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Construction and characterization of solutions converging to solitons for supercritical gKdV equations

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Abstract

We consider the generalized Korteweg-de Vries equation

$$\partial_t u + \partial_x^3 u + \partial_x(u^p) = 0, \quad (t, x) \in \mathbb{R}^2,$$

in the supercritical case $p > 5$, and we are interested in solutions which converge to a soliton in large time in H^1 . In the subcritical case ($p < 5$), such solutions are forced to be exactly solitons by variational characterization [1, 19], but no such result exists in the supercritical case. In this paper, we first construct a "special solution" in this case by a compactness argument, i.e. a solution which converges to a soliton without being a soliton. Secondly, using a description of the spectrum of the linearized operator around a soliton [17], we construct a one parameter family of special solutions which characterizes all such special solutions.

1 Introduction

1.1 The generalized Korteweg-de Vries equation

We consider the generalized Korteweg-de Vries equation:

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^p) = 0 \\ u(0) = u_0 \in H^1(\mathbb{R}) \end{cases} \quad (\text{gKdV})$$

where $(t, x) \in \mathbb{R}^2$ and $p \geq 2$ is integer. The following quantities are formally conserved for solutions of (gKdV):

$$\int u^2(t) = \int u^2(0) \quad (\text{mass}), \quad (1.1)$$

$$E(u(t)) = \frac{1}{2} \int u_x^2(t) - \frac{1}{p+1} \int u^{p+1}(t) = E(u(0)) \quad (\text{energy}). \quad (1.2)$$

Kenig, Ponce and Vega [9] have shown that the local Cauchy problem for (gKdV) is well posed in $H^1(\mathbb{R})$: for $u_0 \in H^1(\mathbb{R})$, there exist $T > 0$ and a solution $u \in C^0([0, T], H^1(\mathbb{R}))$ of (gKdV) satisfying $u(0) = u_0$ which is unique in some class $Y_T \subset C^0([0, T], H^1(\mathbb{R}))$. Moreover, if $T^* \geq T$ is the maximal time of existence of u , then either $T^* = +\infty$ which means that $u(t)$ is a global solution, or $T^* < +\infty$ and then $\|u(t)\|_{H^1} \rightarrow +\infty$ as $t \uparrow T^*$ ($u(t)$ is a finite time blow up solution). Throughout this paper, when referring to an H^1 solution of (gKdV), we mean a solution in the above sense. Finally, if $u_0 \in H^s(\mathbb{R})$ for some $s \geq 1$, then $u(t) \in H^s(\mathbb{R})$ for all $t \in [0, T)$.

In the case where $2 \leq p < 5$, it is standard that all solutions in H^1 are global and uniformly bounded by the energy and mass conservations and the following Gagliardo-Nirenberg inequality:

$$\forall v \in H^1(\mathbb{R}), \quad \int |v|^{p+1} \leq C_{\text{GN}}(p) \left(\int v_x^2 \right)^{\frac{p-1}{4}} \left(\int v^2 \right)^{\frac{p+3}{4}} \quad (1.3)$$

with optimal constant $C_{\text{GN}}(p) > 0$. In the case $p = 5$, the existence of finite time blow up solutions was proved by Merle [16] and Martel and Merle [13]. Therefore $p = 5$ is the critical exponent for the long time behavior of solutions of (gKdV). For $p > 5$, the existence of blow up solutions is an open problem.

We recall that a fundamental property of equations (gKdV) is the existence of a family of explicit traveling wave solutions. Let Q be the only solution (up to translations) of

$$Q > 0, \quad Q \in H^1(\mathbb{R}), \quad Q'' + Q^p = Q, \quad \text{i.e. } Q(x) = \left(\frac{p+1}{2 \cosh^2\left(\frac{p-1}{2}x\right)} \right)^{\frac{1}{p-1}}.$$

Note that Q is the unique minimizer of the Gagliardo-Nirenberg inequality (1.3) (see [2] for the case $p = 5$ for example), i.e. for $v \in H^1(\mathbb{R})$:

$$\|v\|_{L^{p+1}}^{p+1} = C_{\text{GN}}(p) \|v_x\|_{L^2}^{\frac{p-1}{2}} \|v\|_{L^2}^{\frac{p+3}{2}} \iff \exists(\lambda_0, a_0, b_0) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R} : v(x) = a_0 Q(\lambda_0 x + b_0). \quad (1.4)$$

For all $c_0 > 0$ and $x_0 \in \mathbb{R}$, $R_{c_0, x_0}(t, x) = Q_{c_0}(x - x_0 - c_0 t)$ is a solution of (gKdV), where

$$Q_{c_0}(x) = c_0^{\frac{1}{p-1}} Q(\sqrt{c_0}x).$$

We call solitons these solutions though they are known to be solitons only for $p = 2, 3$ (in the sense that they are stable by interaction).

It is well known that solitons are orbitally stable (see definition 2.7) for $p < 5$ and unstable for $p > 5$. An important fact used by Weinstein in [19] to prove their orbital stability when $p < 5$ is the following variational characterization of Q_{c_0} : if u is a solution of (gKdV) such that $E(u) = E(Q_{c_0})$ and $\int u^2 = \int Q_{c_0}^2$ for some $c_0 > 0$, then there exists $x_0 \in \mathbb{R}$ such that $u = R_{c_0, x_0}$. As a direct consequence, if now $u(t)$ is a solution such that

$$\liminf_{t \rightarrow +\infty} \inf_{y \in \mathbb{R}} \|u(t) - Q_{c_0}(\cdot - y)\|_{H^1(\mathbb{R})} = 0 \quad (1.5)$$

(i.e. u converges to Q_{c_0} in the suitable sense), then $u = R_{c_0, x_0}$. For $p = 5$, the same is true for similar reasons (see [20]).

In the present paper, we focus on the supercritical case $p > 5$. Some asymptotic results around solitons have been proved: orbital instability of solitons by Bona et al. [1] (see also [7]) and asymptotic stability (in some sense) by Martel and Merle [15] for example. But available variational arguments do not allow to classify all solutions of (gKdV) satisfying (1.5). In fact, in section 3, we construct a solution of (gKdV) satisfying (1.5) which is not a soliton (we call special solution such a solution). In section 4, by another method, we construct a whole family of such solutions, and we completely characterize solutions satisfying (1.5). This method is strongly inspired of arguments developed by Duyckaerts and Roudenko in [5], themselves an adaptation of arguments developed by Duyckaerts and Merle in [4]. For reader's convenience, we recall in the next section the results in [5] related to our paper.

1.2 The Non-Linear Schrödinger equation case

We recall Duyckaerts and Roudenko's results for (NLS). They consider in [5] the 3d focusing cubic non-linear Schrödinger equation:

$$\begin{cases} i\partial_t u + \Delta u + |u|^2 u = 0, & (x, t) \in \mathbb{R}^3 \times \mathbb{R}, \\ u|_{t=0} = u_0 \in H^1(\mathbb{R}^3). \end{cases} \quad (\text{NLS})$$

This equation is $\dot{H}^{1/2}$ -critical, and so L^2 -supercritical like (gKdV) for $p > 5$, while [4] is devoted to the \dot{H}^1 -critical equation. Similarly to (gKdV), (NLS) is locally well posed in H^1 , and solutions

of (NLS) satisfy the following conservation laws:

$$E_{\text{NLS}}[u](t) = \frac{1}{2} \int |\nabla u(x, t)|^2 dx - \frac{1}{4} \int |u(x, t)|^4 dx = E_{\text{NLS}}[u](0),$$

$$M_{\text{NLS}}[u](t) = \int |u(x, t)|^2 dx = M_{\text{NLS}}[u](0).$$

Moreover, if Q is the unique (in a suitable sense) solution of the non-linear elliptic equation $-Q + \Delta Q + |Q|^2 Q = 0$, then $e^{it}Q(x)$ is a soliton solution of (NLS).

Theorem 2 in [5] states the existence of two radial solutions $Q^+(t)$ and $Q^-(t)$ of (NLS) such that $M_{\text{NLS}}[Q^+] = M_{\text{NLS}}[Q^-] = M_{\text{NLS}}[Q]$, $E_{\text{NLS}}[Q^+] = E_{\text{NLS}}[Q^-] = E_{\text{NLS}}[Q]$, $[0, +\infty)$ is in the time domain of definition of $Q^\pm(t)$, and there exists $e_0 > 0$ such that: $\forall t \geq 0$, $\|Q^\pm(t) - e^{it}Q\|_{H^1} \leq Ce^{-e_0 t}$. Moreover, $Q^-(t)$ is globally defined and scatters for negative time, and the negative time of existence of $Q^+(t)$ is finite.

They also prove the following classification theorem [5, theorem 3]:

Theorem ([5]). Let u be a solution of (NLS) satisfying $E_{\text{NLS}}[u]M_{\text{NLS}}[u] = E_{\text{NLS}}[Q]M_{\text{NLS}}[Q]$.

- (a) If $\|\nabla u_0\|_{L^2}\|u_0\|_{L^2} < \|\nabla Q\|_{L^2}\|Q\|_{L^2}$, then either u scatters or $u = Q^-$ up to the symmetries.
- (b) If $\|\nabla u_0\|_{L^2}\|u_0\|_{L^2} = \|\nabla Q\|_{L^2}\|Q\|_{L^2}$, then $u = e^{it}Q$ up to the symmetries.
- (c) If $\|\nabla u_0\|_{L^2}\|u_0\|_{L^2} > \|\nabla Q\|_{L^2}\|Q\|_{L^2}$ and u_0 is radial or of finite variance, then either the interval of existence of u is of finite length or $u = Q^+$ up to the symmetries.

In particular, if $\lim_{t \rightarrow +\infty} \|u(t) - e^{it}Q\|_{H^1} = 0$, then $u = e^{it}Q$, Q^+ or Q^- up to the symmetries.

Among the various ingredients used to prove results above, one of the most important is a sharp analysis of the spectrum $\sigma(\mathcal{L}_{\text{NLS}})$ of the linearized Schrödinger operator around the ground state solution $e^{it}Q$, due to Grillakis [6] and Weinstein [18]. They prove that $\sigma(\mathcal{L}_{\text{NLS}}) \cap \mathbb{R} = \{-e_0, 0, +e_0\}$ with $e_0 > 0$, and moreover that e_0 and $-e_0$ are simple eigenvalues of \mathcal{L}_{NLS} with eigenfunctions $\mathcal{Y}_+^{\text{NLS}}$ and $\mathcal{Y}_-^{\text{NLS}} = \overline{\mathcal{Y}_+^{\text{NLS}}}$. This structure, which is similar for (gKdV) according to Pego and Weinstein [17], will also be crucial to prove our main result (exposed in the next section).

1.3 Main result and outline of the paper

In this paper, we consider similar questions for the (gKdV) equation in the supercritical case $p > 5$. Recall that similarly to the (NLS) case, Pego and Weinstein have determined in [17] the spectrum of the linearized operator \mathcal{L} around the soliton $Q(x - t)$: $\sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, +e_0\}$ with $e_0 > 0$, and moreover e_0 and $-e_0$ are simple eigenvalues of \mathcal{L} with eigenfunctions \mathcal{Y}_+ and \mathcal{Y}_- which are exponentially decaying (see proposition 4.2 and corollary 4.4). We now state precisely our main result:

Theorem 1.1. Let $p > 5$.

1. (Existence of a family of special solutions). There exists a one-parameter family $(U^A)_{A \in \mathbb{R}}$ of solutions of (gKdV) such that

$$\lim_{t \rightarrow +\infty} \|U^A(t, \cdot + t) - Q\|_{H^1} = 0.$$

Moreover, for all $A \in \mathbb{R}$, there exists $t_0 = t_0(A) \in \mathbb{R}$ such that for all $s \in \mathbb{R}$, there exists $C > 0$ such that

$$\forall t \geq t_0, \quad \|U^A(t, \cdot + t) - Q - Ae^{-e_0 t} \mathcal{Y}_+\|_{H^s} \leq Ce^{-2e_0 t}.$$

2. (Classification of special solutions). If u is a solution of (gKdV) such that

$$\lim_{t \rightarrow +\infty} \inf_{y \in \mathbb{R}} \|u(t) - Q(\cdot - y)\|_{H^1} = 0,$$

then there exist $A \in \mathbb{R}$, $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $u(t) = U^A(t, \cdot - x_0)$ for $t \geq t_0$.

Remark 1.2. From theorem 1.1, there are actually only three different special solutions U^A up to translations in time and in space: U^1 , U^{-1} and $Q(\cdot - t)$ (see proposition 4.12). This is of course related to the three solutions of (NLS) constructed in [5]: $Q^+(t)$, $Q^-(t)$ and $e^{it}Q$.

From section 4.5, we can chose the normalization of \mathcal{Y}_\pm so that for $A < 0$, $\|\partial_x U^A\|_{L^2} < \|Q'\|_{L^2}$. Then $U^{-1}(t)$ is global, i.e. defined for all $t \in \mathbb{R}$. It would be interesting to investigate in more details its behavior as $t \rightarrow -\infty$. On the other hand, the behavior of $U^1(t)$ is not known for $t < t_0$.

Remark 1.3. By scaling, theorem 1.1 extends to Q_c for all $c > 0$ (see corollary 4.11 at the end of the paper).

The paper is organized as follows. In the next section we recall some properties of the solitons, and in particular we recall the proof of their orbital instability when $p > 5$. This result is well known [1], but our proof with an explicit initial data is useful to introduce some suitable tools to the study of solitons of (gKdV) (as modulation, Weinstein's functional, monotonicity, linearized equation, etc.). Moreover, it is the first step to construct one special solution in section 3 by compactness, similarly as Martel and Merle [15]. This proof does not use the precise analysis of the spectrum of \mathcal{L} due to Pego and Weinstein [17], and so can be hopefully adapted to equations for which the spectrum of the linearized operator is not well known. To fully prove theorem 1.1 (existence and uniqueness of a family of special solutions, section 4), we rely on the method introduced in [4] and [5].

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2 Preliminary results

We recall here some well known properties of the solitons and some results of stability around the solitons. We begin by recalling notation and simple facts on the functions $Q(x)$ and $Q_c(x) = c^{\frac{1}{p-1}}Q(\sqrt{c}x)$ defined in section 1.1.

Notation. They are available in the whole paper.

- (a) (\cdot, \cdot) denotes the $L^2(\mathbb{R})$ scalar product, and \perp the orthogonality with respect to (\cdot, \cdot) .
- (b) The Sobolev space H^s is defined by $H^s(\mathbb{R}) = \{u \in \mathcal{D}'(\mathbb{R}) \mid (1 + \xi^2)^{s/2} \hat{u}(\xi) \in L^2(\mathbb{R})\}$, and in particular $H^1(\mathbb{R}) = \{u \in L^2(\mathbb{R}) \mid \|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|u'\|_{L^2}^2 < +\infty\} \hookrightarrow L^\infty(\mathbb{R})$.
- (c) We denote $\frac{\partial}{\partial x}v = \partial_x v = v_x$ the partial derivative of v with respect to x , and $\partial_x^s = \partial^s$ the s -order partial derivative with respect to x when no confusion is possible.
- (d) All numbers C, K appearing in inequalities are real constants (with respect to the context) strictly positive, which may change in each step of an inequality.

Claim 2.1. For all $c > 0$, one has:

- (i) $Q_c > 0$, Q_c is even, Q_c is C^∞ , and $Q'_c(x) < 0$ for all $x > 0$.
- (ii) There exist $K_1, K_2 > 0$ such that: $\forall x \in \mathbb{R}$, $K_1 e^{-\sqrt{c}|x|} \leq Q_c(x) \leq K_2 e^{-\sqrt{c}|x|}$.
- (iii) There exists $C_p > 0$ such that for all $j \geq 0$, $Q_c^{(j)}(x) \sim C_p e^{-\sqrt{c}|x|}$ when $|x| \rightarrow +\infty$.
In particular, for all $j \geq 1$, there exists $C_j > 0$ such that: $\forall x \in \mathbb{R}$, $|Q_c^{(j)}(x)| \leq C_j e^{-\sqrt{c}|x|}$.
- (iv) The following identities hold:

$$\int Q_c^2 = c^{\frac{5-p}{2(p-1)}} \int Q^2 \quad , \quad \int (Q'_c)^2 = c^{\frac{p+3}{2(p-1)}} \int Q'^2. \quad (2.1)$$

2.1 Weinstein's functional linearized around Q

We introduce here the Weinstein's functional F and give an expression of $F(Q + a)$ for a small which will be very useful in the rest of the paper. We recall first that the energy of a function $\varphi \in H^1$ is defined by $E(\varphi) = \frac{1}{2} \int (\partial_x \varphi)^2 - \frac{1}{p+1} \int \varphi^{p+1}$.

Definition 2.2. Weinstein's functional is defined for $\varphi \in H^1$ by $F(\varphi) = E(\varphi) + \frac{1}{2} \int \varphi^2$.

Claim 2.3. If $u_0 \in H^1$ and $u(t)$ solves (gKdV) with $u(0) = u_0$, then for all $t \in [0, T^*)$, $F(u(t)) = F(u_0)$. It is an immediate consequence of (1.1) and (1.2).

Lemma 2.4 (Weinstein's functional linearized around Q). For all $C > 0$, there exists $C' > 0$ such that, for all $a \in H^1$ verifying $\|a\|_{H^1} \leq C$,

$$F(Q + a) = F(Q) + \frac{1}{2}(La, a) + K(a) \quad (2.2)$$

where $La = -\partial_x^2 a + a - pQ^{p-1}a$, and $K : H^1 \rightarrow \mathbb{R}$ satisfies $|K(a)| \leq C'\|a\|_{H^1}^3$.

Proof. Let $a \in H^1$ be such that $\|a\|_{H^1} \leq C$. Then we have

$$\begin{aligned} E(Q + a) &= \frac{1}{2} \int (Q' + \partial_x a)^2 - \frac{1}{p+1} \int (Q + a)^{p+1} \\ &= E(Q) + \frac{1}{2} \int (\partial_x a)^2 + \int Q' \cdot \partial_x a - \frac{1}{p+1} \int \left[(p+1)Q^p a + \frac{(p+1)p}{2} Q^{p-1} a^2 + R(a) \right] \\ &= E(Q) + \frac{1}{2} \int (\partial_x a)^2 - \int Qa - \frac{p}{2} \int Q^{p-1} a^2 - \frac{1}{p+1} \int R(a) \end{aligned}$$

since $Q'' + Q^p = Q$, and where $R(a) = \sum_{k=3}^{p+1} \binom{p+1}{k} Q^{p+1-k} a^k$. Since $\|a\|_\infty \leq C\|a\|_{H^1} \leq C$, then $|R(a)| \leq C|a|^3 \leq C\|a\|_\infty |a|^2$, and so $K(a) = -\frac{1}{p+1} \int R(a)$ verifies $|K(a)| \leq C'\|a\|_{H^1}^3$. Moreover, we have more simply: $\int (Q + a)^2 = \int Q^2 + \int a^2 + 2 \int Qa$. Finally we have

$$F(Q + a) = F(Q) + \frac{1}{2} \int a^2 + \frac{1}{2} \int (\partial_x a)^2 - \frac{p}{2} \int Q^{p-1} a^2 + K(a). \quad \square$$

Claim 2.5 (Properties of L). The operator L defined in lemma 2.4 is self-adjoint and satisfies the following properties:

- (i) First eigenfunction: $LQ^{\frac{p+1}{2}} = -\lambda_0 Q^{\frac{p+1}{2}}$ where $\lambda_0 = \frac{1}{4}(p-1)(p+3) > 0$.
- (ii) Second eigenfunction: $LQ' = 0$, and $\ker L = \{\lambda Q' ; \lambda \in \mathbb{R}\}$.
- (iii) Scaling: If we denote $S = \frac{dQ_\varepsilon}{dc} \Big|_{c=1}$, then $S(x) = \frac{1}{p-1}Q(x) + \frac{1}{2}xQ'(x)$ and $LS = -Q$.
- (iv) Coercivity: There exists $\sigma_0 > 0$ such that for all $u \in H^1(\mathbb{R})$ verifying $(u, Q') = (u, Q^{\frac{p+1}{2}}) = 0$, one has $(Lu, u) \geq \sigma_0 \|u\|_{L^2}^2$.

Proof. The first three properties follow from straightforward computation, except for $\ker L$ which can be determined by ODE techniques, see [18, proposition 2.8]. The property of coercivity follows easily from (i), (ii) and classical results on self-adjoint operators and Sturm-Liouville theory. \square

Lemma 2.6. There exist $K_1, K_2 > 0$ such that for all $\varepsilon \in H^1$ verifying $\varepsilon \perp Q'$:

$$(L\varepsilon, \varepsilon) = \int \varepsilon_x^2 + \int \varepsilon^2 - p \int Q^{p-1} \varepsilon^2 \geq K_1 \|\varepsilon\|_{H^1}^2 - K_2 \left(\int \varepsilon Q^{\frac{p+1}{2}} \right)^2.$$

Proof. By claim 2.5, we already know that there exists $\sigma_0 > 0$ such that for all ε satisfying $\varepsilon \perp Q^{\frac{p+1}{2}}$ and $\varepsilon \perp Q'$, we have $(L\varepsilon, \varepsilon) \geq \sigma_0 \|\varepsilon\|_{L^2}^2$. The first step is to replace the L^2 norm by the H^1 one in this last inequality, which is easy if we choose σ_0 small enough. If we do not suppose $\varepsilon \perp Q^{\frac{p+1}{2}}$, we write $\varepsilon = \varepsilon_1 + aQ^{\frac{p+1}{2}}$ with $a = (\int \varepsilon Q^{\frac{p+1}{2}}) (\int Q^{p+1})^{-1}$ such that $\varepsilon_1 \perp Q^{\frac{p+1}{2}}$ for the L^2 scalar product, but also for the bilinear form $(L\cdot, \cdot)$ since $Q^{\frac{p+1}{2}}$ is an eigenvector for L . Since $Q^{\frac{p+1}{2}} \perp Q'$, we obtain easily the desired inequality from the previous step. \square

2.2 Orbital stability and decomposition of a solution around Q

In this paper, we consider only solutions which stay close to a soliton. So it is important to define properly this notion, and the invariance by translation leads us to consider for $\varepsilon > 0$ the "tube"

$$U_\varepsilon = \{u \in H^1 \mid \inf_{y \in \mathbb{R}} \|u - Q_c(\cdot - y)\|_{H^1} \leq \varepsilon\}.$$

Definition 2.7. The solitary wave Q_c is (orbitally) stable if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $u_0 \in U_\delta$, then the associated solution $u(t) \in U_\varepsilon$ for all $t \in \mathbb{R}$. The solitary wave Q_c is unstable if Q_c is not stable.

Theorem 2.8. Q_c is stable if and only if $p < 5$.

Remark 2.9. 1. This theorem is proved by Bona et al. [1] for $p \neq 5$ and by Martel and Merle [12] for $p = 5$. Nevertheless, we give an explicit proof of the instability of Q when $p > 5$ (i.e. we exhibit an explicit sequence of initial data which contradicts the stability) which will be useful to construct the special solution by the compactness method (section 3).

2. An important ingredient to prove this theorem is the following lemma of modulation close to Q . Its proof is based on the implicit function theorem (see for example [1, lemma 4.1] for details). The orthogonality to Q' obtained by this lemma will be of course useful to exploit the coercivity of the bilinear form $(L \cdot, \cdot)$. Finally, we conclude this section by a simple but useful lemma which describes the effect of small translations on Q .

Lemma 2.10 (Modulation close to Q). There exist $\varepsilon_0 > 0$, $C > 0$ and a unique C^1 map $\alpha : U_{\varepsilon_0} \rightarrow \mathbb{R}$ such that for every $u \in U_{\varepsilon_0}$, $\varepsilon = u(\cdot + \alpha(u)) - Q$ verifies

$$(\varepsilon, Q') = 0 \text{ and } \|\varepsilon\|_{H^1} \leq C \inf_{y \in \mathbb{R}} \|u - Q(\cdot - y)\|_{H^1} \leq C\varepsilon_0.$$

Lemma 2.11. There exist $h_0 > 0$, $A_0 > 0$ and $\beta > 0$ such that:

- (i) if $|h| \leq h_0$ then $\beta h^2 \leq \|Q - Q(\cdot + h)\|_{H^1}^2 \leq 4\beta h^2$,
- (ii) if $|h| > h_0$ then $\|Q - Q(\cdot + h)\|_{H^1}^2 > A_0$.

Proof. It is a simple application of Taylor's theorem to f defined by $f(a) = \|Q - Q(\cdot + a)\|_{H^1}^2$. \square

2.3 Instability of Q for $p > 5$

In this section, we construct an explicit sequence $(u_{0,n})_{n \geq 1}$ of initial data which contradicts the stability of Q :

Proposition 2.12. Let $u_{0,n}(x) = \lambda_n Q(\lambda_n^2 x)$ with $\lambda_n = 1 + \frac{1}{n}$ for $n \geq 1$. Then

$$\int u_{0,n}^2 = \int Q^2 \quad , \quad E(u_{0,n}) < E(Q) \quad \text{and} \quad \|u_{0,n} - Q\|_{H^1} \xrightarrow{n \rightarrow \infty} 0. \quad (2.3)$$

Proof. The first and the last facts are obvious thanks to substitutions and the dominated convergence theorem. For the energy inequality, we compute $E(u_{0,n}) = \frac{\lambda_n^4}{2} \int Q^2 - \frac{\lambda_n^{p-1}}{p+1} \int Q^{p+1}$. But $2 \int Q^2 = \frac{p-1}{p+1} \int Q^{p+1}$ by Pohozaev identities, and so

$$\begin{aligned} E(u_{0,n}) - E(Q) &= \left[\frac{p-1}{4} \times (\lambda_n^4 - 1) - (\lambda_n^{p-1} - 1) \right] \cdot \frac{1}{p+1} \int Q^{p+1} \\ &= \left[\sum_{k=2}^4 \left\{ \frac{p-1}{4} \binom{4}{k} - \binom{p-1}{k} \right\} \frac{1}{n^k} - \sum_{k=5}^{p-1} \binom{p-1}{k} \frac{1}{n^k} \right] \cdot \frac{1}{p+1} \int Q^{p+1}. \end{aligned}$$

To conclude, it is enough to show that $\binom{p-1}{k} > \frac{p-1}{4} \binom{4}{k}$ for $k \in \{2, 3, 4\}$, which is equivalent to show that $\binom{p-2}{k-1} = \frac{k}{p-1} \binom{p-1}{k} > \frac{k}{4} \binom{4}{k} = \binom{3}{k-1}$, which is right since $p > 5$ and $k > 1$. \square

Remark 2.13. We do not really need to know the explicit expression of $u_{0,n}$ to prove the instability of Q : initial data satisfying conditions (2.3) and decay in space would fit. For example, we could have chosen $\lambda_n = 1 - \frac{1}{n}$, so that conditions (2.3) hold for n large (in fact $E(u_{0,n}) - E(Q) \sim \frac{(p-1)(5-p)}{2(p+1)} \int Q^{p+1} \cdot \frac{1}{n^2} < 0$ as $n \rightarrow +\infty$ in this case).

Theorem 2.14. Let u_n be the solution associated to $u_{0,n}$ defined in proposition 2.12. Then

$$\exists \delta > 0, \forall n \geq 1, \exists T_n \in \mathbb{R}_+ \text{ such that } \inf_{y \in \mathbb{R}} \|u_n(T_n) - Q(\cdot - y)\|_{H^1} > \delta. \quad (2.4)$$

• We prove this theorem by contradiction, i.e. we suppose:

$$\forall \varepsilon > 0, \exists n_0 \geq 1, \forall t \in \mathbb{R}_+, \inf_{y \in \mathbb{R}} \|u_{n_0}(t) - Q(\cdot - y)\|_{H^1} \leq \varepsilon,$$

and we apply this assumption to ε_0 given by lemma 2.10. Dropping n_0 for a while, the situation amounts in:

$$\int u_0^2 = \int Q^2, \quad E(u_0) < E(Q) \quad \text{and} \quad \forall t \in \mathbb{R}_+, \inf_{y \in \mathbb{R}} \|u(t) - Q(\cdot - y)\|_{H^1} \leq \varepsilon_0.$$

The last fact implies that $u(t) \in U_{\varepsilon_0}$ for all $t \in \mathbb{R}_+$, so lemma 2.10 applies and we can define $x(t) = \alpha(u(t))$ which is C^1 by standard arguments (see [12] for example), and $\varepsilon(t, x) = u(t, x + x(t)) - Q(x)$ which verifies $(\varepsilon(t), Q') = 0$ and $\|\varepsilon(t)\|_{H^1} \leq C\varepsilon_0$ for all $t \in \mathbb{R}_+$. Note that $x(t)$ is usually called the center of mass of $u(t)$. Before continuing the proof, we give the equation verified by ε and an interesting consequence on x' .

Proposition 2.15. There exists $C > 0$ such that

$$\varepsilon_t - (L\varepsilon)_x = (x'(t) - 1)(Q + \varepsilon)_x + R(\varepsilon),$$

where $\|R(\varepsilon(t))\|_{L^1} \leq C\|\varepsilon(t)\|_{H^1}^2$. As a consequence, one has: $|x'(t) - 1| \leq C\|\varepsilon(t)\|_{H^1}$.

Proof. Since $u(t, x) = Q(x - x(t)) + \varepsilon(t, x - x(t))$ by definition of ε and $-\partial_t u = \partial_x^3 u + \partial_x(u^p)$, we obtain

$$x'(t)(Q + \varepsilon)_x - \varepsilon_t = Q_{xxx} + \varepsilon_{xxx} + (Q^p)_x + p(Q^{p-1}\varepsilon)_x + R(\varepsilon)$$

where

$$R(\varepsilon) = \frac{\partial}{\partial x} \left(\sum_{k=2}^p \binom{p}{k} Q^{p-k} \varepsilon^k \right) = \sum_{k=2}^p \binom{p}{k} [(p-k)Q'Q^{p-k-1}\varepsilon^k + kQ^{p-k}\varepsilon_x \varepsilon^{k-1}].$$

As $\|\varepsilon\|_{\infty} \leq C\|\varepsilon\|_{H^1} \leq C\varepsilon_0$, we have $|R(\varepsilon)| \leq C|\varepsilon|^2 + C'|\varepsilon_x \varepsilon|$, and so $R(\varepsilon)$ is such as expected. Moreover, since $La = -a_{xxx} + a - pQ^{p-1}a$ and $Q'' + Q^p = Q$, we get

$$-\varepsilon_t - \varepsilon_{xxx} - p(Q^{p-1}\varepsilon)_x = Q_{xxx} + (Q^p)_x - x'(t)(Q + \varepsilon)_x + R(\varepsilon)$$

and so $-\varepsilon_t + (L\varepsilon)_x = Q_x - x'(t)(Q + \varepsilon)_x + \varepsilon_x + R(\varepsilon)$.

To obtain the estimate on x' , we multiply the equation previously found by Q' and integrate. Since $(\varepsilon_t, Q') = (\varepsilon, Q')_t = 0$, it gives with an integration by parts:

$$\int (L\varepsilon)Q'' = (x' - 1) \int (Q'^2 + \varepsilon_x Q') + \int R(\varepsilon)Q'.$$

Since L is self-adjoint, we can write $(x' - 1) \int (Q'^2 + \varepsilon_x Q') = \int (LQ'')\varepsilon - \int R(\varepsilon)Q'$. Now, from $|\int \varepsilon_x Q'| \leq \|\varepsilon_x\|_{L^2} \|Q'\|_{L^2} \leq \|\varepsilon\|_{H^1} \|Q'\|_{L^2} \leq C\varepsilon_0 \|Q'\|_{L^2}$, we choose ε_0 small enough so that the last quantity is smaller than $\frac{1}{2} \int Q'^2$; and so we have

$$|x' - 1| \leq \frac{2}{\int Q'^2} \left(\left| \int (LQ'')\varepsilon \right| + \left| \int R(\varepsilon)Q' \right| \right).$$

As $LQ'' \in L^2(\mathbb{R})$ and $Q' \in L^\infty(\mathbb{R})$, then following the estimate on $R(\varepsilon)$, we obtain the desired inequality by the Cauchy-Schwarz inequality. \square

- Return to the proof of theorem 2.14 and now consider

$$\zeta(x) = \int_{-\infty}^x \left(S(y) + \beta Q^{\frac{p+1}{2}}(y) \right) dy$$

for $x \in \mathbb{R}$, where S is defined in claim 2.5 and β will be chosen later. We recall that $S(x) = \frac{1}{p-1}Q(x) + \frac{1}{2}xQ'(x)$ verifies $LS = -Q$, and in particular $S(x) = o(e^{-|x|/2})$ when $|x| \rightarrow +\infty$, since $Q(x), Q'(x) \sim C_p e^{-|x|}$ (see claim 2.1). By integration, we have $\zeta(x) = o(e^{x/2})$ when $x \rightarrow -\infty$, and ζ is bounded on \mathbb{R} .

Now, the main idea of the proof is to consider the functional, defined for $t \in \mathbb{R}_+$,

$$J(t) = \int \varepsilon(t, x) \zeta(x) dx.$$

The first step is to show that J is defined and bounded in time thanks to the following proposition of decay properties of the solutions, and the second one is to show that $|J'|$ has a strictly positive lower bound, which will reach the desired contradiction. Firstly, if we choose ε_0 small enough, we obtain the following proposition.

Proposition 2.16. There exists $C > 0$ such that for all $t \geq 0$ and $x_0 > 0$,

$$\int_{x > x_0} (u^2 + u_x^2)(t, x + x(t)) dx \leq C e^{-x_0/4}. \quad (2.5)$$

Remark 2.17. Inequality (2.5) holds for all solution u_n of (gKdV) associated to the initial data $u_{0,n}$ defined in proposition 2.12, with $C > 0$ independent of n . Indeed, we have $u = u_{n_0}$ for some $n_0 \geq 1$, but the following proof shows that the final constant C does not depend of n_0 .

Proof. It is based on the exponential decay of the initial data, and on monotonicity results that the reader can find in [14, lemma 3]. We recall here their notation and their lemma of monotonicity.

- ◊ Let $\psi(x) = \frac{2}{\pi} \arctan(\exp(x/4))$, so that ψ is increasing, $\lim_{-\infty} \psi = 0$, $\psi(0) = \frac{1}{2}$, $\lim_{+\infty} \psi = 1$, $\psi(-x) = 1 - \psi(x)$ for all $x \in \mathbb{R}$, and $\psi(x) \sim C e^{x/4}$ when $x \rightarrow -\infty$. Now let $x_0 > 0$, $t_0 > 0$ and define for $0 \leq t \leq t_0$: $\psi_0(t, x) = \psi(x - x(t_0)) + \frac{1}{2}(t_0 - t) - x_0$ and

$$\begin{cases} I_{x_0, t_0}(t) = \int u^2(t, x) \psi_0(t, x) dx, \\ J_{x_0, t_0}(t) = \int (u_x^2 + u^2 - \frac{2}{p+1} u^{p+1})(t, x) \psi_0(t, x) dx. \end{cases}$$

Then, if we choose ε_0 small enough, there exists $K > 0$ such that for all $t \in [0, t_0]$, we have

$$\begin{cases} I_{x_0, t_0}(t_0) - I_{x_0, t_0}(t) \leq K \exp\left(-\frac{x_0}{4}\right), \\ J_{x_0, t_0}(t_0) - J_{x_0, t_0}(t) \leq K \exp\left(-\frac{x_0}{4}\right). \end{cases}$$

- ◊ Now, let us prove how this result can preserve the decay of the initial data to the solution for all time, on the right (which means for $x > x_0$ for all $x_0 > 0$). If we apply it to $t = 0$ and replace t_0 by t , we obtain for all $t > 0$:

$$\begin{aligned} & \int (u_x^2 + u^2)(t, x + x(t)) \psi(x - x_0) dx \\ & \leq C' \int (u_{0x}^2 + u_0^2)(x) \psi(x - x(t)) + \frac{1}{2}t - x_0 dx + K' e^{-x_0/4}. \end{aligned}$$

But by proposition 2.15, we have $|x' - 1| \leq C\|\varepsilon\|_{H^1} \leq C\varepsilon_0$, thus if we choose ε_0 small enough, we have $|x' - 1| \leq \frac{1}{2}$, and so we obtain by the mean value inequality (notice that $x(0) = \alpha(u_{0,n_0}) = 0$): $|x(t) - t| \leq \frac{1}{2}t$. We deduce that $-x(t) + \frac{1}{2}t \leq 0$, and since ψ is increasing, we obtain

$$\int (u_x^2 + u^2)(t, x + x(t))\psi(x - x_0) dx \leq C \int (u_{0x}^2 + u_0^2)(x)\psi(x - x_0) dx + Ke^{-x_0/4}.$$

◊ Now we explicit exponential decay of u_0 . In fact, we have clearly $(u_{0x}^2 + u_0^2)(x) \sim Ce^{-2\lambda^2|x|} \leq Ce^{-2|x|}$ when $x \rightarrow \pm\infty$. Moreover, since $\psi(x) \leq Ce^{x/4}$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned} \int (u_{0x}^2 + u_0^2)(x)\psi(x - x_0) dx &\leq C \int (u_{0x}^2 + u_0^2)(x)e^{\frac{x-x_0}{4}} dx \\ &\leq Ce^{-x_0/4} \int (u_{0x}^2 + u_0^2)(x)e^{x/4} dx \leq C'e^{-x_0/4}. \end{aligned}$$

◊ Finally, we have more simply

$$\int (u_x^2 + u^2)(t, x + x(t))\psi(x - x_0) dx \geq \frac{1}{2} \int_{x>x_0} (u_x^2 + u^2)(t, x + x(t)) dx,$$

and so the desired inequality. \square

• Now this proposition is proved, we can easily show the first step of the proof of theorem 2.14.

1st step: We bound $|J(t)|$ independently of time by writing

$$J(t) = \int \varepsilon(t, x)\zeta(x) dx = \int_{x>0} \varepsilon(t, x)\zeta(x) dx + \int_{x<0} \varepsilon(t, x)\zeta(x) dx,$$

so that

$$\begin{aligned} |J(t)| &\leq \|\zeta\|_\infty \int_{x>0} (Q(x) + |u(t, x + x(t))|) dx + \sqrt{\int_{x<0} \varepsilon^2(t, x) dx} \sqrt{\int_{x<0} \zeta^2(x) dx} \\ &\leq \|\zeta\|_\infty \|Q\|_{L^1} + \|\zeta\|_\infty U + \|\varepsilon(t)\|_{L^2} V, \end{aligned}$$

where:

- i) $\|\varepsilon(t)\|_{L^2} \leq \|\varepsilon\|_{H^1} \leq C\varepsilon_0 < +\infty$,
- ii) $V^2 = \int_{x<0} \zeta^2(x) dx < +\infty$ since $\zeta^2(x) = o(e^x)$ when $x \rightarrow -\infty$,
- iii) thanks to (2.5), we finally conclude the first step with:

$$\begin{aligned} U &= \int_{x>0} |u(t, x + x(t))| dx = \sum_{n=0}^{+\infty} \int_n^{n+1} |u(t, x + x(t))| dx \leq \sum_{n=0}^{+\infty} \left(\int_{x>n} u^2(t, x + x(t)) dx \right)^{1/2} \\ &\leq \|u(t, \cdot + x(t))\|_{L^2} + \sum_{n=1}^{+\infty} \left(\int_{x>n} u^2(t, x + x(t)) dx \right)^{1/2} \leq C\varepsilon_0 + \|Q\|_{L^2} + C \sum_{n=1}^{+\infty} e^{-n/8} < +\infty. \end{aligned}$$

2nd step: We evaluate J' by using proposition 2.15 and by integrating by parts:

$$\begin{aligned} J' &= \int \varepsilon_t \zeta = \int (L\varepsilon)_x \zeta + (x' - 1) \int Q_x \zeta + (x' - 1) \int \varepsilon_x \zeta + \int R(\varepsilon) \zeta \\ &= - \int \varepsilon L(\zeta') - (x' - 1) \int Q \zeta' - (x' - 1) \int \varepsilon \zeta' + \int R(\varepsilon) \zeta \\ &= - \int \varepsilon (LS + \beta LQ^{\frac{p+1}{2}}) - (x' - 1) \int Q(S + \beta Q^{\frac{p+1}{2}}) - (x' - 1) \int \varepsilon \zeta' + \int R(\varepsilon) \zeta. \end{aligned}$$

Now we take $\beta = -\frac{\int Q S}{\int Q^{\frac{p+3}{2}}}$ so that the second integral is null. Note that by (iv) of claim 2.1,

$$\frac{d}{dc} \int Q_c^2 = 2 \int Q_c \frac{dQ_c}{dc} = \left(\frac{5-p}{2(p-1)} \right) c^{\frac{5-p}{2(p-1)-1}} \int Q^2 < 0$$

since $p > 5$, and so by taking $c = 1$ we remark that $\beta > 0$. Moreover, since $Q^{\frac{p+1}{2}}$ is an eigenvector for L for an eigenvalue $-\lambda_0$ with $\lambda_0 > 0$ (see claim 2.5), we deduce

$$\begin{aligned} J' &= - \int \varepsilon(-Q - \beta\lambda_0 Q^{\frac{p+1}{2}}) - (x' - 1) \int \varepsilon\zeta' + \int R(\varepsilon)\zeta \\ &= \beta\lambda_0 \int \varepsilon Q^{\frac{p+1}{2}} + \int Q\varepsilon - (x' - 1) \int \varepsilon\zeta' + \int R(\varepsilon)\zeta. \end{aligned}$$

But for the last three terms, we remark that:

- a) the mass conservation $\int u^2(t) = \int u_0^2$ implies that $\int Q^2 + 2 \int \varepsilon Q + \int \varepsilon^2 = \int Q^2$ and so $|\int Q\varepsilon| \leq \frac{1}{2} \int \varepsilon^2 \leq \frac{1}{2} \|\varepsilon\|_{H^1}^2$,
- b) thanks to proposition 2.15, we have $|(x' - 1) \int \varepsilon\zeta'| \leq |x' - 1| \|\varepsilon\|_{L^2} \|\zeta'\|_{L^2} \leq C \|\varepsilon\|_{H^1}^2$,
- c) still thanks to this proposition, we have $|\int R(\varepsilon)\zeta| \leq \|\zeta\|_{\infty} \|R(\varepsilon)\|_{L^1} \leq C \|\varepsilon\|_{H^1}^2$.

We have finally

$$J' = \beta\lambda_0 \int \varepsilon Q^{\frac{p+1}{2}} + K(\varepsilon) \tag{2.6}$$

where $K(\varepsilon)$ verifies $|K(\varepsilon)| \leq C \|\varepsilon\|_{H^1}^2$. We now use identity (2.2) which claims

$$F(u(t)) = F(u_0) = F(Q) + \frac{1}{2}(L\varepsilon, \varepsilon) + K'(\varepsilon)$$

with $|K'(\varepsilon)| \leq C \|\varepsilon\|_{H^1}^3$. In other words, we have $(L\varepsilon, \varepsilon) + 2K'(\varepsilon) = 2[F(u_0) - F(Q)] = 2[F(u_{0,n_0}) - F(Q)] = -\gamma_{n_0}$ with $\gamma_{n_0} > 0$, since $\|u_{0,n_0}\|_{L^2} = \|Q\|_{L^2}$ and $E(u_{0,n_0}) < E(Q)$ by construction of u_{0,n_0} . To estimate the term $(L\varepsilon, \varepsilon)$, we use lemma 2.6, so that if we denote $a(t) = \int \varepsilon Q^{\frac{p+1}{2}}$, we obtain

$$a^2(t) \geq \frac{K_1}{K_2} \|\varepsilon\|_{H^1}^2 - \frac{1}{K_2} (L\varepsilon, \varepsilon) = \frac{\gamma_{n_0}}{K_2} + \frac{K_1}{K_2} \|\varepsilon\|_{H^1}^2 + \frac{2}{K_2} K'(\varepsilon).$$

Since $|K'(\varepsilon)| \leq C \|\varepsilon\|_{H^1}^3$ and $\|\varepsilon\|_{H^1} \leq C\varepsilon_0$, then if we take ε_0 small enough, we have

$$a^2(t) \geq K \|\varepsilon\|_{H^1}^2 + \kappa_{n_0}$$

with $K, \kappa_{n_0} > 0$. In particular, $a^2(t) \geq \kappa_{n_0} > 0$, thus a keeps a constant sign, say positive. Then we have

$$a(t) \geq \sqrt{K \|\varepsilon\|_{H^1}^2 + \kappa_{n_0}} \geq \sqrt{\frac{K}{2}} \|\varepsilon\|_{H^1} + \sqrt{\frac{\kappa_{n_0}}{2}} = K' \|\varepsilon\|_{H^1} + \kappa'_{n_0}.$$

But from (2.6), we also have $J'(t) = \beta\lambda_0 a(t) + K(\varepsilon)$ with $|K(\varepsilon)| \leq C \|\varepsilon\|_{H^1}^2$, and so:

$$J'(t) \geq \beta\lambda_0 K' \|\varepsilon\|_{H^1} + \beta\lambda_0 \kappa'_{n_0} - C \|\varepsilon\|_{H^1}^2 \geq \beta\lambda_0 \kappa'_{n_0} = \theta_{n_0} > 0$$

if we choose as previously ε_0 small enough. But it implies that $J(t) \geq \theta_{n_0} t + J(0) \rightarrow +\infty$ as $t \rightarrow +\infty$, which contradicts the first step and concludes the proof of the theorem. Note that if $a(t) < 0$, it is easy to show by the same arguments that $J'(t) \leq \theta'_{n_0} < 0$, so $\lim_{t \rightarrow +\infty} J(t) = -\infty$ and then the same conclusion.

3 Construction of a special solution by compactness

In this section, we prove the existence of a special solution by a compactness method. This result is of course weaker than theorem 1.1, but it does not require the existence of \mathcal{Y}_\pm proved in [17].

3.1 Construction of the initial data

Now theorem 2.14 is proved, we can change T_n obtained in (2.4) in the first time which realizes this. In other words:

$$\exists \delta > 0, \forall n \geq 1, \exists T_n \in \mathbb{R}_+ \text{ such that } \begin{cases} \inf_{y \in \mathbb{R}} \|u_n(T_n) - Q(\cdot - y)\|_{H^1} = \delta \\ \forall t \in [0, T_n], \inf_{y \in \mathbb{R}} \|u_n(t) - Q(\cdot - y)\|_{H^1} \leq \delta \end{cases} .$$

Remark 3.1. We have $T_n \rightarrow +\infty$. Indeed we would have $T_n < T_0$ for all n otherwise (after passing to a subsequence). But by Lipschitz continuous dependence on the initial data (see [9, corollary 2.18]), we would have for n large enough

$$\sup_{t \in [0, T_0]} \|u_n(t) - Q(\cdot - t)\|_{H^1} \leq K \|u_{0,n} - Q\|_{H^1} .$$

But since $\|u_{0,n} - Q\|_{H^1} \xrightarrow{n \rightarrow \infty} 0$ by (2.3), we would have $\inf_{y \in \mathbb{R}} \|u_n(t) - Q(\cdot - y)\|_{H^1} \leq \frac{\delta}{2}$ for n large enough and for all $t \in [0, T_0]$, which is wrong for $t = T_n \in [0, T_0]$.

Now we can take δ smaller than ε_0 , so that $u_n(t) \in U_{\varepsilon_0}$ for all $t \in [0, T_n]$ and so lemma 2.10 applies: we can define $x_n(t) = \alpha(u_n(t))$ (notice that $x_n(0) = \alpha(u_{0,n}) = 0$) such that $\varepsilon_n(t) = u_n(t, \cdot + x_n(t)) - Q$ verifies

$$\forall t \in [0, T_n], \begin{cases} (\varepsilon_n(t), Q') = 0 \\ \|\varepsilon_n(t)\|_{H^1} \leq C \inf_{y \in \mathbb{R}} \|u_n(t) - Q(\cdot - y)\|_{H^1} \leq C\delta \end{cases} .$$

Moreover, for $t = T_n$, we have more precisely

$$\delta \leq \|\varepsilon_n(T_n)\|_{H^1} \leq C\delta. \quad (3.1)$$

In particular, $\{\varepsilon_n(T_n)\}$ is bounded in H^1 , and so by passing to a subsequence, we can define

$$\varepsilon_n(T_n) \rightharpoonup \varepsilon_\infty \text{ in } H^1 \text{ (weakly) and } v_0 = \varepsilon_\infty + Q.$$

Remark 3.2. 1. As announced in the introduction, one of the most important points in this section is to prove that we have constructed a non trivial object, i.e. v_0 is not a soliton (proposition 3.4). This fact is quite natural since v_0 is the weak limit of $u_n(T_n, \cdot + x_n(T_n))$ which contains a persisting defect $\varepsilon_n(T_n)$.

2. Since the proof of proposition 3.4 is mainly based on evaluating L^2 norms, the following lemma will be useful.

Lemma 3.3. There exists $C_0 > 0$ such that, for n large enough, $\|\varepsilon_n(T_n)\|_{L^2} \geq C_0\delta$.

Proof. It comes from the conservation of the Weinstein's functional F in time. In fact, we can write $F(Q + \varepsilon_n(T_n)) = F(Q + \varepsilon_n(0))$ where $\varepsilon_n(0) = u_{0,n} - Q$ verifies $\|\varepsilon_n(0)\|_{H^1} \xrightarrow{n \rightarrow \infty} 0$ by (2.3). Then by (2.2)

$$F(Q) + \frac{1}{2}(L\varepsilon_n(T_n), \varepsilon_n(T_n)) + K(\varepsilon_n(T_n)) = F(Q) + \frac{1}{2}(L\varepsilon_n(0), \varepsilon_n(0)) + K(\varepsilon_n(0))$$

where $|K(a)| \leq C_1 \|a\|_{H^1}^3$. It comes

$$\int \left[(\partial_x \varepsilon_n(T_n))^2 + \varepsilon_n^2(T_n) - pQ^{p-1} \varepsilon_n^2(T_n) \right] \leq C \|\varepsilon_n(0)\|_{H^1}^2 + K(\varepsilon_n(0)) - K(\varepsilon_n(T_n))$$

and so

$$\|\varepsilon_n(T_n)\|_{H^1}^2 \leq C \int \varepsilon_n^2(T_n) + C\|\varepsilon_n(0)\|_{H^1}^2 + C_1\|\varepsilon_n(0)\|_{H^1}^3 + C_1\|\varepsilon_n(T_n)\|_{H^1}^3.$$

Since $\|\varepsilon_n(0)\|_{H^1} \rightarrow 0$, then by (3.1) we have for n large enough

$$\|\varepsilon_n(T_n)\|_{H^1}^2 \leq C \int \varepsilon_n^2(T_n) + C_1 C \delta \|\varepsilon_n(T_n)\|_{H^1}^2 + \frac{\delta^2}{4}.$$

But if we choose δ small enough so that $C_1 C \delta \leq \frac{1}{2}$, we obtain

$$\frac{\delta^2}{2} \leq \frac{1}{2} \|\varepsilon_n(T_n)\|_{H^1}^2 \leq C \int \varepsilon_n^2(T_n) + \frac{\delta^2}{4}$$

and finally $\int \varepsilon_n^2(T_n) \geq \frac{\delta^2}{4C}$. □

Proposition 3.4. For all $c > 0$, $v_0 \neq Q_c$.

Proof. We proceed by contradiction: suppose that $v_n := u_n(T_n, \cdot + x_n(T_n)) \rightharpoonup v_0 = \varepsilon_\infty + Q = Q_c$ weakly in H^1 for some $c > 0$. We recall that it implies in particular that $v_n \rightarrow Q_c$ strongly in L^2 on compacts as $n \rightarrow +\infty$.

- Decomposition of v_n : Let $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$ equals to 0 on $(-\infty, -1]$ and 1 on $[0, +\infty)$. Now let $A \gg 1$ to fix later and define $\varphi_A(x) = \varphi(x+A)$, so that $\varphi_A(x) = 0$ if $x \leq -A-1$ and 1 if $x \geq -A$. We also define $h_n = (1-\varphi_A)v_n$, $Q_c^A = Q_c \varphi_A$ and $z_n = \varphi_A v_n - \varphi_A Q_c = \varphi_A(v_n - Q_c)$, so that

$$v_n = (1 - \varphi_A)v_n + \varphi_A v_n = h_n + z_n + Q_c^A.$$

- Estimation of $\|z_n\|_{L^2}$:

$$\begin{aligned} \int z_n^2 &= \int (v_n - Q_c)^2 \varphi_A^2 \leq \int_{-A-1}^{A+1} (v_n - Q_c)^2 + \int_{x>A+1} (v_n - Q_c)^2 \\ &\leq \int_{-A-1}^{A+1} (v_n - Q_c)^2 + 2 \int_{x>A+1} v_n^2 + 2 \int_{x>A+1} Q_c^2 = I + J + K. \end{aligned}$$

Notice that $I \xrightarrow{n \rightarrow \infty} 0$ since $v_n \xrightarrow{n \rightarrow \infty} Q_c$ in L^2 on compacts. Moreover, thanks to exponential decay of Q_c , we have $K \leq C e^{-2\sqrt{c}A}$. Finally, we have $J \leq C e^{-A/4}$ with C independent of n by remark 2.17. In summary, there exists $\rho > 0$ such that $\int z_n^2 \leq C e^{-\rho A}$ if $n \geq n(A)$.

- Mass balance: On one hand, we have by (2.3) and mass conservation $\int v_n^2 = \int u_{0,n}^2 = \int Q^2$. On the other hand, we can calculate

$$\int v_n^2 = \int h_n^2 + \int (Q_c^A + z_n)^2 + 2 \int_{-A-1}^{-A} v_n^2 \varphi_A (1 - \varphi_A).$$

But since $v_n \rightarrow Q_c$ on compacts, we have $2 \int_{-A-1}^{-A} v_n^2 \varphi_A (1 - \varphi_A) \xrightarrow{n \rightarrow \infty} 2 \int_{-A-1}^{-A} Q_c^2 \varphi_A (1 - \varphi_A) \leq C e^{-\rho A}$. Consequently,

$$\int Q^2 = \int h_n^2 + \int (Q_c^A)^2 + 2 \int Q_c^A z_n + \int z_n^2 + a_n^A$$

where $a_n^A \geq 0$ verifies $a_n^A \leq C e^{-\rho A}$ for $n \geq n(A)$. Thanks to the previous estimation of $\|z_n\|_{L^2}$ and the Cauchy-Schwarz inequality, we deduce that

$$\int Q^2 = \int h_n^2 + \int (Q_c^A)^2 + a_n^A$$

where a_n^{A} verifies $|a_n^{A}| \leq C e^{-\rho A}$ for $n \geq n(A)$. But

$$\int (Q_c^A)^2 = \int Q_c^2 \varphi_A^2 = \int Q_c^2 + \int Q_c^2 (\varphi_A^2 - 1) \leq \int Q_c^2 + \int_{x < -A} Q_c^2 \leq \int Q_c^2 + C e^{-\rho A}$$

and $\int Q_c^2 = c^{-\beta} \int Q^2$ with $\beta > 0$ since $p > 5$ (see claim 2.1). In conclusion, we have the mass balance

$$(1 - c^{-\beta}) \|Q\|_{L^2}^2 = \|h_n\|_{L^2}^2 + a_n^{A} \quad (3.2)$$

where a_n^{A} still verifies $|a_n^{A}| \leq C e^{-\rho A}$ for $n \geq n(A)$.

- Upper bound of $\|h_n\|_{L^2}$: We remark that for $n \geq n(A)$, $\|h_n\|_{L^2} \leq C_1 \delta$. Indeed, thanks to (3.1), we have

$$\|h_n\|_{L^2} \leq \|(1 - \varphi_A)Q\|_{L^2} + \|\varepsilon_n(T_n)\|_{L^2} \leq C e^{-\rho A} + C \delta \leq C_1 \delta$$

if we definitively fix A large enough so that $e^{-\rho A} \leq \delta^3$ (the power 3 will be useful later in the proof).

- Upper bound of $|c - 1|$: Thanks to the previous point and mass balance (3.2), we have $|1 - c^{-\beta}| \leq C \delta^2$. We deduce that c is close to 1, and so by Taylor's theorem that $|c - 1| \leq K|1 - c^{-\beta}| \leq C \delta^2$.
- Lower bound of $\|h_n\|_{L^2}$: We now prove that for $n \geq n(A)$, $\|h_n\|_{L^2} \geq C_2 \delta$. Firstly, we have by lemma 3.3:

$$\begin{aligned} C_0 \delta &\leq \|\varepsilon_n(T_n)\|_{L^2} = \|v_n - Q\|_{L^2} = \|h_n + Q_c^A + z_n - Q\|_{L^2} \\ &\leq \|h_n\|_{L^2} + \|z_n\|_{L^2} + \|Q_c^A - Q_c\|_{L^2} + \|Q_c - Q\|_{L^2} = \|h_n\|_{L^2} + \|Q_c - Q\|_{L^2} + b_n^A \end{aligned}$$

where $b_n^A = \|z_n\|_{L^2} + \|Q_c^A - Q_c\|_{L^2} \geq 0$ verifies $b_n^A \leq C e^{-\rho A}$ for $n \geq n(A)$. Moreover, if we denote $f(c) = \|Q_c - Q\|_{L^2}^2$ for $c > 0$, then f is C^∞ and $f(c) \geq 0 = f(1)$, hence 1 is a minimum of f , $f'(1) = 0$ and so by Taylor's theorem: $f(c) \leq C(c - 1)^2$, i.e. $\|Q_c - Q\|_{L^2} \leq C|c - 1|$. Thanks to the previous point, we deduce that

$$C_0 \delta \leq \|h_n\|_{L^2} + K \delta^2 + b_n^A \leq \|h_n\|_{L^2} + C \delta^2.$$

Finally, if we choose δ small enough so that $C \delta \leq \frac{C_0}{2}$, we reach the desired inequality.

- Energy balance: We now use the conservation of Weinstein's functional and (2.2) to write

$$F(u_0) = F(v_n) = F(Q + \varepsilon_n(T_n)) = F(Q) + \frac{1}{2}(L\varepsilon_n(T_n), \varepsilon_n(T_n)) + K(\varepsilon_n(T_n))$$

where $|K(\varepsilon_n(T_n))| \leq C \|\varepsilon_n(T_n)\|_{H^1}^3 \leq C \delta^3$ by (3.1). Now we decompose $\varepsilon_n(T_n)$ in

$$\varepsilon_n(T_n) = v_n - Q = h_n + z_n + Q_c^A - Q = (Q_c - Q) + (Q_c^A - Q_c) + (z_n + h_n)$$

in order to expand

$$\begin{aligned} (L\varepsilon_n(T_n), \varepsilon_n(T_n)) &= (L(Q_c - Q), Q_c - Q) + (L(z_n + h_n), z_n + h_n) \\ &\quad + (L(Q_c^A - Q_c), Q_c^A - Q_c) + 2(L(Q_c - Q), z_n + h_n) \\ &\quad + 2(L(Q_c - Q), Q_c^A - Q_c) + 2(L(Q_c^A - Q_c), z_n + h_n). \end{aligned}$$

We recall that $(La, b) = -\int a''b + \int ab - p \int Q^{p-1}ab$, and so by the Cauchy-Schwarz inequality: $|(La, b)| \leq (\|a''\|_{L^2} + C\|a\|_{L^2})\|b\|_{L^2}$. Since we have $\|z_n + h_n\|_{L^2} \leq \|z_n\|_{L^2} + \|h_n\|_{L^2} \leq C e^{-\rho A} + C_1 \delta \leq C \delta$, we can estimate

$$|(L(Q_c - Q), z_n + h_n)| \leq (\|Q_c'' - Q''\|_{L^2} + C\|Q_c - Q\|_{L^2})\|z_n + h_n\|_{L^2} \leq C|c - 1| \cdot C \delta \leq C \delta^3.$$

Similarly, we have

$$\begin{aligned} |(L(Q_c^A - Q_c), z_n + h_n)| &\leq (\|\varphi_A'' Q_c\|_{L^2} + 2\|\varphi_A' Q_c'\|_{L^2} + \|(\varphi_A - 1)Q_c''\|_{L^2} \\ &\quad + C\|Q_c^A - Q_c\|_{L^2})\|z_n + h_n\|_{L^2} \\ &\leq C e^{-\rho A} \cdot C\delta \leq C\delta^3. \end{aligned}$$

Moreover, we have by integrating by parts $(La, b) = \int a'b' + \int ab - p \int Q^{p-1}ab$, and so $|(La, b)| \leq C\|a\|_{H^1}\|b\|_{H^1}$. It implies that

$$\begin{cases} |(L(Q_c - Q), Q_c - Q)| \leq C\|Q_c - Q\|_{H^1}^2 \leq C(c-1)^2 \leq C\delta^3, \\ |(L(Q_c^A - Q_c), Q_c^A - Q_c)| \leq C\|Q_c^A - Q_c\|_{H^1}^2 \leq C e^{-2\rho A} \leq C\delta^3, \\ |(L(Q_c - Q), Q_c^A - Q_c)| \leq C\|Q_c - Q\|_{H^1}\|Q_c^A - Q_c\|_{H^1} \leq C|c-1| \cdot C e^{-\rho A} \leq C\delta^3, \end{cases}$$

thanks to the estimate on $|c-1|$ previously found. For the last term, we have

$$(L(h_n + z_n), h_n + z_n) = \|h_n + z_n\|_{H^1}^2 - p \int Q^{p-1}(h_n + z_n)^2$$

and

$$\begin{aligned} \int Q^{p-1}(h_n + z_n)^2 &\leq 2 \int Q^{p-1}h_n^2 + 2 \int Q^{p-1}z_n^2 \leq 2 \int (1 - \varphi_A)^2 Q^{p-1}v_n^2 + 2\|Q\|_{\infty}^{p-1} \int z_n^2 \\ &\leq 2 \int_{x < -A} Q^{p-1}v_n^2 + 2\|Q\|_{\infty}^{p-1} \int z_n^2. \end{aligned}$$

But $\|v_n\|_{\infty} \leq C\|v_n\|_{H^1} \leq C(\|\varepsilon_n(T_n)\|_{H^1} + \|Q\|_{H^1}) \leq C(K\delta + \|Q\|_{H^1}) = K'$, and so $\int_{x < -A} Q^{p-1}v_n^2 \leq C \int_{x < -A} Q^{p-1} \leq C e^{-\rho A}$. As $\int z_n^2 \leq C e^{-\rho A}$, we have

$$F(u_0) = F(Q) + \frac{1}{2}\|h_n + z_n\|_{H^1}^2 + d_n^A \geq F(Q) + \frac{1}{2}\|h_n + z_n\|_{L^2}^2 + d_n^A$$

where $|d_n^A| \leq C\delta^3$ for $n \geq n(A)$. Moreover we have

$$\|h_n + z_n\|_{L^2}^2 - \|h_n\|_{L^2}^2 \leq \|z_n\|_{L^2}^2 + 2\|z_n\|_{L^2}\|h_n\|_{L^2} \leq C e^{-2\rho A} + 2C e^{-\rho A} \cdot C_1\delta \leq C\delta^3.$$

Finally, energy balance provides us, for some N large enough,

$$F(u_0) \geq F(Q) + \frac{1}{2}\|h_N\|_{L^2}^2 + d'$$

with $|d'| \leq C\delta^3$.

- Conclusion: Since $F(u_0) < F(Q)$ by hypothesis, we obtain $\|h_N\|_{L^2}^2 \leq C\delta^3$. But we also have by the lower bound of $\|h_n\|_{L^2}$: $\|h_N\|_{L^2}^2 \geq C_2^2\delta^2$. Gathering both information, we obtain $\frac{C_2^2}{C} \leq \delta$, which is clearly a contradiction if we choose δ small enough, and so concludes the proof of proposition 3.4. \square

3.2 Weak continuity of the flow

The main idea to obtain the special solution is to reverse the weak convergence of v_n to v_0 in time and in space, using the fact that $u(t, x)$ is a solution of (gKdV) if and only if $u(-t, -x)$ is also a solution. More precisely, we define $w_0 = \check{v}_0 \in H^1(\mathbb{R})$, i.e. for all $x \in \mathbb{R}$, $w_0(x) = v_0(-x)$.

Remark 3.5. For all $c > 0$ and all $x_0 \in \mathbb{R}$, we have

$$w_0 \neq Q_c(\cdot + x_0).$$

In fact, otherwise and since Q_c is even, we would have $v_0(x) = Q_c(x - x_0)$. But $v_n - Q = \varepsilon_n(T_n)$ and $(\varepsilon_n(T_n), Q') = (v_n, Q') = 0$, so by weak convergence in H^1 , $(v_0, Q') = 0$. Thus we would have $\int Q_c(x - x_0)Q'(x) dx = 0$, and if we show that $x_0 = 0$, we shall reach the desired contradiction since we have $v_0 \neq Q_c$ for all $c > 0$ by proposition 3.4. To show this, consider $f(a) = \int Q_c(x - a)Q'(x) dx$ for $a \in \mathbb{R}$, which is odd since Q_c is even and Q' odd. In particular, $f(0) = 0$, and it is enough to show that $f(a) < 0$ for $a > 0$ to conclude (because we shall have $f(a) > 0$ for $a < 0$ by parity). But using again the parity of Q_c and Q' , we have

$$f(a) = \int_0^a [Q_c(a - x) - Q_c(a + x)]Q'(x) dx + \int_a^{+\infty} [Q_c(x - a) - Q_c(x + a)]Q'(x) dx.$$

Since Q' is negative and Q_c is strictly decreasing on \mathbb{R}_+ , both integrals are negative, and so $f(a) < 0$ for $a > 0$, as we desired.

Remark 3.6. 1. Now, w_0 being constructed, we show that the associated solution $w(t)$ is defined for all t positive, and can be seen as a weak limit (proposition 3.8) in order to prove the convergence of $w(t)$ to a soliton.

2. The main ingredient of the proof of proposition 3.8 is the following lemma of weak continuity of the flow, whose proof is inspired by [8, theorem 5]. This proof is long and technical, and thus is not completely written in this paper.

Lemma 3.7. Suppose that $z_{0,n} \rightharpoonup z_0$ in H^1 , and that there exist $T > 0$ and $K > 0$ such that the solution $z_n(t)$ corresponding to initial data $z_{0,n}$ exists for $t \in [0, T]$ and $\sup_{t \in [0, T]} \|z_n(t)\|_{H^1} \leq K$. Then for all $t \in [0, T]$, the solution $z(t)$ such that $z(0) = z_0$ exists, and $z_n(T) \rightharpoonup z(T)$ in H^1 .

Sketch of the proof. Let $T^* = T^*(\|z_0\|_{H^{\frac{3}{4}}}) > 0$ be the maximum time of existence of the solution $z(t)$ associated to z_0 , well defined by [9, corollary 2.18] since $s = \frac{3}{4} > \frac{p-5}{2(p-1)} = s_c(p)$. We distinguish two cases, whether $T < T^*$ or not, and we show that this last case is in fact impossible.

1st case: Suppose that $T < T^*$. As $z(t)$ exists for $t \in [0, T]$ by hypothesis, it is enough to show that $z_n(T) \rightharpoonup z(T)$ in H^1 . But since C_0^∞ is dense in H^{-1} and $\|z_n(T) - z(T)\|_{H^1} \leq \|z_n(T)\|_{H^1} + \|z(T)\|_{H^1} \leq K'$, it is enough to show that $z_n(T) \rightharpoonup z(T)$ in $\mathcal{D}'(\mathbb{R})$. It is the end of this case, very similar to the proof in [8] (but using a H^3 regularization and so using some arguments like in [11, section 3.4]), which is technical and not written in this paper consequently.

2nd case: Suppose that $T^* \leq T$ and let us show that it implies a contradiction. Indeed, there would exist $T' < T^*$ such that $\|z(T')\|_{H^{\frac{3}{4}}} \geq 2K$ (where K is the same constant as in the hypothesis of the lemma). But we can apply the first case with T' instead of T , so that $z_n(T') \rightharpoonup z(T')$ in H^1 , and since $\|z_n(T')\|_{H^1} \leq K$, we obtain by weak convergence $\|z(T')\|_{H^{\frac{3}{4}}} \leq \|z(T')\|_{H^1} \leq K$, and so the desired contradiction. \square

Proposition 3.8. The solution $w(t)$ of (gKdV) such that $w(0) = w_0$ is defined for all $t \geq 0$, and $u_n(T_n - t, x_n(T_n) - \cdot) \rightharpoonup w(t)$ in H^1 .

Proof. As the assumption is clear for $t = 0$, we fix $T > 0$ and we show it for this T . Since $\lim_{n \rightarrow +\infty} T_n = +\infty$ by remark 3.1, then for $n \geq n_0$, we have $T_n \geq T$. As a consequence, for $n \geq n_0$ and for $t \in [0, T]$, $z_n(t) = u_n(T_n - t, x_n(T_n) - \cdot)$ is well defined, solves (gKdV), and has for initial data

$$z_n(0) = u_n(T_n, x_n(T_n) - \cdot) = \check{v}_n \rightharpoonup \check{v}_0 = w_0 \quad \text{in } H^1.$$

Moreover, we have

$$\begin{aligned} \|z_n(t)\|_{H^1} &= \|u_n(T_n - t, x_n(T_n) - \cdot)\|_{H^1} \\ &\leq \|\varepsilon_n(T_n - t, x_n(T_n) - x_n(T_n - t) - \cdot)\|_{H^1} + \|Q(x_n(T_n) - x_n(T_n - t) - \cdot)\|_{H^1} \\ &\leq \|\varepsilon_n(T_n - t)\|_{H^1} + \|Q\|_{H^1} \leq C\delta + \|Q\|_{H^1} = K. \end{aligned}$$

By lemma 3.7, we deduce that w exists on $[0, T]$, and $z_n(T) \rightharpoonup w(T)$ in H^1 . \square

3.3 Exponential decay on the left of w

The goal of this section is to prove an exponential decay on the "left" of w , using the exponential decay of u_n on the right. Indeed, since $\varepsilon_n(T_n - t) = u_n(T_n - t, \cdot + x_n(T_n - t)) - Q$ verifies $(\varepsilon_n(T_n - t), Q') = 0$ and $\|\varepsilon_n(T_n - t)\|_{H^1} \leq C\delta$ for all $t \in [0, T_n]$, then $u_n(T_n - t)$ is in the same situation as the situation of u summed up just before proposition 2.15, with δ instead of ε_0 for the small parameter. In particular, by remark 2.17, inequality (2.5) holds for $u_n(T_n - t)$ with C independent of n if we choose δ small enough. In other words, we have for all $t \geq 0$ and $x_0 > 0$ (and n large enough):

$$\int_{x > x_0} (u_{nx}^2 + u_n^2)(T_n - t, x + x_n(T_n - t)) dx \leq Ce^{-x_0/4}. \quad (3.3)$$

But before passing to the limit, we have to define the "left" of w , i.e. the center of mass $x_w(t)$ of $w(t)$.

Lemma 3.9. There exists $C > 0$ such that, for all $t \geq 0$, $\inf_{y \in \mathbb{R}} \|w(t) - Q(\cdot - y)\|_{H^1} \leq C\delta$.

Proof. Fix $t \geq 0$ and $n_0 \geq 0$ such that for $n \geq n_0$, $T_n \geq t$. Since Q is even, we have

$$\varepsilon_n(T_n - t, x_n(T_n) - x_n(T_n - t) - \cdot) = u_n(T_n - t, x_n(T_n) - \cdot) - Q(\cdot - x_n(T_n) + x_n(T_n - t)).$$

Now if we denote $w_n(t) = u_n(T_n - t, x_n(T_n) - \cdot)$ and $y_n(t) = x_n(T_n) - x_n(T_n - t)$, we have

$$\|w_n(t) - Q(\cdot - y_n(t))\|_{H^1} = \|\varepsilon_n(T_n - t)\|_{H^1} \leq C\delta.$$

But following the remark done at the beginning of this section, proposition 2.15 is still valid, and so $|x'_n(t) - 1| \leq C\delta$ for $t \in [0, T_n]$. We deduce that $y_n(t) = \int_{T_n - t}^{T_n} x'_n(s) ds = \int_{T_n - t}^{T_n} (x'_n(s) - 1) ds + t$ verifies $|y_n(t)| \leq C\delta t + t = Ct$. By passing to a subsequence, we can suppose that $\lim_{n \rightarrow \infty} y_n(t) = y(t)$. But now we can write

$$\|w_n(t) - Q(\cdot - y(t))\|_{H^1} \leq C\delta + \|Q - Q(\cdot + (y_n(t) - y(t)))\|_{H^1} \leq C'\delta$$

for $n \geq N(t, \delta)$ by lemma 2.11. Finally, since $w_n(t) \rightharpoonup w(t)$ in H^1 by proposition 3.8, we obtain by weak convergence $\|w(t) - Q(\cdot - y(t))\|_{H^1} \leq C'\delta$, and the result follows. \square

We can now choose δ small enough so that $C\delta \leq \varepsilon_0$, and so we can define $x_w(t) = \alpha(w(t))$ by lemma 2.10, with notably $\|w(t, \cdot + x_w(t)) - Q\|_{H^1} \leq C\delta$. But to exploit (3.3), we have to show first that $y_n(t) = x_n(T_n) - x_n(T_n - t)$ is close to $x_w(t)$ for all t .

Lemma 3.10. There exists $C > 0$ such that: $\forall t \geq 0, \exists n_0 \geq 0, \forall n \geq n_0, |x_w(t) - y_n(t)| \leq C\delta$.

Proof. Let $t \geq 0$ and n large enough such that $T_n \geq t$. We keep notation $w_n(t)$ and $y_n(t)$ of the previous proof, where we have already remarked that $|y_n(t)| \leq Ct$. For the same reason, we have $|x_w(t) - y_n(t)| \leq \Omega t$. Now choose $A(t) \gg 1$ such that $\|Q\|_{L^2(|x| \geq A(t) - \Omega t)} \leq \delta$. Since $w_n(t) \rightharpoonup w(t)$ in H^1 , then for $n \geq n_0$, we have $\|w_n(t) - w(t)\|_{L^2(|x| \leq A(t))} \leq \delta$. Moreover,

$$\|w(t) - Q(\cdot - x_w(t))\|_{H^1} \leq C\delta \quad \text{and} \quad \|w_n(t) - Q(\cdot - y_n(t))\|_{H^1} \leq C\delta,$$

and so by the triangle inequality: $\|Q(\cdot - x_w(t)) - Q(\cdot - y_n(t))\|_{L^2(|x| \leq A(t))} \leq C\delta$. We deduce that for $n \geq n_0$:

$$\begin{aligned} \|Q - Q(\cdot + x_w(t) - y_n(t))\|_{L^2} &\leq \sqrt{2} \|Q - Q(\cdot + x_w(t) - y_n(t))\|_{L^2(|x| \leq A(t))} \\ &\quad + \sqrt{2} \|Q - Q(\cdot + x_w(t) - y_n(t))\|_{L^2(|x| \geq A(t))} \\ &\leq C\delta + \sqrt{2} \|Q\|_{L^2(|x| \geq A(t))} + \sqrt{2} \|Q(\cdot + x_w(t) - y_n(t))\|_{L^2(|x| \geq A(t))} \\ &\leq C\delta + 2\sqrt{2} \|Q\|_{L^2(|x| \geq A(t) - \Omega t)} \leq C\delta. \end{aligned}$$

We conclude by choosing δ small enough so that $C\delta \leq A_0$, where A_0 is defined in lemma 2.11, and we apply this lemma to reach the desired inequality (note that the lemma holds of course with the L^2 norm instead of the H^1 one). \square

If we choose δ small enough so that $C\delta \leq 1$ (for example) in lemma 3.10, we can now prove the following proposition.

Proposition 3.11. There exists $C > 0$ such that, for all $t \geq 0$ and all $x_0 > 0$,

$$\int_{x < -x_0 - 1} (w_x^2 + w^2)(t, x + x_w(t)) dx \leq Ce^{-x_0/4}.$$

Proof. Let $t \geq 0$, $x_0 > 0$ and $n \geq n_0$ where n_0 is defined in lemma 3.10. From (3.3) and the substitution $y = x_n(T_n) - x_n(T_n - t) - x = y_n(t) - x$, we obtain

$$\int_{x < y_n(t) - x_0} (u_{nx}^2 + u_n^2)(T_n - t, x_n(T_n) - x) dx \leq Ce^{-x_0/4}.$$

If we still denote $w_n(t) = u_n(T_n - t, x_n(T_n) - \cdot)$, we deduce by lemma 3.10 that

$$\int_{x < -x_0 - 1 + x_w(t)} (w_{nx}^2 + w_n^2)(t, x) dx \leq Ce^{-x_0/4}.$$

But $w_n(t) \rightharpoonup w(t)$ in H^1 , so $w_n(t) \rightharpoonup w(t)$ and $w_{nx}(t) \rightharpoonup w_x(t)$ in L^2 . Moreover, since $\psi = \mathbb{1}_{(-\infty, -x_0 - 1 + x_w(t))} \in L^\infty$, then $w_n(t)\psi \rightharpoonup w(t)\psi$ and $w_{nx}(t)\psi \rightharpoonup w_x(t)\psi$ in L^2 , thus by weak convergence $\int_{x < -x_0 - 1 + x_w(t)} w^2(t, x) dx \leq Ce^{-x_0/4}$ and the same inequality for w_x , so the result follows by sum. \square

3.4 Asymptotic stability and conclusion

The final ingredient to prove that $w(t)$ is a special solution is the theorem of asymptotic stability proved by Martel and Merle in [15]. Indeed, thanks to lemma 3.9, we can apply this theorem with $c_0 = 1$ if we choose δ small enough such that $C\delta < \alpha_0$. We obtain $c_+ > 0$ and $t \mapsto \rho(t) \in \mathbb{R}$ such that

$$\|w(t) - Q_{c_+}(\cdot - \rho(t))\|_{H^1(x > t/10)} \xrightarrow{t \rightarrow +\infty} 0. \quad (3.4)$$

Remark 3.12. As usual, $\rho(t)$ and c_+ are defined in [15] by a lemma of modulation close to Q , which gives the estimations: $\|w(t) - Q_{c_+}(\cdot - \rho(t))\|_{H^1} \leq C\delta$, $|\rho'(t) - 1| \leq C\delta$ and $|c_+ - 1| \leq C\delta$. We deduce that

$$\begin{aligned} \|Q - Q(\cdot + \rho(t) - x_w(t))\|_{H^1} &= \|Q(\cdot - \rho(t)) - Q(\cdot - x_w(t))\|_{H^1} \\ &\leq \|Q - Q_{c_+}\|_{H^1} + \|w(t) - Q_{c_+}(\cdot - \rho(t))\|_{H^1} \\ &\quad + \|w(t) - Q(\cdot - x_w(t))\|_{H^1} \\ &\leq K|c_+ - 1| + C\delta + C'\delta \leq C''\delta. \end{aligned}$$

Now if we choose δ small enough, then $C''\delta \leq A_0$ and lemma 2.11 gives $|x_w(t) - \rho(t)| \leq C\delta \leq 1$. Finally, proposition 3.11 becomes

$$\forall t \geq 0, \forall x_0 > 2, \int_{x < -x_0} (w_x^2 + w^2)(t, x + \rho(t)) dx \leq C'e^{-x_0/4}. \quad (3.5)$$

We are now able to prove the main result of this section.

Theorem 3.13 (Existence of one special solution). There exist $w(t)$ solution of (gKdV) defined for all $t \geq 0$, $c_+ > 0$ and $t \mapsto \rho(t)$ such that:

- (i) $\|w(t) - Q_{c_+}(\cdot - \rho(t))\|_{H^1(\mathbb{R})} \xrightarrow{t \rightarrow +\infty} 0$,
- (ii) $\forall c > 0, \forall x_0 \in \mathbb{R}, w(0) \neq Q_c(\cdot + x_0)$.

Proof. By remark 3.5, it is enough to prove (i). We have by the triangle inequality

$$\begin{aligned} \|w(t) - Q_{c_+}(\cdot - \rho(t))\|_{H^1(\mathbb{R})}^2 &\leq \|w(t) - Q_{c_+}(\cdot - \rho(t))\|_{H^1(x > t/10)}^2 + 2\|w(t)\|_{H^1(x < t/10)}^2 \\ &\quad + 2\|Q_{c_+}(\cdot - \rho(t))\|_{H^1(x < t/10)}^2 = \mathbf{I} + \mathbf{II} + \mathbf{III}. \end{aligned}$$

Since $|\rho'(t) - 1| \leq C\delta \leq \frac{1}{10}$ if we choose δ small enough, then $|\rho(t) - t - \rho(0)| \leq \frac{1}{10}t$, and so if we denote $\rho_0 = \rho(0) \in \mathbb{R}$, we have $\frac{t}{10} - \rho(t) \leq -\frac{4}{5}t - \rho_0$. We can now estimate:

• $\mathbf{I} \xrightarrow[t \rightarrow +\infty]{} 0$ by (3.4).

• For t large enough, we have $\frac{4t}{5} + \rho_0 > 2$, and so (3.5) gives

$$\begin{aligned} \frac{1}{2}\mathbf{II} &= \int_{x < t/10} (w_x^2 + w^2)(t, x) dx = \int_{x < t/10 - \rho(t)} (w_x^2 + w^2)(t, x + \rho(t)) dx \\ &\leq \int_{x < -4t/5 - \rho_0} (w_x^2 + w^2)(t, x + \rho(t)) dx \leq Ce^{-t/5} \xrightarrow[t \rightarrow +\infty]{} 0. \end{aligned}$$

• Finally, since $(Q_{c_+}'^2 + Q_{c_+}^2)(x) \leq Ce^{2\sqrt{c_+}x}$ for all $x \in \mathbb{R}$ (see claim 2.1), we have

$$\begin{aligned} \frac{1}{2}\mathbf{III} &= \int_{x < t/10} (Q_{c_+}'^2 + Q_{c_+}^2)(x - \rho(t)) dx = \int_{x < t/10 - \rho(t)} (Q_{c_+}'^2 + Q_{c_+}^2)(x) dx \\ &\leq \int_{x < -4t/5 - \rho_0} (Q_{c_+}'^2 + Q_{c_+}^2)(x) dx \leq C \int_{x < -4t/5 - \rho_0} e^{2\sqrt{c_+}x} dx \leq Ce^{-\frac{8t}{5}\sqrt{c_+}} \xrightarrow[t \rightarrow +\infty]{} 0 \end{aligned}$$

which achieves the proof of theorem 3.13. \square

Corollary 3.14. For all $c > 0$, there exist $w_c(t)$ solution of (gKdV) defined for all $t \geq 0$ and $t \mapsto \rho_c(t)$ such that:

(i) $\|w_c(t, \cdot + \rho_c(t)) - Q_c\|_{H^1(\mathbb{R})} \xrightarrow[t \rightarrow +\infty]{} 0$,

(ii) $\forall c' > 0, \forall x_0 \in \mathbb{R}, w_c(0, \cdot + \rho_c(0)) \neq Q_{c'}(\cdot + x_0)$.

Proof. It is based on the scaling invariance of the (gKdV) equation: if $u(t, x)$ is a solution, then for all $\lambda > 0$, $\lambda^{\frac{2}{p-1}}u(\lambda^3 t, \lambda x)$ is also a solution. For $c > 0$ given, we thus define w_c by $w_c(t) = \lambda_c^{\frac{2}{p-1}}w(\lambda_c^3 t, \lambda_c x)$ with $\lambda_c = \sqrt{\frac{c}{c_+}}$, where w and c_+ are defined above. Since $Q_c(x) = \lambda_c^{\frac{2}{p-1}}Q_{c_+}(\lambda_c x)$, then we have by substitution

$$\begin{aligned} \|w(t) - Q_{c_+}(\cdot - \rho(t))\|_{H^1}^2 &= \lambda_c^{\frac{p-5}{p-1}} \left(\|w_c(t/\lambda_c^3, \cdot + \rho(t)/\lambda_c) - Q_c\|_{L^2}^2 \right. \\ &\quad \left. + \frac{1}{\lambda_c^2} \|\partial_x [w_c(t/\lambda_c^3, \cdot + \rho(t)/\lambda_c) - Q_c]\|_{L^2}^2 \right). \end{aligned}$$

We deduce that

$$\|w(t) - Q_{c_+}(\cdot - \rho(t))\|_{H^1}^2 \geq \begin{cases} \lambda_c^{\frac{p-5}{p-1}} \|w_c(t/\lambda_c^3, \cdot + \rho(t)/\lambda_c) - Q_c\|_{H^1}^2 & \text{if } \lambda_c \leq 1 \\ \lambda_c^{-\frac{p+3}{p-1}} \|w_c(t/\lambda_c^3, \cdot + \rho(t)/\lambda_c) - Q_c\|_{H^1}^2 & \text{if } \lambda_c \geq 1 \end{cases},$$

and so $\lim_{t \rightarrow +\infty} \|w_c(t/\lambda_c^3, \cdot + \rho(t)/\lambda_c) - Q_c\|_{H^1} = 0$ in both cases by theorem 3.13. We finally obtain (i) if we take $\rho_c(t) = \frac{\rho(\lambda_c^3 t)}{\lambda_c}$. For (ii), if we suppose that there exist $c' > 0$ and $x_0 \in \mathbb{R}$ such that $w_c(0, \cdot + \rho_c(0)) = Q_{c'}(\cdot + x_0)$, then we get

$$w_0 = Q_{\frac{c'}{c_+}} \left(\cdot + \left(\sqrt{\frac{c}{c_+}} x_0 - \rho_0 \right) \right)$$

which is a contradiction with remark 3.5. \square

4 Construction and uniqueness of a family of special solutions via the contraction principle

In this section, we prove theorem 1.1. The proof is an extension to (gKdV) of the method by fixed point developed in [4, 5]. To adapt the method to (gKdV), we use first information on the spectrum of the linearized operator around $Q(\cdot - t)$ due to [17] (see proposition 4.2 in the present paper). Secondly, we rely on the Cauchy theory for (gKdV) developed in [9, 10]. Indeed, one of the main difficulties is the lack of a derivative due to the equation, but compensated by a smoothing effect already used in [9, 10].

4.1 Preliminary estimates for the Cauchy problem

Theorem 3.5 of [9] and proposition 2.3 of [10] are summed up and adapted to our situation in proposition 4.1 below. We note $W(t)$ the semigroup associated to the linear equation $\partial_t u + \partial_x^3 u = 0$.

Notation. Let $I \subset \mathbb{R}$ be an interval, $1 \leq p, q \leq \infty$ and $g : \mathbb{R} \times I \rightarrow \mathbb{R}$. Then define

$$\|g\|_{L_x^p L_t^q} = \left(\int_{-\infty}^{+\infty} \left(\int_I |g(x, t)|^q dt \right)^{p/q} dx \right)^{1/p}, \quad \|g\|_{L_t^q L_x^p} = \left(\int_I \left(\int_{-\infty}^{+\infty} |g(x, t)|^p dx \right)^{q/p} dt \right)^{1/q}$$

and $L_x^p L_t^q = \{g \mid \|g\|_{L_x^p L_t^q} < +\infty\}$ and $L_t^q L_x^p = \{g \mid \|g\|_{L_t^q L_x^p} < +\infty\}$. Finally, denote $L_x^p L_t^q = L_x^p L_{\mathbb{R}}^q$ and $L_t^q L_x^p = L_{\mathbb{R}}^q L_x^p$.

Proposition 4.1. There exists $C > 0$ such that for all $g \in L_x^1 L_t^2$ and all $T \in \mathbb{R}$,

$$\left\| \frac{\partial}{\partial x} \int_t^{+\infty} W(t-t')g(x, t') dt' \right\|_{L_{[T, +\infty)}^\infty L_x^2} \leq C \|g\|_{L_x^1 L_{[T, +\infty)}^2}, \quad (4.1)$$

$$\left\| \frac{\partial}{\partial x} \int_t^{+\infty} W(t-t')g(x, t') dt' \right\|_{L_x^5 L_{[T, +\infty)}^{10}} \leq C \|g\|_{L_x^1 L_{[T, +\infty)}^2}. \quad (4.2)$$

Proof. (i) Inequality (4.1) comes from the dual inequality of (3.6) in [9], i.e.

$$\left\| \frac{\partial}{\partial x} \int_{-\infty}^{+\infty} W(-t')g(x, t') dt' \right\|_{L_x^2} \leq C \|g\|_{L_x^1 L_t^2}.$$

Let $t \geq T$, we get for $\tilde{g}(x, t') = \mathbb{1}_{[t, +\infty)}(t')g(x, t')$:

$$\left\| \frac{\partial}{\partial x} \int_t^{+\infty} W(-t')g(x, t') dt' \right\|_{L_x^2} = \left\| \frac{\partial}{\partial x} \int_{-\infty}^{+\infty} W(-t')\tilde{g}(x, t') dt' \right\|_{L_x^2} \leq C \|g\|_{L_x^1 L_{[T, +\infty)}^2}$$

and so the desired inequality since W is unitary on L^2 .

(ii) Inequality (4.2) comes from inequalities (2.6) and (2.8) of [10] with the admissible triples $(p_1, q_1, \alpha_1) = (5, 10, 0)$ and $(p_2, q_2, \alpha_2) = (\infty, 2, 1)$. In fact, if we combine (2.6) cut in time with $[0, +\infty)$ and (2.8), we get

$$\left\| \frac{\partial}{\partial x} \int_t^{+\infty} W(t-t')g(x, t') dt' \right\|_{L_x^5 L_t^{10}} \leq C \|g\|_{L_x^1 L_t^2}.$$

If we apply it to $\tilde{g}(x, t') = \mathbb{1}_{[T, +\infty)}(t')g(x, t')$, we reach the desired inequality since

$$\left\| \frac{\partial}{\partial x} \int_t^{+\infty} W(t-t')g(x, t') dt' \right\|_{L_x^5 L_{[T, +\infty)}^{10}} \leq \left\| \frac{\partial}{\partial x} \int_t^{+\infty} W(t-t')\tilde{g}(x, t') dt' \right\|_{L_x^5 L_t^{10}}. \quad \square$$

4.2 Preliminary results on the linearized equation

4.2.1 Linearized equation

The linearized equation appears if one considers a solution of (gKdV) close to the soliton $Q(x-t)$. More precisely, if $u(t, x) = Q(x-t) + h(t, x-t)$ verifies (gKdV), then h verifies

$$\partial_t h + \mathcal{L}h = R(h) \quad (4.3)$$

where $\mathcal{L}a = -(La)_x$, $La = -\partial_x^2 a + a - pQ^{p-1}a$ is defined in section 2.1, and

$$R(h) = -\partial_x \left(\sum_{k=2}^p \binom{p}{k} Q^{p-k} h^k \right).$$

The spectrum of \mathcal{L} has been calculated by Pego and Weinstein in [17]; their results are summed up here for reader's convenience.

Proposition 4.2 ([17]). Let $\sigma(\mathcal{L})$ be the spectrum of the operator \mathcal{L} defined on $L^2(\mathbb{R})$ and let $\sigma_{\text{ess}}(\mathcal{L})$ be its essential spectrum. Then

$$\sigma_{\text{ess}}(\mathcal{L}) = i\mathbb{R} \quad \text{and} \quad \sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, e_0\} \text{ with } e_0 > 0.$$

Furthermore, e_0 and $-e_0$ are simple eigenvalues of \mathcal{L} with eigenfunctions \mathcal{Y}_+ and $\mathcal{Y}_- = \check{\mathcal{Y}}_+$ which have an exponential decay at infinity, and the null space of \mathcal{L} is spanned by Q' .

4.2.2 Exponential decay

Exponential decay of \mathcal{Y}_+ has been proved in [17], but a generalization of this fact to a larger family of functions will be necessary in the proof of proposition 4.6. For $\lambda > 0$, consider the operator A_λ defined on L^2 by $A_\lambda u = u''' - u' - \lambda u$, and the characteristic equation of $A_\lambda u = 0$,

$$f_\lambda(x) := x^3 - x - \lambda = 0.$$

Note $\sigma_1^\lambda, \sigma_2^\lambda, \sigma_3^\lambda$ the roots of f_λ , eventually complex, and sorted by their real part. A simple study of f_λ shows that σ_3^λ is always real, $\sigma_3^\lambda > \frac{1}{\sqrt{3}}$, and $(\sigma_3^\lambda)_{\lambda>0}$ is increasing. Moreover, we have the three cases:

- (a) If $\lambda > \frac{2}{3\sqrt{3}}$, then σ_1^λ and σ_2^λ are two conjugate roots which verify $\text{Re } \sigma_1^\lambda = \text{Re } \sigma_2^\lambda = -\frac{\sigma_3^\lambda}{2}$.
- (b) If $\lambda = \frac{2}{3\sqrt{3}}$, then $\sigma_1^\lambda = \sigma_2^\lambda = -\frac{1}{\sqrt{3}}$ and $\sigma_3^\lambda = \frac{2}{\sqrt{3}}$.
- (c) If $\lambda < \frac{2}{3\sqrt{3}}$, then $\sigma_1^\lambda, \sigma_2^\lambda$ are real and: $\sigma_1^\lambda \in \left(-\sqrt{3}, -\frac{1}{\sqrt{3}}\right)$; $\sigma_2^\lambda \in \left(-\frac{1}{\sqrt{3}}, 0\right)$. Moreover, $(\sigma_2^\lambda)_\lambda$ is decreasing, and in particular $\sigma_2^\lambda \nearrow 0$ when $\lambda \searrow 0$.

This analysis allows us to define

$$\mu = \frac{1}{4} \min(\sigma_3^\lambda, -\text{Re } \sigma_2^\lambda, e_0, 1) > 0$$

and

$$\mathcal{H} = \{f \in H^\infty(\mathbb{R}) \mid \forall j \geq 0, \exists C_j \geq 0, \forall x \in \mathbb{R}, |f^{(j)}(x)| \leq C_j e^{-\mu|x|}\}.$$

Lemma 4.3. If $u \in L^2$ and $f \in \mathcal{H}$ verify $u''' - u' - \lambda u = f$ with $\lambda \geq e_0$, then $u \in \mathcal{H}$.

Proof. First notice that $u \in H^\infty(\mathbb{R})$ by a simple bootstrap argument. Moreover, the method of variation of constants gives us

$$u(x) = Ae^{\sigma_3^\lambda x} \int_x^{+\infty} e^{-\sigma_3^\lambda s} f(s) ds + Be^{\sigma_2^\lambda x} \int_{-\infty}^x e^{-\sigma_2^\lambda s} f(s) ds + Ce^{\sigma_1^\lambda x} \int_{-\infty}^x e^{-\sigma_1^\lambda s} f(s) ds$$

with $A, B, C \in \mathbb{C}$, if we suppose that $\lambda \neq \frac{2}{3\sqrt{3}}$. We can also notice that u' has the same form as u , except for three terms in $f(x)$ which appear, and which have the expected decay by hypothesis, and so on for $u^{(j)}$ for $j \geq 2$. Hence we only have to check exponential decay for u :

$$|u(x)| \leq A' e^{\sigma_3^\lambda x} \int_x^{+\infty} e^{-\sigma_3^\lambda s} |f(s)| ds + B' e^{\operatorname{Re} \sigma_2^\lambda x} \int_{-\infty}^x e^{-\operatorname{Re} \sigma_2^\lambda s} |f(s)| ds + C' e^{\operatorname{Re} \sigma_1^\lambda x} \int_{-\infty}^x e^{-\operatorname{Re} \sigma_1^\lambda s} |f(s)| ds.$$

By changing x in $-x$ and by the definition of μ , it is enough to show that if

$$v(x) = e^{-ax} \int_{-\infty}^x e^{as} e^{-\mu|s|} ds$$

with $a \geq 2\mu$, then $v(x) \leq e^{-\mu|x|}$. Notice that one half could also have replaced one quarter in the definition of μ , but this gain of 2 allows us to treat the case $\lambda = \frac{2}{3\sqrt{3}}$ (not written here for brevity), which makes appear a polynomial in front of the exponential in the last two terms of the expression of u . Finally, we conclude in both cases, since $a - \mu \geq \mu > 0$:

- If $x < 0$, then $v(x) \leq e^{-ax} \int_{-\infty}^x e^{as} e^{\mu s} ds = C e^{-ax} \cdot e^{(a+\mu)x} = C e^{\mu x} = C e^{-\mu|x|}$.
- If $x \geq 0$, then $v(x) \leq e^{-ax} \int_{-\infty}^x e^{as} e^{-\mu s} ds = C e^{-ax} \cdot e^{(a-\mu)x} = C e^{-\mu x} = C e^{-\mu|x|}$.

The case $\lambda = \frac{2}{3\sqrt{3}}$ is treated similarly. \square

Corollary 4.4. $\mathcal{Y}_+, \mathcal{Y}_- \in \mathcal{H}$.

Proof. Since $\mathcal{Y}_- = \check{\mathcal{Y}}_+$, it is enough to show that $\mathcal{Y}_+ \in \mathcal{H}$. But by definition of \mathcal{Y}_+ in [17], we have $\mathcal{L}\mathcal{Y}_+ = e_0 \mathcal{Y}_+$ with $\mathcal{Y}_+ \in L^2$, i.e.

$$\mathcal{Y}_+''' - \mathcal{Y}_+' - e_0 \mathcal{Y}_+ = -p \partial_x (Q^{p-1} \mathcal{Y}_+) = -p(p-1) Q' Q^{p-2} \mathcal{Y}_+ - p Q^{p-1} \mathcal{Y}_+'.$$

By a bootstrap argument, we have $\mathcal{Y}_+ \in H^\infty(\mathbb{R})$, and in particular $\mathcal{Y}_+^{(j)} \in L^\infty(\mathbb{R})$ for all $j \geq 0$. If we denote $f(x) = -p(p-1) Q' Q^{p-2} \mathcal{Y}_+ - p Q^{p-1} \mathcal{Y}_+'$, then by exponential decay of $Q^{(j)}$ for all $j \geq 0$ and by definition of μ , we have $|f^{(j)}(x)| \leq C e^{-(p-1)|x|} \leq C e^{-\mu|x|}$ and so $f \in \mathcal{H}$. It is enough to apply lemma 4.3 with $\lambda = e_0$ to conclude. \square

4.3 Existence of special solutions

We now prove the following result, which is the first part of theorem 1.1.

Proposition 4.5. Let $A \in \mathbb{R}$. If $t_0 = t_0(A)$ is large enough, then there exists a solution $U^A \in C^\infty([t_0, +\infty), H^\infty)$ of (gKdV) such that

$$\forall s \in \mathbb{R}, \exists C > 0, \forall t \geq t_0, \quad \|U^A(t, \cdot + t) - Q - A e^{-e_0 t} \mathcal{Y}_+\|_{H^s} \leq C e^{-2e_0 t}. \quad (4.4)$$

4.3.1 A family of approximate solutions

The following proposition is similar to [5, proposition 3.4], except for the functional space, which is not the Schwartz space but the space \mathcal{H} described above.

Proposition 4.6. Let $A \in \mathbb{R}$. There exists a sequence $(\mathcal{Z}_j^A)_{j \geq 1}$ of functions of \mathcal{H} such that $\mathcal{Z}_1^A = A \mathcal{Y}_+$, and if $k \geq 1$ and $\mathcal{V}_k^A = \sum_{j=1}^k e^{-j e_0 t} \mathcal{Z}_j^A$, then

$$\partial_t \mathcal{V}_k^A + \mathcal{L} \mathcal{V}_k^A = R(\mathcal{V}_k^A) + \varepsilon_k^A(t), \quad \text{where } \varepsilon_k^A(t) = \sum_{j=k+1}^{pk} e^{-j e_0 t} g_{j,k}^A, \quad g_{j,k}^A \in \mathcal{H}, \quad (4.5)$$

and R is defined in (4.3).

Proof. The proof is very similar to the one in [5], and we write it there for reader's convenience. We prove this proposition by induction, and for brevity, we omit the superscript A .

Define $\mathcal{Z}_1 := A\mathcal{Y}_+$ and $\mathcal{V}_1 := e^{-e_0 t}\mathcal{Z}_1$. Then by the explicit definition of R in (4.3),

$$\partial_t \mathcal{V}_1 + \mathcal{L}\mathcal{V}_1 - R(\mathcal{V}_1) = -R(\mathcal{V}_1) = -R(Ae^{-e_0 t}\mathcal{Y}_+) = \sum_{j=2}^p e^{-je_0 t} A^j \binom{p}{j} \partial_x [Q^{p-j}\mathcal{Y}_+^j]$$

which yields (4.5) for $k = 1$, since $\mathcal{Y}_+, Q \in \mathcal{H}$ by corollary 4.4 and claim 2.1.

Let $k \geq 1$ and assume that $\mathcal{Z}_1, \dots, \mathcal{Z}_k$ are known with the corresponding \mathcal{V}_k satisfying (4.5). Now let $\mathcal{U}_{k+1} := g_{k+1,k} \in \mathcal{H}$, so that

$$\partial_t \mathcal{V}_k + \mathcal{L}\mathcal{V}_k = R(\mathcal{V}_k) + e^{-(k+1)e_0 t} \mathcal{U}_{k+1} + \sum_{j=k+2}^{pk} e^{-je_0 t} g_{j,k},$$

and define $\mathcal{Z}_{k+1} := -(\mathcal{L} - (k+1)e_0)^{-1} \mathcal{U}_{k+1}$. Note that \mathcal{Z}_{k+1} is well defined since $(k+1)e_0$ is not in the spectrum of \mathcal{L} by proposition 4.2, and moreover $\mathcal{Z}_{k+1} \in \mathcal{H}$. Indeed, we have

$$\mathcal{Z}_{k+1}''' - \mathcal{Z}_{k+1}' - (k+1)e_0 \mathcal{Z}_{k+1} = -\mathcal{U}_{k+1} - p(p-1)Q'Q^{p-2}\mathcal{Z}_{k+1} - pQ^{p-1}\mathcal{Z}_{k+1}' \in \mathcal{H}$$

by exponential decay of $Q^{(j)}$ for all $j \geq 0$ and since $\mathcal{Z}_{k+1}^{(j)} \in H^\infty(\mathbb{R})$ by a bootstrap argument. Hence $\mathcal{Z}_{k+1} \in \mathcal{H}$ by lemma 4.3 applied with $\lambda = (k+1)e_0 \geq e_0$.

Then we have

$$\partial_t \left(\mathcal{V}_k + e^{-(k+1)e_0 t} \mathcal{Z}_{k+1} \right) + \mathcal{L} \left(\mathcal{V}_k + e^{-(k+1)e_0 t} \mathcal{Z}_{k+1} \right) = R(\mathcal{V}_k) + \sum_{j=k+2}^{pk} e^{-je_0 t} g_{j,k}.$$

Denote $\mathcal{V}_{k+1} := \mathcal{V}_k + e^{-(k+1)e_0 t} \mathcal{Z}_{k+1}$. Thus we have

$$\partial_t \mathcal{V}_{k+1} + \mathcal{L}\mathcal{V}_{k+1} - R(\mathcal{V}_{k+1}) = R(\mathcal{V}_k) - R(\mathcal{V}_{k+1}) + \sum_{j=k+2}^{pk} e^{-je_0 t} g_{j,k}.$$

We conclude the proof by evaluating

$$\begin{aligned} R(\mathcal{V}_k) - R(\mathcal{V}_{k+1}) &= R(\mathcal{V}_k) - R(\mathcal{V}_k + e^{-(k+1)e_0 t} \mathcal{Z}_{k+1}) \\ &= \partial_x \left[\sum_{j=2}^p \binom{p}{j} Q^{p-j} \left((\mathcal{V}_k + e^{-(k+1)e_0 t} \mathcal{Z}_{k+1})^j - \mathcal{V}_k^j \right) \right] = \sum_{j=k+2}^{p(k+1)} e^{-je_0 t} \tilde{g}_{j,k}, \end{aligned}$$

which yields (4.5) for $k+1$, and thus completes the proof. \square

4.3.2 Construction of special solutions

We now prove proposition 4.5, following the same three steps as in [5]. The main difference comes from step 2, because of the derivative in the error term which forces us to use the sharp smoothing effect developed in [9]. Let $A \in \mathbb{R}$ and $s \geq 1$ integer. Write

$$U^A(t, x+t) = Q(x) + h^A(t, x).$$

First, by a fixed point argument, we construct a solution $h^A \in C^0([t_k, +\infty), H^s)$ of (4.3) for k and t_k large and such that

$$\forall T \geq t_k, \|(h^A - \mathcal{V}_k)(T)\|_{H^s} \leq e^{-(k+\frac{1}{2})e_0 T}. \quad (4.6)$$

Next, the same arguments like in [5] show that h^A does not depend on s and k . For brevity, we omit the superscript A .

Step 1. Reduction to a fixed point problem. If we set $\tilde{h}(t, x) = h(t, x - t)$, equation (4.3) can be written as

$$\partial_t \tilde{h} + \partial_x^3 \tilde{h} = -S(\tilde{h}), \quad S(\tilde{h}) = \frac{\partial}{\partial x} \left[\sum_{k=1}^p \binom{p}{k} Q^{p-k}(x-t) \tilde{h}^k \right]. \quad (4.7)$$

Moreover, we have by (4.5), $\varepsilon_k(t) = \partial_t \mathcal{V}_k + \partial_x^3 \mathcal{V}_k - \partial_x \mathcal{V}_k + \partial_x \left[\sum_{j=1}^p \binom{p}{j} Q^{p-j} \mathcal{V}_k^j \right]$. Now let $v(t, x) = (h - \mathcal{V}_k)(t, x - t)$ and subtract the previous equation from (4.7), so that

$$\partial_t v + \partial_x^3 v = -S[v + \mathcal{V}_k(t, x - t)] + S[\mathcal{V}_k(t, x - t)] - \varepsilon_k(t, x - t).$$

For notation simplicity, we drop the space argument $(x - t)$ for the moment. The equation can then be written as

$$v(t) = \mathcal{M}(v)(t) := \int_t^{+\infty} W(t-t') [S(\mathcal{V}_k(t') + v(t')) - S(\mathcal{V}_k(t')) + \varepsilon_k(t')] dt'. \quad (4.8)$$

Note that (4.6) is equivalent to $\|v(T)\|_{H^s} \leq e^{-(k+\frac{1}{2})e_0 T}$ for $T \geq t_k$. In other words, defining

$$\begin{cases} N_1(v) = \sup_{T \geq t_k} e^{(k+\frac{1}{2})e_0 T} \|v(T)\|_{H^s}, \\ N_2(v) = \sum_{s'=0}^s \sup_{T \geq t_k} e^{(k+\frac{1}{2})e_0 T} \|\partial^{s'} v\|_{L_x^5 L_{[T, +\infty)}^{10}}, \\ \Lambda(v) = \Lambda_{t_k, k, s}(v) = \max(N_1(v), N_2(v)), \end{cases}$$

it is enough to show that \mathcal{M} is a contraction on B defined by

$$B = B(t_k, k, s) = \{v \in C^0([t_k, +\infty), H^s) \mid \Lambda(v) \leq 1\}.$$

Remark 4.7. The choice of the two norms N_1 and N_2 is related to the fact that global well posedness of supercritical (gKdV) with initial data small in H^1 can be proved with the two norms $\widetilde{N}_1(v) = \sup_{t \in \mathbb{R}} \|v(t)\|_{H^1}$ and $\widetilde{N}_2(v) = \|v\|_{L_x^5 L_t^{10}} + \|\partial_x v\|_{L_x^5 L_t^{10}}$, following [10]. We could also have used other norms from [9].

Step 2. Contraction argument. We show that \mathcal{M} is a contraction on B for $s \geq 1$ and k, t_k sufficiently large. Throughout this proof, we denote by C a constant depending only on s , and C_k a constant depending on s and k . To estimate $N_1(\mathcal{M}(v))$ and $N_2(\mathcal{M}(v))$, we have to explicit

$$\begin{aligned} S(\mathcal{V}_k + v) - S(\mathcal{V}_k) &= \frac{\partial}{\partial x} \left[\sum_{i=1}^p \binom{p}{i} Q^{p-i} \left((\mathcal{V}_k + v)^i - \mathcal{V}_k^i \right) \right] \\ &= \frac{\partial}{\partial x} (pQ^{p-1}v) + \frac{\partial}{\partial x} \left[\sum_{i=2}^p \binom{p}{i} Q^{p-i} v \cdot \sum_{l=1}^i \binom{i}{l} \mathcal{V}_k^{i-l} v^{l-1} \right] \\ &= p \frac{\partial \mathbf{I}}{\partial x} + \sum_{\alpha, \beta, \gamma} C_{\alpha, \beta, \gamma} \frac{\partial \mathbf{II}_{\alpha, \beta, \gamma}}{\partial x} \end{aligned}$$

where $\mathbf{I} = Q^{p-1}v$ and $\mathbf{II}_{\alpha, \beta, \gamma} = Q^\alpha \mathcal{V}_k^\beta v^\gamma$, with: $\gamma \geq 1$, $\beta + \gamma \geq 2$, $\alpha + \beta + \gamma = p \geq 6$. We can now write

$$\begin{aligned} \partial^s \mathcal{M}(v) &= p \int_t^{+\infty} W(t-t') \frac{\partial}{\partial x} [\partial^s(\mathbf{I})] dt' + \sum_{\alpha, \beta, \gamma} C_{\alpha, \beta, \gamma} \int_t^{+\infty} W(t-t') \frac{\partial}{\partial x} [\partial^s(\mathbf{II}_{\alpha, \beta, \gamma})] dt' \\ &\quad + \int_t^{+\infty} W(t-t') \partial^s \varepsilon_k(t') dt'. \end{aligned}$$

By (4.1) and (4.2), we obtain

$$\begin{aligned} \max \left(\|\partial^s \mathcal{M}(v)(T)\|_{L_x^2}, \|\partial^s \mathcal{M}(v)\|_{L_x^5 L_{[T,+\infty)}^{10}} \right) &\leq C \|\partial^{s-1} \varepsilon_k\|_{L_x^1 L_{[T,+\infty)}^2} + C \|\partial^s(\mathbf{I})\|_{L_x^1 L_{[T,+\infty)}^2} \\ &+ \sum_{\alpha, \beta, \gamma} C_{\alpha, \beta, \gamma} \|\partial^s(\mathbf{II}_{\alpha, \beta, \gamma})\|_{L_x^1 L_{[T,+\infty)}^2}. \end{aligned} \quad (4.9)$$

We treat the terms ε_k , \mathbf{I} , $\mathbf{II}_{\alpha, \beta, \gamma}$ for $\alpha = p-2, \beta = \gamma = 1$, and for $\alpha = \beta = 0, \gamma = p$. All other terms can be treated similarly: for example, $\mathbf{II}_{0, p-1, 1}$ can be treated like $\mathbf{II}_{p-2, 1, 1}$, etc.

For \mathbf{I} , since Q and his derivatives have the same decay, it is enough to estimate the term $\tilde{\mathbf{I}} = \|Q^{p-1} \partial^s v\|_{L_x^1 L_{[T,+\infty)}^2} \leq C \|e^{-|x-t|} \partial^s v\|_{L_x^1 L_{[T,+\infty)}^2}$:

$$\begin{aligned} \tilde{\mathbf{I}} &\leq C \|e^{x-t} \partial^s v\|_{L_{(-\infty, T]}^1 L_{[T,+\infty)}^2} + C \|e^{t-x} \partial^s v\|_{L_{[T,+\infty)}^1 L_{[T, x]}^2} + C \|e^{x-t} \partial^s v\|_{L_{[T,+\infty)}^1 L_{[x,+\infty)}^2} \\ &\leq C \sqrt{\int_{-\infty}^T e^{2x} dx} \sqrt{\int_x \int_T^{+\infty} e^{-2t} (\partial^s v)^2 dt dx} + C \sqrt{\int_T^{+\infty} e^{-2x} dx} \sqrt{\int_x \int_T^{+\infty} e^{2t} (\partial^s v)^2 dt dx} \\ &\quad + C \sqrt{\int_T^{+\infty} e^{-2x} dx} \sqrt{\int_T^{+\infty} \int_x^{+\infty} e^{4x-2t} (\partial^s v)^2 dt dx} \end{aligned}$$

by the Cauchy-Schwarz inequality. Now, by Fubini's theorem, and since $4x - 2t \leq 2t$ in the last integral, we get

$$\begin{aligned} \tilde{\mathbf{I}} &\leq C e^T N_1(v) \sqrt{\int_T^{+\infty} e^{-(2k+1)e_0 t - 2t} dt} + 2C e^{-T} N_1(v) \sqrt{\int_T^{+\infty} e^{-(2k+1)e_0 t + 2t} dt} \\ &\leq C e^T N_1(v) \frac{e^{-(k+\frac{1}{2})e_0 T - T}}{\sqrt{(2k+1)e_0 + 2}} + 2C e^{-T} N_1(v) \frac{e^{-(k+\frac{1}{2})e_0 T + T}}{\sqrt{(2k+1)e_0 - 2}} \leq C N_1(v) \frac{1}{\sqrt{k}} e^{-(k+\frac{1}{2})e_0 T}. \end{aligned}$$

Note that since k will be chosen large at the end of the argument, we can suppose $(2k+1)e_0 > 2$.

For $\mathbf{II}_{p-2, 1, 1}$, we treat similarly the term $\tilde{\mathbf{II}}_{p-2, 1, 1} = \|Q^{p-2} \mathcal{V}_k \partial^s v\|_{L_x^1 L_{[T,+\infty)}^2}$ since \mathcal{V}_k and his derivatives have the same decay. In fact, we have by Hölder inequality

$$\tilde{\mathbf{II}}_{p-2, 1, 1} \leq C \|\partial^s v\|_{L_x^5 L_{[T,+\infty)}^{10}} \|\mathcal{V}_k\|_{L_x^{5/4} L_{[T,+\infty)}^{5/2}} \leq C N_2(v) e^{-(k+\frac{1}{2})e_0 T} \|\mathcal{V}_k\|_{L_x^{5/4} L_{[T,+\infty)}^{5/2}}.$$

By the definition of \mathcal{V}_k in proposition 4.6, we have by noting $e'_0 = \frac{5}{2}e_0$ and $\mu' = \frac{5}{2}\mu$,

$$\begin{aligned} \|\mathcal{V}_k\|_{L_x^{5/4} L_{[T,+\infty)}^{5/2}} &\leq C_k \|e^{-e_0 t} e^{-\mu|x-t|}\|_{L_x^{5/4} L_{[T,+\infty)}^{5/2}} \\ &\leq C_k \int_{-\infty}^T \sqrt{\int_T^{+\infty} e^{-e'_0 t} e^{-\mu' t} e^{\mu' x} dt dx} + C_k \int_T^{+\infty} \sqrt{\int_T^x e^{-e'_0 t} e^{\mu' t} e^{-\mu' x} dt dx} \\ &\quad + C_k \int_T^{+\infty} \sqrt{\int_x^{+\infty} e^{-e'_0 t} e^{-\mu' t} e^{\mu' x} dt dx} \\ &\leq C_k e^{\frac{\mu'}{2}T} \sqrt{\int_T^{+\infty} e^{-(e'_0 + \mu')t} dt} + C_k e^{-\frac{\mu'}{2}T} \sqrt{\int_T^{+\infty} e^{(\mu' - e'_0)t} dt} \\ &\quad + C_k \int_T^{+\infty} e^{\frac{\mu'}{2}x} \sqrt{\int_x^{+\infty} e^{-(e'_0 + \mu')t} dt dx} \\ &\leq 3C_k e^{-\frac{e'_0}{2}T} \quad \text{since } \mu < e_0 \text{ by definition of } \mu. \end{aligned}$$

We finally deduce that $\|\mathcal{V}_k\|_{L_x^{5/4}L_{[T,+\infty)}^{5/2}} \leq C_k e^{-e_0 T}$ and so $\tilde{\mathbf{II}}_{p-2,1,1} \leq C_k N_2(v) e^{-(k+\frac{3}{2})e_0 T}$.

For $\mathbf{II}_{0,0,p} = v^p$, first remark that

$$\partial^s(v^p) = p\partial^{s-1}(\partial v \cdot v^{p-1}) = p\partial^s v \cdot v^{p-1} + p \sum_{k=0}^{s-2} \binom{s-1}{k} \partial^{k+1} v \cdot \partial^{s-1-k}(v^{p-1})$$

where each term of the sum is a product of p terms like $\partial^{s_j} v$ with $s_j \leq s-1$. Since $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, we can estimate the first term thanks to Hölder's inequality:

$$\begin{aligned} \|\partial^s v \cdot v^{p-1}\|_{L_x^1 L_{[T,+\infty)}^2} &\leq \|v\|_{L_{[T,+\infty)}^\infty}^{p-5} \cdot \|\partial^s v\|_{L_x^5 L_{[T,+\infty)}^{10}} \cdot \|v\|_{L_x^5 L_{[T,+\infty)}^{10}}^4 \\ &\leq C e^{-p(k+\frac{1}{2})e_0 T} N_1(v)^{p-5} N_2(v)^5. \end{aligned}$$

The other terms in the sum can be treated in the same way, and more simply since we can choose any $(p-5)$ terms to take out in $L_{[T,+\infty)}^\infty L_x^\infty$ norm, and any 5 others left in $L_x^5 L_{[T,+\infty)}^{10}$ norm.

For ε_k , we deduce by a similar calculation like above and by the expression of ε_k in (4.5) that

$$\|\partial^{s-1} \varepsilon_k\|_{L_x^1 L_{[T,+\infty)}^2} \leq C_k \int_{\mathbb{R}} \sqrt{\int_T^{+\infty} e^{-2(k+1)e_0 t} e^{-2\mu|x-t|} dt} dx \leq C'_k e^{-(k+1)e_0 T}.$$

Summarizing from (4.9), we have shown

$$\begin{aligned} \max \left(e^{(k+\frac{1}{2})e_0 T} \|\mathcal{M}(v)(T)\|_{H^s}, \sum_{s'=0}^s e^{(k+\frac{1}{2})e_0 T} \|\partial^{s'} v\|_{L_x^5 L_{[T,+\infty)}^{10}} \right) \\ \leq C_k e^{-\frac{e_0}{2} T} + \frac{C N_1(v)}{\sqrt{k}} + C_k N_2(v) e^{-e_0 T} + C e^{-(p-1)(k+\frac{1}{2})e_0 T} N_1(v)^{p-5} N_2(v)^5. \end{aligned}$$

Since $v \in B(t_k, k, s)$, i.e. $\Lambda(v) \leq 1$, then we have

$$\Lambda(\mathcal{M}(v)) \leq C_k e^{-\frac{e_0}{2} t_k} + \left(\frac{C}{\sqrt{k}} + C_k e^{-e_0 t_k} \right) \Lambda(v) \leq \left(\frac{C}{\sqrt{k}} + C_k e^{-\frac{e_0}{2} t_k} \right).$$

First, choose k so that $\frac{C}{\sqrt{k}} \leq \frac{1}{2}$, then take t_k such that $C_k e^{-\frac{e_0}{2} t_k} \leq \frac{1}{2}$. Then \mathcal{M} maps $B = B(t_k, k, s)$ into itself.

It remains to show that \mathcal{M} is a contraction on B . If we $v, w \in B$, we have

$$\mathcal{M}(v) - \mathcal{M}(w) = \int_t^{+\infty} W(t-t') \left[S(\mathcal{V}_k(t') + v(t')) - S(\mathcal{V}_k(t') + w(t')) \right] dt'$$

and

$$\begin{aligned} S(\mathcal{V}_k + v) - S(\mathcal{V}_k + w) &= \frac{\partial}{\partial x} \left[\sum_{j=1}^p \binom{p}{j} Q^{p-j} \left[(\mathcal{V}_k + v)^j - (\mathcal{V}_k + w)^j \right] \right] \\ &= \frac{\partial}{\partial x} \sum_{j=1}^p \binom{p}{j} Q^{p-j} (v-w) \sum_{i=1}^{j-1} (\mathcal{V}_k + v)^i (\mathcal{V}_k + w)^{j-i} \\ &= p \frac{\partial}{\partial x} \left[Q^{p-1} (v-w) \right] + \frac{\partial}{\partial x} \left[(v-w) \cdot \sum_{\alpha, \beta, \gamma, \delta} C_{\alpha, \beta, \gamma, \delta} Q^\alpha \mathcal{V}_k^\beta v^\gamma w^\delta \right]. \end{aligned}$$

Under this form, a similar calculation like above allows us to conclude: the first term is treated like \mathbf{I} , and each $Q^\alpha \mathcal{V}_k^\beta v^\gamma w^\delta$ can be treated like $\mathbf{II}_{\alpha, \beta, \gamma}$ if we systematically take out the term $\Lambda(v-w)$ by Hölder's inequality. Hence we get, since there is no term in ε_k ,

$$\Lambda(\mathcal{M}(v) - \mathcal{M}(w)) \leq \left(\frac{C}{\sqrt{k}} + C_k e^{-e_0 t_k} \right) \Lambda(v-w).$$

Choosing if necessary a larger k , then a larger t_k , we may assume that $\frac{C}{\sqrt{k}} < \frac{1}{2}$ and $C_k e^{-e_0 t_k} \leq \frac{1}{2}$, showing that \mathcal{M} is a contraction on B . Hence, step 2 is complete.

Step 3. End of the proof. By the preceding step with $s = 1$, there exist k_0 and t_0 such that there exists a unique solution U^A of (gKdV) satisfying $U^A \in C^0([t_0, +\infty), H^1)$ and

$$\Lambda_{t_0, k_0, 1} \left(U^A(t, x) - Q(x-t) - \mathcal{V}_{k_0}^A(t, x-t) \right) \leq 1. \quad (4.10)$$

Note that the fixed point argument still holds taking a larger t_0 , and so the uniqueness remains valid, for any $t'_0 \geq t_0$, in the class of solutions of (gKdV) in $C^0([t'_0, +\infty), H^1)$ satisfying (4.10).

Finally, we can show proposition 4.5. Since U^A is a solution of (gKdV), it is sufficient to show that $U^A \in C^0([t_0, +\infty), H^s)$ for any s ; the smoothness in time will follow from the equation. Let $s \geq 1$: by step 2, if k_s is large enough, there exist t_s and $\tilde{U}^A \in C^0([t_s, +\infty), H^s)$ such that

$$\Lambda_{t_s, k_s, s} \left(\tilde{U}^A(t, x) - Q(x-t) - \mathcal{V}_{k_s}^A(t, x-t) \right) \leq 1.$$

Of course, we may choose $k_s \geq k_0 + 1$. But by construction of \mathcal{V}_k^A in proposition 4.6, we have $\mathcal{V}_{k_s}^A(t, x-t) - \mathcal{V}_{k_0}^A(t, x-t) = \sum_{j=k_0+1}^{k_s} e^{-j e_0 t} \mathcal{Z}_j^A(x-t)$ where $\mathcal{Z}_j^A \in \mathcal{H}$, and so by similar calculation like in step 2,

$$\Lambda_{t_s, k_0, s} \left(\mathcal{V}_{k_s}^A(t, x-t) - \mathcal{V}_{k_0}^A(t, x-t) \right) \leq C e^{-\frac{e_0}{2} t_s} \leq \frac{1}{2}$$

for t_s large enough. Moreover, we have by definition of Λ (and since $k_0 \leq k_s - 1$)

$$\Lambda_{t_s, k_0, s}(u) \leq e^{-e_0 t_s} \Lambda_{t_s, k_s, s}(u).$$

Thus, if we choose t_s large enough such that $e^{-e_0 t_s} \leq \frac{1}{2}$, we get by triangle inequality

$$\begin{aligned} \Lambda_{t_s, k_0, 1} \left(\tilde{U}^A(t, x) - Q(x-t) - \mathcal{V}_{k_0}^A(t, x-t) \right) &\leq \Lambda_{t_s, k_0, s} \left(\tilde{U}^A(t, x) - Q(x-t) - \mathcal{V}_{k_0}^A(t, x-t) \right) \\ &\leq \Lambda_{t_s, k_0, s} \left(\tilde{U}^A(t, x) - Q(x-t) - \mathcal{V}_{k_s}^A(t, x-t) \right) + \Lambda_{t_s, k_0, s} \left(\mathcal{V}_{k_s}^A(t, x-t) - \mathcal{V}_{k_0}^A(t, x-t) \right) \leq 1. \end{aligned}$$

In particular, \tilde{U}^A satisfies (4.10) for large t_s . By the uniqueness in the fixed point argument, we have $U^A = \tilde{U}^A$, which shows that $U^A \in C^0([t_s, +\infty), H^s)$. By the persistence of regularity of (gKdV) equation, $U^A \in C^0([t_0, +\infty), H^s)$, where $s \geq 1$. In particular, by compactness on $[t_0, t_s]$, there exists $C = C(s)$ such that

$$\forall t \geq t_0, \quad \|U^A(t, x) - Q(x-t) - \mathcal{V}_{k_0}^A(t, x-t)\|_{H^s} \leq C e^{-(k_0 + \frac{1}{2}) e_0 t}$$

and so (4.4) follows, which achieves the proof of proposition 4.5.

4.4 Uniqueness

Now, the special solution U^A being constructed, we prove its uniqueness, in the sense of the following proposition, which implies the second part of theorem 1.1.

Proposition 4.8. Let u be a solution of (gKdV) such that

$$\inf_{y \in \mathbb{R}} \|u(t) - Q(\cdot - y)\|_{H^1} \xrightarrow{t \rightarrow +\infty} 0. \quad (4.11)$$

Then there exist $A \in \mathbb{R}$, $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $u(t) = U^A(t, \cdot - x_0)$ for all $t \geq t_0$, where U^A is the solution of (gKdV) defined in proposition 4.5.

The proof of proposition 4.8 proceeds in four steps: first we improve condition (4.11) into an exponential convergence and we control the translation parameter, then we improve the exponential convergence up to any order, and finally we adapt step 3 of [5] to (gKdV) to conclude the proof. A crucial argument for the first and third steps is the coercivity of (L, \cdot) under orthogonality to eigenfunctions of the adjoint of \mathcal{L} , proved in [3].

4.4.1 Adjoint of \mathcal{L}

We recall that L is defined by $La = -\partial_x^2 a + a - pQ^{p-1}a$ and \mathcal{L} by $\mathcal{L} = -\partial_x L$. In particular, the adjoint of \mathcal{L} is $L\partial_x$. Moreover \mathcal{L} has two eigenfunctions \mathcal{Y}_\pm , with $\mathcal{L}\mathcal{Y}_\pm = \pm e_0 \mathcal{Y}_\pm$ where $e_0 > 0$.

Lemma 4.9. Let $Z_\pm = L\mathcal{Y}_\pm$. Then the following properties hold:

- (i) Z_\pm are two eigenfunctions of $L\partial_x$: $L(\partial_x Z_\pm) = \mp e_0 Z_\pm$.
- (ii) $(\mathcal{Y}_+, Z_+) = (\mathcal{Y}_-, Z_-) = 0$ and $(Z_+, Q') = (Z_-, Q') = 0$.
- (iii) There exists $\sigma_1 > 0$ such that, for all $v \in H^1$ such that $(v, Z_+) = (v, Z_-) = (v, Q') = 0$, $(Lv, v) \geq \sigma_1 \|v\|_{H^1}^2$.
- (iv) One has $(\mathcal{Y}_+, Z_-) \neq 0$ and $(Q', \mathcal{Y}'_+) \neq 0$. Hence one can normalize \mathcal{Y}_\pm and Z_\pm to have

$$(\mathcal{Y}_+, Z_-) = (\mathcal{Y}_-, Z_+) = 1, \quad (Q', \mathcal{Y}'_+) > 0 \quad \text{and still} \quad L\mathcal{Y}_\pm = Z_\pm.$$

- (v) There exist $\sigma_2 > 0$ and $C > 0$ such that for all $v \in H^1$,

$$(Lv, v) \geq \sigma_2 \|v\|_{H^1}^2 - C(v, Z_+)^2 - C(v, Z_-)^2 - C(v, Q')^2. \quad (4.12)$$

Proof. (i) It suffices to apply L to the equality $-\partial_x(L\mathcal{Y}_\pm) = \pm e_0 \mathcal{Y}_\pm$.

- (ii) We have $(\mathcal{Y}_\pm, Z_\pm) = \mp \frac{1}{e_0} (\partial_x(L\mathcal{Y}_\pm), L\mathcal{Y}_\pm) = 0$ and $(Z_\pm, Q') = (L\mathcal{Y}_\pm, Q') = (\mathcal{Y}_\pm, LQ') = 0$ since $LQ' = 0$ and L is self-adjoint.

- (iii) This fact is assertion (7) proved in [3].

- (iv) If we had $(\mathcal{Y}_+, Z_-) = (Z_+, \mathcal{Y}_-) = 0$, then by (ii) we would have in fact $(\mathcal{Y}_+ + \mathcal{Y}_-) \perp Z_+, Z_-, Q'$ since Q' is odd and $\mathcal{Y}_+ + \mathcal{Y}_-$ is even, and so by (iii): $(L(\mathcal{Y}_+ + \mathcal{Y}_-), \mathcal{Y}_+ + \mathcal{Y}_-) \geq \sigma_1 \|\mathcal{Y}_+ + \mathcal{Y}_-\|_{H^1}^2$. But $(L(\mathcal{Y}_+ + \mathcal{Y}_-), \mathcal{Y}_+ + \mathcal{Y}_-) = (L\mathcal{Y}_+, \mathcal{Y}_+) + (L\mathcal{Y}_-, \mathcal{Y}_-) + 2(L\mathcal{Y}_+, \mathcal{Y}_-) = (Z_+, \mathcal{Y}_+) + (Z_-, \mathcal{Y}_-) + 2(Z_+, \mathcal{Y}_-) = 0$, and so we would get $\|\mathcal{Y}_+ + \mathcal{Y}_-\|_{H^1} = 0$, i.e. $\mathcal{Y}_+ = -\mathcal{Y}_-$, which is a contradiction with the independence of the family $(\mathcal{Y}_+, \mathcal{Y}_-)$.

Similarly, if we had $(Q', \mathcal{Y}'_+) = 0$, we would have $(Q'', \mathcal{Y}_+) = 0$. Moreover we have $(Q, \mathcal{Y}_+) = -\frac{1}{e_0} (Q, (L\mathcal{Y}_+)') = \frac{1}{e_0} (LQ', \mathcal{Y}_+) = 0$, and so we would have

$$(Q, Z_+) = (Q, L\mathcal{Y}_+) = (LQ, \mathcal{Y}_+) = (-Q'' + Q - pQ^p, \mathcal{Y}_+) = -p(Q - Q'', \mathcal{Y}_+) = 0.$$

But we would also have $(Q, Z_-) = 0$ since Q is even and $Z_- = \check{Z}_+$. Since $(Q, Q') = 0$, we would finally have $(LQ, Q) \geq \sigma_1 \|Q\|_{H^1}^2$ by (iii). But a straightforward calculation gives $(LQ, Q) = (1-p) \int Q^{p+1} < 0$, and so a contradiction.

Finally, if we note $\eta = (\mathcal{Y}_+, Z_-) \neq 0$, then the normalization $\widetilde{\mathcal{Y}}_- = \frac{1}{\eta} \mathcal{Y}_-$, $\widetilde{Z}_- = \frac{1}{\eta} Z_- = L\widetilde{\mathcal{Y}}_-$ satisfies the required properties if $(Q', \mathcal{Y}'_+) > 0$. Otherwise, it suffices to change \mathcal{Y}_\pm and Z_\pm in $-\mathcal{Y}_\pm$ and $-Z_\pm$ respectively.

- (v) Let $v \in H^1$, and decompose it as

$$v = \alpha \mathcal{Y}_+ + \beta \mathcal{Y}_- + \gamma Q' + v_\perp$$

with $\alpha = (v, Z_-)$, $\beta = (v, Z_+)$, $\gamma = \|Q'\|_{L^2}^{-2} [(v, Q') - \alpha(\mathcal{Y}_+, Q') - \beta(\mathcal{Y}_-, Q')]$ and v_\perp orthogonal to Z_+, Z_-, Q' by the previous normalization. We have by straightforward calculation $(Lv, v) = (Lv_\perp, v_\perp) + 2\alpha\beta$, and $(Lv_\perp, v_\perp) \geq \sigma_1 \|v_\perp\|_{H^1}^2$ by (iii), so we have $(Lv, v) \geq \sigma_1 \|v_\perp\|_{H^1}^2 - \alpha^2 - \beta^2$. Finally, we have by the previous decomposition of v that

$$\|v\|_{H^1}^2 \leq C(\alpha^2 + \beta^2 + \gamma^2 + \|v_\perp\|_{H^1}^2) \leq C'(\alpha^2 + \beta^2 + (v, Q')^2 + \|v_\perp\|_{H^1}^2)$$

and so $(Lv, v) \geq \sigma_1 \left[\frac{\|v\|_{H^1}^2}{C'} - \alpha^2 - \beta^2 - (v, Q')^2 \right] - \alpha^2 - \beta^2$, as desired. \square

4.4.2 Step 1: Improvement of the decay at infinity

We begin the proof of proposition 4.8 here: let u be a solution of (gKdV) verifying (4.11).

- By lemma 2.10, we can write $\varepsilon(t, x) = u(t, x + x(t)) - Q(x)$ for $t \geq t_0$ with t_0 large enough, where ε verifies $\|\varepsilon(t)\|_{H^1} \rightarrow 0$ and $\varepsilon(t) \perp Q'$ for all $t \geq t_0$. We recall that we have by proposition 2.15:

$$\varepsilon_t - (L\varepsilon)_x = (x' - 1)(Q + \varepsilon)_x + R(\varepsilon) \quad (4.13)$$

where $\|R(\varepsilon)\|_{L^1} \leq C\|\varepsilon\|_{H^1}^2$ and $|x' - 1| \leq C\|\varepsilon\|_{H^1}$.

- Now consider

$$\alpha_+(t) = \int Z_+ \varepsilon(t), \quad \alpha_-(t) = \int Z_- \varepsilon(t)$$

where Z_{\pm} are defined in lemma 4.9. Since $\|\varepsilon(t)\|_{H^1} \rightarrow 0$, we have of course $\alpha_{\pm}(t) \rightarrow 0$. The two remaining points will be to show that $\alpha_{\pm}(t)$ control $\|\varepsilon(t)\|_{H^1}$, and have exponential decay at infinity.

- First, we recall the linearization of Weinstein's functional (lemma 2.4):

$$F(Q + \varepsilon) = F(Q) + \frac{1}{2}(L\varepsilon, \varepsilon) + K(\varepsilon)$$

where $|K(\varepsilon)| \leq C\|\varepsilon\|_{H^1}^3$. But $F(Q + \varepsilon) - F(Q)$ is a constant which tends to 0 at infinity in time, and so is null, hence we get $|(L\varepsilon, \varepsilon)| \leq C\|\varepsilon\|_{H^1}^3$. We now use (4.12), which gives since $(\varepsilon, Q') = 0$:

$$(L\varepsilon, \varepsilon) \geq \sigma_2 \|\varepsilon(t)\|_{H^1}^2 - C\alpha_+^2(t) - C\alpha_-^2(t)$$

and so $\sigma_2 \|\varepsilon(t)\|_{H^1}^2 - C\alpha_+^2(t) - C\alpha_-^2(t) - C\|\varepsilon(t)\|_{H^1}^3 \leq 0$. For t_0 chosen possibly larger, we conclude that

$$\|\varepsilon(t)\|_{H^1}^2 \leq C(\alpha_+^2(t) + \alpha_-^2(t)).$$

- We have now to obtain exponential decay of α_{\pm} to conclude the first step. If we multiply (4.13) by Z_+ and integrate, we obtain

$$\alpha'_+(t) - e_0 \alpha_+(t) = (x' - 1) \int (Q + \varepsilon)_x Z_+ + \int R(\varepsilon) Z_+ = (x' - 1) \int \varepsilon_x Z_+ + \int R(\varepsilon) Z_+$$

by integrating by parts and using (i) and (ii) of lemma 4.9. By the controls of $|x' - 1|$ and $R(\varepsilon)$, we get $|\alpha'_+ - e_0 \alpha_+| \leq C\|\varepsilon\|_{H^1}^2 \leq C(\alpha_+^2 + \alpha_-^2)$. Doing similarly with Z_- , we have finally the differential system

$$\begin{cases} |\alpha'_+ - e_0 \alpha_+| \leq C(\alpha_+^2 + \alpha_-^2), \\ |\alpha'_- + e_0 \alpha_-| \leq C(\alpha_+^2 + \alpha_-^2). \end{cases} \quad (4.14)$$

$$\begin{cases} |\alpha'_+ - e_0 \alpha_+| \leq C(\alpha_+^2 + \alpha_-^2), \\ |\alpha'_- + e_0 \alpha_-| \leq C(\alpha_+^2 + \alpha_-^2). \end{cases} \quad (4.15)$$

- Now define $h(t) = \alpha_+(t) - M\alpha_-^2(t)$ where M is a large constant to define later. Multiplying (4.15) by $|\alpha_-|$ (which can of course be taken less than 1), we get

$$\begin{aligned} h'(t) &= \alpha'_+(t) - 2M\alpha_-(t)\alpha'_-(t) \geq e_0 \alpha_+ - C(\alpha_+^2 + \alpha_-^2) + 2Me_0\alpha_-^2 - 2CM|\alpha_-|(\alpha_+^2 + \alpha_-^2) \\ &\geq e_0 h + 3Me_0\alpha_-^2 - 2Ch^2 - 2CM^2\alpha_-^4 - C^*\alpha_-^2 - 4CMh^2 - 4CM^3|\alpha_-|^5 - 2CM|\alpha_-|^3 \end{aligned}$$

since $\alpha_+^2 = (h + M\alpha_-^2)^2 \leq 2(h^2 + M^2\alpha_-^4)$. We now fix $M = \frac{C^*}{e_0}$, so that

$$h' \geq e_0 h - 2Ch^2 - 4CMh^2 + \alpha_-^2 \left(2Me_0 - 2CM^2\alpha_-^2 - 4CM^3|\alpha_-|^3 - 2CM|\alpha_-| \right).$$

Then for t large enough, the expression in parenthesis is positive, and so

$$h' \geq e_0 h - c_M h^2.$$

Now take t_0 large enough such that for $t \geq t_0$, we have $c_M h^2 \leq \frac{e_0}{2} |h|$, and suppose for the sake of contradiction that there exists $t_1 \geq t_0$ such that $h(t_1) > 0$. Define $T = \sup\{t \geq t_1 \mid h(t) > 0\}$ and suppose that $T < +\infty$: since $h'(t) \geq e_0 \left(h(t) - \frac{|h(t)|}{2} \right)$ for all $t \geq t_0$ and of course $h(T) = 0$, we would have in particular $h'(T) \geq 0$, so h increasing near T , and so $h(t) \leq 0$ for $t \in [T - \varepsilon, T]$, which would be in contradiction with the definition of T . Hence we have $T = +\infty$, and so $h(t) > 0$ for all $t \geq t_1$. Consequently, we would have $h'(t) \geq \frac{e_0}{2} h(t)$ for all $t \geq t_1$, and so $h(t) \geq C e^{\frac{e_0}{2} t}$, which would be a contradiction with $\lim_{t \rightarrow +\infty} h(t) = 0$. Therefore we have $h(t) \leq 0$ for all $t \geq t_0$. Since $-\alpha_+$ satisfies the same differential system, we obtain by the same technique: $\forall t \geq t_0, |\alpha_+(t)| \leq M \alpha_-^2(t)$.

- Reporting this estimate in (4.15), we obtain

$$|\alpha'_-(t) + e_0 \alpha_-(t)| \leq C \alpha_-^2(t) \leq \frac{e_0}{10} |\alpha_-(t)|$$

for t large enough. In other words, we have $|(e^{e_0 t} \alpha_-(t))'| \leq \frac{e_0}{10} |e^{e_0 t} \alpha_-(t)|$, and so by integration: $|\alpha_-(t)| \leq C e^{-\frac{9}{10} e_0 t}$. By a bootstrap argument we get $|\alpha'_-(t) + e_0 \alpha_-(t)| \leq C e^{-\frac{9}{10} e_0 t} |\alpha_-(t)|$, and so still by integration, we get $|e^{e_0 t} \alpha_-(t)| \leq C$ for all $t \geq t_0$, i.e. $|\alpha_-(t)| \leq C e^{-e_0 t}$. By the previous point, we also obtain

$$|\alpha_+(t)| \leq C e^{-2e_0 t} \quad (4.16)$$

and finally $\|\varepsilon(t)\|_{H^1}^2 \leq C(\alpha_+^2(t) + \alpha_-^2(t)) \leq C e^{-2e_0 t}$.

For clarity, we summarize the results obtained so far.

Lemma 4.10. If u is a solution of (gKdV) which verifies $\inf_{y \in \mathbb{R}} \|u(t) - Q(\cdot - y)\|_{H^1} \xrightarrow{t \rightarrow +\infty} 0$, then there exist a C^1 map $x : t \in \mathbb{R} \mapsto x(t) \in \mathbb{R}$, $t_0 \in \mathbb{R}$ and $C > 0$ such that

$$\forall t \geq t_0, \quad \|u(t, \cdot + x(t)) - Q\|_{H^1} \leq C e^{-e_0 t}.$$

4.4.3 Step 2: Removing modulation

- From the previous point, we have in fact $|(e^{e_0 t} \alpha_-(t))'| \leq C e^{-e_0 t} \in L^1([t_0, +\infty))$, and so there exists

$$\lim_{t \rightarrow +\infty} e^{e_0 t} \alpha_-(t) =: A \in \mathbb{R}$$

with $|e^{e_0 t} \alpha_-(t) - A| \leq C e^{-e_0 t}$ for $t \geq t_0$ by integration. Similarly, since $|x'(t) - 1| \leq C \|\varepsilon(t)\|_{H^1} \leq C e^{-e_0 t}$, then $\exists \lim_{t \rightarrow +\infty} x(t) - t =: x_0 \in \mathbb{R}$ with $|x(t) - t - x_0| \leq C e^{-e_0 t}$.

- Now consider the special solution U^A constructed in proposition 4.5, defined for a t_0 chosen possibly larger, and still write $U^A(t, x + t) = Q(x) + h^A(t, x)$. Let

$$v(t, x) = u(t, x + t + x_0) - Q(x) - h^A(t, x) = u(t, x + t + x_0) - U^A(t, x + t).$$

So we want to prove $v = 0$ to complete the proof of proposition 4.8. We first give estimates on v using the previous estimates on ε .

- Since $v(t, x) = \varepsilon(t, x - (x(t) - t - x_0)) - h^A(t, x) + Q(x - (x(t) - t - x_0)) - Q(x)$, then we simply obtain exponential decay for v for t_0 large enough, by lemma 2.11 and exponential decay of h^A :

$$\begin{aligned} \|v(t)\|_{H^1} &\leq \|\varepsilon(t)\|_{H^1} + \|h^A(t)\|_{H^1} + \|Q - Q(\cdot - (x(t) - t - x_0))\|_{H^1} \\ &\leq C e^{-e_0 t} + C |x(t) - t - x_0| \leq C e^{-e_0 t}. \end{aligned}$$

- Moreover, we can write

$$u(t, x) = Q(x - x(t)) + \varepsilon(t, x - x(t)) = Q(x - t - x_0) + h^A(t, x - t - x_0) + v(t, x - t - x_0).$$

If we denote $\omega(t, x) = Q(x - (x(t) - t - x_0)) - Q(x) - (x(t) - t - x_0)Q'(x)$, we have $\|\omega(t)\|_{L^\infty} \leq C(x(t) - t - x_0)^2 \leq Ce^{-2e_0t}$ by Taylor-Lagrange inequality, and

$$v(t, x) = (x(t) - t - x_0)Q'(x) - h^A(t, x) + \varepsilon(t, x - (x(t) - t - x_0)) + \omega(t, x).$$

Moreover, we have for all $x \in \mathbb{R}$ and $t \geq t_0$:

$$\begin{aligned} |\varepsilon(t, x - (x(t) - t - x_0)) - \varepsilon(t, x)| &= \left| \int_x^{x - (x(t) - t - x_0)} \partial_x \varepsilon(t, s) ds \right| \\ &\leq \sqrt{|x(t) - t - x_0|} \cdot \|\varepsilon(t)\|_{H^1} \leq Ce^{-\frac{3}{2}e_0t} \end{aligned}$$

by the Cauchy-Schwarz inequality. We have finally

$$v(t, x) = (x(t) - t - x_0)Q'(x) - h^A(t, x) + \varepsilon(t, x) + \omega(t, x) \quad (4.17)$$

where ω verifies $\|\omega(t)\|_{L^\infty} \leq Ce^{-\frac{3}{2}e_0t}$.

- Following the proof (v) in lemma 4.9, we now decompose

$$v(t, x) = \alpha_+^A(t)\mathcal{Y}_-(x) + \alpha_-^A(t)\mathcal{Y}_+(x) + \beta(t)Q'(x) + v_\perp(t, x) \quad (4.18)$$

with

$$\alpha_+^A(t) = \int Z_+ v(t), \quad \alpha_-^A(t) = \int Z_- v(t), \quad \beta(t) = \|Q'\|_{L^2}^{-2} \int (v(t) - \alpha_+^A(t)\mathcal{Y}_- - \alpha_-^A(t)\mathcal{Y}_+)Q'.$$

Hence we have $(v_\perp, Q') = (v_\perp, Z_+) = (v_\perp, Z_-) = 0$, and so by (iii) of lemma 4.9:

$$(Lv_\perp, v_\perp) \geq \sigma_1 \|v_\perp\|_{H^1}^2. \quad (4.19)$$

- Multiplying (4.17) by Z_\pm , we obtain information on α_\pm^A . Indeed, since $(Z_\pm, Q') = 0$, then we have

$$\alpha_\pm^A = -(h^A, Z_\pm) + \alpha_\pm + (\omega, Z_\pm).$$

But $|(h^A, Z_+)| \leq Ce^{-2e_0t}$ since $(\mathcal{Y}_+, Z_+) = 0$, and $|\alpha_+| \leq Ce^{-2e_0t}$ by (4.16), hence $|\alpha_+^A| \leq Ce^{-\frac{3}{2}e_0t}$. Similarly, $(\mathcal{Y}_+, Z_-) = 1$ implies that $|(h^A, Z_-) - Ae^{-e_0t}| \leq Ce^{-2e_0t}$, and since $|\alpha_- - Ae^{-e_0t}| \leq Ce^{-2e_0t}$, we also get $|\alpha_-^A| \leq Ce^{-\frac{3}{2}e_0t}$. To sum up this step, we have (4.18) with the following estimates for $t \geq t_0$:

$$|\alpha_+^A(t)| \leq Ce^{-\frac{3}{2}e_0t}, \quad |\alpha_-^A(t)| \leq Ce^{-\frac{3}{2}e_0t}, \quad \|v(t)\|_{H^1} \leq Ce^{-e_0t}. \quad (4.20)$$

In (4.20), it is essential to have obtained estimates better than Ce^{-e_0t} for α_\pm^A (see next step).

4.4.4 Step 3: Exponential decay at any order

- We want to prove in this section that v decays exponentially at any order to 0. In other words, we prove:

$$\forall \gamma > 0, \exists C_\gamma > 0, \forall t \geq t_0, \quad \|v(t)\|_{H^1} \leq C_\gamma e^{-\gamma t}. \quad (4.21)$$

It has been proved for $\gamma = e_0$, so that it is enough to prove it by induction on $\gamma \geq e_0$: suppose that $\|v(t)\|_{H^1} \leq Ce^{-\gamma t}$ and let us prove that it implies $\|v(t)\|_{H^1} \leq C'e^{-(\gamma + \frac{1}{2}e_0)t}$.

- Since u and U^A are solutions of (gKdV), v verifies the following equation:

$$\partial_t v - \partial_x v + \partial_x^3 v + \partial_x \left[(Q + h^A + v)^p - (Q + h^A)^p \right] = 0. \quad (4.22)$$

But

$$\begin{aligned} (Q + h^A + v)^p - (Q + h^A)^p &= p(Q + h^A)^{p-1}v + \sum_{k=2}^p \binom{p}{k} (Q + h^A)^{p-k} v^k \\ &= pQ^{p-1}v + \omega_1(t, x)v + \omega_2(t, x)v^2 \end{aligned}$$

where $\omega_1(t, x) = p \left(\sum_{k=1}^{p-1} \binom{p-1}{k} Q^{p-1-k} (h^A)^k \right)$ and $\omega_2(t, x) = \sum_{k=2}^p \binom{p}{k} (Q + h^A)^{p-k} v^{k-2}$. Since $\|h^A(t)\|_{L^\infty} \leq C \|h^A(t)\|_{H^1} \leq Ce^{-e_0 t}$ and $\|v(t)\|_{L^\infty} \leq C \|v(t)\|_{H^1} \leq C$, we have the estimates

$$\|\omega_1(t)\|_{L^\infty} \leq Ce^{-e_0 t}, \quad \|\omega_2(t)\|_{L^\infty} \leq C, \quad (4.23)$$

and (4.22) can be rewritten

$$\partial_t v + \mathcal{L}v + \partial_x [\omega_1(t, x)v] + \partial_x [\omega_2(t, x)v^2] = 0. \quad (4.24)$$

- If we multiply (4.24) by Z_+ and integrate, we get $\alpha_+^{A'} - e_0 \alpha_+^A = \int \omega_1 v Z_+' + \int \omega_2 v^2 Z_+'$, and so

$$\begin{aligned} |\alpha_+^{A'} - e_0 \alpha_+^A| &\leq \|\omega_1(t)\|_{L^\infty} \|v(t)\|_{L^\infty} \|Z_+' \|_{L^1} + \|\omega_2(t)\|_{L^\infty} \|v(t)\|_{L^\infty}^2 \|Z_+' \|_{L^1} \\ &\leq Ce^{-(\gamma+e_0)t} + Ce^{-2\gamma t} \leq Ce^{-(\gamma+e_0)t}. \end{aligned}$$

Consequently, we have $|(e^{-e_0 t} \alpha_+^A)'| \leq Ce^{-(\gamma+2e_0)t}$, and since $e^{-e_0 t} \alpha_+^A(t) \xrightarrow[t \rightarrow +\infty]{} 0$ by (4.20), we get by integration $|\alpha_+^A(t)| \leq Ce^{-(\gamma+e_0)t}$.

Multiplying (4.24) by Z_- , we obtain similarly $|\alpha_-^{A'} + e_0 \alpha_-^A| \leq Ce^{-(\gamma+e_0)t}$, and so $|\alpha_-^A(t)| \leq Ce^{-(\gamma+e_0)t}$, since $|e^{e_0 t} \alpha_-^A(t)| \leq Ce^{-\frac{1}{2}e_0 t} \xrightarrow[t \rightarrow +\infty]{} 0$ still by (4.20).

- We now want to estimate $|(Lv, v)|$. To do this, we rewrite (4.22) as

$$\partial_t v + \partial_x \left[\partial_x^2 v - v + (Q + h^A + v)^p - (Q + h^A)^p \right] = 0,$$

multiply this equality by the expression in the brackets and integrate, to obtain $\int \partial_t v \cdot [\partial_x^2 v - v + (Q + h^A + v)^p - (Q + h^A)^p] = 0$. In other words, if we define

$$F(t) = \frac{1}{2} \int v_x^2 + \frac{1}{2} \int v^2 - \int \frac{1}{p+1} (Q + h^A + v)^{p+1} + \int v (h^A + Q)^p + \int \frac{1}{p+1} (h^A + Q)^{p+1},$$

we have: $F'(t) = - \int \partial_t h^A \cdot \left[(Q + h^A + v)^p - (Q + h^A)^p - pv(Q + h^A)^{p-1} \right]$.

But h^A verifies (4.3) by definition, so $\partial_t h^A = -\partial_x^3 h^A + \partial_x h^A - p\partial_x(Q^{p-1}h^A) + R(h^A)$. Moreover, by proposition 4.5, there exists $C > 0$ such that for all $t \geq t_0$, we have $\|h^A(t)\|_{H^4} \leq Ce^{-e_0 t}$. We deduce that

$$\|\partial_t h^A\|_\infty \leq C \|\partial_t h^A\|_{H^1} \leq C \|h^A(t)\|_{H^4} \leq Ce^{-e_0 t}.$$

Therefore $|F'(t)| \leq C \|\partial_t h^A\|_\infty \|v(t)\|_{L^2}^2 \leq Ce^{-(2\gamma+e_0)t}$, and so $|F(t)| \leq Ce^{-(2\gamma+e_0)t}$ by integration, since $\lim_{t \rightarrow +\infty} F(t) = 0$. Moreover, by developing $(Q + h^A + v)^{p+1}$ in the expression of F , we get

$$\begin{aligned} F(t) &= \frac{1}{2} \int v_x^2 + \frac{1}{2} \int v^2 - \frac{p}{2} \int (Q + h^A)^{p-1} v^2 - \frac{1}{p+1} \sum_{k=3}^{p+1} \binom{p+1}{k} \int (Q + h^A)^{p+1-k} v^k \\ &= \frac{1}{2} (Lv, v) - \frac{1}{2} \int \omega_1(t, x)v^2 - \int \widetilde{\omega}_2(t, x)v^3 \end{aligned}$$

where ω_1 defined above and $\widetilde{\omega}_2(t, x) = \frac{1}{p+1} \sum_{k=3}^{p+1} \binom{p+1}{k} (Q + h^A)^{p+1-k} v^{k-3}$ verify the estimates $\|\omega_1(t)\|_{L^\infty} \leq C e^{-e_0 t}$ and $\|\widetilde{\omega}_2(t)\|_{L^\infty} \leq C$. Hence we have

$$\begin{aligned} \left| F(t) - \frac{1}{2} (Lv, v) \right| &\leq \frac{1}{2} \|\omega_1(t)\|_{L^\infty} \|v(t)\|_{L^2}^2 + \|\widetilde{\omega}_2(t)\|_{L^\infty} \|v(t)\|_{H^1}^3 \\ &\leq C e^{-(2\gamma+e_0)t} + C e^{-3\gamma t} \leq C e^{-(2\gamma+e_0)t}. \end{aligned}$$

Thus, we finally obtain $|(Lv, v)| \leq C e^{-(2\gamma+e_0)t}$.

- The previous points allow us to estimate $\|v_\perp\|_{H^1}$. Indeed, we have by straightforward calculation from (4.18) the identity

$$(Lv, v) = (Lv_\perp, v_\perp) + 2\alpha_+^A \alpha_-^A,$$

and so $|(Lv_\perp, v_\perp)| \leq |(Lv, v)| + 2|\alpha_+^A| \cdot |\alpha_-^A| \leq C e^{-(2\gamma+e_0)t} + C e^{-(2\gamma+2e_0)t} \leq C e^{-(2\gamma+e_0)t}$. But from (4.19), we deduce that $\sigma_1 \|v_\perp\|_{H^1}^2 \leq C e^{-(2\gamma+e_0)t}$, and so $\|v_\perp\|_{H^1} \leq C e^{-(\gamma+\frac{1}{2}e_0)t}$.

- To conclude this step, it is now enough to estimate $|\beta(t)|$, since the conclusion will then immediately follow from decomposition (4.18). To do this, we first multiply (4.24) by Q' and integrate, so that

$$\begin{aligned} |(\partial_t v, Q') + (\mathcal{L}v, Q')| &\leq \|\omega_1(t)\|_{L^\infty} \|v(t)\|_{L^\infty} \|Q''\|_{L^1} + \|\omega_2(t)\|_{L^\infty} \|v(t)\|_{L^\infty} \|Q''\|_{L^1} \\ &\leq C e^{-(\gamma+e_0)t} + C e^{-2\gamma t} \leq C e^{-(\gamma+e_0)t}. \end{aligned}$$

Moreover, by applying \mathcal{L} to (4.18), we get $\mathcal{L}v = -e_0 \alpha_+^A \mathcal{Y}_- + e_0 \alpha_-^A \mathcal{Y}_+ + \mathcal{L}v_\perp$, and so

$$\begin{aligned} \|Q'\|_{L^2}^2 \beta'(t) &= (\partial_t v - \alpha_+^{A'} \mathcal{Y}_- - \alpha_-^{A'} \mathcal{Y}_+, Q') \\ &= (\partial_t v + \mathcal{L}v, Q') - (-e_0 \alpha_+^A \mathcal{Y}_- + e_0 \alpha_-^A \mathcal{Y}_+ + \alpha_+^{A'} \mathcal{Y}_- + \alpha_-^{A'} \mathcal{Y}_+, Q') - (\mathcal{L}v_\perp, Q') \\ &= (\partial_t v + \mathcal{L}v, Q') - (\alpha_+^{A'} - e_0 \alpha_+^A) (\mathcal{Y}_-, Q') - (\alpha_-^{A'} + e_0 \alpha_-^A) (\mathcal{Y}_+, Q') + (v_\perp, LQ''). \end{aligned}$$

Finally, we obtain thanks to all previous estimates:

$$\begin{aligned} |\beta'(t)| &\leq C |(\partial_t v + \mathcal{L}v, Q')| + C |\alpha_+^{A'} - e_0 \alpha_+^A| + C |\alpha_-^{A'} + e_0 \alpha_-^A| + C \|v_\perp\|_{L^2} \\ &\leq C e^{-(\gamma+e_0)t} + C e^{-(\gamma+e_0)t} + C e^{-(\gamma+e_0)t} + C e^{-(\gamma+\frac{1}{2}e_0)t} \leq C e^{-(\gamma+\frac{1}{2}e_0)t} \end{aligned}$$

and so $|\beta(t)| \leq C e^{-(\gamma+\frac{1}{2}e_0)t}$ by integration.

4.4.5 Step 4: Conclusion of uniqueness argument by contraction

- The final argument, which corresponds to step 3 in [5], is an argument of contraction in short time. In other words, we want to reproduce the contraction argument developed in section 4.3.2 on a short interval of time, with suitable norms.

Define $w(t, x) = v(t, x - t)$, so that (4.22) can be rewritten

$$\partial_t w + \partial_x^3 w = -\partial_x \left[(Q(x-t) + h^A(t, x-t) + w)^p - (Q(x-t) + h^A(t, x-t))^p \right].$$

If we denote $\Omega_w(t, x) = \sum_{k=1}^p \binom{p}{k} (Q(x-t) + h^A(t, x-t))^{p-k} w^k(t, x)$, then the equation on w can be rewritten

$$\partial_t w + \partial_x^3 w = -\partial_x (\Omega_w).$$

Moreover, we have by previous steps: $\forall \gamma > 0, \exists C_\gamma > 0, \forall t \geq t_0, \|w(t)\|_{H^1} \leq C_\gamma e^{-\gamma t}$.

- Now let $t_1 \geq t_0$, $\tau > 0$ to fix later, and $I = (t_1, t_1 + \tau)$. Moreover, consider the non-linear equation in \tilde{w} :

$$\begin{cases} \partial_t \tilde{w} + \partial_x^3 \tilde{w} = -\partial_x(\Omega_{\tilde{w}}), \\ \tilde{w}(t_1 + \tau) = w(t_1 + \tau). \end{cases} \quad (4.25)$$

Note that w is of course a solution of (4.25), associated to a solution u of (gKdV) in the sense of [9].

- Then for $t \in I$, we have the following Duhamel's formula:

$$\tilde{w}(t) = \mathcal{M}^I(\tilde{w})(t) := W(t - t_1 - \tau)w(t_1 + \tau) + \int_t^{t_1 + \tau} W(t - t')\partial_x[\Omega_{\tilde{w}}(t')] dt'.$$

Similarly as in section 4.3.2, we consider

$$\begin{cases} N_1^I(\tilde{w}) = \sup_{t \in I} \|\tilde{w}(t)\|_{H^1}, & N_2^I(\tilde{w}) = \|\tilde{w}\|_{L_x^5 L_I^{10}} + \|\partial_x \tilde{w}\|_{L_x^5 L_I^{10}}, \\ \Lambda^I(\tilde{w}) = \max(N_1^I(\tilde{w}), N_2^I(\tilde{w})), \end{cases}$$

and we prove that for t_1 large enough, τ small enough independently of t_1 , and $K > 1$ to determine, $\tilde{w} \mapsto \mathcal{M}^I(\tilde{w})$ is a contraction on

$$B = \{\tilde{w} \in C^0(I, H^1) \mid \Lambda^I(\tilde{w}) \leq 3K\|w(t_1 + \tau)\|_{H^1}\}.$$

In other words, we want to estimate $\Lambda^I(\mathcal{M}^I(\tilde{w}))$ in terms of $\Lambda^I(\tilde{w})$, and as in section 4.3.2, we estimate only the term

$$\partial_x \mathcal{M}^I(\tilde{w})(t) = W(t - t_1 - \tau)\partial_x w(t_1 + \tau) + \frac{\partial}{\partial x} \int_t^{t_1 + \tau} W(t - t')\partial_x[\Omega_{\tilde{w}}(t')] dt'$$

in $L_I^\infty L_x^2$ and $L_x^5 L_I^{10}$ norms. The term $\mathcal{M}^I(\tilde{w})(t)$ is treated similarly.

- Firstly, for the linear term, we have

$$\begin{cases} \|W(t - t_1 - \tau)\partial_x w(t_1 + \tau)\|_{L^2} = \|\partial_x w(t_1 + \tau)\|_{L^2} \leq \|w(t_1 + \tau)\|_{H^1}, \\ \|W(t - t_1 - \tau)\partial_x w(t_1 + \tau)\|_{L_x^5 L_I^{10}} \leq C\|\partial_x w(t_1 + \tau)\|_{L^2} \leq C\|w(t_1 + \tau)\|_{H^1}, \end{cases}$$

since W is unitary on L^2 and by the linear estimate (2.3) of [10]: $\|W(t)u_0\|_{L_x^5 L_t^{10}} \leq C\|u_0\|_{L^2}$.

- For the non linear term, we have to use estimates similar to (4.1) and (4.2). We obtain easily by a similar proof that for all $g \in L_x^1 L_I^2$,

$$\left\| \frac{\partial}{\partial x} \int_t^{t_1 + \tau} W(t - t')g(x, t') dt' \right\|_{L_I^\infty L_x^2} + \left\| \frac{\partial}{\partial x} \int_t^{t_1 + \tau} W(t - t')g(x, t') dt' \right\|_{L_x^5 L_I^{10}} \leq C\|g\|_{L_x^1 L_I^2}.$$

Hence we get

$$\begin{cases} \left\| \frac{\partial}{\partial x} \int_t^{t_1 + \tau} W(t - t')\partial_x[\Omega_{\tilde{w}}(t')] dt' \right\|_{L_I^\infty L_x^2} \leq C\|\partial_x(\Omega_{\tilde{w}})\|_{L_x^1 L_I^2}, \\ \left\| \frac{\partial}{\partial x} \int_t^{t_1 + \tau} W(t - t')\partial_x[\Omega_{\tilde{w}}(t')] dt' \right\|_{L_x^5 L_I^{10}} \leq C\|\partial_x(\Omega_{\tilde{w}})\|_{L_x^1 L_I^2}. \end{cases}$$

We deduce that we only have to estimate $\|\partial_x(\Omega_{\tilde{w}})\|_{L_x^1 L_I^2}$. There are many terms to estimate, so as in section 4.3.2, we only treat three typical terms: $\mathbf{A} = \|\partial_x \tilde{w} \cdot \tilde{w}^4 \cdot \tilde{w}^{p-5}\|_{L_x^1 L_I^2}$, $\mathbf{B} = \|\partial_x \tilde{w} \cdot (h^A)^{p-1}(t, x - t)\|_{L_x^1 L_I^2}$, $\mathbf{D} = \|\partial_x \tilde{w} \cdot Q^{p-1}(x - t)\|_{L_x^1 L_I^2}$.

For \mathbf{A} , we have by Hölder's inequality:

$$\mathbf{A} \leq \|\tilde{w}\|_{L_I^\infty L_x^\infty}^{p-5} \|\partial_x \tilde{w}\|_{L_x^5 L_I^{10}} \|\tilde{w}\|_{L_x^5 L_I^{10}}^4 \leq C e^{-e_0 t_1} N_2^I(\tilde{w})^5 \leq C' e^{-e_0 t_1} N_2^I(\tilde{w}).$$

Indeed, we have

$$\Lambda^I(\tilde{w}) \leq 3K\|w(t_1 + \tau)\|_{H^1} \leq Ce^{-e_0 t_1} \leq 1$$

for t_1 large enough, by exponential decay of w in H^1 . In particular, we have $N_2^I(\tilde{w}) \leq 1$ and $\|\tilde{w}\|_{L_I^\infty L_x^\infty}^{p-5} \leq CN_1^I(\tilde{w})^{p-5} \leq Ce^{-e_0 t_1}$ since $p-5 \geq 1$.

For **B**, we write similarly

$$\mathbf{B} \leq \|h^A\|_{L_I^\infty L_x^\infty}^{p-5} \|\partial_x \tilde{w}\|_{L_x^5 L_I^{10}} \|h^A(t, x-t)\|_{L_x^5 L_I^{10}}^4.$$

Moreover, we have by construction of h^A (see section 4.3.2), $\|h^A\|_{L_I^\infty L_x^\infty}^{p-5} \leq Ce^{-e_0 t_1}$ since $\|h^A(t)\|_{H^1} \leq Ce^{-e_0 t} \leq Ce^{-e_0 t_1}$ for $t \geq t_1$ and $p-5 \geq 1$, and

$$\begin{aligned} \|h^A(t, x-t)\|_{L_x^5 L_I^{10}} &\leq \|h^A(t, x-t)\|_{L_x^5 L_{[t_1, +\infty)}^{10}} \\ &\leq \|(h^A - \mathcal{V}_{k_0}^A)(t, x-t)\|_{L_x^5 L_{[t_1, +\infty)}^{10}} + \|V_{k_0}^A(t, x-t)\|_{L_x^5 L_{[t_1, +\infty)}^{10}} \\ &\leq Ce^{-(k_0 + \frac{1}{2})e_0 t_1} + Ce^{-e_0 t_1} \leq Ce^{-e_0 t_1}. \end{aligned}$$

Note that the estimate $\|V_{k_0}^A(t, x-t)\|_{L_x^5 L_{[t_1, +\infty)}^{10}} \leq Ce^{-e_0 t_1}$ follows from the paragraph on **II** _{$p-2, 1, 1$} in section 4.3.2.

For **D**, we use exponential decay of Q to write

$$\begin{aligned} \mathbf{D} &\leq C \int_{\mathbb{R}} \sqrt{\int_I e^{-2|x-t|} (\partial_x \tilde{w})^2 dt} dx \leq C \int_{-\infty}^{t_1} e^x \sqrt{\int_I e^{-2t} (\partial_x \tilde{w})^2 dt} dx \\ &\quad + C \int_{t_1+\tau}^{+\infty} e^{-x} \sqrt{\int_I e^{2t} (\partial_x \tilde{w})^2 dt} dx + C \int_I \sqrt{\int_I (\partial_x \tilde{w})^2 dt} dx = \mathbf{D}_1 + \mathbf{D}_2 + \mathbf{D}_3. \end{aligned}$$

But by the Cauchy-Schwarz inequality, we get

$$\begin{cases} \mathbf{D}_1 \leq Ce^{t_1} \sqrt{\int_I e^{-2t} \int_{\mathbb{R}} (\partial_x \tilde{w})^2 dx dt} \leq Ce^{t_1} N_1^I(\tilde{w}) \sqrt{\int_I e^{-2t} dt} \leq C\sqrt{\tau} N_1^I(\tilde{w}), \\ \mathbf{D}_2 \leq Ce^{-(t_1+\tau)} \sqrt{\int_I e^{2t} \int_{\mathbb{R}} (\partial_x \tilde{w})^2 dx dt} \leq Ce^{-(t_1+\tau)} N_1^I(\tilde{w}) \sqrt{\int_I e^{2t} dt} \leq C\sqrt{\tau} N_1^I(\tilde{w}), \\ \mathbf{D}_3 \leq C\sqrt{\tau} \sqrt{\int_I \int_I (\partial_x \tilde{w})^2 dx dt} \leq C\tau N_1^I(\tilde{w}). \end{cases}$$

Hence we obtain $\mathbf{D} \leq C\sqrt{\tau} N_1^I(\tilde{w})$.

- In conclusion, we have shown that there exist $K, C_1, C_2 > 0$ such that

$$\Lambda^I(\mathcal{M}^I(\tilde{w})) \leq K \left[\|w(t_1 + \tau)\|_{H^1} + C_1 e^{-e_0 t_1} \Lambda^I(\tilde{w}) + C_2 \sqrt{\tau} \Lambda^I(\tilde{w}) \right].$$

Now fix $\tau = \frac{1}{9C_2^2 K^2}$ and t_1 such that $C_1 e^{-e_0 t_1} \leq \frac{1}{3K}$, thus we get

$$\Lambda^I(\mathcal{M}^I(\tilde{w})) \leq K \|w(t_1 + \tau)\|_{H^1} + \frac{2}{3} \Lambda^I(\tilde{w}).$$

We conclude that \mathcal{M}^I maps B into itself for this choice of t_1, τ, K . We prove similarly that \mathcal{M}^I is a contraction on B , and so there exists a unique solution $\tilde{w} \in B$ of (4.25).

- Now we identify w and \tilde{w} . It is well known for (gKdV) that for regular solutions (H^2), uniqueness holds by energy method. Since w and \tilde{w} are both obtained by fixed point, we get $w = \tilde{w}$ by continuous dependence, persistence of regularity and density. In particular, $w \in B$, and so

$$\|w(t_1)\|_{H^1} \leq N_1^I(w) \leq \Lambda^I(w) \leq 3K\|w(t_1 + \tau)\|_{H^1}.$$

To conclude the proof, we fix $t \geq t_1$, and we remark that a simple iteration argument and the exponential decay at any order of w show that for all $n \in \mathbb{N}$, we have

$$\|w(t)\|_{H^1} \leq (3K)^n \|w(t + n\tau)\|_{H^1} \leq C_\gamma (3K)^n e^{-\gamma t} e^{-\gamma n\tau} = C_\gamma e^{-\gamma t} (3K e^{-\gamma\tau})^n.$$

We finally choose γ large enough so that $3K e^{-\gamma\tau} \leq \frac{1}{2}$. Thus,

$$\|w(t)\|_{H^1} \leq \frac{C}{2^n} \xrightarrow{n \rightarrow +\infty} 0,$$

i.e. $\|w(t)\|_{H^1} = 0$. This finishes the proof of proposition 4.8.

4.5 Corollaries and remarks

Corollary 4.11. Let $c > 0$.

1. There exists a one-parameter family $(U_c^A)_{A \in \mathbb{R}}$ of solutions of (gKdV) such that

$$\forall A \in \mathbb{R}, \exists t_0 \in \mathbb{R}, \forall s \in \mathbb{R}, \exists C > 0, \forall t \geq t_0, \quad \|U_c^A(t, \cdot + ct) - Q_c\|_{H^s} \leq C e^{-e_0 c^{3/2} t}.$$

2. If u_c is a solution of (gKdV) such that $\lim_{t \rightarrow +\infty} \inf_{y \in \mathbb{R}} \|u_c(t) - Q_c(\cdot - y)\|_{H^1} = 0$, then there exist $A \in \mathbb{R}$, $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $u_c(t) = U_c^A(t, \cdot - x_0)$ for $t \geq t_0$.

Proof. The proof, based on the scaling invariance, is very similar to the proof of corollary 3.14. We recall that if $u(t, x)$ is a solution of (gKdV), then $\lambda^{\frac{2}{p-1}} u(\lambda^3 t, \lambda x)$ with $\lambda > 0$ is also a solution.

1. We define U_c^A by $U_c^A(t, x) = c^{\frac{1}{p-1}} U^A(c^{3/2} t, \sqrt{c} x)$, where U^A is defined in theorem 1.1. Since $U^A(c^{3/2} t, \sqrt{c} x + c^{3/2} t) = Q(\sqrt{c} x) + A e^{-e_0 c^{3/2} t} \mathcal{Y}_+(\sqrt{c} x) + O(e^{-2e_0 c^{3/2} t})$ and $Q_c(x) = c^{\frac{1}{p-1}} Q(\sqrt{c} x)$, then U_c^A satisfies

$$U_c^A(t, x + ct) = Q_c(x) + A c^{\frac{1}{p-1}} e^{-e_0 c^{3/2} t} \mathcal{Y}_+(\sqrt{c} x) + O(e^{-2e_0 c^{3/2} t}).$$

2. Let u be the solution of (gKdV) defined by $u(t, x) = c^{-\frac{1}{p-1}} u_c\left(\frac{t}{c^{3/2}}, \frac{x}{\sqrt{c}}\right)$. Then we have

$$u(t, x) - Q(x - y) = c^{-\frac{1}{p-1}} u_c\left(\frac{t}{c^{3/2}}, \frac{x}{\sqrt{c}}\right) - c^{-\frac{1}{p-1}} Q_c\left(\frac{x - y}{\sqrt{c}}\right)$$

for all $y \in \mathbb{R}$, and so like in the proof of corollary 3.14,

$$\inf_{y \in \mathbb{R}} \|u(t) - Q(\cdot - y)\|_{H^1} \leq K(c) \inf_{y \in \mathbb{R}} \left\| u_c\left(\frac{t}{c^{3/2}}, \frac{x}{\sqrt{c}}\right) - Q_c\left(\cdot - \frac{y}{\sqrt{c}}\right) \right\|_{H^1} \xrightarrow{t \rightarrow +\infty} 0.$$

Therefore by theorem 1.1, there exist $A \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $u(t, x) = U^A(t, x - x_0)$, and so finally $u_c(t, x) = U_c^A\left(t, x - \frac{x_0}{\sqrt{c}}\right)$. \square

Proposition 4.12. Up to translations in time and in space, there are only three special solutions: U^1 , U^{-1} and Q . More precisely, one has (for t large enough in each case):

- (a) If $A > 0$, then $U^A(t) = U^1(t + t_A, \cdot + t_A)$ for some $t_A \in \mathbb{R}$.

(b) If $A = 0$, then $U^0(t) = Q(\cdot - t)$.

(c) If $A < 0$, then $U^A(t) = U^{-1}(t + t_A, \cdot + t_A)$ for some $t_A \in \mathbb{R}$.

Proof. (a) Let $A > 0$ and denote $t_A = -\frac{\ln A}{e_0}$. Then by proposition 4.5,

$$U^1(t + t_A, x + t + t_A) = Q(x) + e^{-e_0(t+t_A)}\mathcal{Y}_+(x) + O(e^{-2e_0t}) = Q(x) + Ae^{-e_0t}\mathcal{Y}_+(x) + O(e^{-2e_0t}).$$

In particular, we have $\lim_{t \rightarrow +\infty} \inf_{y \in \mathbb{R}} \|U^1(t + t_A) - Q(\cdot - y)\|_{H^1} = 0$, and so by proposition 4.8, there exist $\tilde{A} \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $U^1(t + t_A) = U^{\tilde{A}}(t, \cdot - x_0)$. But still by proposition 4.5, we have $U^1(t + t_A, x + t + t_A) = U^{\tilde{A}}(t, x + t + t_A - x_0) = Q(x + t_A - x_0) + \tilde{A}e^{-e_0t}\mathcal{Y}_+(x + t_A - x_0) + O(e^{-2e_0t})$, and so

$$Q(x + t_A - x_0) + \tilde{A}e^{-e_0t}\mathcal{Y}_+(x + t_A - x_0) + O(e^{-2e_0t}) = Q(x) + Ae^{-e_0t}\mathcal{Y}_+(x) + O(e^{-2e_0t}).$$

The first order imposes $x_0 = t_A$, since $\|Q - Q(\cdot + t_A - x_0)\|_{H^1} \leq Ce^{-e_0t}$ and so lemma 2.11 applies for t large. Similarly, the second order imposes $\tilde{A} = A$, as expected.

(b) Since $\inf_{y \in \mathbb{R}} \|Q(\cdot - t) - Q(\cdot - y)\|_{H^1} = 0$, then proposition 4.8 applies, so there exist $A \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $Q(x - t) = U^A(t, x - x_0)$. Hence we have by proposition 4.5

$$U^A(t, x + t) = Q(x - x_0) = Q(x) + Ae^{e_0t}\mathcal{Y}_+(x) + O(e^{-2e_0t}).$$

As in the previous case, it follows first that $x_0 = 0$, then $A = 0$, and so the result.

(c) For $A < 0$, the proof is exactly the same as $A > 0$, with $-A$ instead of A . \square

We conclude this paper by two remarks, based on the following claim. The first one is the fact that $U^{-1}(t)$ is defined for all $t \in \mathbb{R}$, and the second one is the identification of the special solution $w(t)$ constructed in section 3 among the family (U^A) constructed in section 4.

Claim 4.13. For all $c > 0$, $\|\partial_x U_c^A(t)\|_{L^2}^2 - \|Q'_c\|_{L^2}^2$ has the sign of A as long as $U_c^A(t)$ exists.

Proof. • From corollary 4.11, we have

$$\partial_x U_c^A(t, x + ct) = Q'_c(x) + Ac^{\frac{p+1}{2(p-1)}}e^{-e_0c^{3/2}t}\mathcal{Y}'_+(\sqrt{cx}) + O(e^{-2e_0c^{3/2}t})$$

and so

$$\|\partial_x U_c^A(t)\|_{L^2}^2 - \|Q'_c\|_{L^2}^2 = 2Ac^{\frac{p+1}{2(p-1)}}e^{-e_0c^{3/2}t} \int Q'_c(x)\mathcal{Y}'_+(\sqrt{cx}) dx + O(e^{-2e_0c^{3/2}t}).$$

But $\int Q'_c(x)\mathcal{Y}'_+(\sqrt{cx}) dx = c^{\frac{1}{p-1}} \int Q'(y)\mathcal{Y}'_+(y) dy > 0$ by the substitution $y = \sqrt{cx}$ and the normalization chosen in lemma 4.9, and so $\|\partial_x U_c^A(t)\|_{L^2}^2 - \|Q'_c\|_{L^2}^2$ has the sign of A for t large enough.

- It remains to show that this fact holds as long as $U_c^A(t)$ exists. For example, suppose that $A > 0$ and so $\|\partial_x U_c^A(t)\|_{L^2}^2 - \|Q'_c\|_{L^2}^2 > 0$ for $t \geq t_1$, and suppose for the sake of contradiction that there exists $T < t_1$ such that $U^A(T)$ is defined and $\|\partial_x U_c^A(T)\|_{L^2}^2 = \|Q'_c\|_{L^2}^2$. Since $\|U_c^A(t, \cdot + ct) - Q_c\|_{H^1} \rightarrow 0$, then by (1.1) and (1.2), we also have $\|U_c^A(T)\|_{L^2} = \|Q_c\|_{L^2}$ and $E(U_c^A(T)) = E(Q_c)$. In other words, we would get by scaling

$$\|U^A(T)\|_{L^2} = \|Q\|_{L^2}, \quad \|\partial_x U^A(T)\|_{L^2} = \|Q'\|_{L^2} \quad \text{and} \quad E(U^A(T)) = E(Q).$$

But the two last identities give in particular $\int U^A(T)^{p+1} = \int Q^{p+1}$, and so by (1.4)

$$\|U^A(T)\|_{L^{p+1}}^{p+1} \geq \|Q\|_{L^{p+1}}^{p+1} = C_{GN}(p)\|Q'\|_{L^2}^{\frac{p-1}{2}}\|Q\|_{L^2}^{\frac{p+3}{2}} = C_{GN}(p)\|\partial_x U^A(T)\|_{L^2}^{\frac{p-1}{2}}\|U^A(T)\|_{L^2}^{\frac{p+3}{2}}.$$

Still by (1.4), we get $(\lambda_0, a_0, b_0) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}$ such that $U^A(T, x) = a_0 Q(\lambda_0 x + b_0)$. But $\|U^A(T)\|_{L^2} = \|Q\|_{L^2}$ and $\|\partial_x U^A(T)\|_{L^2} = \|Q'\|_{L^2}$ impose $\lambda_0 = 1$ and $a_0 \in \{-1, 1\}$. Thus, by uniqueness in (gKdV), $U^A(t, x) = \pm Q(x - t + T + b_0)$ for all $t \geq T$. In particular, $\|\partial_x U_c^A(t)\|_{L^2}^2 = \|Q_c'\|_{L^2}^2$ for $t \geq t_1$, which is a contradiction. The cases $A = 0$ and $A < 0$ are treated similarly. \square

Remark 4.14. Let us now notice that U^{-1} is globally defined, i.e. $U^{-1}(t)$ exists for all $t \in \mathbb{R}$. By the blow up criterion and the mass conservation, it is enough to remark that $\|\partial_x U^{-1}(t)\|_{L^2}$ is bounded uniformly on its interval of existence, which is an immediate consequence of claim 4.13 since $\|\partial_x U^{-1}(t)\|_{L^2} < \|Q'\|_{L^2}$ for all t .

Remark 4.15. As noticed in remark 2.13, we can chose $\lambda_n = 1 - \frac{1}{n}$ in the definition of $u_{0,n}$ in section 3. We still call $w(t)$ the special solution obtained by this method for this new initial data. In this remark, we prove that $w = U_{c_+}^{-1}$ up to translations in time and in space. We do not know if U^1 can be obtained similarly by a compactness method. We recall that $u_{0,n}(x) = \lambda_n Q(\lambda_n^2 x)$, $u_n(T_n, \cdot + x_n(T_n)) \rightharpoonup \check{w}_0 \neq Q_{c_+}$ and $\|w(t, \cdot + \rho(t)) - Q_{c_+}\|_{H^1} \rightarrow 0$.

- First note that $\int u_{0,n}^2 = \lambda_n^4 \int Q^2 < \int Q^2$ for $n \geq 2$, and let us prove that $\|\partial_x(u_n(T_n))\|_{L^2} < \|Q'\|_{L^2}$ for n large enough. Otherwise, there would exist n large and $T \in [0, T_n]$ such that $\|\partial_x(u_n(T))\|_{L^2} = \|Q'\|_{L^2}$ and $E(u_{0,n}) < E(Q)$. But we have by (1.2),

$$\begin{aligned} E(u_{0,n}) &= E(u_n(T)) = \frac{1}{2} \int (\partial_x(u_n(T)))^2 - \frac{1}{p+1} \int u_n^{p+1}(T) = \frac{1}{2} \int Q'^2 - \frac{1}{p+1} \int u_n^{p+1}(T) \\ &< E(Q) = \frac{1}{2} \int Q'^2 - \frac{1}{p+1} \int Q^{p+1}. \end{aligned}$$

Hence, as $\|u_n(T)\|_{L^2} = \|u_{0,n}\|_{L^2} = \|Q\|_{L^2}$ by (1.1),

$$\begin{aligned} \|u_n(T)\|_{L^{p+1}}^{p+1} &\geq \int u_n^{p+1}(T) > \int Q^{p+1} = C_{\text{GN}}(p) \left(\int Q'^2 \right)^{\frac{p-1}{4}} \left(\int Q^2 \right)^{\frac{p+3}{4}} \\ &= C_{\text{GN}}(p) \left(\int (\partial_x(u_n(T)))^2 \right)^{\frac{p-1}{4}} \left(\int u_n^2(T) \right)^{\frac{p+3}{4}}, \end{aligned}$$

which would be a contradiction with the Gagliardo-Nirenberg inequality (1.3).

- Since $u_n(T_n, \cdot + x_n(T_n)) \rightharpoonup \check{w}_0$ in H^1 , we obtain $\|w'_0\|_{L^2} \leq \|Q'\|_{L^2}$ and $\|w_0\|_{L^2} \leq \|Q\|_{L^2}$ by weak convergence. But $\|w(t, \cdot + \rho(t)) - Q_{c_+}\|_{H^1} \rightarrow 0$ implies by (1.1) and (2.1) that

$$\|w_0\|_{L^2}^2 = \|w(t)\|_{L^2}^2 = \|Q_{c_+}\|_{L^2}^2 = c_+^{\frac{5-p}{2(p-1)}} \|Q\|_{L^2}^2 \leq \|Q\|_{L^2}^2,$$

thus $c_+ \geq 1$, and so $\|w'_0\|_{L^2}^2 \leq \|Q'\|_{L^2}^2 = c_+^{-\frac{p+3}{2(p-1)}} \|Q'_{c_+}\|_{L^2}^2 \leq \|Q'_{c_+}\|_{L^2}^2$ by (2.1).

- Finally, since $\|w(t, \cdot + \rho(t)) - Q_{c_+}\|_{H^1} \rightarrow 0$, corollary 4.11 applies, and so there exists $A \in \mathbb{R}$ such that $w = U_{c_+}^A$ up to a translation in space. But the conclusion of the previous point and claim 4.13 impose $A < 0$ (note that $A \neq 0$ since $w_0 \neq Q_{c_+}$), i.e. $w = U_{c_+}^{-1}$ up to translations in time and in space by proposition 4.12.

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