Invariant Measures and Decay of Correlations of a Class of Ergodic Probabilistic Cellular Automata

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Abstract

Using an extended version of the duality concept between two stochastic processes, we give new ergodicity conditions for two states probabilistic cellular automata (PCA) of any dimensions and any radius. Under these assumptions, in the one dimensional case, we study some properties of the unique invariant measure and show that it is shift mixing. Also, the decay of correlation is studied in detail. In this sense, the extended concept of duality gives exponential decay of correlation. When the extended concept of duality can not be applied we are able to get, once again, exponential decay of correlation using well known results from the theory of branching processes.

keywords Probabilistic cellular automata, Invariant measures, Duality, Decay of correlations.

1 Introduction and known results

Probabilistic cellular automata (PCA) are discrete time Markov processes which have been intensely studied since at least Stavskaja and Pjatetskii-Shapiro [13] (1968). This kind of processes have as state space a product space $X = A^G$ where A is any finite set and G is any locally finite and connected graph. On this work we will focus our attention on $G = \mathbb{Z}^d$ and $A = \{0, ..., n\}$ for some integer $n \geq 1$. We may regard a PCA as an interacting particle system where particles update its states simultaneously and independently. Recall that a

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PCA is ergodic if there exists only one invariant measure μ and starting from any initial measure μ_0 the system converges to μ.

The aim of this paper is to use the duality principle to study the ergodicity of two-states PCA. More precisely our work gives new ergodicity conditions for the expression of the PCA's local transition probabilities (see Proposition 2) and show that under these conditions the invariant measure is shift mixing with exponential decay of correlation. Relations between the PCA and the dual process (see Lemma 3 and Proposition 1) also allow us to give a very simple expression of the constant of the decay of correlation as a function of the radius (of the PCA) and the transition probabilities of the PCA. (see Theorem 2) and show in detail the way to compute the value of the invariant measure on cylinders (see Remark 3). Moreover, the comparison between the dual process and a Branching process allows to show that a certain class of dual process eventually enters in an absorbing state which implies (see Theorem 1) that the associated PCA is ergodic and its invariant measure is shift-mixing with exponential decay of correlation (see Proposition 5). In this case, the conditions on the transition probabilities for the dual process can be expressed as conditions on the probabilities of transition of the associated PCA using Lemma 2. Results about the decay of correlations is an answer to one question raised in [12].

The existence of a dual process that satisfy the duality equation (see Definition 1 and Liggett [9]) gives useful information (problems in uncountable sets can be reformulated in term of countable sets) on the PCA but is not always sufficient to prove that a PCA is ergodic. In [12], Lopez, Sanz and Sobottka introduced an extended concept of duality (see Definition 2) and give general results about ergodicity (see Theorem 1 and Remark 1). They used this powerful general theory to give results on multi-states one-dimensional PCA of radius one and extend for example previous results about the *Domany-Kinzel model* (see [3] for an introduction and [7] for extensions) in the case there is no null transition probability. Previously, in [8] Konno has given ergodicity conditions for multi-states one-dimensional PCA using a self-duality equations.

Even if, in some cases, the existence of null transition probabilities allows to prove the ergodicity of a certain class of PCA (see [7] and [8]), it had benn conjectured that in the one dimensional case positive noise cellular automata are ergodic. However, P. Gacks, in 2000 , showed a very complex counterexample (see [4] and [5]) for noisy deterministic cellular automata. In that case, the noisy one-dimensional cellular automata does not forget the past and starting from different initial distribution, the PCA may converge to different invariant measures. His result can be extended to noisy PCA with positive rates. This conjecture was formulated only in the one dimensional case since in dimension 2 or higher, it is easier to show the existence of at least two invariant measures. For example the two dimensional Ising model [5] or the Toom example (see [14]) that exhibit *eroder* properties.

In [12], the authors explore some ergodic conditions on multi-state PCA. When the number of states is greater than 2, the conditions of ergodicity are rather restrictive in order to be able to give general results. More general ergodicity conditions are interesting (see [12], Section 3.2) but seems to be very

complex when the radius grow. In this paper, we restrict the study to the 2 states case, which allows to show more easily general results for PCA of any radius. In the case of d-dimensional PCA we are able to give explicit ergodicity conditions even if in this case ergodicity seems to be a less common phenomenon.

Finally, we show that the set of all invariant measures associated to one dimensional with any radius ergodic PCA that satisfy our conditions is a connected set if we endow this set with the weak* topology.

1.1 Probabilistic Cellular Automata

Let A a be a finite set, $d \geq 1$ an integer, R a finite subset of $\mathbb{Z}^d$ of cardinal $|R|$ and f a map from $A^{|R|+1}$ to $[0,1]$. A d-dimensional PCA is a discrete time Markov process $\eta_\cdot = \{\eta_t(z) \in A | t \in \mathbb{N}, z \in \mathbb{Z}^d\}$ such that its evolution satisfy

$$\mathbb{P}\left[\eta_{t+1}(z) = a \text{ if } \eta_t(z+i) = b_i, \ \forall\ i \in R\right] = f\left(a, (b_i)_{i \in R}\right), \tag{1.1}$$

for all $t \in \mathbb{N}$ and $z \in \mathbb{Z}^d$.

Let μ_0 be the initial distribution of the PCA. For any $t \geq 0$, we call μ_t the distribution of the process at time t. The relation between μ_t and μ_0 is given by

$$\mu_t(u) = \sum \mu_0(v) \mathbb{P}_{\eta_0 \in v}\{\eta_t \in u\},$$

where u and v are cylinders on $A^{\mathbb{Z}^d}$ ($u = \{\xi \in F^{\mathbb{Z}^d} | \xi(x) = \phi(x)\ \forall\ x \in \Lambda\}$ for some fixed $\phi \in A^{\mathbb{Z}^d}$ and $\Lambda \subset \mathbb{Z}^d, |\Lambda| < \infty$). In the one dimensional case we adopt the following notation. For any sequence of letters $U = (u_0 \dots u_n) \in A^{n+1}$, the set $[U]_s = [u_0 \dots u_n]_s := \{x \in A^{\mathbb{Z}} | x(s) = u_0, \dots, x(s+n) = u_n\}$ will be called cylinder.

1.2 Duality

The notion of duality allows the study of properties of a given stochastic process by studying the properties of another one, whose analysis should be easier than the previous one in order to get some advances. Now we give the (classical) definition of duality taken from [9].

Definition 1 *Let $\eta_\cdot$ and $\zeta_\cdot$ be two Markov processes with state spaces X and Y respectively, and let $H(\eta, \zeta)$ be a bounded measurable function on $X \times Y$. The processes $\eta_\cdot$ and $\zeta_\cdot$ are said to be dual to one another with respect to H if*

$$\mathbb{E}^\eta\left[H(\eta_t, \zeta)\right] = \mathbb{E}^\zeta\left[H(\eta, \zeta_t)\right] \tag{1.2}$$

for all $\eta \in X$ and $\zeta \in Y$.

Unfortunately, is not true that every process accepts a dual. Recently, Lopez et al [12] gave a new notion of duality which extends the previous one. More precisely, they gave the following definition.

Definition 2 *Given two discrete time Markov processes, η_t with state space X and ζ_t with state space Y and $H : X \times Y \to \mathbb{R}$ and $\mathcal{D} : Y \to [0, \infty)$ bounded measurable functions, the process $\eta_.$ and $\zeta_.$ are said dual to one another with respect to $(H, \mathcal{D})$ if*

$$\mathbb{E}_{\eta_0=x}\left[H\left(\eta_1, y\right)\right] = \mathcal{D}(y)\mathbb{E}_{\zeta_0=y}\left[H\left(x, \zeta_1\right)\right]. \tag{1.3}$$

This extension allows to show the following theorem.

Theorem 1 *[12] Suppose η_t is a Markov process with state space X and ξ_t is a markov chain with countable state space Y, which are dual to one another with respect to $(H, \mathcal{D})$. If $0 \leq \mathcal{D}(y) < 1$ for all $y \in Y$, then there exist a stochastic process $\tilde{\xi}_t$ with state space $\tilde{Y} = Y \cup \mathcal{P}$ with $\mathcal{P}$ an extra state and a bounded measurable function $\tilde{H} : X \times \tilde{Y} \to \mathbb{R}$ such that $\eta_.$ and $\tilde{\xi}_.$ are dual to one another with respect to H. Furthermore, denoting by Θ the set of all absorbing states of $\xi_.$, if*

i) the linear combinations of $\{H(., y) : y \in Y\}$ is a dense set of $C\left(X\right)$, the set of continuous map from X to $\mathbb{R}$;

ii) $\mathcal{D}(y) < 1$ for any $y \notin \Theta$, and $\mathbf{d} = \sup_{y \in Y : \mathcal{D}(y) < 1}\{\mathcal{D}(y)\} < 1$;

iii) $H(., \theta) \equiv c(\theta)$ for all $\theta \in \Theta$ with $\mathcal{D}(\theta) = 1$;

then $\eta_.$ is ergodic and its unique invariant measure is determined for any $y \in Y$ by

$$\tilde{\nu}(y) = \sum_{\theta \in \Theta d(\theta)=1} c(\theta)\mathbb{P}_{\tilde{\xi}_0=y}\left[\tilde{\xi}_\tau = \theta\right], \tag{1.4}$$

where τ is the hitting time of $\{\theta \in \Theta : \mathcal{D}(\theta) = 1\} \cup \{\mathcal{P}\}$ for $\tilde{\xi}_t$ and $\hat{\mu} = \lim_{t \to \infty} \hat{\mu}_t$ with

$$\hat{\mu}_t(y) = \int_X H(x, y) d\mu_t(x).$$

Sketch of the proof. In order to state our results in section 3, we need to give the spirit and some elements of the proof of Theorem 1. First recall that τ is the hitting time of the process $\eta_.$ entering an absorbing state θ. If there exists a dual process $\tilde{\zeta}$ and a function $\tilde{H}$ that satisfies the following (classical) duality equation

$$\mathbb{E}_{\eta_0=x}\left[\tilde{H}\left(\eta_1, y\right)\right] = \mathbb{E}_{\tilde{\zeta}_0=y}\left[\tilde{H}\left(x, \tilde{\zeta}_1\right)\right], \tag{1.5}$$

it is possible to show that $\hat{\mu}_s(y) = \hat{\mu}_s(y) = \int_X H(x, y) d\mu_s(x) = \mathbb{E}_{\eta_0=y}[\hat{\mu}(\eta_s)]$. If $\mathbb{P}\{\tau < \infty\} = 1$, it follows that

$$\begin{array}{ll} \lim_{s \to \infty} \hat{\mu}_s(y) & = \lim_{s \to \infty} \sum_\theta \mathbb{E}[\hat{\mu}(\eta_s) | \eta_t = \theta, \tau \leq s]\mathbb{P}\{\eta_t = \theta, \tau \leq s\} \\ & + \lim_{s \to \infty} \mathbb{E}[\hat{\mu}(\eta_s) | \tau > s]\mathbb{P}\{\tau > s\} \\ & = \sum c(\theta)\mathbb{P}\{\eta_\tau = \theta\}. \end{array}$$

Finally, when the set of linear combinations of the set $\{\tilde{H}(., y) | y \subset \mathbb{Z}^d\}$ is dense in $C(X)$ (the set of continuous functions from X to $\mathbb{R}$) the sequence

$(\mu_n)_{n\in\mathbb{N}}$ converges in the weak* topology. Also, the limit measure μ does not depend on μ_0.

Hence, we have seen that the key point is to prove that $\mathbb{P}\{\tau < \infty\} = 1$. One way to show this, is to introduce the new type of duality (see Equation 1.3). If there exists a dual process with state space Y that verifies Equation 1.3 then we can define a standard dual process $\tilde{\xi}_.$ with state space $\tilde{Y}$ such that $\tilde{Y} = Y \cup \mathcal{P}$. Here $\mathcal{P}$ is an extra absorbing state and the transition probabilities verify

$$\mathbb{P}_{\tilde{\xi}_0=\tilde{y}_0}\{\tilde{\xi}_1 = \tilde{y}_1\} = \begin{cases} \mathcal{D}(\tilde{y}_0)\mathbb{P}\{\xi_1 = \tilde{y}_1\} & , \text{ if } \tilde{y}_0, \tilde{y}_1 \in Y \\ 1 - \mathcal{D}(\tilde{y}_0) & , \text{ if } \tilde{y}_0 \in Y, \tilde{y}_1 = \mathcal{P} \\ 1 & , \text{ if } \tilde{y}_0 = \tilde{y}_1 = \mathcal{P}. \end{cases}$$

Taking $\tilde{H}(x, y) = H(x, y)$ when $y \in Y$ and $\tilde{H}(x, y) = 0$ when $y = \mathcal{P}$ we obtain that the dual process $\tilde{\xi}_.$ verify the standart duality equation 1.2. Note that since $\mathbf{d} = \sup_{y\in Y:\mathcal{D}(y)<1}\{\mathcal{D}(y)\} < 1$, at each iteration the probability to enter the special "neutral" absorbing state $\mathcal{P}$ is positive and this imply the following result:

Lemma 1 *Under the conditions of Theorem 1, for all integer $i \geq 1$ one has $\mathbb{P}(\tau \geq i) \leq \frac{\mathbf{d}^{i+1}}{1-\mathbf{d}}$.*

Proof. It is sufficient to show that $\mathbb{P}_{\xi_0=y_0}\{\tau = i\} \leq \mathbb{P}_{\xi_{i-1}\neq\mathcal{P}}\{\xi_i \neq \mathcal{P}\} \times \mathbb{P}_{\xi_0=y_0}\{\xi_{i-1} \neq \mathcal{P}\} = \mathbf{d} \times \mathbb{P}_{\xi_0=y_0}\{\xi_{i-1} \neq \mathcal{P}\} \leq \mathbf{d}^i$. □

Note that Lemma 1 implies that $\mathbb{P}\{\tau < \infty\} = 1$ which finishes the proof of Theorem 1.

Remark 1 *[12] Substituting conditions (ii) by $\mathcal{D}(y) = 1$ for all $y \in Y$ and $\mathbb{P}\{\tau < \infty\} = 1$ in Theorem 1, we obtain again the ergodicity for $\eta_.$.*

2 Class of two states ergodic PCA

In order to simplify the notation we will focus our attention on two states PCA, that is to say PCA $\eta_.$ on $\{0, 1\}^{\mathbb{Z}^d}$. In this case, for all sites $z \in \{0, 1\}^{\mathbb{Z}^d}$, the transition function of $\eta_.$ is characterized by the equation

$$\mathbb{P}\{\eta_{t+1}(z) = 1 | \eta_t(z + R(i)) = a_i | i \in I_R, R(i) \in R \subset \mathbb{Z}^d\} = \mathbf{p}\left(a_1, \ldots, a_{|R|}\right),$$

where R is a finite subset of $\mathbb{Z}^d$, $R[1], \ldots, R[|R|]$ are its elements listed in the lexicographical order and $I_R = [1..|R|]|$. Note that any PCA with state space $\{0, 1\}^{\mathbb{Z}^d}$ is completely characterized by the set R and the probabilities $\mathbf{p}(a_1, \ldots, a_{|R|})$. To simplify the equations we can rewrite the definition of a PCA as the set R and the set of probabilities $\{p(\emptyset) \cup p(i_0, \ldots, i_k) | 0 \leq k \leq |R| - 1\}$ where $p(i_0, \ldots, i_k) = \mathbf{p}(a_1, \ldots, a_{|R|})$ if $a_t = 1$ when $t \in \{i_0, \ldots, i_k\}$ else $a_t = 0$.

2.1 Duality and ergodicity

In [12], the authors give ergodicity conditions for one-dimensional multi-states PCA of radius one (window of size 3) acting on $\mathbb{Z}$. Here, we will use an analog dual process to give ergodicity conditions for two-states multi-dimensional PCA with any size of window R using the following duality equation:

$$\mathbb{E}_{\eta_0=x}[H(\eta_1, y)] = \mathcal{D}(y)\mathbb{E}_{\xi_0=y}H[(x, \xi_1)], \tag{1.6}$$

where $\eta_.$ is a process with state space $\{0,1\}^{\mathbb{Z}^d}$ and the state space of the dual process $\xi_.$ are all the finite subsets of $\mathbb{Z}^d$. As in [12] we define the function H as:

$$H(x, Y) = \begin{cases} 1 \text{ if } x(z) = 1, \forall z \in Y \\ 0 \text{ Otherwise.} \end{cases}$$

We choose a dual process such that the evolution of points belonging only to one component are independent of the rest. The evolution rule of the process ξ_t is given by $\xi_{t+1} = \cup_{z\in\xi_t} B(z)$ such that for all $0 \leq k \leq |R|$ and $i_0, \dots i_k \in \{0, \dots |R|\}$ one has

$$\mathbb{P}\Big[B(z) = \{z + R(i_0), \dots, z + R(i_k)\}\Big] = \pi(i_0, \dots i_k)$$

and

$$\mathbb{P}\big[B(z) = \emptyset\big] = \pi(\emptyset).$$

Next, we take the function $\mathcal{D}$ such that for all finite subset $Y \subset \mathbb{Z}^d$ one has $\mathcal{D}(Y) = \mathbf{D}^{|Y|}$ with $\mathbf{D} \in [0, 1]$. Note that $\mathcal{D}(\emptyset) = 1$ and $\emptyset$ is the unique absorbing state for this dual process. Then for the right hand of equation (1.4) we obtain

$$\mathbb{E}_{\eta_0=x}[H(\eta_1, Y)] = \mathbb{P}_{\eta_0=x}\{\eta_1(z) = 1 \ \forall z \in Y_1\}.$$

Hence, using the independence property of $\eta_.$ we get that

$$\mathbb{P}_{\eta_0=x}\{\eta_1(z) = 1 \ \forall z \in Y_1\} = \Pi_{z\in Y_1}\mathbb{P}_{\eta_0=x}\{\eta_1(z) = 1\}.$$

For the left hand of equation 1.6 we have

$$\mathbb{E}_{Y_0=Y}[H(x, Y_1)] = \mathbb{P}_{Y_0=Y}\{x(z) = 1 \ , \forall z \in Y_1\}.$$

Denoting by I_x the set $\{z \in \mathbb{Z} | x(z) = 1\}$, we obtain

$$\mathbb{P}_{Y_0=Y}\{x(z) = 1 \ \forall z \in Y_1\} = \mathbb{P}_{Y_0=Y}\{Y_1 \subset I_x\}.$$

Using the independent properties of the dual process we can assert that

$$\mathbb{P}_{Y_0=Y}\{Y_1 \subset R_x\} = \Pi_{z\in Y_0}\mathbb{P}_{Y_0=Y}\{B(z) \subset I_x\}.$$

Finally we can rewrite equation 1.6 as

$$\begin{aligned} \Pi_{z\in Y_1}\mathbb{P}_{\eta_0=x}\{\eta_1(z) = 1\} &= \mathbf{D}^{|Y_0|}\Pi_{z\in Y_0} \times \mathbb{P}_{Y_0=Y}\{B(z) \subset I_x\} \\ &= \Pi_{z\in Y_0}\mathbf{D} \times \mathbb{P}_{Y_0=Y}\{B(z) \subset I_x\} \end{aligned} \tag{1.7}$$

which implies that

$$\begin{aligned}
&\Pi_{z\in Y_0}\mathbf{p}\Big(x(z+R[1]),\ldots,x(z+R[|R|])\Big)\\
&=\Pi_{z\in Y_0}p(i_0,\ldots,i_k) \text{ such that } z+R(i)=\begin{cases} 1 \text{ if } i\in\{i_0,\ldots,i_k\}\\ 0 \text{ Otherwise.}\end{cases}\\
&=\Pi_{z\in Y_0}\mathbf{D}\times\left[\begin{array}{l}\pi(\emptyset)+\sum_{i\in[1..|R|]}\mathbf{1}_{x(z+R[i])}(1))\pi(i)\\ +\sum_{i,j\in[1..|R|]}\mathbf{1}_{x(z+R[i])x(z+R[j])}(11)\pi(i,j)+\ldots\\ +\sum_{i_1,\ldots i_k\in[1..|R|]}\mathbf{1}_{x(z+R[i_1])\ldots x(z+R[i_k])}(1^k)\pi(i_1,\ldots,i_k)\\ +\ldots+\mathbf{1}_{x(z+R[1])\ldots x(z+R[|R|])}(1^{|R|})\pi(1,\ldots,|R|)\end{array}\right],
\end{aligned}\tag{1.8}$$

where 1^n is an n-uple with all its entries equal to one and 1_A is indicator function of the set A.

Since equation 1.8 is true for all $x\in\{0,1\}^{\mathbb{Z}^d}$ and since it is always possible to define the PCA η in such a way that the set R is an hypercube in $\mathbb{Z}^d$, we obtain the following equations for $\pi(.)$,

$$\begin{array}{ll}
p(\emptyset) & =\mathbf{D}\pi(\emptyset)\\
p(i) & =\mathbf{D}[\pi(\emptyset)+\pi(i)] \qquad (1\le i\le R-1)\\
p(i,j) & =\mathbf{D}[\pi(\emptyset)+\pi(i)+\pi(j)+\pi(i,j)]\\
p(i,j,k) & =\mathbf{D}[\pi(\emptyset)+\pi(i)+\pi(j)+\pi(i,j)+\pi(i,k)+\pi(j,k)+\pi(i,j,k)]\\
\ldots\ldots & \\
 & \text{More generally for } 0\le k\le|R|-1,\\
p(i_0,...,i_k) & =\mathbf{D}\Big[\pi(\emptyset)+\sum_{l=0}^{k}\pi(l)+\ldots+\sum_{i=0}^{k-1}\sum_{l_0,\ldots,l_i\in\{i_0,\ldots,i_k\}}\pi(l_0,...,l_i)\\
 & +\pi(l_0,l_1,...,l_k)\Big].
\end{array}\tag{1.9}$$

Since

$$\pi(\emptyset)+\sum_{k=0}^{R-1}\left(\sum_{l_0,l_1,\ldots l_k\in\{i_0,\ldots,i_R\}}\pi(l_0,l_1,\ldots l_k)\right)+\pi(0,1,\ldots,|R|-1)=1,$$

we have $\mathbf{D}=p(0,1,2,\ldots,|R|-1)=\mathbf{p}(1,1\ldots,1)$.

2.1.1 Relation between the function $\hat{\mu}$ and the invariant measure μ

Using this particular function H we observe that $\hat{\mu}(\mathbb{Z}^d)=\int_X H(x,\mathbb{Z}^d)d\mu(x)=\mu(^\infty 1^\infty)=0$ and $\hat{\mu}(\emptyset)=\int_X H(x,\emptyset)d\mu(x)=\mu(\{0,1\}^{\mathbb{Z}^d})=1$, where $X=\{0,1\}^{\mathbb{Z}^d}$ and $^\infty 1^\infty$ is the all one configuration $(1_{\mathbb{Z}^d}(x))_{x\in\mathbb{Z}^d}$. The first citation of the first part of the following proposition appears in [11]. The second part, which follows from the proof of the first part, is used in Remark 3 and Corollary 1 below. Also, it is used in Section 3.

Proposition 1 *The set of linear combinations of $\{H(.,y)|y\in\mathbb{Z}^d\}$ is dense in $C\left(\{0,1\}^{\mathbb{Z}^d},\mathbb{R}\right)$, the set of continuous function of $\{0,1\}^{\mathbb{Z}^d}\to\mathbb{R}$. For all*

cylinder $U = N_\Lambda(\varphi) \subset \{0,1\}^{\mathbb{Z}^d}$ *with* ($\Lambda \in \mathbb{Z}^d$ *and* $\varphi \in A^{\mathbb{Z}^d}$), *one has* $\mu(U) = \sum_{Y(i)} \alpha_i \hat{\mu}(Y(i))$ *where* $\alpha_i \in \mathbb{Z}$, $Y(i) \subset \mathbb{Z}^d$ *and* $\max\{|Y(i)|\} \leq |\Lambda|$.

Proof. For simplification, we will only give the proof for the two states, one-dimensional case. The key point of the proof consist in showing that any cylinder $[U]_t := [u_0 \dots u_n]_t$, ($u_i \in \{0,1\}$ and $t, n \in \mathbb{N}$) can be decompose into a non commutative sequence of subtraction and union of intersections of cylinders of the type $[1]_t, t \in \mathbb{Z}$. We denote by $T([U]_t)$ this decomposition. One way to realize this decomposition is to follow the following rules:

$$T([1]_t) = [1]_t, \qquad T([0]_t) = 1 - [1]_t$$

and for all $t, n \in \mathbb{Z}$ and $U = u_0 \dots u_n$ one has

$$T([U1]_t) = T([U]_t) \cap [1]_{t+n+2} \text{ and}$$
$$T([U0]_t) = T([U]_t) - T([U]_t) \cap [1]_{t+2+n}.$$

For example

$$\begin{aligned} T([100]_0) &= T([10]_0) - T([101]_0) \\ &= (T([1]_0) - T([11]_0)) - T([10]_0) \cap [1]_2 \\ &= [1]_0 - [11]_0 - ([1]_0 - [11]_0) \cap [1]_2 \\ &= [1]_0 - [11]_0 - [1]_0 \cap [1]_2 \cup [111]_0. \end{aligned}$$

Then, note that $\mathbf{1}_{[1000]_0}$ the characteristic function of the cylinder $[1000]_0$ can be written as $\mathbf{1}_{[1000]_0}(x) = \mathbf{1}_{[1]_0}(x) + \mathbf{1}_{[111]_0}(x) - \mathbf{1}_{[1]_0 \cap [1]_2}(x) - \mathbf{1}_{[11]]_0}(x) = H(x, \{0\}) + H(x, \{0,1,2\}) - H(x, \{0,2\}) - H(x, \{0,1\})$. Since for all finite subset $Y \subset \mathbb{Z}$ one has $\mathbf{1}_{\cap_{i \in Y}[1]_i}(x) = H(x, Y)$ we have that for all $n \in \mathbb{N}$, $t \in \mathbb{Z}$ and $U \in \{0,1\}^n$, $\mathbf{1}_{[U]_t} = \sum \alpha_i H(x, Y(i))$, which implies that the set of linear combinations of the set $\{H(., Y) | Y \in \mathbb{Z}^d\}$ is dense in $C(\{0,1\}^{\mathbb{Z}^d})$. We finish the proof by observing that for all cylinders $[U]_t$, we have $\mu([U]_t) = \int \mathbf{1}_{[U]_t}(x) d\mu(x) = \int \sum \alpha_i H(x, Y(i)) d\mu(x) = \sum \alpha_i \hat{\mu}(Y(i))$.

□

Remark 2 *Using the definition of H taken in [12] which take into consideration the multi-state case, it is possible to prove Proposition 1 for more general PCA acting on* $A^{\mathbb{Z}^d}$.

2.2 A class of ergodic PCA

Let $\mathbf{1}_{\{i_0,\dots,i_n\}}(e_0, \dots, e_{|R|-1}) = \begin{cases} 1 \text{ if } e_j = 1 \text{ for all } j \in \{i_0, \dots i_n\} \\ 0 \text{ Otherwise.} \end{cases}$ Then,

Proposition 2 *Let* $\eta_.$ *be a d-dimensional probabilistic cellular automaton defined thanks to a fix windows* $R = \{R(e_0), \dots R(e_{|R|-1})\} \in \mathbb{Z}^d$ *with transition probabilities* $\mathbf{p}(e_0, \dots, e_{|R|-1})$. *Consider*

$$\mathbf{p}(e_0, \dots, e_l) = \sum_{n=1}^{l} \left(\sum_{i_1 \leq \dots \leq i_n \in [-l..l]} \lambda(i_1, \dots, i_n) \times \mathbf{1}_{\{i_1,\dots,i_n\}}(e_1, \dots, e_l) \right) + p.$$

If for all $l, n \in [0..|R|-1]$ *one has* $0 \leq p$, $\lambda(i_0, \ldots, i_n) \leq 1$,

$$\sum_{n=0}^{|R|-1} \sum_{i_0,\ldots,i_n \in \{i_0,\ldots,i_{|R|-1}\}} \lambda(i_0, \ldots, i_n) \leq 1$$

and

$$\mathbf{p}(1, 1 \ldots, 1) = \sum_{n=0}^{|R|-1} \left(\sum_{i_0 \leq \ldots \leq i_n \in [1..|R|-1]} \lambda(i_0, \ldots, i_n) \right) + p < 1,$$

then $\eta_\cdot$ *is an ergodic PCA.*

Proof. By definition, the dual process is completely determined by the parameters $0 \leq \pi(\emptyset)$, $\pi(i_0), \ldots, \pi(i_0, \ldots, i_{|R|} - 1) \leq 1$ which satisfy

$$\pi(\emptyset) + \sum_{k=0}^{|R|-1} \left(\sum_{i_0,\ldots,i_k \in [0..|R|]} \pi(i_0, \ldots i_k) \right) = 1.$$

For all $k \in [0, |R| - 1]$ with $i_0, \ldots, i_k \in [0..|R| - 1]$ we take the parameters $\pi(i_0, \ldots, i_k) = \frac{1}{\mathbf{D}} \lambda(i_0, \ldots, i_k)$. In this case, the equations 1.9 are trivially satisfied and then the dual process is well defined.

To show that the PCA is ergodic we need to verify the three conditions of Theorem 1.

From Proposition 1, the set of linear combinations of functions belonging to $\{H(., Y), |Y \in \mathbb{Z}^d\}$ is dense in $C\left(\{0,1\}^{\mathbb{Z}^d}, \mathbb{R}\right)$. Then, condition $i)$ is verified. Since $H(., \emptyset) = 1$ and $\mathcal{D}(\emptyset) = \mathbf{D}^{|\emptyset|} = \mathbf{1}$ condition $iii)$ follows. Finally, condition $ii)$ is verified because $\mathbf{D} = p(1, 1, \ldots, 1) < 1$ implies $\sup_{Y \neq \emptyset} \{\mathcal{D}(Y)\} < 1$. □

Remark 3 *When a PCA verifies the conditions of Proposition 2, the value of the unique invariant measure* μ *on any cylinder* U *in* $A^{\mathbb{Z}^d}$*:* $\mu(U)$ *can be compute explicitly using Proposition 1 (*$\mu(U) = \alpha_i \hat{\mu}(Y(i))$*) and the fact that for all* $Y(i) \in \mathbb{Z}^d$ *one has* $\hat{\mu}(Y(i)) = \sum_{k=0}^{\infty} \mathbb{P}_{Y_0 = Y(i)}\{Y_k = \emptyset\}$ *(see Theorem 1 and Proposition 2).*

Corollary 1 *Under the conditions of Proposition 2, when* $p = 0$ *we have that* $\mu = \delta_0$*. Also, when* $p = 1$ *we have that* $\mu = \delta_1$*, where* δ_0 *and* δ_1 *are respectively the Dirac measure on* $^\infty 0^\infty$ *and on* $^\infty 1^\infty$*.*

Proof. When $p = 1$, clearly starting from any initial measure μ_0, we obtain that $\mu_1 = \delta_1$. When $p = 0$, Proposition 1 and Remark 3 imply that for each cylinder U that does not contain the point $^\infty 0^\infty$ one has $\mu(U) = \sum \alpha_i \left(\sum_{k=0}^{\infty} \mathbb{P}_{Y_0 = Y(i)}\{Y_k = \emptyset\} \right) = 0$ since $\pi(\emptyset) = p = 0$. Finally we get that $\mu(^\infty 0^\infty) = 1 - \mu(\{0,1\}^{\mathbb{Z}^d} -^\infty 0^\infty) = 1$ which permit to conclude. □

Remark 4 *Proposition 2 could be reformulated as follows. If* $\eta_\cdot$ *is a PCA with a window* R *of cardinal* $|R|$*, every finite sequence* $\left(p(i_0, i_1, \ldots, i_n)\right)_{n \in [0..|R|-1]}$ *induces a convex curve in* $[0..|R| - 1] \times [0, 1]$*.*

2.3 Existence of a Dual Process

Note that Proposition 2 may be useful to construct ergodic PCA but is not suitable to give ergodicity conditions when the rules of a PCA are still defined. From the relations between the probabilities $p()$ and $\pi()$ given before we obtain:

Lemma 2 *The probabilities of transitions $\pi()$ of the dual process verify*

$$\begin{array}{ll}
\pi(\emptyset) & = \frac{p(\emptyset)}{\mathbf{D}} \\
\pi(i) & = \frac{p(i)-p(\emptyset)}{\mathbf{D}} \\
\pi(i,j) & = \frac{1}{\mathbf{D}}[p(i,j)+p(\emptyset)-p(i)-p(j)] \\
\pi(i,j,k) & = \frac{1}{\mathbf{D}}[p(i,j,k)-p(\emptyset)+p(i)+p(j)+p(k)-p(i,j)-p(i,k)-p(j,k)] \\
\pi(i,j,k,l) & = \frac{1}{\mathbf{D}}[p(i,j,k,l)+p(\emptyset)-\sum_{l_0\in\{i,j,k,l\}} p(l_0)+\sum_{(l_0,l_1)\in\{i,j,k,l\}} p(l_0,l_1) \\
 & \quad -\sum_{(l_0,l_1,l_2)\in\{i,j,k,l\}} p(l_0,l_1,l_2)]
\end{array}$$

More generally for $0 \leq k \leq |R|-1$

$$\pi(i_0,\ldots,i_k) = \frac{1}{\mathbf{D}}\left[(-1)^{k+1}p(\emptyset)+\sum_{j=0}^{k}(-1)^{k-j}\sum_{(l_0,\ldots,l_j)\in\{i_0,\ldots,i_k\}} p(l_0,\ldots,l_j)\right].$$

Proof. We can prove the general equation by mathematical induction For the two first iterations it is easily seen that $\pi(\emptyset) = \frac{p(\emptyset)}{\mathbf{D}}$, $\pi(i) = \frac{p(i)-p(0)}{\mathbf{D}}$ and $\pi(i,j) = \frac{1}{\mathbf{D}}[p(i,j)+p(0)-p(i)-p(j)]$. Then suppose that the order k is true:

$$\pi(i_0,\ldots i_k) = \frac{1}{\mathbf{D}}\left[(-1)^{k+1}p(\emptyset)+\sum_{j=0}^{k}(-1)^{k-j}\sum_{(l_0,\ldots,l_j)\in\{i_0,\ldots,i_k\}} p(l_0,\ldots l_j)\right].$$

Using equation 1.9 we obtain that $\pi(i_0,\ldots,i_{k+1})$

$$= \frac{1}{\mathbf{D}}\left[p(i_0,\ldots,i_{k+1}) - d\pi(\emptyset) - \mathbf{D}\sum_{j=0}^{k}\left(\sum_{(l_0,\ldots,l_j)\in\{i_0,\ldots,i_{k+1}\}}\pi(l_0,\ldots,l_j)\right)\right]. \tag{1.10}$$

Then we suppose the rank k true and use equation 1.10 to obtain that the term in $p(\emptyset)$ in $\pi(i_0,\ldots,i_{k+1})$ is

$$-p(\emptyset)-\sum_{j=0}^{k}\left(\sum_{l_0,\ldots,l_j\in\{i_0,\ldots,i_{k+1}\}}(-1)^{j+1}p(\emptyset)\right) = p(\emptyset)\left(-1-\sum_{j=0}^{k}C_{j+1}^{k+2}(-1)^{j+1}\right)$$

$$= p(\emptyset)\Big(-1+C_0^{k+2}(-1)^0+C_{k+2}^{k+2}(-1)^{k+2}-(1-1)^{k+2}\Big) = (-1)^{k+2}p(\emptyset)$$

where the constants C_j^k represent the binomial coefficients. Next we obtain that the term in $\sum_{l_0\in\{i_0,\dots,i_{k+1}\}} p(l_0)$ in $\pi(i_0,\dots,i_{k+1})$ is equal to

$$-\sum_{j=0}^{k}\sum_{(l_0,\dots,l_j)\in\{i_0,\dots,i_{k+1}\}}\left(\sum_{h_0\in\{l_0,\dots,l_j\}} p(h_0)\right)(-1)^j$$

$$=-\sum_{l_0\in\{i_0,\dots,i_{k+1}\}} p(l_0)\left(\sum_{j=0}^{k} C_j^{k+1}(-1)^j\right)$$

$$=-\sum_{l_0\in\{i_0,\dots,i_{k+1}\}} p(l_0)\left((1-1)^{k+1}-C_{k+1}^{k+1}(-1)^{k+1}\right)=\sum_{l_0\in\{i_0,\dots,i_{k+1}\}} p(l_0)(-1)^{k+1}.$$

Note that C_j^{k+1} represents the number of ways to choose $l_1,\dots,l_j$ in $i_1,\dots,i_{k+1}$ when we have chosen l_0 and i_0. More generally, for $0\le M\le k$, the term in $\sum_{(l_0,\dots,l_M)\in\{i_0,\dots,i_{k+1}\}} p(l_0,\dots,l_M)$ in $\pi(i_0,\dots,i_{k+1})$ is equal to

$$-\sum_{j=M}^{k}\sum_{(l_0,\dots,l_j)\in\{i_0,\dots,i_{k+1}\}}\left(\sum_{(h_0,\dots,h_M)\in\{l_0,\dots,l_j\}} p(h_0,\dots,h_M)\right)(-1)^{j-M}$$

$$=-\sum_{(l_0,\dots,l_M)\in\{i_0,\dots,i_{k+1}\}} p(l_0,\dots,l_M)\left(\sum_{j=0}^{k-M} C_j^{k+1-M}(-1)^j\right)$$

$$=-\sum_{(l_0,\dots,l_M)\in\{i_0,\dots,i_{k+1}\}} p(l_0,\dots,l_M)\left((1-1)^{k+1-M}-(-1)^{k+1-M}\right)$$

$$=\sum_{(l_0,\dots,l_M)\in\{i_0,\dots,i_{k+1}\}} p(l_0,\dots,l_M)(-1)^{k+1-M}.$$

□

Since for all $i_0,\dots,i_k\in R$ one has $\pi(i_0,\dots i_k)\ge 0$, we get the following

Proposition 3 *If the rules of the PCA $\eta_\cdot$ verify*

$p(i)\ge p(\emptyset)$
$p(i,j)\ge p(i)+p(j)-p(\emptyset)$
$p(i,j,k)\ge p(i,j)+p(i,k)+p(j,k)+p(\emptyset)-p(i)-p(j)-p(k)$
$p(i,j,k,l)\ge -p(\emptyset)+\sum_{l_0\in\{i,j,k,l\}} p(l_0)-\sum_{l_0,l_1\in\{i,j,k,l\}} p(l_0,l_1)$
$+\sum_{l_0,l_1,l_2\in\{i,j,k,l\}} p(l_0,l_1,l_2)$
More generally for $0\le k\le R-1$
$p(i_0,\dots,i_k)\ge(-1)^k p(\emptyset)-\sum_{j=1}^{k}(-1)^{k+1-j}\sum_{l_0,\dots l_j} p(l_0,\dots l_j)$

with $p(1,\dots,1)<1$ then $\eta_\cdot$ is an ergodic PCA.

Note that in some case it is possible to exchange the role of the two states $0\leftrightarrow 1$ in order to show ergodicity using the previous results.

2.4 Duality and branching Process

When the PCA under consideration does not verify the conditions of Proposition 3, the probability of the hitting time being finite may verify $\mathbb{P}\{\tau < \infty\} = 1$ which imply that the PCA is still ergodic (see Remark 1). Using basic and well known results from the Branching Process Theory (see [1] for general results and [6], [12] for special applications to cellular automata) we can expand the set of ergodic two states PCA to the following case: $\sum_{i=0}^{k} \sum_{(l_0,\ldots,l_i)} (i+1) \times \pi(l_0,\ldots,l_i) := m < 1$. Using the branching process theory, we can used the standard version of duality which lead to a simplified version of equations 1.9 taking $\mathbf{D} = 1$ and the conditions on $\pi(l_0,\ldots,l_i)$ can be rewritten as conditions on $p(l_0,\ldots,l_i)$ using Lemma 2. When $m < 1$ we still have that $\mathbb{P}\{\tau < \infty\} = 1$. Indeed, the dual process of the PCA is a set valued Markov chain. In this context, a usual discrete time branching process $\mathcal{B}_t$ is coupled to the set valued Markov chain ξ_t in such a way that $|\xi_t| \leq \mathcal{B}_t$. Then, if the branching process $\mathcal{B}_t$ does not survive, the process ξ_t will be absorbed by the empty set. Since this happens with probability one when the mean number of offsprings of a single individual is less or equal than one we are able to prove the ergodicity. This generalize results given in [6] using dual process for particular two state one dimensional PCA call spin flip mixing automata.

3 Some properties of the invariant measure in the one-dimensional case

For the sake of simplification we only give results for PCA on $\{0,1\}^{\mathbb{Z}}$. Extensions to the multi-dimensional case require too much notation. In the one dimensional case, $R = \{i \in \mathbb{Z}\} \cap [-r, r]$ where r is the so called radius of the PCA. We denote by $E(\mathbb{Z}, r)$ the set of all ergodic PCA $\eta_\cdot$ of radius r from $\{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ which satisfy Proposition 3. We call $M(\mathbb{Z}, r)$ the set of all invariant measures of ergodic PCA $\eta_\cdot$ in $E(\mathbb{Z}, r)$. Also, let $M(\mathbb{Z})$ denote the set $\lim_{r\to\infty} \cup M(\mathbb{Z}, r)$. The following Proposition seems to be well known. However, it's proof can not be found or at least it is quite hard to be found.

Proposition 4 *Every measure $\mu \in M(\mathbb{Z})$ is shift-invariant.*

Remark 5 *This result can be easily extended to PCA on $A^{\mathbb{Z}^d}$.*

Proof. It is sufficient to show that for all cylinders $[U]_t$ ($U \in \{0,1\}^l$, $l \in \mathbb{N}$), one has $\mu(\sigma^{-1}[U]_t) = \mu([U]_t)$. Since $\mu \in M(\mathbb{Z})$ then there exits a sequence $(\mu_i)_{i\in\mathbb{N}}$ which converges in the weak* topology to μ, where μ_i is the distribution of a PCA $\eta_\cdot$ at time i starting from an initial distribution μ_0. It follows that for all cylinders $[U]_t$, one has $\lim_{n\to\infty} \mu_n([U]_t) = \mu([U]_t)$. Since for all positive integer i one has

$$\mu_i([U]_t) = \sum_{V_j \in \{0,1\}^{n+1+2ir}} \mu_0([V_j]_{t-ir}) \mathbb{P}_{\eta_0 \in [V_j]_{t-ir}}\{\eta_i \in [U]_t\},$$

we can choose μ_0 as a shift-invariant probability measure. Hence, for any positive integer i and any cylinder $[U]_t$ we have $\mu_i([U]_t) = \mu_i(\sigma^{-1}[U]_t)$, which finishes the proof.

□

The following Lemma is a key property of the function $\hat{\mu}$, which will be used in the proof of Theorem 2.

Lemma 3 *Let $[U]_0$ and $[V]_0$ be two cylinders. If $\mu([U]_0) = \sum \alpha_i \hat{\mu}(A_i), \mu([V]_0) = \sum \beta_i \hat{\mu}(B_i)$ and $t \geq |U| + |V|$, then*

$$\mu([U]_0 \cap \sigma^{-t}[V]_0) = \mu([U]_0 \cap [V]_t) = \sum \alpha_i \hat{\mu}(A_i)(*,t) \sum \beta_i \hat{\mu}(B_i),$$

where $\sum \alpha_i \hat{\mu}(A_i)(*,t) \sum \beta_i \hat{\mu}(B_i) := \sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{B_i + t\})$.

Proof. We prove the theorem by using the principle of mathematical induction. Let's prove first the case $|U| = 1$ and $|V| = 1$. Note that for all finite sets $\{i_0, \ldots, i_k\}$, $(i_0, \ldots, i_k \in \mathbb{Z})$, $\hat{\mu}(\{i_0, \ldots, i_k\}) = \int H(\{i_0, \ldots, i_k\}, x) d\mu(x) = \mu(\{\cap [1]_j | j \in \{i_0, \ldots, i_k\}\})$ and observe that for all $k \geq 1 \in \mathbb{N}$ one has $\mu([1]_0 \cap [1]_k) = \hat{\mu}(\{0\} \cup \{k\})$.

Since $\mu([0]_k) = 1 - \hat{\mu}(\{k\}) = \hat{\mu}(\emptyset) - \hat{\mu}(\{k\})$ and $\mu([1]_0 \cap [0]_k) = \mu([1]_0) - \mu([1]_0 \cap [1]_k) = \hat{\mu}(\{0\}) - \hat{\mu}(\{0, k\})$ we get, again, that $\mu([1]_0 \cap [0]_k) = \hat{\mu}(\{0\} \cup \emptyset) - \hat{\mu}(\{0\} \cup \{k\})$. Furthermore, we have $\mu([0]_0 \cap [1]_k) = \hat{\mu}(\{1\}) - \hat{\mu}(\{0, k\})$. To finish the case $|U| = |V| = 1$ note that

$$\begin{aligned}
\mu([0]_0 \cap [0]_k) &= 1 - \mu([1]_0 \cap [1]_k) - \mu([1]_0 \cap [0]_k) - \mu([0]_0 \cap [1]_k) \\
&= \hat{\mu}(\emptyset) - \hat{\mu}(\{0\} \cup \{k\}) - \hat{\mu}(\{0\}) + \hat{\mu}(\{0\} \cup \{k\}) - \hat{\mu}(\{k\}) \\
&+ \hat{\mu}(\{0\} \cup \{k\}) \\
&= \hat{\mu}(\emptyset) + \hat{\mu}(\{0\} \cup \{k\}) - \hat{\mu}(\{0\}) - \hat{\mu}(\{k\}) \\
&= [\hat{\mu}(\emptyset) - \hat{\mu}(\{0\})] (*,t) [\hat{\mu}(\emptyset) - \hat{\mu}(\{k\})] .
\end{aligned}$$

Now, suppose that $\mu([U]_0 \cap \sigma^{-t}[V]_0) = \sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{B_i + t\})$ is true for $|U| = |V| = n$. Consider $t \geq 2n + 1$ and let $[U]_0, [V]_0$ be two cylinders such that $\mu([U]_0) = \sum \alpha_i \hat{\mu}(A_i)$ and $\mu([V]_0) = \sum \beta_j \hat{\mu}(B_j)$. Since $\mu([U1]_0) = \sum \alpha_i \hat{\mu}(A_i \cup \{|U|\})$, we get

$$\mu([U1]_0 [V1]_t) = \sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{B_j + t\} \cup \{|U|, |V| + t\}).$$

Noting that $\mu([V1]_t) = \sum \alpha_i \hat{\mu}(\{B_i + t\} \cup \{|V| + t\})$, we get the desired result for the case $[U1]_0 \cap [V1]_t$.

The result for the case $[U1]_0 \cap [V0]_t$ follows by noting that

$$\begin{aligned}
\mu([U1]_0 \cap [V0]_t) &= \mu([U1]_0 \cap [V]_t) - \mu([U1]_0 \cap [V1]_t) \\
&= \sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{|U|\} \cup \{B_j + t\}) \\
&- \sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{B_j + t\} \cup \{|U|, |V| + t\}) \\
&= \sum \alpha_i \hat{\mu}(A_i \cup \{|U|\})(*, t) \sum \alpha_i \hat{\mu}(\{B_i + t\}) \\
&- \sum \alpha_i \hat{\mu}(\{B_i + t\} \cup \{|V| + t\}).
\end{aligned}$$

It can also be shown that

$$\mu([U0]_0 \cap [V1]_t) = [\sum \alpha_i \hat{\mu}(A_i) - \sum \alpha_i \hat{\mu}(A_i \cup \{|U|\})](*, t)[\sum \alpha_i \hat{\mu}(\{B_i + t\} \cup \{|V| + t\})].$$

Finally, using that $\mu([U0]_0 \cap [V0]_t) = \mu([U]_0 \cap [V]_t) - \mu([U1]_0 \cap [V0]_t) - \mu([U0]_0 \cap [V1]_t) - \mu([U1]_0 \cap [V1]_t)$ we can show that

$$\begin{aligned}
\mu([U0]_0 \cap [V0]_t) &= [\sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{B_i + t\}) \\
&- \sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{B_i + t\} \cup \{|U|\})](*, t) \\
&[\sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{B_i + t\}) \\
&- \sum_{i,j} \alpha_i \beta_j \hat{\mu}(A_i \cup \{B_i + t\}) \cup \{|V| + t\}]
\end{aligned}$$

which finishes the proof.

□

Theorem 2 *Let $\eta_\cdot$ be a PCA $\in M(\mathbb{Z})$ of radius r with $p(1, 1 \ldots, 1) = \mathbf{D} \in [0, 1)$. Then, the unique invariant measure μ is shift-mixing. Also, if $\mathbf{D} \neq 0$, for all pairs of cylinders $[U]_0 = [u_0 \ldots u_k]$, $[V]_0 = [v_0 \ldots v_{k'}]$ and $t \geq |U| + |V|$ we have*

$$|\mu([U]_0 \cap \sigma^{-t}[V]_0) - \mu([U]_0) \times \mu([V]_0)| \leq \exp(-a \times t) \times K(U, V),$$

where $a = 1/2r \times \ln(1/\mathbf{D})$ and $K(U, V)$ is a constant depending only on U, V, $\mathbf{D}$ and r.

Proof. From Proposition 4 the invariant measure is shift-invariant and the exponential decay of correlations of cylinders imply the mixing property. If $\mathbf{D} = 0$, then $p = \mathbf{p}(0, \ldots 0) = 0$ and $\mu = \delta_0$ (see Corollary 1). Hence, suppose that $0 < \mathbf{D} = p(1, 1 \ldots, 1) < 1$. For all finite subset E of $\mathbb{Z}$ and $s \in \mathbb{Z}$ define $\{E + s\} = \{x + s | x \in E\}$. We claim that for all finite subset E and F, if $t \geq 2Nr + |E| + |F|$ we have

$$|\hat{\mu}(E \cup \{F + t\}) - \hat{\mu}(E) \times \hat{\mu}(F)| \leq \mathbf{D}^{N+1} \frac{1}{1 - \mathbf{D}}.$$

The proof of this claim uses Theorem 1 and Proposition 2 which say that for any finite subset $E \subset \mathbb{Z}$, $\hat{\mu}(E) = \mathbb{P}_{\eta_0=E}\{\eta_\tau = \emptyset\}$, which in turn implies that

$$\hat{\mu}(E) = \sum_{k=0}^{\infty} \mathbb{P}_{\eta_0=E}\{\tau = k\},$$

where τ is the hitting time for the process $\eta_\cdot$. By Lemma 1, for any integer $N > 0$ we have

$$\left| \hat{\mu}(E) - \sum_{k=0}^{N} \mathbb{P}_{\eta_0=E}\{\tau = k\} \right| \leq \mathbf{D}^{N+1} \frac{1}{1-\mathbf{D}}.$$

Then, note that for all positive integer i, if $s \geq 2ri + |E| + |F|$

$$\begin{aligned} \mathbb{P}_{\eta_0=E\cup\{F+s\}}\{\tau = i\} &= \mathbb{P}_{\eta_0=E}\{\tau = i\} \times \sum_{j=0}^{i} \mathbb{P}_{\eta_0=\{F+s\}}\{\tau = j\} \\ &+ \mathbb{P}_{\eta_0=\{F+s\}}\{\tau = i\} \times \sum_{j=0}^{i} \mathbb{P}_{\eta_0=E\cup\{F+s\}}\{\tau = j\}. \end{aligned}$$

It follows that if $s \geq |E| + |F| + 2N \times r$ we get

$$\sum_{i=0}^{N} \mathbb{P}_{E_0=E\cup\{F+s\}}\{\tau = i\} = \sum_{i=0}^{N} \mathbb{P}_{E_0=E}\{\tau = i\} \times \sum_{i=0}^{N} \mathbb{P}_{E_0=\{F+s\}}\{\tau = i\}$$

which implies the claim : when $s \geq |E| + |F| + 2N \times r$ one has

$$|\hat{\mu}(E \cup \{F+s\}) - \hat{\mu}(E) \times \hat{\mu}(F)| \leq \mathbf{D}^{N+1} \frac{1}{1-\mathbf{D}}. \tag{1.11}$$

By Proposition 1, for any pair of cylinders $[U]_0$ and $[V]_0$ there exist finite sequences of sets (A_i) and (B_i) and finite sequences of real numbers α_i and β_i such that $\mu([U]_0) = \sum \alpha_i \hat{\mu}(A_i)$ and $\mu([V]_0) = \sum \beta_i \hat{\mu}(B_i)$. By inequality 1.11, for any pair of subsets of $\mathbb{Z}$, A_i and B_j with $s \geq |U| + |V| + 2N \times r$ we get

$$\left| \alpha_i \beta_j \hat{\mu}(A_i \cup \{B_i + s\} - \alpha_i \hat{\mu}(A_i) \times \beta_j \hat{\mu}(B_j)) \right| \leq |\alpha_i \beta_j| \times \mathbf{D}^{N+1} \frac{1}{1-\mathbf{D}},$$

which implies that

$$\Big| \sum_{i,j} \alpha_i \beta_j \hat{\mu}\big(A_i \cup \{B_i + s\}\big) - \sum_i \alpha_i \hat{\mu}(A_i) \times \sum_j \beta_j \hat{\mu}(B_j) \Big| \leq F(U,V) \times \mathbf{D}^N,$$

where $F(U,V) = \sum_{i,j} |\alpha_i \beta_j| \times \frac{\mathbf{D}}{1-\mathbf{D}}$. Then, if $0 < \mathbf{D} < 1$ and $t \geq |U| + |V|$ we have, by Lemma 3, that

$$\left| \mu([U]_0 \cap \sigma^{-t}[V]_0 - \mu([U]_0)\mu([V]_0)) \right| \leq K(U,V) \times \exp\Big(- t \times \frac{\ln(1/\mathbf{D})}{2r} \Big),$$

where $K(U,V) = F(U,V) \times D^{-(\frac{|U|+|V|}{2r})}$.

When $\mathbf{D} = 0$, then $p = 0$ and from Corollary $\mu = \delta_0$ and μ has exponential decay of correlations.

□

Remark 6 *When the transition rules of the PCA do not verify the hypothesis of Proposition 3 but verifies the condition* $m := \sum_{i=0}^{k} \sum_{(l_0,\ldots,l_i)} (i+1) \times \pi(l_0,\ldots,l_i) < 1$ *we can use the theory of branching processes to study the decay of correlations of the PCA. The relation between* $p(l_0,...,l_i)$ *and* $\pi(l_0,\ldots,l_i)$ *given by Lemma 2 when* $\mathbf{D} = 1$ *is left to the reader.*

If $(\mathcal{B}_n)_{n\in\mathbb{N}}$ is a branching process with mean number of offspring given by $m < 1$, then its extinction occurs exponentially fast implying that the correlation of the PCA decays exponentially fast. If a PCA $\eta_\cdot$ satisfies the conditions of proposition 3 with $p(1,\ldots,1)$ not necessarily strictly less than one, we say that $\eta_\cdot$ satisfies the duality condition. We prove the following

Proposition 5 *Let* $\eta_\cdot$ *be a PCA satisfying the duality condition and such that the transition probabilities of its dual verify* $m < 1$*. Then, for any pair of cylinders* $[U]_0 = [u_0 \ldots u_k]$, $[V]_0 = [v_0 \ldots v_{k'}]$ *and integer* $t \geq |U| + |V|$

$$|\mu([U]_0 \cap \sigma^{-t}[V]_0) - \mu([U]_0) \times \mu([V]_0)| \leq K(U,V,m) \exp(-a \times t),$$

where $a = \frac{\ln(1/m)}{2r}$.

Proof. Since the transition probabilities satisfy equation 1.9, the dual process exists. As pointed out in [12], consider a branching process $\mathcal{B}_n$ defined as follows. As usual, $\mathcal{B}_n$ is the number of particles or individuals in the n-th generation starting at time zero with a single particle. Then, each individual, independently of everithing else produce 0 offspring with probability $\pi(\emptyset)$, 1 offspring with probability $\sum_{k_0} \pi(l_{k_0})$ and so on. Then, the mean number of offspring of a single individual is m. The key point here is that the the dual and the branching process can be coupled in such a way that $\mathcal{B}_n$ dominates the dual process ξ_n in the sense that $|\xi_n| \leq \mathcal{B}_n \ \forall n \geq 0$.

Notice that $\mathbb{P}[\mathcal{B}_n > 0] \leq \mathbb{E}[\mathcal{B}_n] = m^n$. Then, if $t \geq |E| + |F| + 2Nr$ we have that

$$|\hat{\mu}(E \cup \{F+t\}) - \hat{\mu}(E) \times \hat{\mu}(F)| \leq m^{N+1} \frac{1}{1-m}. \tag{1.12}$$

The proof of the inequality above follows in the same lines of the proof of Theorem 2. Then, using Lemma 3 we can conclude, exactly as in the proof of theorem 2, that

$$|\mu([U]_0 \cap \sigma^{-t}[V]_0) - \mu([U]_0) \times \mu([V]_0)| \leq K(U,V) \times e^{-at}$$

where $a = \frac{\ln(1/m)}{2r}$ and $K(U,V)$ is a positive constant depending only on U, V, m and r. □

Remark 7 *When Proposition 3 does not apply and the mean number of offspring of the branching process dominating the dual process is greater than one, we have that the probability of surviving for the process $\mathcal{B}_n$ is positive. In this case, nothing can be said about the ergodicity nor about the decay of correlations of the PCA using the tools from branching processes theory. Also, the result of Proposition 5 can not be directly extended to the case $m = 1$ because the probability of surviving is bounded above by a non summable quantity.*

4 The set of invariant measures

Proposition 2 and 3 guarantee the existence of a correspondence between the set of parameters of the transition function and the unique invariant measure belonging to $M(\mathbb{Z}, r)$. We remark that $M(\mathbb{Z}, r)$ is endowed with the weak* topology For any PCA $\eta_\cdot \in E(\mathbb{Z}, r)$ we denote by $\lambda(\eta_\cdot)$ the set of all parameters or transition probabilities needed to define the evolution of $\eta_\cdot$. That it is to say,

$$\lambda(\eta_\cdot) = \{\mathbf{p}^\lambda(e_{-r}, \ldots, e_r) | e_{-r}, \ldots, e_r \in \{0,1\}^{2r+1}\}.$$

Also, let $\mu^{\eta_\cdot}$ denote the invariant measure associated to $\eta_\cdot$ and $P(\mathbb{Z}, r)$ denote the set $\{\lambda(\eta_t) | \eta_t \in E(\mathbb{Z}, r)\}$. Then, for each pairs of PCA $\eta_\cdot^1$ and $\eta_\cdot^2$ in $E(\mathbb{Z}, r)$, we call $\Delta\lambda(\eta_\cdot^1, \eta_\cdot^2)$ the value of

$$\sup\{|\mathbf{p}^{\eta_t^1}(e_{-r}, \ldots e_r) - \mathbf{p}^{\eta_t^2}(e_{-r}, \ldots e_r) | e_{-r}, \ldots, e_r \in \{0,1\}^{2r+1}\}.$$

Proposition 6 *Let $[U]_0$ be any cylinder and $\eta_\cdot^1, \eta_\cdot^2$ two given PCA. Then,*

$$\lim_{\Delta\lambda(\eta_1, \eta_2) \to 0} \mu^{\eta_t^1}([U]_0) - \mu^{\eta_t^2}([U]_0) = 0.$$

Proof. From Proposition 2 and 3 it follows that the set $P(\lambda, r)$ is a closed subset of $[0,1]^{2r+1}$. Since $P(\mathbb{Z}, r)$ has no isolated points, it follows from Proposition 3 that if $\lambda(\eta_\cdot^1), \lambda(\eta_\cdot^2) \in P(\mathbb{Z}, r)$ then, for any cylinder $[U]_0 = [u_0 \ldots u_n]_0, i \in \{0,1\}$ we have $\lim_{n\to\infty} \mu_n^i([U]_0) = \mu^i([U]_0)$ where

$$\mu_n^i([U]_0) = \sum_{[V_j] \in \{0,1\}^{n+1+2nr}} \mu_0([V_j]_{-nr}) \times \mathbb{P}_{\eta_0^i \in [V_j]_{-nr}}\{\eta_n^i \in [U]_0\}$$

and μ_0 is the so called uniform product measure $B(\frac{1}{2}, \frac{1}{2})$. For any cylinder $[U]_0$, $i \in \{1,2\}$ and $\epsilon > 0$ there exists an integer N such that for all $n, m \geq N$, $|\mu_n^i([U]_0) - \mu_m^i([U]_0)| \leq \frac{\epsilon}{2}$. To finish the proof we need to show that there exists $\gamma > 0$ such that if $\Delta\lambda(\eta_1, \eta_2) \leq \gamma$ then for all $n \geq N$ we have $|\mu_n^1([U]_0) - \mu_n^2([U]_0)| \leq \frac{\epsilon}{2}$. Now, for all $2 \leq j \leq n$ and $i \in \{1,2\}$ we have

$$\mu_j^i([u_0, \ldots, u_n]_0) = \sum_{v_{-r}, \ldots, v_{n+r} \in \{0,1\}^{n+2r+1}} \mu_{j-1}^i([v_{-r}, \ldots, v_{n+r}]_0) \Pi_{l=0}^n \mathcal{P}[l].$$

Note that $\mathcal{P}[l] = \mathcal{P}(u_l|v_{l-r},\dots v_{l+r}) = \mathbf{p}(v_{l-r},\dots,v_{l+r})$ if $u_l = 1$ and that $\mathcal{P}[l] := 1 - \mathbf{p}(v_{l-r},\dots,v_{l+r})$ when $u_l = 0$. Also,

$$\mu_1^i([u_{-ir},\dots,u_{n+ir}]) = C(i,r) \sum_{v_0,\dots,v_{n+2ir+1}\in J} \left(\prod_{l=-ir}^{n+ir} \mathcal{P}(u_l|v_{l-r},\dots v_{l+r}) \right),$$

where $C(i,r) = \left(\frac{1}{2}\right)^{n+1+2(i+1)r}$ and $J = \{0,1\}^{n+2ir+1}$. It follows that the maps $\mu_n^i([U]_0)$ are polynomial functions on the sets of parameters $\lambda(\eta_\cdot^i)$. Then, the existence of γ follows, which finishes the proof.

□

Corollary 2 *The sets $M(\mathbb{Z},r)$ are compact and connected sets and $M(\mathbb{Z}) = \lim_{r\to\infty} \cup M(\mathbb{Z},r)$ is a connected set.*

Proof. The sets $P(\mathbb{Z},r)$ are compact and connected sets which implies that $M(\mathbb{Z},r)$ are compact and connected too.

□

Final questions

i) Is there exist an ergodic PCA such that the unique invariant measure is not shift-mixing?

ii) If there exist a negative answer to the first question, is there exist an ergodic PCA such that the invariant measure has non exponential decay of correlation?

iii) Under which conditions the set $M(d) = \lim_{r\to\infty} \cup M(r,d)$ is a compact set?

iv) Is there anyway to extend the sets $M(\mathbb{Z}^d)$?

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