Another bridge between Nim and Wythoff

Eric Duchene, Aviezri S. Fraenkel, Sylvain Gravier, Richard J. Nowakowski

Another bridge between Nim and Wythoff

E. Duchêne,
ERTé "Maths à modeler", University of Liège, Belgium

A. S. Fraenkel,
Dept. of Computer Science and Applied Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel

S. Gravier,
ERTé "Maths à modeler", Institut Fourier, Grenoble, France

R. J. Nowakowski
Dept of Mathematics, Dalhousie University, Halifax, Canada

Abstract

The $\mathcal{P}$ positions of both two-heap Nim and Wythoff's game are easy to describe, more so in the former than in the latter. Calculating the actual $\mathcal{G}$ values is easy for Nim but seemingly hard for Wythoff's game. We consider what happens when the rules for removing from both heaps are modfied in various ways.

Key words: Nim, Wythoff's game, Sprague-Grundy function, impartial combinatorial game.

AMS subject classification: 91A46

Abbreviated Title: Another Bridge between Nim and Wythoff.

1 Introduction

In the game of Nim, played on two heaps of tokens, the two players alternate in choosing a heap and removing any positive number of tokens from that heap. Wythoff's game is also played with two heaps, as in Nim, a player may remove any positive number from a single heap or the same positive number from both heaps, subject to the proviso that every heap remains of nonnegative size at all times. For both games, and in all other games considered in the sequel, the player first unable to move loses.

Definition 1 *For a position v of an impartial game, let* $\mathrm{Op}(v)$ *be the set of the options of v. That is* $\mathrm{Op}(v)$ *is the set of all the positions that can be reached from v in one move.*

Definition 2 *A $\mathcal{P}$ position is one in which the next player has no winning move and in an $\mathcal{N}$ position, the next player does have a winning move.*

In WYTHOFF and in NIM, the position in which both heaps are empty is a $\mathcal{P}$ position and any position that has a move to a $\mathcal{P}$ position is an $\mathcal{N}$ position. Every move from a $\mathcal{P}$ position leads to an $\mathcal{N}$ position.

Let $U \subset \mathbb{Z}_{\geq 0}$, $U \neq \mathbb{Z}_{\geq 0}$. The *Minimum EXcluded value* of U, denotd by $Mex(U)$, is the smallest nonnegative integer not in U. In particular, $Mex(\emptyset) = 0$.

The $\mathcal{P}$ and $\mathcal{N}$ classsification of positons can be refined. To each game position v of an impartial game we associate a nonnegative integer value $\mathcal{G}(v)$, called the $\mathcal{G}$-value of v. This function $\mathcal{G}$ is called the *Sprague-Grundy function*. It can be defined recursively as follows:

$$\mathcal{G}(v) = Mex(\{\mathcal{G}(u) : u \in Op(v)\}).$$

It is well-known that the 0s of the Grundy function constitute the $\mathcal{P}$ positions of a game. (See e.g. [1, 5, 14] for more information on the $\mathcal{G}$-function.) Note that this function exists uniquely for any finite impartial acyclic game. In an acyclic game, each game position is reached at most once.

Definition 3 *Given two positive integers a and b, $a \bmod b$ denotes the smallest non-negative remainder of the division of a by b.*

Definition 4 *We denote the nim-sum of a and b by $a \oplus b$, that is, addition in binary without carries. (Also known as XOR or addition over GF(2) of a and b.)*

In two heap NIM, the $\mathcal{G}$-value of (a, b) is $a \oplus b$ and is a $\mathcal{P}$ position precisely when $a = b$. In WYTHOFF's game, the $\mathcal{P}$ positions are $(\lfloor n\tau \rfloor, \lfloor n\tau^2 \rfloor)$, where $\tau = (1 + \sqrt{5})/2$ is the golden ratio. The non-zero $\mathcal{G}$-values appear to be difficult to calculate (cf. [2, 16]), however they exhibit additive periodicity. See [6]. A much simplified proof is given in [15].

In the literature, several variations of WYTHOFF's game were investigated, some concerning its $\mathcal{P}$ positions, others its Sprague-Grundy function. The variations can be subdivided mainly into two categories: (i) extensions, i.e., adjoining new rules to those of WYTHOFF, and (ii), restrictions, where only certain subsets of WYTHOFF's moves are permitted. Most investigations concern (i). Examples are [7], where the "diagonal" move (taking from both piles) is relaxed to taking $k > 0$ from one pile, $\ell > 0$ from the other, subject to $|k - \ell| < a$, where a is a fixed positive integer parameter. In [8], this rule is further extended to permit diagonal moves of the form $0 \leq k - \ell < (s-1)k + t$, $k \in \mathbb{Z}_{\geq 1}$, where $s, t \in \mathbb{Z}_{\geq 1}$ are fixed parameters. See also [4, 9, 13]. Still other extensions are generalizations to more than 2 piles [11], [18], [17].

Examples of (ii) are [13], where $m \geq 2$ piles are considered to be components of a vector, and removals can be made only from the first and the last end piles of the vector. In [12], the diagonal moves of WYTHOFF are restricted in certain ways. In [10] the moves from a single pile are restricted, and the diagonal moves are both restricted and extended!

In this paper, we define a new variation of WYTHOFF's game called Wyt_K. The rules are more restrictive than in classical WYTHOFF's game but are in the same spirit as [12], which appears to constitute the first bridge between NIM and WYTHOFF's game. In [12], the authors deal with games where the "diagonal move" (i.e., taking $k > 0$ from one pile and $\ell > 0$ from the other) is subject to a relation $R(k, \ell)$. Such games are called NIMHOFF games. WYTHOFF's game is NIMHOFF's game where $R(k, k)$ for all $k > 0$, whereas NIM is the game where no pair (k, ℓ) satisfies R. The main objective there is to a find a closed formula for the $\mathcal{G}$-values of NIMHOFF games, for some particular relations R. The family of *cyclic* NIMHOFF games is widely studied in [12]. This family contains games of the type $R(k, \ell)$ if $0 < k+l < h$, where h is a fixed positive integer. In addition to these results, the cases $R(1, 1)$ and $R(k, k)$ for all k being a power of 2 are investigated. A generalized *Nim sum* is provided to ensure the polynomiality of the $\mathcal{G}$-function of such games.

In the present paper, the games Wyt_K that we investigate are exactly the subset of NIMHOFF games corresponding to restrictions of WYTHOFF's game. Unlike the previous paper, we here focus on the *regularity* of $\mathcal{G}$-functions (defined in Sect. 2). The set Wyt_K is an illustration of games having a certain regular $\mathcal{G}$-function that we call p-Nim regular. For that purpose, the games $R(1, 1)$ and $R(2^q, 2^q)$ of [12] are also considered here as instances of Wyt_K having such a regular $\mathcal{G}$-function. Besides, we deal with the conditions for which a game does or does not have a p-Nim regular $\mathcal{G}$-function.

Definition 5 *Let K be a subset of the positive integers. The game Wyt_K is* *played with two heads of tokens and*

$$\begin{aligned}\mathrm{Op}(a,b) &= \{(a-i,b) : 0 < i \le a\} \cup \{(a,b-j) : 0 < j \le b\} \\ &\quad \cup\{(a-k,b-k) : 0 < k \le \min\{a,b\}, k \in K\}.\end{aligned}$$

That is, for a given K, Wyt_K is WYTHOFF's Nim but with a restricted set K of moves along the diagonal.

Specifically, we focus on the following questions:

1. What are the $\mathcal{P}$ positions for Wyt_K?
2. For any non-negative integer j, is there an a_j such that $(a_j, a_j + j)$ is a $\mathcal{P}$ position?
3. What are the $\mathcal{G}$-values and do they exhibit any regularity?

The interest in the first question is clear. The second is an indication of how close the game is to WYTHOFF's game. The third is clear but needs a little explanation. Subtraction games have periodic Sprague-Grundy functions; many infinite octal games (including one-heap NIM) have arithmetic periodic Sprague-Grundy functions. As noted, the rows of the Sprague-Grundy function of WYTHOFF's game are ultimately additive periodic [6], [15]. For many other games, when a player is analyzing a new game, hand calculations are usually tried first, varying the value of just one or two heaps, say of size k, and calculating the corresponding Sprague-Grundy function, call it $\mathcal{G}(k)$, Even though, initially, this sequence can hold the promise of regularity, the appearance of values k' where $\mathcal{G}(k') \ge k'$ is a typical indicator of impending chaos. This is the motivation behind our definition of nim-regularity in section 2, where we give an automatic test (one suitable for computers) for checking for this regularity. This test forms the basis for our positive results, but it also leads, later on, to conditions where games do not have any of the aforementioned periodicities, though it may have other regularities. This negative result does not appear (explicitly) in [12, 2, 3].

In section 3 we show that when $K = \{k\}$, the $\mathcal{P}$ positions of Wyt_K are nim-regular. For k even, this is (essentially) Lemma 10 of [12]. We complete the picture for k odd in this section.

In section 4 we consider the case where K contains only powers of 2. In 4.1 we handle the case $K = \{1\}$, followed by stating, in the present language, the case $K = \{2^k\}$ for fixed $k > 0$, already given in [12]. We then state and prove the negative result alluded to earlier. In section 4.2

we deal with the case where $|K|$ is an infinite set of powers of 2. The case $K = \{1, 2^i, i \in I \subseteq \mathbb{Z}_{\geq 1}\}$, turns out to be equivalent to the game where $K = \{1\}$. In that case, we show a surprising regularity of the $\mathcal{G}$-values modulo 3. We wrap up with a brief final section 5.

2 Closed p-Nim Regularity Check

Our basic definitions are the following.

Definition 6 *Let A be a doubly, semi-infinite matrix and A_p be the finite matrix consisting of the first p rows and first p columns of A. The matrix A is called* p**-nim-regular** *if*

$$A(a, b) = p\left(\left\lfloor \frac{a}{p} \right\rfloor \oplus \left\lfloor \frac{b}{p} \right\rfloor\right) + A_p(a \bmod p, b \bmod p) \text{ for } a, b \in \mathbb{Z}_{\geq 0};$$

if, in addition, each row and each column of A_p contains all the integers 0 through $p - 1$ then A is called **closed** p**-nim-regular**. *A game whose $\mathcal{G}$-values constitute a (closed) p-nim-regular matrix, is said to be a* **(closed)** p**-nim-regular game**.

Figure 1 illustrates this definition with $p = 4$. Roughly speaking, one can say that a p-nim-regular matrix is obtained by tiling the quarter-plane with copies of A_p scaled according to the Nim matrix.

[a/p]		0				1				2				3				
[b/p]	b \ a	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14	17
	2	2	3	1	0	6	7	5	4	10	11	9	8	14	15	13	12	18
	3	3	2	0	1	7	6	4	5	11	10	8	9	15	14	12	13	19
	4	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11	20
1	5	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10	21
	6	6	7	5	4	2	3	1	0	14	15	13	12	10	11	9	8	22
	7	7	6	4	5	3	2	0	1	15	14	12	13	11	10	8	9	23
	8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7	24
2	9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6	25
	10	10	11	9	8	14	15	13	12	2	3	1	0	6	7	5	4	26
	11	11	10	8	9	15	14	12	13	3	2	0	1	7	6	4	5	27
	12	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3	28
	13	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2	29

$A_K(p)$

Figure 1: The first $\mathcal{G}$-values of $\mathrm{Wyt}_K$when $K = \{2\}$: an example of a closed 4-nim-regular matrix

Notation 1 *For a given Wyt_K and $p \in \mathbb{Z}_{\geq 1}$, denote by $A_K(p)$ a $p \times p$ matrix where $A_K(p)(i,j) = \mathcal{G}(i,j)$, i.e. the (i,j) entry of $A_K(p)$ is the $\mathcal{G}$-value of the position with heaps of size i and j, for all $0 \leq i,j < p$.*

Note that if A is a closed p-nim-regular matrix, then in the matrix $A_{2^k p}$, every row and column contains all the values 0 through $2^k p - 1$; i.e., this matrix is a *latin square.*

Theorem 1 of [12] shows that the following game is p-nim-regular: given 2 heaps and allowing the subtraction of a_i from heap i, $i \leq 2$ where $a_1 + a_2 < p$. Here, Theorem 5 notes that not all Wyt_K with $|K| = 1$ are closed p-nim-regular.

We present an automatic check for closed p-nim-regularity for any finite set K.

Lemma 1 *Let $K \subseteq \mathbb{Z}_{\geq 1}$ be a finite set and A be the matrix with entries $\mathcal{G}(a,b)$. If there is a positive integer $p > \max K$ such that*

(i) each row and each column of $A_K(p)$ contains all the integers 0 through $p-1$; and

(ii) $\mathcal{G}(a+p-k, b+p-k) \neq \mathcal{G}(a,b)$ for $0 \leq a+p-k, b+p-k < p$, $0 \leq a,b < p$ and $k \in K$,

then A is closed p-nim regular.

Figure 1 above shows the first few $\mathcal{G}$-values of Wyt_K for $K = \{2\}$. In that case, the value $p = 4$ makes $A_K(p)$ satisfy both conditions of Lemma 1.

Proof: Assume there exists some $p > \max K$ satisfying both conditions (i) and (ii).

Now, let (M_n) be the following sequence of matrices:

$$M_n = \begin{bmatrix} M_{n-1} & M_{n-1} + 2^{n-1}p \\ M_{n-1} + 2^{n-1}p & M_{n-1} \end{bmatrix}$$

for all $n \geq 1$. Set $M_0 = A_K(p)$.

We will now prove four properties about the sequence (M_n):

1. $M_n = A_K(2^n p)$.
2. $M_n(a,b) = p(\lfloor \frac{a}{p} \rfloor \oplus \lfloor \frac{b}{p} \rfloor) + A_K(p)(a \bmod p, b \bmod p)$ for $0 \leq a,b < 2^n p$.

3. Each row and each column of M_n contains all the integers 0 through $2^n p - 1$.

4. $M_n(a + 2^n p - k, b + 2^n p - k) \neq M_n(a, b)$ for $0 \leq a + 2^n p - k, b + 2^n p - k < 2^n p$, $0 \leq a, b < 2^n p$ and $k \in K$.

One can check that these properties are true for $M_0 = A_K(p)$. Now suppose that they are true for some matrix M_{n-1} with $n \geq 1$, and consider the matrix M_n.

1. We will prove that $M_n = A_K(2^n p)$.

   Let $0 \leq a, b < 2^{n-1}p$. By the induction hypothesis, we have $\mathcal{G}(a, b) = M_n(a, b) = M_{n-1}(a, b)$.

   Now consider the position $(a, b + 2^{n-1}p)$. According to the rules of the game, we have

$$\begin{aligned}\mathcal{G}(a, b + 2^{n-1}p) &= \mathrm{mex}(\{\mathcal{G}(a - i, b + 2^{n-1}p) : 0 < i \leq a\} \\ &\quad \cup \{\mathcal{G}(a, b + 2^{n-1}p - j) : 0 < j \leq b + 2^{n-1}p\} \\ &\quad \cup \{\mathcal{G}(a - k, b + 2^{n-1}p - k) : 0 < k \leq a,\ k \in K\}).\end{aligned}$$

   One may assume inductively that $\mathcal{G}(s, t) = M_n(s, t)$ for all the pairs $(s, t) \neq (a, b + 2^{n-1}p)$ satisfying $0 \leq s \leq a$ and $0 \leq t \leq b + 2^{n-1}p$. With this hypothesis and by construction of M_n, we have $\{\mathcal{G}(a - i, b + 2^{n-1}p) : 0 < i \leq a\} = \{\mathcal{G}(a - i, b) + 2^{n-1}p : 0 < i \leq a\}$.

   Similarly

$$\begin{aligned}&\{\mathcal{G}(a, b + 2^{n-1}p - j) : 0 < j \leq b + 2^{n-1}p\} \\ &= \{\mathcal{G}(a, b + 2^{n-1}p - j) : 0 < j \leq b\} \\ &\quad \cup \{\mathcal{G}(a, b + 2^{n-1}p - j) : b < j \leq b + 2^{n-1}p\} \\ &= \{\mathcal{G}(a, b - j) + 2^{n-1}p : 0 < j \leq b\} \\ &\quad \cup \{0, 1, 2, \ldots, 2^{n-1}p - 1\}\end{aligned}$$

   and

$$\begin{aligned}&\{\mathcal{G}(a - k, b + 2^{n-1}p - k) : 0 < k \leq a,\ k \in K\} \\ &= \{\mathcal{G}(a - k, b + 2^{n-1}p - k) : 0 < k \leq a, b,\ k \in K\} \\ &\quad \cup \{\mathcal{G}(a - k, b + 2^{n-1}p - k) : b < k \leq a,\ k \in K\} \\ &= \{\mathcal{G}(a - k, b - k) + 2^{n-1}p : 0 < k \leq a, b,\ k \in K\} \\ &\quad \cup \{\mathcal{G}(a - k, b + 2^{n-1}p - k) : b < k \leq a,\ k \in K\}.\end{aligned}$$

Since

$\{\mathcal{G}(a-k, b+2^{n-1}p-k) : b < k \leq a,\ k \in K\} \subseteq \{0, 1, 2, \ldots, 2^{n-1}p-1\}$,

we have that

$$\begin{aligned}\mathcal{G}(a, b+2^{n-1}p) &= \mathrm{mex}(\{\mathcal{G}(\mathrm{Op(a,b)}) + 2^{n-1}\mathrm{p}\} \cup \{0, 1, 2, \ldots, 2^{n-1}\mathrm{p}-1\}) \\ &= \mathcal{G}(a,b) + 2^{n-1}p.\end{aligned}$$

Now consider the position $(a+2^{n-1}p, b+2^{n-1}p)$. Then

$$\begin{aligned}&\mathcal{G}(a+2^{n-1}p, b+2^{n-1}p) \\ &\quad= \mathrm{mex}(\{\mathcal{G}(\mathrm{a}+2^{n-1}\mathrm{p}-\mathrm{i}, \mathrm{b}+2^{n-1}\mathrm{p}) : 0 < \mathrm{i} \leq \mathrm{a}+2^{n-1}\mathrm{p}\} \\ &\qquad \cup\{\mathcal{G}(a+2^{n-1}p, b+2^{n-1}p-j) : 0 < j \leq b+2^{n-1}p\} \\ &\qquad \cup\{\mathcal{G}(a+2^{n-1}p-k, b+2^{n-1}p-k) : 0 < k \leq \min(a+2^{n-1}p, b+2^{n-1}p),\ k \in K\}).\end{aligned}$$

As previously, suppose that $\mathcal{G}(s,t) = M_n(s,t)$ for all the pairs $(s,t) \neq (a+2^{n-1}p, b+2^{n-1}p)$ satisfying $0 \leq s \leq a+2^{n-1}p$ and $0 \leq t \leq b+2^{n-1}p$.
Then we have:

$$\begin{aligned}&\{\mathcal{G}(a+2^{n-1}p-i, b+2^{n-1}p) : 0 < i \leq a+2^{n-1}p\} \\ &\quad= \{\mathcal{G}(a-i, b) : 0 < i \leq a\} \cup \{2^{n-1}p, \ldots, 2^n p - 1\} \\ &\{\mathcal{G}(a+2^{n-1}p, b+2^{n-1}p-j) : 0 < j \leq b+2^{n-1}p\} \\ &\quad= \{\mathcal{G}(a, b-j) : 0 < j \leq b\} \cup \{2^{n-1}p, \ldots, 2^n p - 1\} \\ &\{\mathcal{G}(a+2^{n-1}p-k, b+2^{n-1}p-k) : 0 < k \leq \min(a+2^{n-1}p, b+2^{n-1}p),\ k \in K\} \\ &\quad= \{\mathcal{G}(a-k, b-k) : 0 < k \leq \min(a,b),\ k \in K\} \\ &\qquad \cup\{\mathcal{G}(a+2^{n-1}p-k, b+2^{n-1}p-k) : k > \min(a,b),\ k \in K\}.\end{aligned}$$

Hence

$$\begin{aligned}&\mathcal{G}(a+2^{n-1}p, b+2^{n-1}p) \\ &\quad= \mathrm{mex}(\{\mathcal{G}(\mathrm{Op(a,b)})\} \\ &\qquad \cup\{2^{n-1}p, \ldots, 2^n p - 1\} \\ &\qquad \cup\{\mathcal{G}(a+2^{n-1}p-k, b+2^{n-1}p-k) : k > \min(a,b),\ k \in K\}).\end{aligned}$$

We already know that $\mathcal{G}(a,b) \notin \{2^{n-1}p, \ldots, 2^n p - 1\}$, since $\mathcal{G}(a,b) \in M_{n-1}$.

Moreover, $\mathcal{G}(a,b) \notin \{\mathcal{G}(a+2^{n-1}p-k, b+2^{n-1}p-k) : k > \min(a,b),\ k \in K\}$. Indeed, if $a+2^{n-1}p-k \geq 2^{n-1}p$ or $b+2^{n-1}p-k \geq 2^{n-1}p$, then

$\mathcal{G}(a + 2^{n-1}p - k, b + 2^{n-1}p - k) \geq 2^{n-1}p > \mathcal{G}(a, b)$. Otherwise, if $a + 2^{n-1}p - k < 2^{n-1}p$ and $b + 2^{n-1}p - k < 2^{n-1}p$, it is true since M_{n-1} satisfies Property (4).

Therefore, we conclude that $\mathcal{G}(a + 2^{n-1}p, b + 2^{n-1}p) = \mathcal{G}(a, b)$.

2. Let $0 \leq a, b < 2^n p$. We will prove that the formula $M_n(a, b) = p(\lfloor a/p \rfloor \oplus \lfloor b/p \rfloor) + A_K(p)(a \bmod p, b \bmod p)$ is satisfied.

   - If $0 \leq a, b < 2^{n-1}p$, then it is true by induction.
   - If $2^{n-1}p \leq a, b < 2^n p$, then by construction of M_n, we have

   $$M_n(a, b) = M_n(a - 2^{n-1}p, b - 2^{n-1}p). \tag{2A}$$

   By the induction hypothesis, we expand (2A):

   $$\begin{aligned} M_n(a, b) &= M_n(a - 2^{n-1}p, b - 2^{n-1}p) \\ &= p(\lfloor a/p \rfloor - 2^{n-1} \oplus \lfloor b/p \rfloor - 2^{n-1}) + A_K(a \bmod p, b \bmod p) \\ &= p(\lfloor a/p \rfloor \oplus \lfloor b/p \rfloor) + A_K(a \bmod p, b \bmod p). \end{aligned}$$

   - If $a \geq 2^{n-1}p$ and $b < 2^{n-1}p$, then by construction:

   $$M_n(a, b) = M_n(a - 2^{n-1}p, b) + 2^{n-1}p. \tag{2B}$$

   By the induction hypothesis, we know that

   $$\begin{aligned} &M_n(a - 2^{n-1}p, b) \\ &\quad = p(\lfloor a/p \rfloor - 2^{n-1} \oplus \lfloor b/p \rfloor) + A_K(a \bmod p, b \bmod p). \end{aligned}$$

   Finally, since $a \geq 2^{n-1}p$ and $b < 2^{n-1}p$, and according to the properties of the nim-sum, we get from (2B)

   $$\begin{aligned} &M_n(a, b) \\ &\quad = 2^{n-1}p + p(\lfloor a/p \rfloor - 2^{n-1} \oplus \lfloor b/p \rfloor) + A_K(a \bmod p, b \bmod p) \\ &\quad = p(\lfloor a/p \rfloor \oplus \lfloor b/p \rfloor) + A_K(a \bmod p, b \bmod p). \end{aligned}$$

   - If $b \geq 2^{n-1}p$ and $a < 2^{n-1}p$, then we reduce to the previous case by symmetry.

3. By construction of M_n from M_{n-1}, and since M_{n-1} satisfies property (3) by the induction hypothesis, one can easily check that each row and each column of M_n contains all the integers 0 through $2^n p - 1$.

4. We will show that $M_n(a+2^np-k, b+2^np-k) \neq M_n(a,b)$ for $0 \leq a+2^np-k, b+2^np-k < 2^np$, $0 \leq a, b < 2^np$ and $k \in K$.

   The condition $0 \leq a+2^np-k, b+2^np-k < 2^np$ implies $a, b < k$. And since $k < p$, we now consider that $0 \leq a, b < p$.

   We now define two integers A and B such that:

$$\begin{aligned} A &= a + (2^n - 1)p \\ B &= b + (2^n - 1)p. \end{aligned}$$

   By the formula proved in (2), we have $M_n(a,b) = M_n(A,B)$.

   Let (X, Y) be a position defined as follows:

$$\begin{aligned} X &= a + 2^np - k \\ Y &= b + 2^np - k. \end{aligned}$$

   Thus $M_n(X,Y) = M_n(A+p-k, B+p-k) = M_n(a+p-k, b+p-k)$ according to the formula. It now suffices to prove that $M_n(a+p-k, b+p-k) \neq M_n(a,b)$.

   Since $a+2^np-k, b+2^np-k < 2^np$, we have $a-k+p, b-k+p < p$, which means that the positions (a,b) and $(a+p-k, b+p-k)$ both belong to the square $[0,\ldots,p-1] \times [0,\ldots,p-1]$. As $A_K(p)$ satisifes condition (ii), we deduce that $M_n(a+p-k, b+p-k) \neq M_n(a,b)$.

■

3 $\mathcal{P}$ positions

Theorem 2 *Let $K = \{k\}$. Then in Wyt_K,*

1. *For $k = 2j$, the $\mathcal{P}$ positions are $(i+2kp, i+2kp)$, $i = 0, 1, \ldots, k-1$, $p \in \mathbb{Z}_{\geq 0}$ and $(k+2i+2kp, k+2i+2kp+1)$, $i = 0, 1, \ldots, j-1$, $p \in \mathbb{Z}_{\geq 0}$.*

2. *For $k = 2j+1$, the $\mathcal{P}$ positions are $(i+(2k+1)p, i+(2k+1)p)$, $i = 0, 1, \ldots, k-1$, $p \in \mathbb{Z}_{\geq 0}$ and $(k+2i+(2k+1)p, k+2i+(2k+1)p+1)$, $i = 0, 1, \ldots, j$, $p \in \mathbb{Z}_{\geq 0}$.*

Proof: Denote by $S \subset \mathbb{Z}^2$ the set of positions described by the theorem. Denote by A the set $\mathbb{Z}^2 \setminus S$. We must prove that any move from S lands in a position of A, and that from any position of A, there exists a move leading to a position of S.

For any subset T of $\mathbb{Z}^2$, denote by $T|_x$ the set $\{(i,j) \in T : 0 \leq i,j < x\}$. Denote by B the value $2k$ (resp. $2k+1$) if k is even (resp. odd).

Figure 2 depicts the set S, which is the diagonal concatenation, modulo B, of the pattern $S|_B$.

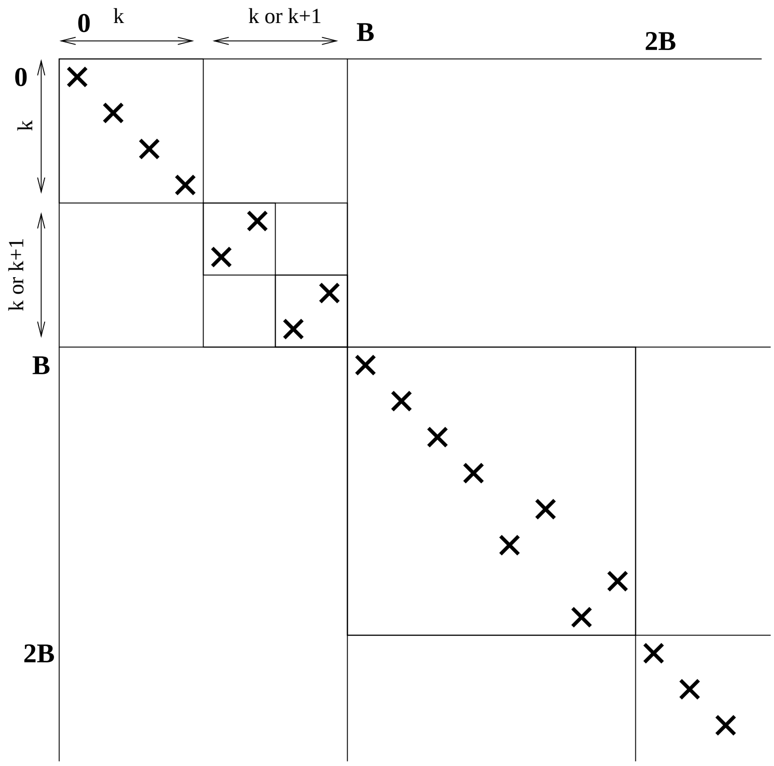

Figure 2: A view of the structure S.

We first remark that there is exactly one position of S in each row and in each column. It is then straightforward to see that it is not possible to move from a position of $S|_B$ to another one. Besides, from each position of $A|_B$, one can move to a position of $S|_B$. Therefore, from any position of the sets $\{(i,j) : i \geq B, j < B\}$ and $\{(i,j) : j \geq B, i < B\}$, one can reach a position of $S|_B$.

Now consider a position of $S \cap \{(i,j) : B \leq i,j < 2B\}$. Moves in a single heap clearly land in A. A move of length k in both heaps may land in $\mathbb{Z}^2|_B$, but not in $S|_B$ (see figure 2). Also, from any position of $A \cap \{(i,j) : B \leq i, B \leq j < 2B \vee B \leq j, B \leq i < 2B\}$, one can move to a position of $S \cap \{(i,j) : B \leq i, j < 2B\}$.

It now suffices to iterate the result for the sets $S \cap \{(i,j) : pB \leq i, j < (p+1)B\}$ and $A \cap \{(i,j) : (pB \leq i, pB \leq j < (p+1)B)$ or $(pB \leq j, pB \leq i < (p+1)B)\}$, with $p > 1$. ■

Remark 1 *When $K = \{k\}$, this result ensures that the only integers j for which there exists an a_j such that $(a_j, a_j + j)$ is a $\mathcal{P}$ position are* 0 *and* 1.

This remark leads us to the following conjecture:

Conjecture 1 *Let K be a finite set, then there exists an integer $J_K > 0$ such that if (a,b) is a $\mathcal{P}$ position for Wyt$_K$ then $|a - b| < J_K$.*

4 $\mathcal{G}$-Values

4.1 K is finite

Theorem 3 *Let $K = \{1\}$. Then in Wyt$_K$*

$$\mathcal{G}(3m+i, 3n+j) = 3(m \oplus n) + A_K(3)(i,j),\ 0 \leq i,j < 3,\ \forall m,n \in \mathbb{Z}_{\geq 0}$$

where,

$$A_K(3) \quad = \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Figure 3 illustrates this result by depicting the table of the first $\mathcal{G}$-values.

Proof: One can check that $A_K(3)$ contains the first $\mathcal{G}$-values of the game Wyt$_K$with $K = \{1\}$. With $p = 3 > \max K$, it is now straightforward to see that the conditions of Lemma 1 are satisfied. ■

For completeness, we report the cases where $K = \{2^j\}$ since the language used in [12] does not immediately lend itself to this interpretation.

Theorem 4 *Let $K = \{2^k\}$ for fixed $k > 0$. Then in Wyt$_K$*

$$\mathcal{G}(2^{k+1}m + i, 2^{k+1}n + j) = 2^{k+1}(m \oplus n) + A_K(2^{k+1})(i,j),$$
$$0 \leq i,j < 2^{k+1},\ \forall m,n \in \mathbb{Z}_{\geq 0}$$

and, as a 2×2 array of matrices,

m		0			1			2			3			
n	j \ i	0	1	2	3	4	5	6	7	8	9	10	11	12
	0	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	2	0	4	5	3	7	8	6	10	11	9	13
	2	2	0	1	5	3	4	8	6	7	11	9	10	14
	3	3	4	5	0	1	2	9	10	11	6	7	8	15
1	4	4	5	3	1	2	0	10	11	9	7	8	6	16
	5	5	3	4	2	0	1	11	9	10	8	6	7	17
	6	6	7	8	9	10	11	0	1	2	3	4	5	18
2	7	7	8	6	10	11	9	1	2	0	4	5	3	19

Figure 3: The first $\mathcal{G}$-values of Wyt$_K$for $K = \{1\}$.

$$A_K(2^{k+1}) \quad = \quad \begin{bmatrix} A_K(2^k) & A_K(2^k) \oplus 2^k \\ A_K(2^k) \oplus 2^k & A_K(2^k) \oplus 1 \end{bmatrix},$$

where $A_K(2^k)$ is the $2^k \times 2^k$ matrix of the $\mathcal{G}$-values of Nim *with two heaps of sizes lower than 2^k.*

As an illustration of that case with $k = 1$, one can refer to Figure 1.

Theorem 5 *Let $K = \{k : k \neq 2^q \neq k + 1\ \forall q \in \mathbb{Z}_{\geq 1}\}$ (for any integer $q > 0$). Then there is no p such that $A_K(p)$ satisfies Lemma 1, condition (i), i.e. Wyt$_K$is not closed p-nim-regular for any p.*

Proof: We suppose that such a p exists.

First, let $K = \{2j\}$, $j \neq 2^q$. From Theorem 2, the P-positions repeat with period $4j$, consequently, then p would be also be a multiple of $4j$. There exists an i such that $2^i < 2j$ and with the property that $2^i \oplus 2j = 2^i + 2j$. Moreover, since $2j$ is not a power of 2 then $2^{i+1} < 2j$ also holds and there is no diagonal move available from $(2^{i+1}, 4j - 1)$. Hence the $\mathcal{G}$-value of the position is $2^{i+1} \oplus 4j - 1$—this is 2-heap nim. Then we have $2^{i+1} \oplus 4j - 1 = 2^{i+1} + 4j - 1 \geq 4j$. Therefore, since $A_K(4j)$ contains a number greater than $4j - 1$ not every row and column can contain all the numbers 0 through $4j - 1$.

For $K = \{2j + 1\}$, the argument is similar. ¿From Theorem 2, p would be a multiple of $4j + 3$. Since $2j + 2$ is not a power of 2 then there exists $2^i \notin 2j + 1$, $1 < 2^i < 2j + 1$ so that $2^i \oplus 2j + 1 = 2^i + 2j + 1$. Therefore,

$2^{i+1} \oplus 4j + 2 = 2^{i+1} + 4j + 2 > 4j + 3$. The position $(2^{i+1}, 4j + 2)$ has a $\mathcal{G}$-value equal to $2^{i+1} \oplus 4j + 2$. Therefore, since $A_K(4j + 3)$ contains a number greater than $4j + 3$ not every row and column can contain all the numbers 0 through $4j + 2$. ∎

4.2 $|K|$ is an infinite set of powers of 2

When K is a subset of the powers of two including 1, we show that the $\mathcal{G}$-values of $\mathrm{Wyt}_K$are those described by Theorem 3 (see Figure 3).

Theorem 6 *Let $K = \{1, 2^i : i \in I \subseteq \mathbb{Z}_{\geq 1}\}$. Then in Wyt_K, $\mathcal{G}(3m+i, 3n+j) = 3(m \oplus n) + A_K(3)(i, j)$, $0 \leq i, j < 3$, $\forall m, n \in \mathbb{Z}_{\geq 0}$, where $A_K(3)$ is as defined in Theorem 3.*

Proof: Denote by $\mathcal{G}_1(a, b)$ the $\mathcal{G}$-value of the position (a, b) for $\mathrm{Wyt}_K$with $K = \{1\}$. The function $\mathcal{G}_1$ is described in Theorem 3. We aim at proving that $\mathcal{G}_1$ and $\mathcal{G}$ for $\mathrm{Wyt}_K$where $K = \{1, 2^i : i \in I \subseteq \mathbb{Z}_{\geq 1}\}$ are identical.

Since the moves of $\mathrm{Wyt}_K$with $K = \{1\}$ are included in those of $\mathrm{Wyt}_K$with $K = \{1, 2^i : i \in I \subseteq \mathbb{Z}_{\geq 1}\}$, it suffices to check that $\mathcal{G}_1(a, b) \neq \mathcal{G}_1(a - k, b - k)$ for all k in $\{2^i : i \in I \subseteq \mathbb{Z}_{\geq 1}\}$.

Suppose that there exists a position (a, b) and an integer $k = 2^i, i \geq 1$ such that $\mathcal{G}_1(a, b) = \mathcal{G}_1(a - k, b - k)$. Then we also have $\mathcal{G}_1(a, b) \bmod 3 = \mathcal{G}_1(a - k, b - k) \bmod 3$.

According to Theorem 3, we can assert that $\mathcal{G}_1(x, y) \bmod 3 = A_K(3)(x \bmod 3, y \bmod 3)$ for any position (x, y). This implies that $A_K(3)(a \bmod 3, b \bmod 3) = A_K(3)((a - k) \bmod 3, (b - k) \bmod 3)$. When looking at the matrix $A_K(3)$, we notice that each value $0, 1, 2$ appears exactly once in each diagonal (fig. 4).

Therefore, $A_K(3)(a \bmod 3, b \bmod 3) = A_K(3)((a-k) \bmod 3, (b-k) \bmod 3)$ if and only if $a \bmod 3 = (a-k) \bmod 3$ and $b \bmod 3 = (b-k) \bmod 3$. These equalities now imply $k \bmod 3 = 0$, which is impossible since k is a power of 2. ∎

Remark 2 *From Theorem 6, one can see that in $Wyt_K$with $K = \{2^i : i \geq 0\}$, we have $\mathcal{G}(3m+i, 3n+j) = 3(m \oplus n) + A_K(3)(i, j)$, $0 \leq i, j < 3$, $\forall m, n \in \mathbb{Z}_{\geq 0}$.*

Conjecture 2 *Let $K = \{2^i : i > 0\}$. Then for all non-negative integers j, there is an a_j such that $(a_j, a_j + j)$ is a $\mathcal{P}$ position.*

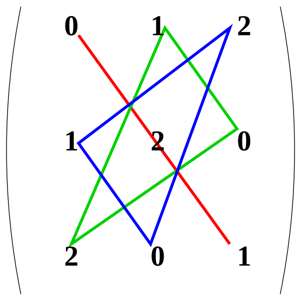

Figure 4: The three diagonals of $A_K(3)$ modulo 3

5 Concluding remarks

As we pointed out in the Introduction, this work is about *restrictions* of WYTHOFF's game. Most previous papers about WYTHOFF concerned *extensions* thereof. The Wythoff variation defined in [10] contains both a restriction and an extension. It depends on two given positive integer parameters a, b: (i) Remove a positive multiple of b tokens from a *single* pile (restriction), or (ii) remove $k > 0$, $\ell > 0$ tokens from the 2 piles, subject to the constraints $k - \ell \equiv 0 \pmod{b}$ (restriction), $|k - \ell| < ab$ (extension). Other games that are both an extension and a restriction of WYTHOFF are suggested by the present paper. For example, let $a \in \mathbb{Z}_{\geq 1}$, $K \subset \mathbb{Z}_{\geq 1}$. The diagonal move is extended as follows: take $k > 0$ from one pile and $\ell > 0$ from the other subject to $|k - \ell| < a$ (extension – see [7]) and $k \in K$ (restriction). The extension [8] can be restricted similarly. (Although in [13] there are $m \geq 2$ piles, this is not a genuine extension, since all moves are restricted to taking from at most 2 piles.)

References

[1] E.R. Berlekamp, J.H. Conway and R.K. Guy, *Winning Ways for Your Mathematical Plays* (in 4 volumes), 2nd edition, A K Peters, Wellesley, MA(2001-2004).

[2] U. Blass and A.S. Fraenkel, *The Sprague-Grundy function for Wythoff's game*, Theor. Comput. Sci. 75 (1990), 311-333.

[3] Blass, U.; Fraenkel, A. S. & Guelman, R., *How far can Nim in diguise be stretched?* J. Combin. Theory (Ser. A) 84 (1998), 145-156

[4] I.G. Connell, *A generalization of Wythoff's game*, Canad. Math. Bull. 2 (1959), 181-190.

[5] J.H. Conway, *On Numbers and Games*, 2nd edition, A K Peters, Wellesley, MA, 2001.

[6] A. Dress, A. Flammenkamp and N. Pink, *Additive periodicity of the Sprague-Grundy function of certain Nim games*, Adv. in Appl. Math. 22 (1999), 249–270.

[7] A.S. Fraenkel, *How to beat your* WYTHOFF *games' opponent on three fronts*, Amer. Math. Monthly 89 (1982), 353-361.

[8] A.S. Fraenkel, *Heap games, numeration systems and sequences,* Annals of Combinatorics 2 (1998), 197-210.

[9] A.S. Fraenkel, *Euclid and Wythoff games*, Discrete Math. 304 (2005), 65–68.

[10] A.S. Fraenkel and I. Borosh, *A generalization of Wythoff's game*, J. Combin. Theory (Ser. A) 15 (1973), 175–191.

[11] A.S. Fraenkel and D. Krieger, *The structure of complementary sets of integers: a 3-shift theorem*, Internat. J. Pure and Appl. Math. 10 (2004), 1-49.

[12] A.S. Fraenkel and M. Lorberbom, *Nimhoff games*, J. Combin. Theory (Ser. A) 58 (1991), 1-25.

[13] A.S. Fraenkel and E. Reisner, *The game of End-Wythoff*, to appear in Games of No Chance III.

[14] A.S. Fraenkel and Y. Yesha, *The generalized Sprague-Grundy function and its invariance under certain mappings*, J. Combin. Theory (Ser. A) 43 (1986), 165-177.

[15] H. Landman, *A simple FSM-based proof of the additive periodicity of the Sprague-Grundy function of Wythoff's game*, in: More Games of No Chance, Proc. MSRI Workshop on Combinatorial Games, July, 2000, Berkeley, CA, MSRI Publ. (R.J. Nowakowski, ed.), Vol. 42, Cambridge University Press, Cambridge, 2002, pp. 383–386.

[16] Gabriel Nivasch, *More on the Sprague Grundy function for Wythoff's game*, to appear in Games of No Chance III.

[17] X. Sun, *Wythoff's sequence and N-heap Wythoff's conjectures*, Discrete Math. 300 (2005), 180–195.

[18] X. Sun and D. Zeilberger, *On Fraenkel's N-heap* WYTHOFF*'s conjectures*, Ann. Comb. 8 (2004), 225–238.

[19] W. Wythoff, *A modification of the game of Nim*, Nieuw Arch. Wisk. 7 (1907), 199-202.