j \wedge \nabla_{e_j} + \sum_{j=1}^{m} e_j \cdot \nabla_{e_j}$$

Then, it can be shown that the operator $\sum_{j=1}^{m} e_j \wedge \nabla_{e_j}$ satisfies the properties that define in a unique way the exterior derivative $d$ on $\bigwedge T^*X$. Moreover, computing the adjoint of $\sum_{j=1}^{m} e_j \wedge \nabla_{e_j}$, we obtain $\sum_{j=1}^{m} e_j \cdot \nabla_{e_j}$ which ends the proof. □

As $\Delta$ preserves the subbundles $\bigwedge^k T^*X$ of $\bigwedge T^*X$ for $k \in \{0 \cdots m\}$, we deduce that $D^2$ preserves the subbundles $\bigwedge^k TX$ of $Cl(X,g)$.

Proposition 3.5. *Let $E$ be a vector bundle over a manifold $X$ equipped with a connection $\nabla^E$. Let $V, W \in \Gamma(TX)$. The operator*

$$R_{V,W} = \nabla^E_V \nabla^E_W - \nabla^E_W \nabla^E_V - \nabla^E_{[V,W]}$$

*is a zero-order operator on $\Gamma(E)$, i.e. $R_{V,W} \in \Gamma(End(E))$. It is called the* ***curvature transformation*** *associated to $V$ and $W$.*

PROPOSITION 3.6. *Let $Cl(X,g)$ be the Clifford bundle of a Riemannian manifold $(X,g)$ equipped with the connection $\nabla^C$. The Clifford-Hodge operator $D^2$ is a generalized Laplacian.*

*Proof.* [11] We first remind that the definition of $D$ does not depend of the choice of local orthonormal frame fields. Let us fix a point $x \in X$ and choose a local orthonormal frame field $(e_1, \cdots, e_m)$ such that $(\nabla e_j)_x = 0$ for each $j$, where $\nabla$ is the Levi-Cevita connection of $TX$. Then, for $\varphi \in \Gamma(Cl(X,g))$ we have at $x$

$$\begin{aligned} D^2\varphi &= \sum_{ij} e_i \nabla^C_{e_i}(e_j \nabla^C_{e_j}\varphi) \\ &= \sum_{ij} e_i e_j \nabla^C_{e_i}\nabla^C_{e_j}\varphi \end{aligned}$$

since $\nabla^C_{e_i}(e_j\nabla^C_{e_j}\varphi) = \nabla^C_{e_i}e_j\,\nabla^C_{e_j}\varphi + e_j\nabla^C_{e_i}\nabla^C_{e_j}\varphi$, and $(\nabla^C_{e_i}e_j)_x = (\nabla_{e_i}e_j)_x = 0$. It may rewritten as

$$\begin{aligned} D^2\varphi &= -\sum_i \nabla^C_{e_i}\nabla^C_{e_i}\varphi + \sum_{i<j} e_i e_j(\nabla^C_{e_i}\nabla^C_{e_j} - \nabla^C_{e_i}\nabla^C_{e_j})\varphi \\ &= -\sum_i \nabla^C_{e_i}\nabla^C_{e_i}\varphi + \sum_{i<j} e_i e_j R_{e_i,e_j}(\varphi) \end{aligned}$$

Indeed, as $TX$ is endowed with the Levi-Cevita connection, $[e_i,e_j] = \nabla_{e_i}e_j - \nabla_{e_j}e_i$ and $([e_i,e_j])_x = 0$ by property of $(e_1,\cdots,e_m)$. As a consequence, $(\nabla^C_{[e_i,e_j]})_x = 0$ and we deduce from the previous proposition that $(\nabla^C_{e_i}\nabla^C_{e_j} - \nabla^C_{e_i}\nabla^C_{e_j})_x = (R_{e_i,e_j})_x$. Therefore we have

$$D^2 = \Delta^C + F$$

where $\Delta^C$ is the connection Laplacian on $Cl(X,g)$ with respect to the connection $\nabla^C$ and $F = \sum_{i<j} e_i e_j R_{e_i,e_j}$ is a section of $\mathrm{End}(Cl(X,g))$. □

**3.2. Application to functions, vector fields and oriented orthonormal frame fields smoothing.** As the operator $D^2$ is a generalized Laplacian, we may consider the corresponding heat equation, whose solution arises from the convolution of the initial condition with the heat kernel $K_t(x,y,D^2)$ associated to $D^2$.

DEFINITION 3.7. *Let $s_0 \in \Gamma(Cl(X,g))$. The* ***Clifford-Hodge flow*** *of $s_0$ is the solution $s_t$ of the heat equation*

$$\frac{\partial s_t}{\partial t} = D^2 s_t$$

*of initial condition $s_0$.*

As $D^2$ preserves the subbundles $\bigwedge^k TX$ for $k \in \{0,\cdots,m\}$, we deduce that the Clifford-Hodge flow preserves the subbundles $\bigwedge^k TX$ too.

Let $x \in X$. From the embeddings of $T_xX$ and $\mathbb{R}$ into $Cl(T_xX, g(x))$, both tangent vector fields and functions on a Riemannian manifold may be viewed as sections of its Clifford bundle. More precisely, we have the identifications

$$C^\infty(X) \simeq \Gamma(\textstyle\bigwedge^0 TX) \qquad \Gamma(TX) \simeq \Gamma(\textstyle\bigwedge^1 TX)$$

Let us denote by $\Gamma(\mathrm{P\,SO}(X))$ the set of smooth oriented orthonormal frame fields of $TX$. On a local trivialization $(U, \Phi)$ of $Cl(X, g)$, we have the identifications

$$\Gamma(\mathrm{P\,SO}(U)) \simeq C^\infty(U, \mathrm{SO}(m)) \qquad \Gamma(\textstyle\bigwedge^2 TU) \simeq C^\infty(U, \mathfrak{so}(m))$$

the latter arising from the Lie algebra isomorphism $\mathfrak{spin}(m) \simeq \mathfrak{so}(m)$. We call such fields generators of orthonormal frame fields.

As the Clifford-Hodge flow preserves the subbundles $\bigwedge^k TX$, it preserves the structures of functions, tangent vector fields and generators of orthonormal frame fields. As a consequence, it provides a tool to regularize such fields.

Through the regularization of sections of degree 2, the Clifford-Hodge flow provides also a tool to smooth orthonormal frame fields, as follows. Given a mapping $f$ from $U$ to $\mathrm{SO}(m)$, we construct $\tilde{f} \in \Gamma(\bigwedge^2 TU)$ such that $f = \rho \circ exp \circ \tilde{f}$ where $\rho$ is the projection map of the covering $\mathrm{Spin}(m) \longrightarrow \mathrm{SO}(m)$ and $exp$ the exponential map from $\mathfrak{spin}(m)$ to $\mathrm{Spin}(m)$. Then, the Clifford-Hodge flow provides a regularization of $\tilde{f}$ given by sections $\tilde{f}_{t,t\geq 0} \in \Gamma(\bigwedge^2 TU)$. Computing $\rho \circ exp \circ \tilde{f}_t$, we obtain mappings $f_{t,t\geq 0} \in C^\infty(U,\mathrm{SO}(m))$. In this sense, we obtain a regularization of $f$.

Let us give some precisions concerning the construction of $\tilde{f}$ for $m \geq 3$.
Two methods may be envisaged to construct $\tilde{f}$. The first one consists in characterizing the rotations $f(x), x \in U$, by their angles and invariant planes from the computation of eigenvalues and eigenvectors. Then $\tilde{f}$ follows in a straightforward way using the fact that bivectors in $\mathbb{R}_{n,0}$ represent oriented planes in $\mathbb{R}^n$. The second one uses the matrix logarithm $log\colon\ \mathrm{SO}(m) \longrightarrow \mathfrak{so}(m)$ with values in the matrix representation of the Lie algebra $\mathfrak{so}(m)$, i.e. the antisymmetric matrices. We may write

$$log \circ f(x) = \sum_{i<j} a_{ij}(x) E_{ij}$$

where $E_{ij}$ is the elementary $m \times m$ antisymmetric matrix such that $E_{ij}(i, j) = -1$ and $E_{ij}(j, i) = 1$. The Lie algebra isomorphism $\mathfrak{so}(m) \longrightarrow \mathfrak{spin}(m)$ maps $E_{ij}$ to $\dfrac{1}{2} e_i e_j$. Hence we obtain

$$\tilde{f}(x) = \sum_{i<j} \frac{1}{2} a_{ij}(x)\, e_i e_j$$

For $m = 3$, the first method is suitable since the spectrum of a rotation is easily computable from geometric calculus methods [9]. Indeed, let us suppose that $f(x)$ is

the rotation of angle $\theta$ in the oriented plane $\mathcal{P}$ in $\mathbb{R}^3$, given by the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

in the orthonormal basis $(e_1, e_2, e_3)$.

Then, considering the Clifford algebra $Cl(\mathbb{R}^3, -\| \ \|_2)$, we have

$$\begin{cases} a_{11} + a_{22} + a_{33} &= 1 + 2cos(\theta) \\ (a_{12} - a_{21})e_1e_2 + (a_{13} - a_{31})e_1e_3 + (a_{23} - a_{21})e_1e_2 &= 2sin(\theta)\dfrac{B}{\|B\|} \end{cases}$$

where $B$ stands for the bivector representing the oriented plane $\mathcal{P}$ of the rotation. A proof and a generalization for $m > 3$ may be found in [9].

From the two equalities, we may construct $\tilde{f}(x) = \dfrac{\theta}{2}\dfrac{B}{\|B\|}$, $\theta \in [0, 2\pi[$, which satisfies $\rho \circ \exp \circ \tilde{f}(x) = f(x)$.

Note that $\tilde{f}(x)$ is not defined in a unique way since

$$\rho \circ exp \circ \Big(\frac{\theta}{2}\frac{B}{\|B\|}\Big) = \rho \circ exp \circ \Big((\theta/2 + k\pi)\frac{B}{\|B\|}\Big) \quad \text{for any } k \in \mathbb{Z}$$

More generally, for $B = \theta_1 e_1e_2 + \cdots + \theta_{C_m^2} e_{m-1}e_m$, we have

$$\rho \circ exp \circ \Big(\theta_1 e_1e_2 + \cdots + \theta_{C_m^2} e_{m-1}e_m\Big) = \rho \circ exp \circ \Big((\theta_1 + k\pi)e_1e_2 + \cdots + (\theta_{C_m^2} + k\pi)e_{m-1}e_m\Big)$$

The $\pi$ periodicity raises a problem when convolving $\tilde{f}$ with a kernel $K_t^N(x, y, D^2)$. It may be solved by defining $\tilde{f}$ locally on the normal neighborhoods determined by the function $\Psi$ of Section 2.3. Let $x_0 \in U$ and $\theta_j$'s defined as above. Let $\Omega_{x_0} \subset U$ be such a neighborhood of $x_0$. We state $\widetilde{\theta_j}(x) = \theta_j(x) - \pi$ if $\theta_j(x) - \theta_j(x_0) \geq \pi/2$, $\widetilde{\theta_j}(x) = \theta_j(x) + \pi$ if $\theta_j(x) - \theta_j(x_0) < -\pi/2$, and $\widetilde{\theta_j}(x) = \theta_j(x)$ otherwise. This makes the functions $\widetilde{\theta_j}$ be valued in $[\theta_j(x_0) - \pi/2, \theta_j(x_0) + \pi/2[$ on $\Omega_{x_0}$. Then we define $\tilde{f} = \widetilde{\theta_1}e_1e_2 + \cdots \widetilde{\theta_{C_m^2}} e_{m-1}e_m$ on $\Omega_{x_0}$. Extending this construction for each $x_0 \in U$, we construct $\tilde{f}$ for which the convolution with the kernel $K_t^N(x, y, D^2)$ makes sense.

At last, let us treat the particular case $m = 2$ since $\mathrm{Spin}(2) \simeq \mathrm{SO}(2)$.
Given $f \in C^\infty(U, \mathrm{SO}(2))$, we construct $\tilde{f} = \widetilde{\theta} e_1e_2$ such that $exp(\widetilde{\theta} e_1e_2) = cos(\theta)e_1 + sin(\theta)e_2$. There is a $2\pi$ periodicity in the choice of $\tilde{f}$, that may be solved by defining $\tilde{f}$ locally on the neighborhoods $\Omega_{x_0}$. Indeed, we state $\widetilde{\theta}(x) = \theta(x) - 2\pi$ if $\theta(x) - \theta(x_0) \geq \pi$, $\widetilde{\theta}(x) = \theta(x) + 2\pi$ if $\theta(x) - \theta(x_0) < -\pi$ and $\widetilde{\theta}(x) = \theta(x)$ otherwise. Then we extend this construction for each $x_0 \in U$ as mentionned above.

The heat kernel of a generalized Laplacian $H$ on a vector bundle $E$ equipped with a connection $\nabla^E$ is determined by geodesic distances on the base manifold, transport

parallel map associated to $\nabla^E$ along geodesic curves, and $H$ itself. In the rest of the paper we give explicit formulae of the operator $D^2$ and parallel transport map on $Cl(X,g)$ for $X$ of dimension 2 and 3. We present applications in the contexts of image and video processing.

## 4. The case $m = 2$.

**4.1. The operator $D^2$ and parallel transport map on $Cl(X,g)$.** Let us first determine the connection $\nabla^C$ in an orthonormal frame $(e_1, e_2)$. Let $\Gamma_{ij}^{k'}$ be the Levi-Cevita connection's symbols of $(X,g)$ with respect to the frame $(e_1, e_2)$.

$$\nabla^C_{e_1} 1 = 0 \qquad \nabla^C_{e_2} 1 = 0$$

$$\nabla^C_{e_1} e_1 = \Gamma_{11}^{2'} e_2 \qquad \nabla^C_{e_1} e_2 = -\Gamma_{11}^{2'} e_1$$
$$\nabla^C_{e_2} e_1 = \Gamma_{21}^{2'} e_2 \qquad \nabla^C_{e_2} e_2 = -\Gamma_{21}^{2'} e_1$$

$$\nabla^C_{e_1} e_1 e_2 = 0 \qquad \nabla^C_{e_2} e_1 e_2 = 0$$

Proposition 4.1. *Let $(X,g)$ be a Riemannian manifold of dimension 2. Let $(e_1, e_2)$ be an orthonormal frame on $X$ and $\varphi = \varphi_1 1 + \varphi_2 e_1 + \varphi_3 e_2 + \varphi_4 e_1 e_2 \in \Gamma(Cl(X,g))$. Then*

$$\begin{aligned} D^2(\varphi_1 1) &= \Big( - d^2_{e_1,e_1}\varphi_1 - d^2_{e_2,e_2}\varphi_1 - \Gamma_{21}^{2'} d_{e_1}\varphi_1 + \Gamma_{11}^{2'} d_{e_2}\varphi_1 \Big) 1 \\ &= -\Delta_g(\varphi_1) 1 \end{aligned}$$

$$\begin{aligned} D^2(\varphi_2 e_1 + \varphi_3 e_2) = \Big( & - d^2_{e_1,e_1}\varphi_2 - d^2_{e_2,e_2}\varphi_2 - \Gamma_{21}^{2'} d_{e_1}\varphi_2 + \Gamma_{11}^{2'} d_{e_2}\varphi_2 + 2\,\Gamma_{11}^{2'} d_{e_1}\varphi_3 \\ & + 2\,\Gamma_{21}^{2'} d_{e_2}\varphi_3 + \varphi_2(d_{e_2}\Gamma_{11}^{2'} - d_{e_1}\Gamma_{21}^{2'}) + \varphi_3(d_{e_1}\Gamma_{11}^{2'} + d_{e_2}\Gamma_{21}^{2'}) \Big) e_1 \\ & + \Big( - d^2_{e_1,e_1}\varphi_3 - d^2_{e_2,e_2}\varphi_3 - \Gamma_{21}^{2'} d_{e_1}\varphi_3 + \Gamma_{11}^{2'} d_{e_2}\varphi_3 - 2\,\Gamma_{11}^{2'} d_{e_1}\varphi_2 \\ & - 2\,\Gamma_{21}^{2'} d_{e_2}\varphi_2 + \varphi_2(-d_{e_2}\Gamma_{21}^{2'} - d_{e_1}\Gamma_{11}^{2'}) + \varphi_3(-d_{e_1}\Gamma_{21}^{2'} + d_{e_2}\Gamma_{11}^{2'}) \Big) e_2 \end{aligned}$$

$$\begin{aligned} D^2(\varphi_4 e_1 e_2) &= \Big( - d^2_{e_1,e_1}\varphi_4 - d^2_{e_2,e_2}\varphi_4 - \Gamma_{21}^{2'} d_{e_1}\varphi_4 + \Gamma_{11}^{2'} d_{e_2}\varphi_4 \Big) e_1 e_2 \\ &= -\Delta_g(\varphi_4) e_1 e_2 \end{aligned}$$

*where $\Delta_g$ stands for the Laplace-Beltrami operator on $(X,g)$ (see Sect. 2.4).*

*Proof.* We obtain $D^2(\varphi)$ from (3.2) and the relations above defining $\nabla^C$ in the frame $(e_1, e_2)$. Then we simplify the expression using some properties of the Levi-Cevita connection:
(i) In an orthonormal frame, symbols satisfy $\Gamma_{ij}^{k'} = -\Gamma_{ik}^{j'}$.
(ii) $[e_1, e_2] = -\Gamma_{11}^{2'} e_1 - \Gamma_{21}^{2'} e_2$ □

Proposition 4.2 (Parallel transport on $Cl(X,g)$). *Let $(X,g)$ be a Riemannian manifold of dimension 2. Let $(e_1, e_2)$ be an orthonormal frame on $X$, and $Y_0 = Y_0^1 1(y) + Y_0^2 e_1(y) + Y_0^3 e_2(y) + Y_0^4 e_1 e_2(y) \in Cl(X,g)_y$. Let $\gamma$ be a $C^1$ curve*

*in $X$ such that $\gamma(0) = y$. The parallel transport $Y$ of $Y_0$ along $\gamma$ is*

$$Y(t) = Y_0^1\, 1(\gamma(t)) + \Big[f_1 \cdot \Big((Y_0^2 f_1 + Y_0^3 f_2)S(t)\Big)\Big] e_1(\gamma(t)) + \Big[f_2 \cdot \Big((Y_0^2 f_1 + Y_0^3 f_2)S(t)\Big)\Big] e_2(\gamma(t)) + Y_0^4\, e_1e_2(\gamma(t))$$

*where $(f_1, f_2)$ is an orthonormal basis generating $Cl(\mathbb{R}^2, -\| \|_2)$ and $S(t) \in Spin(2) \subset Cl(\mathbb{R}^2, -\| \|_2)$ defined by*

$$S(t) = exp\Big(f_1 f_2 \int_0^t \dot\gamma_1(s)\, \Gamma_{11}^{2'}(s) + \dot\gamma_2(s)\, \Gamma_{21}^{2'}(s)\, ds\Big)$$

*Proof.* The parallel transport of $Y_0$ along $\gamma$ is the solution $Y(t) = Y_1(t)\, 1(\gamma(t)) + Y_2(t)\, e_1(\gamma(t)) + Y_3(t)\, e_2(\gamma(t)) + Y_4(t)\, e_1e_2(\gamma(t))$ of the differential equation

$$\begin{cases} \nabla^C_{\dot\gamma}\, Y(t) = 0 \\ \\ Y(0) = Y_0 \end{cases} \tag{4.1}$$

$$\nabla^C_{\dot\gamma}\, Y(t) = \nabla^C_{\dot\gamma} Y_1\, 1 + Y_2\, e_1 + Y_3\, e_2 + Y_4\, e_1e_2\,(t)$$

$$\begin{aligned} &= \frac{\partial Y_1}{\partial t}(t)\, 1(t) + Y_1(t)(\dot\gamma_1(t)\nabla^C_{e_1} 1\,(t) + \dot\gamma_2(t)\nabla^C_{e_2} 1\,(t)) \\ &\quad + \frac{\partial Y_2}{\partial t}(t)\, e_1(t) + Y_2(t)(\dot\gamma_1(t)\nabla^C_{e_1} e_1\,(t) + \dot\gamma_2(t)\nabla^C_{e_2} e_1\,(t)) \\ &\quad + \frac{\partial Y_3}{\partial t}(t)\, e_2(t) + Y_3(t)(\dot\gamma_1(t)\nabla^C_{e_1} e_2\,(t) + \dot\gamma_2(t)\nabla^C_{e_2} e_2\,(t)) \\ &\quad + \frac{\partial Y_4}{\partial t}(t)\, e_1e_2(t) + Y_4(t)(\dot\gamma_1(t)\nabla^C_{e_1} e_1e_2\,(t) + \dot\gamma_2(t)\nabla^C_{e_2} e_1e_2\,(t)) \end{aligned}$$

$$\begin{aligned} &= \frac{\partial Y_1}{\partial t}(t)\, 1(t) \\ &\quad + \frac{\partial Y_2}{\partial t}(t)\, e_1(t) + Y_2(t)(\dot\gamma_1(t)\, \Gamma_{11}^{2'}(t)\, e_2(t) + \dot\gamma_2(t)\, \Gamma_{21}^{2'}(t)\, e_2(t)) \\ &\quad + \frac{\partial Y_3}{\partial t}(t)\, e_2(t) + Y_3(t)(-\dot\gamma_1(t)\, \Gamma_{11}^{2'}(t)\, e_1(t) - \dot\gamma_2(t)\, \Gamma_{21}^{2'}(t)\, e_1(t)) \\ &\quad + \frac{\partial Y_4}{\partial t}(t)\, e_1e_2(t) \end{aligned}$$

Finally, we obtain a differential equation on $\mathbb{R}^4$.

$$\begin{pmatrix} dY_1/dt \\ \\ dY_2/dt \\ \\ dY_3/dt \\ \\ dY_4/dt \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \\ 0 & 0 & \dot\gamma_1\, \Gamma_{11}^{2'} + \dot\gamma_2\, \Gamma_{21}^{2'} & 0 \\ \\ 0 & -\dot\gamma_1\, \Gamma_{11}^{2'} - \dot\gamma_2\, \Gamma_{21}^{2'} & 0 & 0 \\ \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Y_1 \\ \\ Y_2 \\ \\ Y_3 \\ \\ Y_4 \end{pmatrix}$$

of initial condition $Y_1(0) = Y_0^1$, $Y_2(0) = Y_0^2$, $Y_3(0) = Y_0^3$ and $Y_4(0) = Y_0^4$.

It leads to $Y_1(t) = Y_0^1$, $Y_4(t) = Y_0^4$ and a differential equation on $\mathbb{R}^2$

$$\begin{pmatrix} dY_2/dt \\ \\ dY_3/dt \end{pmatrix} = \begin{pmatrix} 0 & \dot{\gamma_1}\Gamma_{11}^{2'} + \dot{\gamma_2}\Gamma_{21}^{2'} \\ \\ -\dot{\gamma_1}\Gamma_{11}^{2'} - \dot{\gamma_2}\Gamma_{21}^{2'} & 0 \end{pmatrix} \begin{pmatrix} Y_2 \\ \\ Y_3 \end{pmatrix} \tag{4.2}$$

of initial condition $Y_2(0) = Y_0^2$ and $Y_3(0) = Y_0^3$.

Under the identification between $\mathbb{R}^2$ and $\mathbb{C}$, (4.2) becomes

$$\begin{cases} \partial(Y_2 + i\,Y_3)/\partial t &= i\,(\dot{\gamma_1}\,\Gamma_{11}^{2'} + \dot{\gamma_2}\,\Gamma_{21}^{2'})(Y_2 + i\,Y_3) \\ \\ Y_2(0) + i\,Y_3(0) &= Y_0^2 + i\,Y_0^3 \end{cases} \tag{4.3}$$

The solution of (4.3) is

$$Y_2(t) + i\,Y_3(t) = exp\Big(i\int_0^t \dot{\gamma_1}(s)\,\Gamma_{11}^{2'}(s) + \dot{\gamma_2}(s)\,\Gamma_{21}^{2'}(s)\;ds\Big)(Y_0^2 + i\,Y_0^3) \tag{4.4}$$

Under the identification between $\mathbb{R}^2$ and $\mathbb{C}$, the right term of (4.4) is the rotation of angle $\int_0^t \dot{\gamma_1}(s)\,\Gamma_{11}^{2'}(s) + \dot{\gamma_2}(s)\,\Gamma_{21}^{2'}(s)\,ds$ applied to the vector $(Y_0^2, Y_0^3)$.

Embedding $\mathbb{R}^2$ of orthonormal basis $(f_1, f_2)$ into the Clifford algebra $Cl(\mathbb{R}^2, -\|\,\|_2)$ of basis $(1, f_1, f_2, f_1 f_2)$, (4.4) may be written into the Clifford algebras context as

$$Y_2(t)f_1 + Y_3(t)f_2 = exp\Big(f_1 f_2 \int_0^t \dot{\gamma_1}(s)\,\Gamma_{11}^{2'}(s) + \dot{\gamma_2}(s)\,\Gamma_{21}^{2'}(s)\;ds\Big)(Y_0^2 f_1 + Y_0^3 f_2)$$

Then $Y_2(t) = f_1 \cdot (Y_2(t)f_1 + Y_3(t)f_2)$ and $Y_3(t) = f_2 \cdot (Y_2(t)f_1 + Y_3(t)f_2)$, where $\cdot$ is the inner product of $Cl(\mathbb{R}^2, -\|\,\|_2)$. □

**4.2. The particular context of images.** Let us consider a nD image defined by a function $I\colon (x, y) \longmapsto (I_1(x, y), \cdots, I_n(x, y))$ on a domain $\Omega \subset \mathbb{R}^2$. $I$ determines a surface $S$ embedded in $\mathbb{R}^{n+2}$ parametrized by

$$\varphi\colon (x, y) \longmapsto (x, y, I_1(x, y), \cdots, I_n(x, y))$$

Then endow $\mathbb{R}^{n+2}$ of a metric $h$ of the form

$$h(p) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} h_1(p) & 0 & 0 & \cdots\cdots & 0 \\ 0 & h_2(p) & 0 & \cdots\cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots\cdots & 0 & h_n(p) \end{pmatrix}$$

where $h_1, \cdots, h_n$ are positive functions and denote by $g$ the metric on $S$ induced by $h$ makes the couple $(S, g)$ be a Riemannian manifold of dimension 2 of global chart $(\Omega, \varphi)$.

On the manifold $S$ the natural frame is $(\partial/\partial x, \partial/\partial y)$ induced by the cartesian coordinates system $(x, y)$. However, the Clifford-Hodge operator $D^2$ is defined with

respect to orthonormal frame fields on the base manifold (see Sect.3.1). In what follows, we construct an oriented orthonormal frame field $(e_1, e_2)$ on $S$ and compute the transformation of Levi-Cevita connection's symbols with respect to the frame change from $(\partial/\partial x, \partial/\partial y)$ to $(e_1, e_2)$.

PROPOSITION 4.3 (Positively oriented orthonormal basis of $T_pS$).
*Let* $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ *be the matrix representation of* $g$ *at* $p$ *in the basis* $(\partial/\partial x, \partial/\partial y)(p)$. *Let* $\lambda^+$ *and* $\lambda^-$ $(\lambda^+ \geq \lambda^-)$ *be the two eigenvalues of the induced endomorphism. Then a positively oriented orthonormal basis* $(e_1, e_2)$ *of* $T_pS$ *may be constructed from eigenvectors, distinguishing four cases:*

*(i) if* $F \neq 0$, *take* $(e_1, e_2) =$

$$\left( \begin{pmatrix} \dfrac{\lambda^+ - G}{\sqrt{\lambda^+}\sqrt{F^2 + (\lambda^+ - G)^2}} \\ \dfrac{F}{\sqrt{\lambda^+}\sqrt{F^2 + (\lambda^+ - G)^2}} \end{pmatrix}, sign(F) \begin{pmatrix} \dfrac{\lambda^- - G}{\sqrt{\lambda^-}\sqrt{F^2 + (\lambda^- - G)^2}} \\ \dfrac{F}{\sqrt{\lambda^-}\sqrt{F^2 + (\lambda^- - G)^2}} \end{pmatrix} \right)$$

*in the basis* $(\partial/\partial x, \partial/\partial y)(p)$.

*(ii) if* $F = 0$ *and* $E > G$, *take*

$$(e_1, e_2) = \Big(\frac{\partial/\partial x(p)}{\sqrt{E}}, \frac{\partial/\partial y(p)}{\sqrt{G}}\Big)$$

*(iii) if* $F = 0$ *and* $E < G$, *take*

$$(e_1, e_2) = \Big(\frac{\partial/\partial y(p)}{\sqrt{G}}, -\frac{\partial/\partial x(p)}{\sqrt{E}}\Big)$$

*(iv) if* $F = 0$ *and* $E = G$, *the whole space* $T_pS$ *is eigenspace. Then for any* $\theta$,

$$\left( \begin{pmatrix} \dfrac{\cos(\theta)}{\sqrt{E}} \\ \dfrac{\sin(\theta)}{\sqrt{E}} \end{pmatrix}, \begin{pmatrix} \dfrac{-\sin(\theta)}{\sqrt{E}} \\ \dfrac{\cos(\theta)}{\sqrt{E}} \end{pmatrix} \right)$$

*in the basis* $(\partial/\partial x, \partial/\partial y)(p)$ *is a positively oriented orthonormal basis of* $T_pS$.

*Proof.* As unit eigenvectors, they form an orthonormal basis of $T_pS$. The orientation is clearly positive in the cases $(ii), (iii)$ and $(iv)$. For the case $(i)$, one just need to compute the 2-form

$$\omega = \Big[\Big(\frac{\lambda^+ - G}{\sqrt{\lambda^+}\sqrt{F^2 + (\lambda^+ - G)^2}}\Big)dx + \Big(\frac{F}{\sqrt{\lambda^+}\sqrt{F^2 + (\lambda^+ - G)^2}}\Big)dy\Big] \wedge$$

$$\Big[\Big(\frac{\lambda^- - G}{\sqrt{\lambda^-}\sqrt{F^2 + (\lambda^- - G)^2}}\Big)dx + \Big(\frac{F}{\sqrt{\lambda^-}\sqrt{F^2 + (\lambda^- - G)^2}}\Big)dy\Big]$$

Then

$$\omega = \frac{F(\lambda^+ - \lambda^-)}{\sqrt{\lambda^+\lambda^-}\sqrt{F^2 + (\lambda^+ - G)^2}\sqrt{F^2 + (\lambda^- - G)^2}}\, dx \wedge dy$$

and the sign of the scalar term is the sign of $F$. □

To complete the construction, we state $(e_1, e_2) = \Big(\frac{\partial/\partial x(p)}{\sqrt{E}}, \frac{\partial/\partial y(p)}{\sqrt{E}}\Big)$ when the case *(iv)* holds.

Following this construction for each $p \in S$, we obtain a positively oriented orthonormal frame field $(e_1, e_2)$ on $S$, where $e_1$ is the unit vector field of highest variations (eigenvectors associated to the eigenvalues $\lambda^+$) and $e_2$ the unit vector field of lowest variations (eigenvectors associated to the eigenvalues $\lambda^-$).

By the antisymmetry property of its symbols $\Gamma_{ij}^{k'}$ in an orthonormal frame, the Levi-Cevita connection is entirely determined by the symbols $\Gamma_{11}^{2'}$ and $\Gamma_{21}^{2'}$ in such frames. In the next proposition, we determine the expressions of these two symbols in a frame $(v_1, v_2)$ in function of the symbols $\Gamma_{ij}^{k}$ of the connection in the frame $(\partial/\partial x, \partial/\partial y)$.

Proposition 4.4. *Let $(v_1, v_2)$ be a frame such that $v_1 = a\,\partial/\partial x + b\,\partial/\partial y$ and $v_2 = c\,\partial/\partial x + d\,\partial/\partial y$. Then*

$$\Gamma_{11}^{2'} = 1/(ad-bc)\times$$

$$\Big(-ab\frac{\partial a}{\partial x} - a^2 b\,\Gamma_{11}^1 - 2ab^2\,\Gamma_{12}^1 - b^2\frac{\partial a}{\partial y} - b^3\,\Gamma_{22}^1 + a^3\,\Gamma_{11}^2 + a^2\frac{\partial b}{\partial x} + 2a^2 b\,\Gamma_{12}^2 + ab\frac{\partial b}{\partial y} + ab^2\,\Gamma_{22}^2\Big)$$

$$\Gamma_{21}^{2'} = 1/(ad-bc)\times$$

$$\Big(-bc\frac{\partial a}{\partial x} - acb\,\Gamma_{11}^1 - (bc+ad)b\,\Gamma_{12}^1 - bd\frac{\partial a}{\partial y} - b^2 d\,\Gamma_{22}^1 + a^2 c\,\Gamma_{11}^2 + ac\frac{\partial b}{\partial x} + (bc+ad)a\,\Gamma_{12}^2$$

$$+ad\frac{\partial b}{\partial y} + abd\,\Gamma_{22}^2\Big)$$

*Proof.* By definition, we have

$$\nabla_{v_1} v_1 = \Gamma_{11}^{1'} v_1 + \Gamma_{11}^{2'} v_2 \qquad \nabla_{v_2} v_1 = \Gamma_{21}^{1'} v_1 + \Gamma_{21}^{2'} v_2$$

With respect to the frame $(\partial_x, \partial_y) := (\partial/\partial x, \partial/\partial y)$, we obtain

$$\begin{aligned}
\nabla_{v_1} v_1 &= \nabla_{a\partial_x + b\partial_y} a\,\partial_x + b\,\partial_y \\
&= a\nabla_{\partial_x} a\,\partial_x + a\nabla_{\partial_x} b\,\partial_y + b\nabla_{\partial_y} a\,\partial_x + b\nabla_{\partial_y} b\,\partial_y \\
&= a\Big[\frac{\partial a}{\partial x}\,\partial_x + a\Big(\Gamma_{11}^1\,\partial_x + \Gamma_{11}^2\,\partial_y\Big)\Big] + a\Big[\frac{\partial b}{\partial x}\,\partial_y + b\Big(\Gamma_{12}^1\,\partial_x + \Gamma_{12}^2\,\partial_y\Big)\Big] \\
&\quad + b\Big[\frac{\partial a}{\partial y}\,\partial_x + a\Big(\Gamma_{21}^1\,\partial_x + \Gamma_{21}^2\,\partial_y\Big)\Big] + b\Big[\frac{\partial b}{\partial y}\,\partial_y + b\Big(\Gamma_{22}^1\,\partial_x + \Gamma_{22}^2\,\partial_y\Big)\Big]
\end{aligned}$$

$$= \Big[a\frac{\partial a}{\partial x} + a^2\,\Gamma_{11}^1 + 2ab\,\Gamma_{12}^1 + b\frac{\partial a}{\partial y} + b^2\,\Gamma_{22}^1\Big]\partial_x + \Big[a\frac{\partial b}{\partial x} + a^2\,\Gamma_{11}^2 + 2ab\,\Gamma_{12}^2 + b\frac{\partial b}{\partial y} + b^2\,\Gamma_{22}^2\Big]\partial_y$$

Then, since

$$\partial_x = \frac{1}{ad-bc}(d\,v_1 - b\,v_2) \qquad \partial_y = \frac{1}{ad-bc}(-c\,v_1 + a\,v_2)$$

we obtain

$$\begin{aligned}\nabla_{v_1} v_1 = \frac{1}{ad-bc}\Big[&ad\frac{\partial a}{\partial x} + a^2 d\,\Gamma_{11}^1 + 2abd\,\Gamma_{12}^1 + bd\frac{\partial a}{\partial y} + b^2 d\,\Gamma_{22}^1 - a^2 c\,\Gamma_{11}^2 - ac\frac{\partial b}{\partial x}\\ &-2abc\,\Gamma_{12}^2 - bd\frac{\partial b}{\partial y} - b^2 c\,\Gamma_{22}^2\Big] v_1\\ &+\frac{1}{ad-bc}\Big[-ab\frac{\partial a}{\partial x} - a^2 b\,\Gamma_{11}^1 - 2ab^2\,\Gamma_{12}^1 - b^2\frac{\partial a}{\partial y} - b^3\,\Gamma_{22}^1 + a^3\,\Gamma_{11}^2 + a^2\frac{\partial b}{\partial x}\\ &+2a^2 b\,\Gamma_{12}^2 + ab\frac{\partial b}{\partial y} + ab^2\,\Gamma_{22}^2\Big] v_2\end{aligned}$$

from which we deduce $\Gamma_{11}^{2'}$,

and

$$\begin{aligned}\nabla_{v_2} v_1 = \frac{1}{ad-bc}\Big[&cd\frac{\partial a}{\partial x} + acd\,\Gamma_{11}^1 + (bc+ad)d\,\Gamma_{12}^1 + d^2\frac{\partial a}{\partial y} + bd^2\,\Gamma_{22}^1 - ac^2\,\Gamma_{11}^2 - c^2\frac{\partial b}{\partial x}\\ &-(bc+ad)c\,\Gamma_{12}^2 - cd\frac{\partial b}{\partial y} - cbd\,\Gamma_{22}^2\Big] v_1\\ &+\frac{1}{ad-bc}\Big[-bc\frac{\partial a}{\partial x} - acb\,\Gamma_{11}^1 - (bc+ad)b\,\Gamma_{12}^1 - bd\frac{\partial a}{\partial y} - b^2 d\,\Gamma_{22}^1 + a^2 c\,\Gamma_{11}^2\\ &+ac\frac{\partial b}{\partial x} + (bc+ad)a\,\Gamma_{12}^2 + ad\frac{\partial b}{\partial y} + abd\,\Gamma_{22}^2\Big] v_2\end{aligned}$$

from which we deduce $\Gamma_{21}^{2'}$. □

**4.3. Experiments.** We show three applications of the Clifford-Hodge flow in dimension 2.

The Clifford-Hodge flow of functions on Riemannian manifolds of dimension 2 may be devoted to anisotropic regularization of nD images. Let us detail the context of the regularization. Let $I = (I_1, \cdots, I_n)$ be a nD image. As mentionned above, it induces a Riemannian manifold $(S, g)$ of dimension 2. Then we consider each component $I_i$ as a section of the Clifford bundle $Cl(S, g)$ of degree 0, and apply the Clifford-Hodge flow on each one of them. Under the identification between sections of $Cl(S, g)$ of degree 0 and functions of $S$, we get $n$ PDE's

$$\frac{\partial I_i}{\partial t} + \Delta I_i = 0$$

where $\Delta$ is the scalar Laplacian on $C^\infty(S, g)$.

The Beltrami framework of Sochen et al. in the context of image regularization (see e.g. [14],[15]) yields to define each component $I_i$ as a function over $(S, g)$, and then to solve the n PDE's $\partial I_i/\partial t = \Delta_g I_i$. From Sect. 2.4, it corresponds to solve heat equations associated to the scalar Laplacian over the Riemannian manifold $(S, g)$. This makes the image regularization process induced by Clifford-Hodge and Beltrami

flows be equivalent. In [14], the diffusion process arises from the Euler scheme (2.5). In [15], the diffusion process arises from a convolution of the initial data with a mask corresponding to the discretization of the kernel $K_t^0$ mentionned in Sect. 2.3.

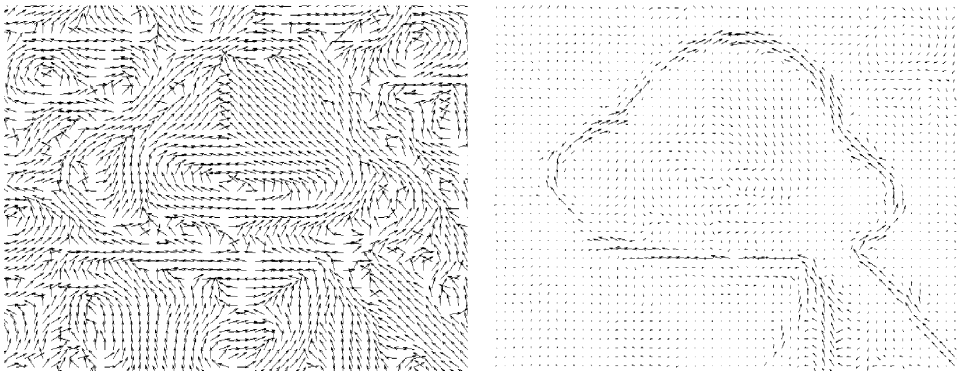
Fig. 4.1 is an illustration of the anisotropic regularization of a color image ($n = 3$) induced by the Clifford-Hodge flow. It is computed from the convolution with a 5x5 mask discretizing the kernel $K_t^0$. Fig. 4.1(a) is taken from the Berkeley image segmentation database [12]. Fig. 4.1(b) is the result of the regularization after 10 iterations for $h_1 = h_2 = h_3 = 0.01$ and $t = 0.3$, the metric $g$ being updated at each iteration. We obtain a smoothing of the image on regions of low color variations whereas high edges are preserved.

(a) Original image (b) Anisotropic Clifford-Hodge flow

FIG. 4.1. *Clifford-Hodge flow of functions*

The Clifford-Hodge flow of tangent vector fields on Riemannian manifolds of dimension 2 may be devoted to anisotropic regularization of vector fields related to nD images, that is illustrated on Fig. 4.2. Fig. 4.2(a) is the unit vector field $v = (v_1, v_2)$ indicating edge orientations in a region around the hat of Fig. 4.1(b). Let $(S, g)$ be the Riemannian manifold associated to the color image of Fig. 4.1(b), and $(\Omega, \varphi)$ be the global chart of $S$. Using the chart, $v$ may be viewed as the tangent vector field $v = (v_1, v_2)$ in the frame $(\partial/\partial x, \partial/\partial y)$. Then under the frame change from $(\partial/\partial x, \partial/\partial y)$ to $(e_1, e_2)$ (see Section 4.2), we may consider the Clifford-Hodge flow of $v$. Fig. 4.2(b) is the result of the Clifford-Hodge flow of $v$ after 99 iterations for $h_1 = h_2 = h_3 = 0.01$ and $t = 0.3$. It is computed from the convolution with a 5x5 mask discretizing the kernel $K_t^0$. We see that it tends to preserve the vector field on high edges of the image, conversely to vanish it on low edges.

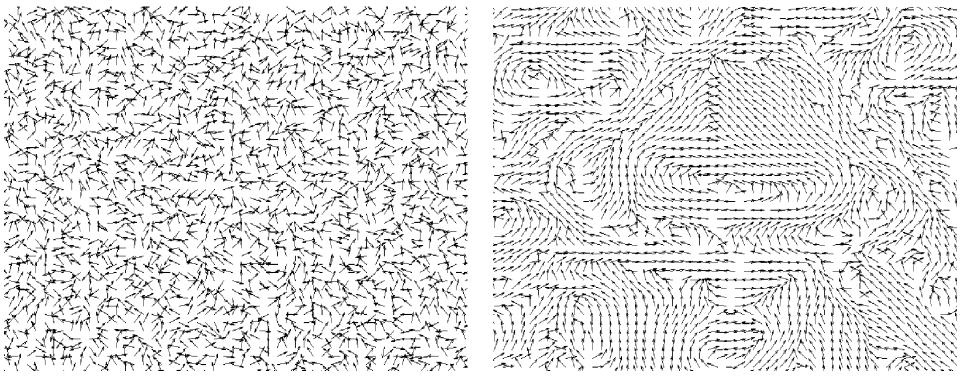
In dimension 2, the Clifford-Hodge flow of oriented orthonormal frame fields on Riemannian manifolds may be devoted to anisotropic regularization of orientation fields related to nD images. Let $v = (v_1, v_2)$ be the unit vector field on $\Omega \subset \mathbb{R}^2$ of Fig. 4.2(a). For each $p \in S$, $v(p)$ may be represented by polar coordinates $(1, \theta(p))$, $\theta(p) \in [0, 2\pi[$. Let us corrupt the function $\theta$ by adding to it a random map with values in $[-\pi/2, \pi/2]$. The corresponding unit vector field $u$ is on Fig. 4.3(a). Using the chart $(\Omega, \varphi)$ of $S$, $u$ is a tangent vector field of $S$ that may be written $u = \lambda(\cos(\psi)e_1 + \sin(\psi)e_2)$, for some functions $\lambda, \psi$. Then it may be viewed (up to a normalization) as the



(a) Unit vector field of edge orientations (b) Anisotropic Clifford-Hodge flow

FIG. 4.2. *Clifford-Hodge flow of tangent vector field*

result of a rotation field of angle $\psi$ applied to $e_1$. Following Section 3.2, we construct $\widetilde{\psi}e_1e_2 \in \Gamma(\bigwedge^2 TX)$ such that the rotation field of angle $\psi$ equals $exp \circ \widetilde{\psi}e_1e_2$, and we regularize it through the Clifford-Hodge flow of $\widetilde{\psi}e_1e_2$. From Section 4.1, it produces the scalar Laplacian/Beltrami flow of the function $\widetilde{\psi}$. Hence we obtain a unit vector field $u_t$ in the frame $(e_1, e_2)$. By the inverse frame change from $(e_1, e_2)$ to $(\partial/\partial x, \partial/\partial y)$, we obtain (up to normalization) a unit vector field in the frame $(\partial/\partial x, \partial/\partial y)$, and consequently on the chart. Fig. 4.2(b) is the result of this process after 1000 iterations for $h_1 = h_2 = h_3 = 0.1$ and $t = 0.3$. It is computed from the convolution with a 5x5 mask discretizing the kernel $K_0^t$. Up to details, we refind the original unit vector field of Fig. 4.2(a).



(a) Noisy unit vector field of edge orientations (b) Anisotropic Clifford-Hodge flow

FIG. 4.3. *Clifford-Hodge flow of oriented orthonormal frame field*

The results concerning regularization of images and orientation fields should have been foreseen. Indeed, it is a well-known fact that the Beltrami flow tends to diffuse in the direction of edges on regions located on high edges, and behaves as a gaussian diffusion on homogeneous regions. By the result of Fig. 4.2. where the unit vector field of edge orientations is preserved on high edges and where its components are averaged on regions of low color variations, we may deduce more generally that the

Clifford-Hodge flow acts in this way.

**5. The case $m = 3$.**

**5.1. The operator $D^2$ and parallel transport map on $Cl(X, g)$.** Let us first determine the connection $\nabla^C$ in an orthonormal frame $(e_1, e_2, e_3)$. Let $\Gamma_{ij}^{k'}$ be the Levi-Cevita connection's symbols of $(X, g)$ with respect to the frame $(e_1, e_2, e_3)$.

$$\nabla^C_{e_1} 1 = 0 \qquad \nabla^C_{e_2} 1 = 0 \qquad \nabla^C_{e_3} 1 = 0$$

$$\begin{array}{lll}
\nabla^C_{e_1} e_1 = \Gamma^{2'}_{11} e_2 + \Gamma^{3'}_{11} e_3 & \nabla^C_{e_2} e_1 = \Gamma^{2'}_{21} e_2 + \Gamma^{3'}_{21} e_3 & \nabla^C_{e_3} e_1 = \Gamma^{2'}_{31} e_2 + \Gamma^{3'}_{31} e_3 \\
\nabla^C_{e_1} e_2 = \Gamma^{1'}_{12} e_1 + \Gamma^{3'}_{12} e_3 & \nabla^C_{e_2} e_2 = \Gamma^{1'}_{22} e_1 + \Gamma^{3'}_{22} e_3 & \nabla^C_{e_3} e_2 = \Gamma^{1'}_{32} e_1 + \Gamma^{3'}_{32} e_3 \\
\nabla^C_{e_1} e_3 = \Gamma^{1'}_{13} e_1 + \Gamma^{2'}_{13} e_2 & \nabla^C_{e_2} e_3 = \Gamma^{1'}_{23} e_1 + \Gamma^{2'}_{23} e_2 & \nabla^C_{e_3} e_3 = \Gamma^{1'}_{33} e_1 + \Gamma^{2'}_{33} e_2
\end{array}$$

$$\begin{array}{lll}
\nabla^C_{e_1} e_1 e_2 = \Gamma^{3'}_{12} e_1 e_3 - \Gamma^{3'}_{11} e_2 e_3 & \nabla^C_{e_2} e_1 e_2 = \Gamma^{3'}_{22} e_1 e_3 - \Gamma^{3'}_{21} e_2 e_3 & \nabla^C_{e_3} e_1 e_2 = \Gamma^{3'}_{32} e_1 e_3 - \Gamma^{3'}_{31} e_2 e_3 \\
\nabla^C_{e_1} e_1 e_3 = \Gamma^{2'}_{13} e_1 e_2 + \Gamma^{2'}_{11} e_2 e_3 & \nabla^C_{e_2} e_1 e_3 = \Gamma^{2'}_{23} e_1 e_2 + \Gamma^{2'}_{21} e_2 e_3 & \nabla^C_{e_3} e_1 e_3 = \Gamma^{2'}_{33} e_1 e_2 + \Gamma^{2'}_{31} e_2 e_3 \\
\nabla^C_{e_1} e_2 e_3 = \Gamma^{1'}_{12} e_1 e_3 - \Gamma^{1'}_{13} e_1 e_2 & \nabla^C_{e_2} e_2 e_3 = \Gamma^{1'}_{22} e_1 e_3 - \Gamma^{1'}_{23} e_1 e_2 & \nabla^C_{e_3} e_2 e_3 = \Gamma^{1'}_{32} e_1 e_3 - \Gamma^{1'}_{33} e_1 e_2
\end{array}$$

$$\nabla^C_{e_1} e_1 e_2 e_3 = 0 \qquad \nabla^C_{e_2} e_1 e_2 e_3 = 0 \qquad \nabla^C_{e_3} e_1 e_2 e_3 = 0$$

PROPOSITION 5.1. *Let $(X, g)$ be a Riemannian manifold of dimension 3. Let $(e_1, e_2, e_3)$ be an orthonormal frame on $X$ and $\varphi = \varphi_1 1 + \varphi_2 e_1 + \varphi_3 e_2 + \varphi_4 e_3 + \varphi_5 e_1 e_2 + \varphi_6 e_1 e_3 + \varphi_7 e_2 e_3 + \varphi_8 e_1 e_2 e_3 \in \Gamma(Cl(X, g))$. Then*

$$\begin{aligned}
D^2(\varphi_1 1) = \Big( & - d^2_{e_1,e_1} \varphi_1 - d^2_{e_2,e_2} \varphi_1 - d^2_{e_3,e_3} \varphi_1 + \Gamma^{2'}_{11} d_{e_2} \varphi_1 + \Gamma^{3'}_{11} d_{e_3} \varphi_1 - \Gamma^{2'}_{21} d_{e_1} \varphi_1 \\
& + \Gamma^{3'}_{22} d_{e_3} \varphi_1 - \Gamma^{3'}_{31} d_{e_1} \varphi_1 - \Gamma^{3'}_{32} d_{e_2} \varphi_1 \Big) 1 \\
= & -\Delta_g(\varphi_1) 1
\end{aligned}$$

$$D^2(\varphi_2 e_1 + \varphi_3 e_2 + \varphi_4 e_3) =$$

$$\Big( -d^2_{e_1,e_1} \varphi_2 - d^2_{e_2,e_2} \varphi_2 - d^2_{e_3,e_3} \varphi_2 + d_{e_1} \varphi_2 (-\Gamma^{2'}_{21} - \Gamma^{3'}_{31}) + d_{e_2} \varphi_2 (\Gamma^{2'}_{11} - \Gamma^{3'}_{32}) + d_{e_3} \varphi_2 (\Gamma^{3'}_{22} + \Gamma^{3'}_{11})$$

$$+ 2\Gamma^{2'}_{11}\, d_{e_1} \varphi_3 + 2\Gamma^{2'}_{21}\, d_{e_2} \varphi_3 + 2\Gamma^{2'}_{31}\, d_{e_3} \varphi_3 + 2\Gamma^{3'}_{11}\, d_{e_1} \varphi_4 + 2\Gamma^{3'}_{21}\, d_{e_2} \varphi_4 + 2\Gamma^{3'}_{31}\, d_{e_3} \varphi_4$$

$$+ \varphi_2 (-d_{e_1} \Gamma^{2'}_{21} - d_{e_1} \Gamma^{3'}_{31} + d_{e_2} \Gamma^{2'}_{11} + d_{e_3} \Gamma^{3'}_{11} + (\Gamma^{3'}_{21})^2 + (\Gamma^{2'}_{31})^2 - 2\Gamma^{3'}_{21} \Gamma^{2'}_{31} - \Gamma^{3'}_{11} \Gamma^{3'}_{22} + \Gamma^{3'}_{32} \Gamma^{2'}_{11})$$

$$+ \varphi_3 (d_{e_1} \Gamma^{2'}_{11} - d_{e_1} \Gamma^{3'}_{32} + d_{e_2} \Gamma^{2'}_{21} + d_{e_3} \Gamma^{3'}_{12} + d_{e_3} \Gamma^{2'}_{31} - 2\Gamma^{3'}_{22} \Gamma^{2'}_{31} - \Gamma^{3'}_{12} \Gamma^{3'}_{22} + \Gamma^{3'}_{21} \Gamma^{3'}_{22} + \Gamma^{3'}_{32} \Gamma^{2'}_{21})$$

$$+ \varphi_4 (d_{e_1} \Gamma^{3'}_{11} + d_{e_1} \Gamma^{3'}_{22} - d_{e_2} \Gamma^{3'}_{12} + d_{e_2} \Gamma^{3'}_{21} + d_{e_3} \Gamma^{3'}_{31} + 2\Gamma^{3'}_{21} \Gamma^{3'}_{32} - \Gamma^{3'}_{22} \Gamma^{3'}_{31} - \Gamma^{3'}_{32} \Gamma^{3'}_{12} - \Gamma^{2'}_{31} \Gamma^{3'}_{32}) \Big) e_1$$

$$+\Big(-d^2_{e_1,e_1}\varphi_3-d^2_{e_2,e_2}\varphi_3-d^2_{e_3,e_3}\varphi_3+d_{e_1}\varphi_3(-\Gamma^{2'}_{21}-\Gamma^{3'}_{31})+d_{e_2}\varphi_3(\Gamma^{2'}_{11}-\Gamma^{3'}_{32})+d_{e_3}\varphi_3(\Gamma^{3'}_{22}+\Gamma^{3'}_{11})$$

$$-2\Gamma^{2'}_{11}\,d_{e_1}\varphi_2-2\Gamma^{2'}_{21}\,d_{e_2}\varphi_2-2\Gamma^{2'}_{31}\,d_{e_3}\varphi_2+2\Gamma^{3'}_{12}\,d_{e_1}\varphi_4+2\Gamma^{3'}_{22}\,d_{e_2}\varphi_4+2\Gamma^{3'}_{32}\,d_{e_3}\varphi_4$$

$$+\varphi_2(-d_{e_1}\Gamma^{2'}_{11}-d_{e_2}\Gamma^{2'}_{21}-d_{e_2}\Gamma^{3'}_{31}+d_{e_3}\Gamma^{3'}_{21}-d_{e_3}\Gamma^{2'}_{31}+2\Gamma^{3'}_{11}\Gamma^{2'}_{31}+\Gamma^{3'}_{12}\Gamma^{3'}_{11}-\Gamma^{3'}_{11}\Gamma^{3'}_{21}-\Gamma^{3'}_{31}\Gamma^{2'}_{11})$$

$$+\varphi_3(-d_{e_1}\Gamma^{2'}_{21}+d_{e_2}\Gamma^{2'}_{11}-d_{e_2}\Gamma^{3'}_{32}+d_{e_3}\Gamma^{3'}_{22}+(\Gamma^{3'}_{12})^2+(\Gamma^{2'}_{31})^2+2\Gamma^{3'}_{12}\Gamma^{2'}_{31}-\Gamma^{3'}_{11}\Gamma^{3'}_{22}-\Gamma^{3'}_{31}\Gamma^{2'}_{21})$$

$$+\varphi_4(d_{e_1}\Gamma^{3'}_{12}-d_{e_1}\Gamma^{3'}_{21}+d_{e_2}\Gamma^{3'}_{11}+d_{e_2}\Gamma^{3'}_{22}+d_{e_3}\Gamma^{3'}_{32}+2\Gamma^{3'}_{12}\Gamma^{3'}_{31}-\Gamma^{3'}_{11}\Gamma^{3'}_{32}-\Gamma^{3'}_{31}\Gamma^{3'}_{21}+\Gamma^{2'}_{31}\Gamma^{3'}_{31})\Big)e_2$$

$$+\Big(-d^2_{e_1,e_1}\varphi_4-d^2_{e_2,e_2}\varphi_4-d^2_{e_3,e_3}\varphi_4+d_{e_1}\varphi_4(-\Gamma^{2'}_{21}-\Gamma^{3'}_{31})+d_{e_2}\varphi_4(\Gamma^{2'}_{11}-\Gamma^{3'}_{32})+d_{e_3}\varphi_4(\Gamma^{3'}_{22}+\Gamma^{3'}_{11})$$

$$-2\Gamma^{3'}_{11}\,d_{e_1}\varphi_2-2\Gamma^{3'}_{21}\,d_{e_2}\varphi_2-2\Gamma^{3'}_{31}\,d_{e_3}\varphi_2-2\Gamma^{3'}_{12}\,d_{e_1}\varphi_3-2\Gamma^{3'}_{22}\,d_{e_2}\varphi_3-2\Gamma^{3'}_{32}\,d_{e_3}\varphi_3$$

$$+\varphi_2(-d_{e_1}\Gamma^{3'}_{11}-d_{e_2}\Gamma^{3'}_{21}+d_{e_2}\Gamma^{2'}_{31}-d_{e_3}\Gamma^{2'}_{21}-d_{e_3}\Gamma^{3'}_{31}+2\Gamma^{2'}_{11}\Gamma^{3'}_{21}-\Gamma^{3'}_{12}\Gamma^{2'}_{11}-\Gamma^{2'}_{11}\Gamma^{2'}_{31}-\Gamma^{2'}_{21}\Gamma^{3'}_{11})$$

$$+\varphi_3(-d_{e_1}\Gamma^{3'}_{12}-d_{e_1}\Gamma^{2'}_{31}-d_{e_2}\Gamma^{3'}_{22}+d_{e_3}\Gamma^{2'}_{11}-d_{e_3}\Gamma^{3'}_{32}-2\Gamma^{3'}_{12}\Gamma^{2'}_{21}+\Gamma^{2'}_{11}\Gamma^{3'}_{22}+\Gamma^{3'}_{21}\Gamma^{2'}_{21}-\Gamma^{2'}_{21}\Gamma^{2'}_{31})$$

$$+\varphi_4(-d_{e_1}\Gamma^{3'}_{31}-d_{e_2}\Gamma^{3'}_{32}+d_{e_3}\Gamma^{3'}_{11}+d_{e_3}\Gamma^{3'}_{22}+(\Gamma^{3'}_{12})^2+(\Gamma^{3'}_{21})^2-2\Gamma^{3'}_{12}\Gamma^{3'}_{21}+\Gamma^{2'}_{11}\Gamma^{3'}_{32}-\Gamma^{2'}_{21}\Gamma^{3'}_{31})\Big)e_3$$

$$D^2(\varphi_5\,e_1e_2+\varphi_6\,e_1e_3+\varphi_7\,e_2e_3)=$$

$$\Big(-d^2_{e_1,e_1}\varphi_5-d^2_{e_2,e_2}\varphi_5-d^2_{e_3,e_3}\varphi_5+d_{e_1}\varphi_5(-\Gamma^{2'}_{21}-\Gamma^{3'}_{31})+d_{e_2}\varphi_5(\Gamma^{2'}_{11}-\Gamma^{3'}_{32})+d_{e_3}\varphi_5(\Gamma^{3'}_{22}+\Gamma^{3'}_{11})$$

$$+2\Gamma^{3'}_{12}\,d_{e_1}\varphi_6+2\Gamma^{3'}_{22}\,d_{e_2}\varphi_6+2\Gamma^{3'}_{32}\,d_{e_3}\varphi_6-2\Gamma^{3'}_{11}\,d_{e_1}\varphi_7-2\Gamma^{3'}_{21}\,d_{e_2}\varphi_7-2\Gamma^{3'}_{31}\,d_{e_3}\varphi_7$$

$$+\varphi_5(-d_{e_1}\Gamma^{3'}_{31}-d_{e_2}\Gamma^{3'}_{32}+d_{e_3}\Gamma^{3'}_{11}+d_{e_3}\Gamma^{3'}_{22}+(\Gamma^{3'}_{21})^2+(\Gamma^{3'}_{12})^2-2\Gamma^{3'}_{21}\Gamma^{3'}_{12}+\Gamma^{2'}_{11}\Gamma^{3'}_{32}-\Gamma^{2'}_{21}\Gamma^{3'}_{31})$$

$$+\varphi_6(d_{e_1}\Gamma^{3'}_{12}+d_{e_1}\Gamma^{2'}_{31}+d_{e_2}\Gamma^{3'}_{22}-d_{e_3}\Gamma^{2'}_{11}+d_{e_3}\Gamma^{3'}_{32}+2\Gamma^{2'}_{21}\Gamma^{3'}_{12}-\Gamma^{2'}_{11}\Gamma^{3'}_{22}+\Gamma^{2'}_{21}\Gamma^{2'}_{31}-\Gamma^{3'}_{21}\Gamma^{2'}_{21})$$

$$+\varphi_7(-d_{e_1}\Gamma^{3'}_{11}-d_{e_2}\Gamma^{3'}_{21}+d_{e_2}\Gamma^{2'}_{31}-d_{e_3}\Gamma^{2'}_{21}-d_{e_3}\Gamma^{3'}_{31}+2\Gamma^{2'}_{11}\Gamma^{3'}_{21}-\Gamma^{2'}_{11}\Gamma^{2'}_{31}-\Gamma^{3'}_{12}\Gamma^{2'}_{11}-\Gamma^{2'}_{21}\Gamma^{3'}_{11})\Big)e_1e_2$$

$$+\Big(-d^2_{e_1,e_1}\varphi_6-d^2_{e_2,e_2}\varphi_6-d^2_{e_3,e_3}\varphi_6+d_{e_1}\varphi_6(-\Gamma^{2'}_{21}-\Gamma^{3'}_{31})+d_{e_2}\varphi_6(\Gamma^{2'}_{11}-\Gamma^{3'}_{32})+d_{e_3}\varphi_6(\Gamma^{3'}_{22}+\Gamma^{3'}_{11})$$

$$-2\Gamma_{12}^{3'}\,d_{e_1}\varphi_5 - 2\Gamma_{22}^{3'}\,d_{e_2}\varphi_5 - 2\Gamma_{32}^{3'}\,d_{e_3}\varphi_5 + 2\Gamma_{11}^{2'}\,d_{e_1}\varphi_7 + 2\Gamma_{21}^{2'}\,d_{e_2}\varphi_7 + 2\Gamma_{31}^{2'}\,d_{e_3}\varphi_7$$

$$+\varphi_5(-d_{e_1}\Gamma_{12}^{3'} + d_{e_1}\Gamma_{21}^{3'} - d_{e_2}\Gamma_{11}^{3'} - d_{e_2}\Gamma_{22}^{3'} - d_{e_3}\Gamma_{32}^{3'} - 2\Gamma_{12}^{3'}\Gamma_{31}^{3'} + \Gamma_{32}^{3'}\Gamma_{11}^{3'} - \Gamma_{31}^{2'}\Gamma_{31}^{3'} + \Gamma_{31}^{3'}\Gamma_{21}^{3'})$$

$$+\varphi_6(-d_{e_1}\Gamma_{21}^{2'} + d_{e_2}\Gamma_{11}^{2'} - d_{e_2}\Gamma_{32}^{3'} + d_{e_3}\Gamma_{22}^{3'} + (\Gamma_{12}^{3'})^2 + (\Gamma_{31}^{2'})^2 + 2\Gamma_{12}^{3'}\Gamma_{31}^{2'} - \Gamma_{11}^{3'}\Gamma_{22}^{3'} - \Gamma_{31}^{3'}\Gamma_{21}^{2'})$$

$$+\varphi_7(d_{e_1}\Gamma_{11}^{2'} + d_{e_2}\Gamma_{21}^{2'} + d_{e_2}\Gamma_{31}^{3'} - d_{e_3}\Gamma_{21}^{3'} + d_{e_3}\Gamma_{31}^{2'} - 2\Gamma_{11}^{3'}\Gamma_{31}^{2'} + \Gamma_{11}^{3'}\Gamma_{21}^{3'} - \Gamma_{12}^{3'}\Gamma_{11}^{3'} + \Gamma_{31}^{3'}\Gamma_{11}^{2'})\Big)e_1e_3$$

$$+\Big(-d_{e_1,e_1}^2\varphi_7 - d_{e_2,e_2}^2\varphi_7 - d_{e_3,e_3}^2\varphi_7 + d_{e_1}\varphi_7(-\Gamma_{21}^{2'} - \Gamma_{31}^{3'}) + d_{e_2}\varphi_7(\Gamma_{11}^{2'} - \Gamma_{32}^{3'}) + d_{e_3}\varphi_7(\Gamma_{22}^{3'} + \Gamma_{11}^{3'})$$

$$+2\Gamma_{11}^{3'}\,d_{e_1}\varphi_5 + 2\Gamma_{21}^{3'}\,d_{e_2}\varphi_5 + 2\Gamma_{31}^{3'}\,d_{e_3}\varphi_5 - 2\Gamma_{11}^{2'}\,d_{e_1}\varphi_6 - 2\Gamma_{21}^{2'}\,d_{e_2}\varphi_6 - 2\Gamma_{31}^{2'}\,d_{e_3}\varphi_6$$

$$+\varphi_5(d_{e_1}\Gamma_{11}^{3'} + d_{e_1}\Gamma_{22}^{3'} - d_{e_2}\Gamma_{12}^{3'} + d_{e_2}\Gamma_{21}^{3'} + d_{e_3}\Gamma_{31}^{3'} + 2\Gamma_{21}^{3'}\Gamma_{32}^{3'} - \Gamma_{22}^{3'}\Gamma_{31}^{3'} - \Gamma_{31}^{2'}\Gamma_{32}^{3'} - \Gamma_{32}^{3'}\Gamma_{12}^{3'})$$

$$+\varphi_6(-d_{e_1}\Gamma_{11}^{2'} + d_{e_1}\Gamma_{32}^{3'} - d_{e_2}\Gamma_{21}^{2'} - d_{e_3}\Gamma_{12}^{3'} - d_{e_3}\Gamma_{31}^{2'} + 2\Gamma_{22}^{3'}\Gamma_{31}^{2'} - \Gamma_{21}^{3'}\Gamma_{22}^{3'} + \Gamma_{22}^{3'}\Gamma_{12}^{3'} - \Gamma_{32}^{3'}\Gamma_{21}^{2'})$$

$$+\varphi_7(-d_{e_1}\Gamma_{21}^{2'} - d_{e_1}\Gamma_{31}^{3'} + d_{e_2}\Gamma_{11}^{2'} + d_{e_3}\Gamma_{11}^{3'} + (\Gamma_{21}^{3'})^2 + (\Gamma_{31}^{2'})^2 - 2\Gamma_{21}^{3'}\Gamma_{31}^{2'} - \Gamma_{22}^{3'}\Gamma_{11}^{3'} + \Gamma_{32}^{3'}\Gamma_{11}^{2'})\Big)e_2e_3$$

$$\begin{aligned} D^2(\varphi_8\, e_1e_2e_3) = \Big( & -d_{e_1,e_1}^2\varphi_8 - d_{e_2,e_2}^2\varphi_8 - d_{e_3,e_3}^2\varphi_8 + \Gamma_{11}^{2'}d_{e_2}\varphi_8 + \Gamma_{11}^{3'}d_{e_3}\varphi_8 - \Gamma_{21}^{2'}d_{e_1}\varphi_8 \\ & + \Gamma_{22}^{3'}d_{e_3}\varphi_8 - \Gamma_{31}^{3'}d_{e_1}\varphi_8 - \Gamma_{32}^{3'}d_{e_2}\varphi_8\Big)e_1e_2e_3 \\ = & -\Delta_g(\varphi_8)e_1e_2e_3 \end{aligned}$$

*Proof.* We obtain $D^2(\varphi)$ from (3.2) and the relations above defining $\nabla^C$ in the frame $(e_1, e_2, e_3)$. Then we simplify the expression using some properties of the Levi-Cevita connection:
(i) In an orthonormal frame, symbols satisfy $\Gamma_{ij}^{k'} = -\Gamma_{ik}^{j'}$.
(ii) $[e_1, e_2] = -\Gamma_{11}^{2'}e_1 - \Gamma_{21}^{2'}e_2 + \Gamma_{12}^{3'}e_3 - \Gamma_{21}^{3'}e_3$

$[e_3, e_1] = \Gamma_{11}^{3'}e_1 + \Gamma_{31}^{2'}e_2 + \Gamma_{12}^{3'}e_2 + \Gamma_{31}^{3'}e_3$

$[e_2, e_3] = -\Gamma_{21}^{3'}e_1 + \Gamma_{31}^{2'}e_1 - \Gamma_{22}^{3'}e_2 - \Gamma_{32}^{3'}e_3$ □

PROPOSITION 5.2 (Parallel transport on $Cl(X, g)$). *Let $(X, g)$ be a Riemannian manifold of dimension 3. Let $(e_1, e_2, e_3)$ be an orthonormal frame on $X$, and $Y_0 = Y_0^1 1(y) + Y_0^2 e_1(y) + Y_0^3 e_2(y) + Y_0^4 e_3(y) + Y_0^5 e_1e_2(y) + Y_0^6 e_1e_3(y) + Y_0^7 e_2e_3(y) + Y_0^8 e_1e_2e_3(y) \in Cl(X, g)_y$. Let $\gamma$ be a $C^1$ curve in $X$ such that $\gamma(0) = y$. The parallel*

*transport $Y$ of $Y_0$ along $\gamma$ is obtained from the solution of the following differential equation on $\mathbb{R}^8$.*

$$dY_1/dt = 0, \begin{pmatrix} dY_2/dt \\ dY_3/dt \\ dY_4/dt \end{pmatrix} = A \begin{pmatrix} Y_2 \\ Y_3 \\ Y_4 \end{pmatrix}, \begin{pmatrix} dY_5/dt \\ dY_6/dt \\ dY_7/dt \end{pmatrix} = B \begin{pmatrix} Y_5 \\ Y_6 \\ Y_7 \end{pmatrix}, dY_8/dt = 0 \tag{5.1}$$

*where* $A =$

$$\begin{pmatrix} 0 & \dot\gamma_1\Gamma_{11}^{2'} + \dot\gamma_2\Gamma_{21}^{2'} + \dot\gamma_3\Gamma_{31}^{2'} & \dot\gamma_1\Gamma_{11}^{3'} + \dot\gamma_2\Gamma_{21}^{3'} + \dot\gamma_3\Gamma_{31}^{2'} \\ -\dot\gamma_1\Gamma_{11}^{2'} - \dot\gamma_2\Gamma_{21}^{2'} - \dot\gamma_3\Gamma_{31}^{2'} & 0 & \dot\gamma_1\Gamma_{12}^{3'} + \dot\gamma_2\Gamma_{22}^{3'} + \dot\gamma_3\Gamma_{32}^{3'} \\ -\dot\gamma_1\Gamma_{11}^{3'} - \dot\gamma_2\Gamma_{21}^{3'} - \dot\gamma_3\Gamma_{31}^{2'} & -\dot\gamma_1\Gamma_{12}^{3'} - \dot\gamma_2\Gamma_{22}^{3'} - \dot\gamma_3\Gamma_{32}^{3'} & 0 \end{pmatrix}$$

*and* $B =$

$$\begin{pmatrix} 0 & \dot\gamma_1\Gamma_{12}^{3'} + \dot\gamma_2\Gamma_{22}^{3'} + \dot\gamma_3\Gamma_{32}^{3'} & -\dot\gamma_1\Gamma_{11}^{3'} - \dot\gamma_2\Gamma_{21}^{3'} - \dot\gamma_3\Gamma_{31}^{2'} \\ -\dot\gamma_1\Gamma_{12}^{3'} - \dot\gamma_2\Gamma_{22}^{3'} - \dot\gamma_3\Gamma_{32}^{3'} & 0 & \dot\gamma_1\Gamma_{11}^{2'} + \dot\gamma_2\Gamma_{21}^{2'} + \dot\gamma_3\Gamma_{31}^{2'} \\ \dot\gamma_1\Gamma_{11}^{3'} + \dot\gamma_2\Gamma_{21}^{3'} + \dot\gamma_3\Gamma_{31}^{2'} & -\dot\gamma_1\Gamma_{11}^{2'} - \dot\gamma_2\Gamma_{21}^{2'} - \dot\gamma_3\Gamma_{31}^{2'} & 0 \end{pmatrix}$$

*of initial condition $Y_i(0) = Y_0^i$, for $i \in \{1, \cdots, 8\}$.*

The solutions $Y_1(t)$ and $Y_8(t)$ are given by $Y_1(t) = Y_0^1$ and $Y_8(t) = Y_0^8$. To the best of knowledge of the author, there exist no explicit formulae for $(Y_2(t), Y_3(t), Y_4(t))$ and $(Y_5(t), Y_6(t), Y_7(t))$. In practice, we suppose the matrices $A$ and $B$ to be locally constant. This gives explicit solutions of (5.1).

Corollary 5.3. *Let us suppose that Christoffel's symbols $\Gamma_{ij}^{k'}$ and coordinates $\dot\gamma_i$ of $\dot\gamma$ are constant on some open set $\Omega_y \supset y$. Let $(f_1, f_2, f_3)$ be an orthonormal basis of $\mathbb{R}^3$ generating the Clifford algebra $Cl(\mathbb{R}^3, -\| \|_2)$ and $S_1(t), S_2(t) \in Spin(3) \subset Cl(\mathbb{R}^3, -\| \|_2)$ defined by*

$$S_1(t) = exp\Big[\frac{t\sqrt{\alpha^2+\beta^2+\delta^2}}{2}\Big(\frac{\alpha f_1f_2 + \beta f_1f_3 + \delta f_2f_3}{\sqrt{\alpha^2+\beta^2+\delta^2}}\Big)\Big]$$

*and*

$$S_2(t) = exp\Big[\frac{t\sqrt{\alpha^2+\beta^2+\delta^2}}{2}\Big(\frac{\delta f_1f_2 - \beta f_1f_3 + \alpha f_2f_3}{\sqrt{\alpha^2+\beta^2+\delta^2}}\Big)\Big]$$

*where* $\alpha = \dot\gamma_1\Gamma_{11}^2 + \dot\gamma_2\Gamma_{21}^2 + \dot\gamma_3\Gamma_{31}^2$, $\beta = \dot\gamma_1\Gamma_{11}^3 + \dot\gamma_2\Gamma_{21}^3 + \dot\gamma_3\Gamma_{31}^2$ *and* $\delta = \dot\gamma_1\Gamma_{12}^3 + \dot\gamma_2\Gamma_{22}^3 + \dot\gamma_3\Gamma_{32}^3$.

*The solution $Y(t)$ of (5.1) is given on $\Omega_y$ by Y(t)=*

$$
\begin{aligned}
&Y_0^1 1(\gamma(t)) + \Big[f_1 \cdot \Big(S_1(t)^\dagger (Y_0^2 f_1 + Y_0^3 f_2 + Y_0^4 f_3) S_1(t)\Big)\Big] e_1(\gamma(t)) \\
&+ \Big[f_2 \cdot \Big(S_1(t)^\dagger (Y_0^2 f_1 + Y_0^3 f_2 + Y_0^4 f_3) S_1(t)\Big)\Big] e_2(\gamma(t)) \\
&+ \Big[f_3 \cdot \Big(S_1(t)^\dagger (Y_0^2 f_1 + Y_0^3 f_2 + Y_0^4 f_3) S_1(t)\Big)\Big] e_3(\gamma(t)) \\
&+ \Big[f_1 \cdot \Big(S_2(t)^\dagger (Y_0^5 f_1 + Y_0^6 f_2 + Y_0^7 f_3) S_2(t)\Big)\Big] e_1 e_2(\gamma(t)) \\
&+ \Big[f_2 \cdot \Big(S_2(t)^\dagger (Y_0^5 f_1 + Y_0^6 f_2 + Y_0^7 f_3) S_2(t)\Big)\Big] e_1 e_3(\gamma(t)) \\
&+ \Big[f_3 \cdot \Big(S_2(t)^\dagger (Y_0^5 f_1 + Y_0^6 f_2 + Y_0^7 f_3) S_2(t)\Big)\Big] e_2 e_3(\gamma(t)) + Y_0^8 e_1 e_2 e_3(\gamma(t))
\end{aligned}
$$

*Proof.* We aim at solving the differential equations

$$
\begin{pmatrix} dY_2/dt \\ \\ dY_3/dt \\ \\ dY_4/dt \end{pmatrix} = A \begin{pmatrix} Y_2 \\ \\ Y_3 \\ \\ Y_4 \end{pmatrix} \text{ and } \begin{pmatrix} dY_5/dt \\ \\ dY_6/dt \\ \\ dY_7/dt \end{pmatrix} = B \begin{pmatrix} Y_5 \\ \\ Y_6 \\ \\ Y_7 \end{pmatrix} \tag{5.2}
$$

where $A$ and $B$ are antisymmetric of the form

$$
A = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \delta \\ -\beta & -\delta & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & \delta & -\beta \\ -\delta & 0 & \alpha \\ \beta & -\alpha & 0 \end{pmatrix}
$$

They got the same eigenvalues, i.e. $0$, $i\sqrt{\alpha^2+\beta^2+\delta^2}$ and its conjugate. The eigenspaces associated to 0 are $\mathbb{R}(\delta, -\beta, \alpha)$ for $A$, and $\mathbb{R}(\alpha, \beta, \delta)$ for $B$. The two-dimensional eigenspaces of the complex eigenvalue $i\sqrt{\alpha^2+\beta^2+\delta^2}$ are their orthogonal supplementary in $\mathbb{R}^3$. Let $u_1$ resp. $v_1$ be an eigenvector of $A$ resp. $B$ relative to the eigenvalue 0, let $(u_2, u_3)$ resp. $(v_2, v_3)$ be a basis of the eigenspace relative to the eigenvalue $i\sqrt{\alpha^2+\beta^2+\delta^2}$, and $P_1$ resp. $P_2$ be the basis change matrix from $(f_1, f_2, f_3)$ to $(u_1, u_2, u_3)$ resp. $(v_1, v_2, v_3)$.

In the basis $(u_1, u_2, u_3)$ of coordinates $(Z_2, Z_3, Z_4)$, the first differential equation of (5.2) is

$$
\begin{pmatrix} dZ_2/dt \\ \\ d(Z_3 + i\, Z_4)/dt \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \\ 0 & i\sqrt{\alpha^2+\beta^2+\delta^2} \end{pmatrix} \begin{pmatrix} Z_2 \\ \\ Z_3 + i\, Z_4 \end{pmatrix} \tag{5.3}
$$

of initial condition

$$
\begin{pmatrix} Z_2(0) \\ \\ Z_3(0) \\ \\ Z_4(0) \end{pmatrix} = P_1^{-1} \begin{pmatrix} Y_0^2 \\ \\ Y_0^3 \\ \\ Y_0^4 \end{pmatrix}
$$

In the basis $(v_1, v_2, v_3)$ of coordinates $(Z_5, Z_6, Z_7)$, the second differential equation of (5.2) is

$$
\begin{pmatrix} dZ_5/dt \\ \\ d(Z_6 + i\, Z_7)/dt \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \\ 0 & i\sqrt{\alpha^2+\beta^2+\delta^2} \end{pmatrix} \begin{pmatrix} Z_5 \\ \\ Z_6 + i\, Z_7 \end{pmatrix} \tag{5.4}
$$

of initial condition

$$\begin{pmatrix} Z_5(0) \\ Z_6(0) \\ Z_7(0) \end{pmatrix} = P_2^{-1} \begin{pmatrix} Y_0^5 \\ Y_0^6 \\ Y_0^7 \end{pmatrix}$$

Solutions of (5.3) and (5.4) are

$$\begin{cases} Z_2(t) &= Z_2(0) \\ Z_3(t) + iZ_4(t) &= exp(i\,t\,\sqrt{\alpha^2+\beta^2+\delta^2})(Z_3(0)+iZ_4(0)) \end{cases}$$

$$\begin{cases} Z_5(t) &= Z_5(0) \\ Z_6(t) + iZ_7(t) &= exp(i\,t\,\sqrt{\alpha^2+\beta^2+\delta^2})(Z_6(0)+iZ_7(0)) \end{cases}$$

and general solutions of (5.2) are

$$\begin{pmatrix} Y_2(t) \\ Y_3(t) \\ Y_4(t) \end{pmatrix} = P_1 \begin{pmatrix} Z_2(t) \\ Z_3(t) \\ Z_4(t) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Y_5(t) \\ Y_6(t) \\ Y_7(t) \end{pmatrix} = P_2 \begin{pmatrix} Z_5(t) \\ Z_6(t) \\ Z_7(t) \end{pmatrix}$$

We see that solutions of (5.2) are given by rotations of angle $t\sqrt{\alpha^2+\beta^2+\delta^2}$ in $\mathbb{R}^3$ with respect to the planes that are eigenspaces for the eigenvalue $i\sqrt{\alpha^2+\beta^2+\delta^2}$ applied to the initial condition vectors. As in the case $m = 2$, using the Clifford algebras context, such operations take a straightforward form. For this, we embed $\mathbb{R}^3$ of basis $(f_1, f_2, f_3)$ into $Cl(\mathbb{R}^3, -\| \|_2)$.

Indeed, oriented planes in $\mathbb{R}^3$ are represented by bivectors in $Cl(\mathbb{R}^3, -\| \|_2)$. In particular, the bivector representing the orthogonal of the vector $a_1f_1 + a_2f_2 + a_3f_3$ results from the product $(a_1f_1 + a_2f_2 + a_3f_3)f_1f_2f_3$. Hence the rotation of angle $t\sqrt{\alpha^2+\beta^2+\delta^2}$ in the oriented plane orthogonal to $\delta f_1 - \beta f_2 + \alpha f_3$ resp. $\alpha f_1 + \beta f_2 + \delta f_3$ is given by the spinor

$$S_1(t) = exp\Big[\frac{t\sqrt{\alpha^2+\beta^2+\delta^2}}{2}\Big(\frac{\alpha f_1f_2 + \beta f_1f_3 + \delta f_2f_3}{\sqrt{\alpha^2+\beta^2+\delta^2}}\Big)\Big]$$

resp.

$$S_2(t) = exp\Big[\frac{t\sqrt{\alpha^2+\beta^2+\delta^2}}{2}\Big(\frac{\delta f_1f_2 - \beta f_1f_3 + \alpha f_2f_3}{\sqrt{\alpha^2+\beta^2+\delta^2}}\Big)\Big]$$

Then we have

$$Y_2(t)f_1 + Y_3(t)f_2 + Y_4(t)f_3 = S_1(t)^\dagger(Y_0^2f_1 + Y_0^3f_2 + Y_0^4f_3)S_1(t)$$

and

$$Y_5(t)f_1 + Y_6(t)f_2 + Y_7(t)f_3 = S_2(t)^\dagger(Y_0^5f_1 + Y_0^6f_2 + Y_0^7f_3)S_2(t)$$

This gives the parallel transport $Y(t)$ of $Y_0$. □

**5.2. The particular context of videos.** Let us consider a nD video defined by a function $I\colon (x,y,z) \longmapsto (I_1(x,y,z),\cdots,I_n(x,y,z))$ on a domain $\Omega \subset \mathbb{R}^3$. $I$ determines a submanifold $S$ of $\mathbb{R}^{n+3}$ of dimension 3 parametrized by

$$\varphi\colon (x,y,z) \longmapsto (x,y,z,I_1(x,y,z),\cdots,I_n(x,y,z))$$

Then endow $\mathbb{R}^{n+3}$ of a metric $h$ of the form

$$h(p) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \beta \end{pmatrix} \oplus \begin{pmatrix} h_1(p) & 0 & 0 & \cdots\cdots & 0 \\ 0 & h_2(p) & 0 & \cdots\cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots\cdots & 0 & h_n(p) \end{pmatrix} \tag{5.5}$$

where $h_1,\cdots,h_n$ are positive functions, $\lambda,\beta > 0$ and denote by $g$ the metric on $S$ induced by $h$ makes the couple $(S,g)$ be a Riemannian manifold of dimension 3 of global chart $(\Omega,\varphi)$.

As for the case $m = 2$, an oriented orthonormal frame field $(e_1,e_2,e_3)$ may be constructed on $(S,g)$ from the matrix representation of $g$ in the frame $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$. Indeed, for each $p \in S$, a positively oriented orthonormal basis of $T_pS$ may be constructed from its eigenvectors.

Then we need to compute the transformation of Levi-Cevita connection's symbols with respect to the frame change from $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ to $(e_1,e_2,e_3)$. Remind that by the antisymmetry property of its symbols $\Gamma_{ij}^{k'}$ in an orthonormal frame, the Levi-Cevita connection is determined by the 9 symbols $\Gamma_{11}^{2'}, \Gamma_{11}^{3'}, \Gamma_{21}^{2'}, \Gamma_{21}^{3'}, \Gamma_{31}^{2'}, \Gamma_{31}^{3'}, \Gamma_{12}^{3'}, \Gamma_{22}^{3'}$ and $\Gamma_{32}^{3'}$ in such frames. In the next proposition, we determine the expression of the symbol $\Gamma_{11}^{2'}$ in a frame $(v_1,v_2,v_3)$ in function of the symbols $\Gamma_{ij}^{k}$ of the connection in the frame $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$. For the sake of shortness, we give the expression of just one symbol.

PROPOSITION 5.4. *Let $(v_1,v_2,v_3)$ be a frame on $(S,g)$ and*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

*be the change frame field from $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ to $(v_1,v_2,v_3)$. Then*

$$\begin{aligned}
\Gamma_{11}^{2'} = \\
&\frac{1}{\det A}(a_{31}a_{23} - a_{21}a_{33})\Big(a_{11}\frac{\partial a_{11}}{\partial x} + (a_{11})^2\Gamma_{11}^1 + 2a_{11}a_{21}\Gamma_{12}^1 + 2a_{11}a_{31}\Gamma_{13}^1 + a_{21}\frac{\partial a_{11}}{\partial y} \\
&+(a_{21})^2\Gamma_{22}^1 + 2a_{21}a_{31}\Gamma_{23}^1 + a_{31}\frac{\partial a_{11}}{\partial z} + (a_{31})^2\Gamma_{33}^1\Big) + \frac{1}{\det A}(a_{11}a_{33} - a_{13}a_{31})\Big((a_{11})^2\Gamma_{11}^2 \\
&+a_{11}\frac{\partial a_{21}}{\partial x} + 2a_{11}a_{21}\Gamma_{12}^2 + 2a_{11}a_{31}\Gamma_{31}^2 + a_{21}\frac{\partial a_{21}}{\partial y} + (a_{21})^2\Gamma_{22}^2 + 2a_{21}a_{31}\Gamma_{23}^2 + a_{31}\frac{\partial a_{21}}{\partial z} \\
&+(a_{31})^2\Gamma_{33}^2 + \frac{1}{\det A}(a_{11}a_{23} - a_{13}a_{21})\Big((a_{11})^2\Gamma_{11}^3 + a_{11}\frac{\partial a_{31}}{\partial x} + 2a_{11}a_{21}\Gamma_{12}^3 + 2a_{11}a_{31}\Gamma_{13}^3 \\
&+a_{21}\frac{\partial a_{31}}{\partial y} + (a_{21})^2\Gamma_{32}^2 + 2a_{21}a_{31}\Gamma_{23}^3 + a_{31}\frac{\partial a_{31}}{\partial z} + (a_{31})^2\Gamma_{33}^3\Big)
\end{aligned}$$

(a) Original video (left) / Anisotropic Clifford-Hodge flow (right) for $\beta = 1$

(b) Original video (left) / Anisotropic Clifford-Hodge flow (right) for $\beta = 10$

FIG. 5.1. *Clifford-Hodge flow of functions*

**5.3. Experiments.** In dimension 3, the Clifford-Hodge flow on functions may be devoted to anisotropic regularization of nD videos. The process is the 3D counterpart of the 2D process devoted to anisotropic regularization of nD images mentionned above . Let $I = (I_1, \cdots, I_n)$ be a nD video. It induces a Riemannian manifold $(S, g)$ of dimension 3. Then we consider each component $I_i$ as a section of the Clifford bundle $Cl(S, g)$ of degree 0, and apply the Clifford-Hodge flow on each one of them. Under the identification between sections of $Cl(S, g)$ of degree 0 and functions on $S$, we get $n$ PDE's

$$\frac{\partial I_i}{\partial t} + \Delta I_i = 0$$

where $\Delta$ is the scalar Laplacian on $C^\infty(S, g)$.

In this application, we aim at showing the role of the parameter $\beta$ of (5.5) in the regularization process. By construction of the Riemannian manifold $(S, g)$ , $\beta$ controls the norms of tangent vector fields in the direction $\partial/\partial z$. More precisely, it controls the measure of video's variations in the temporal direction.

Fig. 5.1. is an illustration of anisotropic regularizations of a color video (n=3) induced by the Clifford-Hodge flow. Fig. 5.1(a) (left) and Fig. 5.1(b) (left) is a capture

of a video taken in the DynTex database [13]. Fig. 5.1(a) (right) and Fig. 5.1(b) (right) are corresponding captures of anisotropic regularizations of the original video. They are computed from a convolution with a 3x3x3 mask discretizing the kernel $K_t^1$ (see Sect. 2.3). They result from 3 iterations for $h_1 = h_2 = h_3 = 0.01$, $t = 0.3$ and $\lambda = 1$, where the metric $g$ is updated at each iteration. They differ from the parameter $\beta$, where $\beta = 1$ on Fig. 5.1(a) (right) and $\beta = 10$ on Fig. 5.1(b) (right).

As mentionned above, the more $\beta$ is taken high, the more variations of the video in the temporal direction are considered as high. From the property of the Beltrami flow of diffusing in the direction of edges, we conclude that the more $\beta$ is taken high, the more the diffusion process behaves as a frame by frame diffusion. This is illustrated on Fig. 5.1. where some details of the original video have been removed (see inside the red ellipses) by the diffusion process for $\beta = 1$, conversely only smoothed for $\beta = 10$.

**6. Conclusion.** In this paper, we have proposed a new application of Clifford algebras to image processing. We have shown that it provides a common framework to regularize images (videos), vector fields and oriented orthonormal frame fields. As we were concerned with anisotropic regularization, we introduced Clifford bundles of Riemannian manifolds. Then images (videos), vector fields and generators of orthonormal frame fields may be viewed as sections of such vector bundles, respectively of degree 0,1,2. Regularization process arises from the theory of heat equations associated to generalized Laplacians on vector bundles over Riemannian manifolds, where the solution is given by the convolution of the initial condition with a corresponding heat kernel. We have considered the Clifford-Hodge operator on Clifford bundles. This generalized Laplacian has the property of preserving the degree of sections, and consequently so does the corresponding flow. It generalizes the Laplace-Beltrami operator in the sense that the Clifford-Hodge, restricted to functions, is minus the Laplace-Beltrami.
For the applications we had in mind to image and video processing, we were concerned with base manifolds of dimension 2 and 3. We have computed transport parallel map and transformation formulae of Levi-Cevita connection's symbols under the frame change from the holonomic frame to an orthonormal frame, needed to compute an approximate solution of the heat equation. Applications to color images (videos), 2D vector fields and orientation fields regularization were shown. By the application on 2D vector fields, it appears that the Clifford-Hodge flow possesses the same properties as the Beltrami flow it extends, i.e. it behaves as a gaussian diffusion on homogeneous regions and tends to diffuse only in the direction of edges on regions of high edges. Thus, we may apply the Clifford-Hodge flow of 3D vector fields to optical flow regularization. Applications of the Clifford-Hodge flow of 3D oriented orthonormal frame fields to DTI dataset regularization, for a well-chosen Riemannian manifold of dimension 3, may also be envisaged. These last two points will be addressed in further works.

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