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CUTWIDTH AND DEGENERACY OF GRAPHS

by

Benoît Kloeckner

Abstract. — We prove an inequality involving the degeneracy, the cutwidth and the sparsity of graphs. It implies a quadratic lower bound on the cutwidth in terms of the degeneracy for all graphs and an improvement of it for clique-free graphs.

1. Introduction

The starting point of the author’s interest in cutwidth is a lecture by Misha Gromov at the “Glimpses of Geometry” conference held in Lyon in mai 2008. During this lecture, Gromov introduced a concept similar to cutwidth in the realm of differential topology. Although this paper will stick to combinatorics of graphs, let us give an idea of this important motivation to this work.

Gromov’s question is the following: given a manifold X and a continuous map $F : X \rightarrow \mathbb{R}$, how can one relate the topological complexity of X to the maximum topological complexity of the level sets of F ? It turns out that the answer depends heavily on the dimension of X . If X is an orientable surface of arbitrarily high genus, it is always possible to design F so that its levels are a point, a circle, a couple of circles, a figure eight or empty (see figure 1). The complexity of X is therefore not bounded by that of level sets of F . The picture gets different in some higher dimension but it is not our purpose to detail this here.

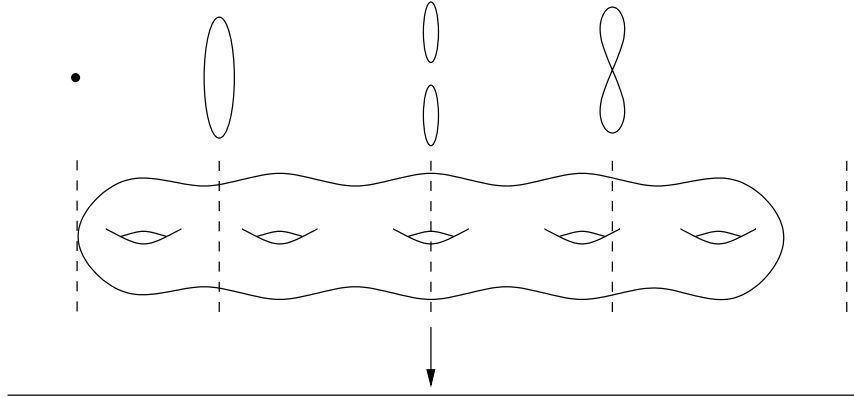


FIGURE 1. A map from a surface of arbitrarily high genus to the line, with level sets of bounded complexity.

2. Some graph invariants

2.1. Cutwidth. — The complexity of a level set makes sense in graphs: if a simple graph G is identified with its topological realization, one can consider the infimum, over all continuous maps $f : G \rightarrow \mathbb{R}$, of the maximal multiplicity (that is, number of inverse images) of points of \mathbb{R} . This turns out to be exactly the *cutwidth* of G , usually defined as follows.

Given a linear ordering $\mathcal{O} = (x_1 < \dots < x_n)$ of the vertices, one defines the cutwidth of the ordering as

$$\text{cw}(G, \mathcal{O}) = \max_{1 \leq i \leq n} \#\{(uv) \in E \mid u \leq x_i < v\}$$

Then, the cutwidth of G is defined as

$$\text{cw}(G) = \min_{\mathcal{O}} \text{cw}(G, \mathcal{O})$$

where \mathcal{O} runs over all linear orderings of V .

Now, given an ordering \mathcal{O} of V that is optimal with respect to cutwidth, an optimal map is obtained by mapping each vertex to the integer corresponding to its rank, and mapping edges monotonically between their endpoints. Its maximal multiplicity is then the cutwidth of the ordering. Conversely, it can be shown that any continuous map $G \rightarrow \mathbb{R}$ can be modified into a map that is one-to-one on V and monotonic on edges without raising its maximal multiplicity.

2.2. Degeneracy. — What we need next is to define the complexity of a graph. There are many invariant that can play this role; here we use the *degeneracy*, defined as follows.

Given an integer k , the k -core G_k of G is the subgraph obtained by recursively pruning the vertices of degree strictly less than k . The degeneracy $\delta(G)$ of the graph is the largest k such that G_k is not empty. It is to be understood that a graph with big degeneracy is in some sense thick.

One of the features of degeneracy is that it is an upper bound for the chromatic (and even list-chromatic) number:

$$\chi(G) \leq \chi_\ell(G) \leq \delta(G) + 1.$$

The proof of this is classical: if the graph G' obtained by pruning any one vertex of degree at most $\delta(G)$ is colorable by $\delta(G)+1$ colors, so is G : since the pruned vertex has at most $\delta(G)$ neighbors, any $(\delta(G)+1)$ -coloring of G' can be completed into a coloring of G with the same number of color. By induction, since the $(\delta(G)+1)$ -core is empty and therefore colorable with any number of color, we get $\chi(G) \leq \delta(G)+1$. The same reasoning obviously applies to the list chromatic number.

For each inequality below, we therefore get an inequality involving chromatic number that shall not be stated explicitly. One of our first goal was in fact to prove an optimal bound on chromatic number in terms of cutwidth, but it turned out that this is rather trivial although not quoted in the litterature we found.

2.3. Sparsity. — In order to get a more interesting inequality, we need to involve another invariant of graphs. We shall use a uniform variant of sparsity, which can be controlled for clique-free graphs, and will enable us to deduce some kind of expanding property for G .

A graph G on n vertices is said to be λ -sparse (where $\lambda \geq 1$) if it as at most $n(n-1)/(2\lambda)$ edges. We shall say that G is (ρ, λ) -uniformly sparse if all subgraphs of G that contain at least ρn vertices are λ -sparse.

Note that we cannot ask for sparsity of all subgraphs of G , since a subgraph consisting of two adjacent vertices is not λ -sparse for any $\lambda > 1$.

3. A quadratic inequality

3.1. The main result. — We start by a general lower bound on the cutwidth of a graph in terms of its degeneracy and uniform sparsity.

Theorem 3.1. — For all (ρ, λ) -uniformly sparse graph G on n vertices we have

$$(1) \quad \text{cw}(G) \geq \lfloor \rho n \rfloor \left(\delta(G) - \frac{\lfloor \rho n \rfloor - 1}{\lambda} \right).$$

Moreover if $2n\rho \leq \delta(G)\lambda - 1$ then

$$(2) \quad \text{cw}(G) > \frac{(\delta(G)\lambda + 1)^2}{4\lambda} - \frac{1}{\lambda}.$$

It may seem strange that in (2) the cutwidth is bounded from below by an *increasing* function of the sparsity; this simply translates the fact that when (uniform) sparsity increases, the degeneracy decreases more than the cutwidth.

Note that in some class of graph, the latitude in the choice of ρ enables to get a bound that is quadratic in δ from (1) too. As a matter of fact, (2) is simply an optimization of (1) when ρ can be taken small enough. Since it can be difficult to prove (ρ, λ) -sparsity with good constants, it is not obvious that there is a need for such a general statement. Our main reason to state it is Corollaries 4.1 and 4.3 on clique-free graphs.

Proof. — We consider a simple graph G on n vertices that is assumed to be (ρ, λ) -uniformly sparse. Let G' be the $\delta(G)$ -core of G : its minimal degree is $\delta(G)$ and since it is a subgraph of G , $\text{cw}(G) \geq \text{cw}(G')$.

Let $\mathcal{O} = (x_1 < \dots < x_{n'})$ be a linear ordering of the vertices of G' that minimizes $\text{cw}(G', \mathcal{O})$. For all i let $n_i = \#\{(uv) \in E' \mid u \leq x_i < v\}$ and denote by $G(i)$ the subgraph of G' consisting of the vertices $\{x_1, \dots, x_i\}$. By assumption, for all $i \geq \rho n$, the graph $G(i)$ has at most $i(i-1)/(2\lambda)$ edges. The total sum of the degrees in G' of the vertices of $G(i)$ is at least $i\delta(G)$, so that

$$n_i \geq i\delta(G) - \frac{i^2 - i}{\lambda}.$$

If $2n\rho \leq \delta(G)\lambda - 1$, we can evaluate this inequality at the optimal point $i = \lfloor (\delta(G)\lambda + 1)/2 \rfloor$ since it satisfies $i \geq \rho n$. Letting $\varepsilon = (\delta(G)\lambda + 1)/2 -$

$\lfloor (\delta(G)\lambda + 1)/2 \rfloor$ we then get

$$\begin{aligned} \text{cw}(G) &\geq \left(\frac{\delta(G)\lambda + 1}{2} - \varepsilon \right) \left(\delta(G) - \frac{\left(\frac{\delta(G)\lambda + 1}{2} - \varepsilon \right) - 1}{\lambda} \right) \\ &\geq \frac{\delta(G)\lambda + 1 - 2\varepsilon}{2} \left(\frac{\delta(G)\lambda + 2\varepsilon + 1}{2\lambda} \right) \\ &= \frac{(\delta(G)\lambda + 1)^2}{4\lambda} - \frac{\varepsilon^2}{\lambda} \end{aligned}$$

which gives the desired inequality since $\varepsilon < 1$.

If $2n\rho > \delta(G)\lambda - 1$, we can nevertheless consider the point $i = \lfloor \rho n \rfloor$ and get (1). \square

3.2. Application to general graphs. — If we let down any information on G , we get the following.

Corollary 3.2. — *For all simple graph, we have*

$$(3) \quad \text{cw}(G) \geq \frac{1}{4}\delta(G)^2 + \frac{1}{2}\delta(G).$$

Proof. — Since every graph is $(0, 1)$ -uniformly sparse, from (2) we deduce that $\text{cw}(G) > (\delta(G) + 1)^2/4 - 1$. If $\delta(G)$ is odd, then it follows that $\text{cw}(G) \geq (\delta(G) + 1)^2/4$ but otherwise, writing $\delta(G) = 2k$ we see that $\text{cw}(G) > k^2 + k - 3/4$ so that $\text{cw}(G) \geq k^2 + k$. \square

This is a positive answer to our version of Gromov's question: all continuous maps from a high-complexity graph to the line have high multiplicity. Of course, in many cases this estimate is rather poor: for example trees have degeneracy 1 and unbounded cutwidth (so that there is no lower bound of $\delta(G)$ in terms of $\text{cw}(G)$) and hypercubes have exponential cutwidth but linear degeneracy. However it is sharp for complete graphs, and a better bound would have to involve much more information on the graph.

As pointed out to me by professors Raspaud and Gravier, the consequence in terms of chromatic number is in fact easy to prove directly: consider an optimal coloring as a morphism $G \rightarrow K_{\chi(G)}$ (which must be onto the edge set), and use that $\text{cw}(K_n) = \lfloor n^2/4 \rfloor$. However Corollary 3.2 is stronger in the sense that it applies to the degeneracy, and in particular implies a bound on list chromatic number.

4. The case of clique-free graphs

We shall deduce the following from Theorem 3.1.

Corollary 4.1. — *For all simple graph G without triangle, one has*

$$(4) \quad \text{cw}(G) \geq \frac{1}{2}\delta(G)^2.$$

The main point is to show that triangle-free graphs are somewhat sparse.

Proposition 4.2. — *Any graph without triangle on n vertices is $2\frac{n-1}{n}$ -sparse.*

Note that this estimate is sharp, as shown by the balanced complete bipartite graphs $K_{m,m}$. This last result is not new at all, since it is a particular case of Turán's Theorem [3], see for example [1]. However, we provide a simple proof for the sake of completeness.

Proof. — Let $G = (V, E)$ be a simple graph with n vertices such that

$$\#E > \frac{n^2}{4}$$

and let us prove that it contains a triangle. We define the degree of an edge as the sum of the degrees of its endpoints: $d(xy) := d(x) + d(y)$. Using the convexity of $x \mapsto x^2$, we get

$$\sum_{(xy) \in E} d(xy) = \sum_{x \in V} d(x)^2 \geq \frac{1}{n} \left(\sum_{x \in V} d(x) \right)^2 = \frac{4(\#E)^2}{n}$$

It follows that the mean value of $d(xy)$ is at least $4\#E/n$. Using the density assumption, this gives that there is an edge (ab) whose degree is strictly larger than n . By the pigeon-hole principle, there is at least one vertex c in G that is connected both to a and b , so that G contains a triangle abc . \square

Proof of Corollary 4.1. — All subgraphs of G are triangle-free, so by the preceding proposition G is $(\rho, 2(\rho n - 1)/(\rho n))$ -uniformly sparse for all ρ such that ρn is an integer. Applying the second part of the main theorem,

we get that

$$\begin{aligned} \text{cw}(G) &\geq \rho n \left(\delta(G) - \frac{\rho n - 1}{2 \frac{\rho n - 1}{\rho n}} \right) \\ &\geq \rho n \left(\delta(G) - \frac{1}{2} \rho n \right) \end{aligned}$$

and taking $\rho = \delta/n$, the desired inequality follows. \square

With the same argument, one can deduce the following from Turán's Theorem.

Corollary 4.3. — *For all simple graph G without subgraph isomorphic to K_{k+1} , one has*

$$(5) \quad \text{cw}(G) \geq \frac{k}{k-1} \frac{\delta(G)^2}{4} - \frac{k-1}{k}.$$

Let us show that this gives an asymptotically sharp result for Turán's graph $T(n, k)$, defined as the most balanced complete k -partite graph on n vertices. On the one hand Corollary 4.3 gives

$$(6) \quad \text{cw}(T(n, k)) \geq \frac{k-1}{k} \frac{n^2}{4} - \frac{n}{2} - \frac{3k}{4(k-1)}$$

but on the other hand, one can give an explicit ordering of vertices of $T(n, k)$ with cutwidth of the same order of magnitude. Indeed, if one considers the graph whose vertices are the integers $\{1, 2, \dots, n\}$ and where two vertices a, b are connected by an edge if and only if $a \not\equiv b \pmod{k}$, it is isomorphic to $T(n, k)$ and endowed with a natural ordering. The number of edges that cross the vertex i and are issued from any fixed vertex $j < i$ is at most

$$n - i - \left\lceil \frac{n - i}{k} \right\rceil \leq \frac{k-1}{k} (n - i) + 1$$

so that the total number of edges crossing i is at most

$$c(i) = i \left((n - i) \frac{k-1}{k} + 1 \right).$$

Now the function c takes its maximal value at $x = (n + k/(k - 1))/2$ so that we get

$$(7) \quad \text{cw}(T(n, k)) \leq \frac{k-1}{k} \frac{n^2}{4} + \frac{n}{2} + \frac{k}{4(k-1)}.$$

As a final remark, let us observe that the previous approach applies to all solved forbidden subgraph extremal problems, for example to $T(rt, r)$ -free graphs [2], see also [1, Theorem VI.3.1].

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