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MODEL THEORETIC PROPERTIES OF METRIC VALUED FIELDS

ITAÏ BEN YAACOV

Abstract. We study model theoretic properties of valued fields (equipped with a real-valued multiplicative valuation), viewed as metric structures in continuous first order logic.

For technical reasons we prefer to consider not the valued field $(K, |\cdot|)$ directly, but rather the associated projective spaces $K\mathbf{P}^n$, as bounded metric structures.

We show that the class of (projective spaces over) metric valued fields is elementary, with theory MVF, and that the projective spaces $\mathbf{P}^n$ and $\mathbf{P}^m$ are biïnterpretable for every $n, m \geq 1$. The theory MVF admits a model completion $ACMVF$, the theory of algebraically closed metric valued fields (with a non trivial valuation). This theory is strictly stable (even up to perturbation).

Similarly, we show that the theory of real closed metric valued fields, $RCMVF$, is the model companion of the theory of formally real metric valued fields, and that it is dependent.

1. The theory of metric valued fields

Let us recall some terminology from Berkovich [Ber90]. A *semi-normed ring* is a unital commutative ring R equipped with a mapping $|\cdot|\colon R \to \mathbf{R}^{\geq 0}$ such that

(i) $|1| = 1$,
(ii) $|xy| \leq |x||y|$,
(iii) $|x + y| \leq |x| + |y|$.

If $|x| = 0 \Longrightarrow x = 0$ then $|\cdot|$ is a *norm*. A semi-norm is *multiplicative* if $|xy| = |x||y|$. A multiplicative norm is also called a *valuation*. Thus, a valued field is equipped with a natural metric structure $d(x, y) = |x - y|$. In some contexts, a valuation is allowed to take values in $\Gamma \cup \{0\}$ where $(\Gamma, \cdot)$ is an arbitrary ordered Abelian group and $0 < \Gamma$, but this will not be the case in the present text. When we wish to make this explicit we shall refer to our fields as *metric valued fields*.

If K is a complete valued field then either $K \in \{\mathbf{R}, \mathbf{C}\}$ and $|\cdot|$ is the usual absolute value to some power (in which case $|\cdot|$ is *Archimedean*) or $|x + y| \leq |x| \vee |y|$ ($|\cdot|$ is *non Archimedean*, or *ultra-metric*). From a model theoretic point of view, Archimedean valued fields, being locally compact, resemble finite structures of classical logic and are thus far less interesting than their ultra-metric counterparts. On the other hand, while everything we do here applies to arbitrary valued fields, including Archimedean ones, restricting our attention to the ultra-metric case does allow us many simplifications. Thus, with very little loss of generality, we shall only consider ultra-metric valued fields.

Convention 1.1. Throughout, unless explicitly stated otherwise, by a *valued field* we mean a non Archimedean one.

The valuation is said to be *trivial* if $|x| = 1$ for every $x \neq 0$. It is *discrete* if the image of $|\cdot|$ on $K^\times$ is discrete. Clearly every trivial valuation is discrete. On the other hand, a non trivial valuation on an algebraically (or separably) closed field cannot be discrete.

A non trivially valued field is unbounded as a metric space, and therefore does not fit in the framework of standard bounded continuous logic. One device we use quite often with Banach space structures (Banach spaces, Banach lattices, and so on) is to restrict our attention to the structure formed by the closed unit ball. This approach may seem natural for valued fields as well, since the unit ball is simply the corresponding

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valuation ring. However, in the case of a non discrete valuation this approach is not adequate, as shown by the following result.

Proposition 1.2. *Let $(K, |\cdot|)$ be a field equipped with a non discrete valuation, and let $R = (R, 0, 1, -, +, \cdot, |\cdot|)$ be its valuation ring. Then R cannot be saturated as a metric structure (i.e., in the sense of continuous logic). In fact, it cannot even realise every type over $\varnothing$.*

Proof. Since R is not discrete we can find for each n an element $a_n \in R$ such that $1 - 2^{-n} < |a_n| < 1$. Such an element is not invertible in R, and worse, for every $b \in R$ we have $|a_n b| < |b| \leq 1$, whereby $|a_n b - 1| = 1$. In other words, each a_n satisfies the assertion that $\inf_y |xy - 1| = 1$. Thus in an ultra-power of R there exists a such that $|a| = \inf_y |ay - 1| = 1$. Since every element of R of value 1 is invertible, such an element cannot exist in R. $\blacksquare_{1.2}$

Therefore, if we are to hope for a reasonable model theoretic treatment of valued fields, the entire field should be considered as an unbounded structure. Unbounded metric structures are discussed in [Ben08a], where we also introduce an *emboundment* process whereby unbounded structures can be turned into bounded ones through the addition of a single point at infinity. In the case of a valued field, the resulting structure can be naturally identified (as a set of points) with the projective line, which is a natural object in itself. For our purposes it will be more convenient to consider the projective line directly, rather than as the emboundment of the field (and one can check that the two structures are interdefinable). As in the general case of emboundment, even though the field language contains function symbols, these do not pass on to the projective line. Indeed, the addition map $\big([x : 1], [y : 1]\big) \mapsto [x + y : 1]$ is ill defined at $\big([1 : 0], [1 : 0]\big)$, and similarly $\big([x : 1], [y : 1]\big) \mapsto [xy : 1]$ is ill defined at $\big([0 : 1], [1 : 0]\big)$. We shall therefore have to do, at least for the time being, with a purely relational language (this will be remedied later on when we consider projective spaces of higher dimension).

We recall that the projective n-space over a field K is the quotient $(K^{n+1} \smallsetminus \{0\})/K^\times$. The class of $(a) = (a_i) = (a_0, \ldots, a_n)$ is denoted $\mathbf{a} = [a] = [a_i] = [a_0 : \ldots : a_n]$. Dividing by a coordinate with maximal value we see that any member of $K\mathbf{P}^n$ can be written as $[a_i]$ where $\bigvee |a_i| = 1$. From now on we shall assume that all the representatives are of this form, which determines them up to a multiplicative factor from the group $\{x \in K \colon |x| = 1\} = \ker |\cdot|$.

Notation 1.3. Let $\bar{X} = (X_0, \ldots, X_{n-1})$ denote n formal unknowns. We let $\bar{X}^*$ denote a copy of $\bar{X}$, and let $\mathbf{Z}^h[\bar{X}] \subseteq \mathbf{Z}[\bar{X}, \bar{X}^*]$ denote the ring of polynomials in $\bar{X}$, $\bar{X}^*$ which are homogeneous in each pair (X_i, X_i^*) separately (which is stronger than being homogeneous in all the variables simultaneously). For a polynomial $Q(\bar{X}, \bar{X}^*) \in \mathbf{Z}^h[\bar{X}]$ let $\bar{Q}(\bar{X}) = Q(\bar{X}, \bar{1}) \in \mathbf{Z}[\bar{X}]$.

For $P(\bar{X}) \in \mathbf{Z}[\bar{X}]$ let $\deg_{\bar{X}} P = (\deg_{X_0} P, \ldots, \deg_{X_{n-1}} P) \in \mathbf{N}^n$ and let $P^*(\bar{X}^*) = (\bar{X}^*)^{\deg_{\bar{X}} P} = \prod (X_i^*)^{\deg_{X_i} P} \in \mathbf{Z}[\bar{X}^*]$, $P^h(\bar{X}, \bar{X}^*) = P(\frac{\bar{X}}{\bar{X}^*}) P^*(\bar{X}^*)$. Then $P^h \in \mathbf{Z}[\bar{X}, \bar{X}^*]$ is unique such that $P = \overline{P^h}$ and no X_i^* can be factored out of P^h. We call P^h the *homogenisation* of P and observe that $P \mapsto P^h$ is multiplicative. Conversely, every $Q \in \mathbf{Z}^h[\bar{X}]$ can be written uniquely as $\bar{Q}^h \cdot (\bar{X}^*)^{\alpha(Q)}$, where $\alpha(\bar{Q}) \in \mathbf{N}^n$ is a multi-exponent.

We now have everything we need to define the language and theory of (projective lines of) metric valued fields in ordinary (i.e., bounded) continuous logic, as presented in [BU10] or [BBHU08].

Definition 1.4. We define the language $\mathcal{L}_{\mathbf{P}^1}$ to consist of a constant symbol ∞ and one n-ary, $[0, 1]$-valued predicate symbol $\|P(\bar{x})\|$ for each n and each polynomial $P \in \mathbf{Z}[X_0, \ldots, X_{n-1}]$. (There is some abuse of notation here, since P does not determine n but this will not cause any problems.)

Definition 1.5. For a valued field $(K, |\cdot|)$, we view $K\mathbf{P}^1$ as an $\mathcal{L}_{\mathbf{P}^1}$-pre-structure by:

$$\infty := [1 : 0], \qquad \|P(\bar{\mathbf{a}})\| := |P^h(\bar{a}, \bar{a}^*)|, \qquad d(\mathbf{a}, \mathbf{b}) := \|\mathbf{a} - \mathbf{b}\| = |ab^* - a^*b|.$$

This is independent of the choice of representatives, keeping mind that we only consider representatives for $[a : a^*] \in K\mathbf{P}^1$ such that $|a| \vee |a^*| = 1$.

We observe that $|a^*| = \|\mathbf{a} - \infty\| = d(\mathbf{a}, \infty)$, and we shall use $\|x^*\|$ as an abbreviation for the formula $d(x, \infty)$. For $P(\bar{X}) \in Z[\bar{X}]$ we have $|P^*(\bar{a}^*)| = \prod |a_i^*|^{\deg_{X_i} P}$, and we shall similarly use $\|P^*(\bar{x})\|$ as an abbreviation for $\prod \|x_i^*\|^{\deg_{X_i} P}$. We notice that $\|P(\bar{\mathbf{a}})\| = |P(\bar{\mathbf{a}})| \|P^*(\bar{\mathbf{a}})\|$ (if $\mathbf{a}_i \in K \subseteq K\mathbf{P}^1$ whenever $\deg_{X_i} P > 0$ then this makes sense, and otherwise $\|P^*(\bar{\mathbf{a}})\| = 0$, and the identity still makes sense).

Definition 1.6. We define MVF, the theory of projective lines over metric valued fields, to consist of the following axioms. In axiom (Perm), $\sigma \in \mathfrak{S}_n$ is a permutation and $(X_0, \ldots, X_{n-1})^\sigma = (X_{\sigma 0}, \ldots, X_{\sigma(n-1)})$.

(Norm)	$\|x\| \vee \|x^*\| = 1$	
(Perm)	$\|P(\bar{x})\| = \|Q(\bar{x}^\sigma, \bar{y})\|$	$\big(P(\bar{X}) = Q(\bar{X}^\sigma, \bar{Y})\big)$
(Ult)	$\|\bar{x}^*\|^\alpha \|P(\bar{x})\| \leq \|\bar{x}^*\|^\beta \|Q(\bar{x})\| \vee \|\bar{x}^*\|^\gamma \|R(\bar{x})\|$	$\big((\bar{X}^*)^\alpha P^h = (\bar{X}^*)^\beta Q^h - (\bar{X}^*)^\gamma R^h\big)$
(Prod)	$\|(PQ)(\bar{x})\| = \|P(\bar{x})\| \|Q(\bar{x})\|$	
(Dist)	$d(x, y) = \|x - y\|$	
(Lin)	$\exists y \, \|P(\bar{x}, y)\| = 0$	$\big(\deg_Y P(\bar{X}, Y) = 1\big)$

Axioms are universally quantified, so axiom (Norm), for example, should be understood as the sentence $\sup_x \big|1 - \|x\| \vee \|x^*\|\big|$ (where we recall the convention of continuous logic, that zero is "True"), and similarly for the other axioms which appear quantifier-free. In the last axiom, the existential quantifier should be understood in the approximate sense: there exists y such that $\|P(\bar{x}, y)\|$ is as close as desired to zero, or formally, $\sup_{\bar{x}} \inf_y \|P(\bar{x}, y)\|$. In continuous logic one simply cannot express directly the existence of some y such that something holds precisely (e.g., such that $\|P(\bar{x}, y)\|$ is precisely zero), although in concrete situations one can prove that approximate existence implies precise existence, as is the case with Lemma 1.7 below.

It follows immediately from the axioms that $\|P\| = \|-P\|$ and $\|\infty\| = 1$.

Lemma 1.7. *Assume that $\mathcal{M} \vDash MVF$. Then for every $P, Q \in \mathbf{Z}[\bar{X}]$ and every $\bar{a} \in M^n$, if $\|P(\bar{a})\| \|Q^*(\bar{a})\| \neq 0$ then there exists a unique $b \in M$ such that $\|P(\bar{a})b - Q(\bar{a})\| = 0$, i.e., $\|R(\bar{a}, b)\| = 0$ where $R = PY - Q$. Moreover, this b is distinct from ∞.*

Proof. Let $\alpha = \deg_{\bar{X}} Q \dot{-} \deg_{\bar{X}} P$, $\beta = \deg_{\bar{X}} P \dot{-} \deg_{\bar{X}} Q$, so $R^h = (\bar{X}^*)^\alpha P^h Y - (\bar{X}^*)^\beta Y^* Q^h$. Then

$$(\bar{X}^*)^\alpha \big[(Y - Z)P\big]^h = Z^* R(\bar{X}, Y)^h - Y^* R(\bar{X}, Z)^h.$$

By the ultra-metric axiom (Ult):

$$d(y, z) \leq \frac{\|R(\bar{a}, y)\| \|z^*\| \vee \|R(\bar{a}, z)\| \|y^*\|}{\|P(\bar{a})\| \|\bar{a}^*\|^\alpha} \leq \frac{\|R(\bar{a}, y)\| \vee \|R(\bar{a}, z)\|}{\|P(\bar{a})\| \|Q^*(\bar{a})\|}.$$

Uniqueness follows. By the linear solution axiom (Lin) there exists a sequence (b_n) such that $\|P(\bar{a})b_n - Q(\bar{a})\| \to 0$. It follows from our argument above that this is a Cauchy sequence, and its limit b is a solution. Finally, $\|R(\bar{a}, \infty)\| = \|P(\bar{a})\| \|\bar{a}^*\|^\alpha \neq 0$, so $b \neq \infty$. $\blacksquare_{1.7}$

When b is as in the lemma we write $b = \frac{Q(\bar{a})}{P(\bar{a})}$, and if $P = 1$ we write $b = Q(\bar{a})$.

Theorem 1.8. *An $\mathcal{L}_{\mathbf{P}^1}$-structure is a model of MVF if and only if it is isomorphic to $K\mathbf{P}^1$ for some complete valued field K.*

Proof. Only one direction requires a proof. Assume therefore that $\mathcal{M} \vDash MVF$. Let $K = M \smallsetminus \{\infty\}$. For $a, b \in K$, and with the notation above, $a + b = \frac{a+b}{1}$ is the unique solution for $\|Y - a - b\| = 0$. We may similarly define ab, $-a$, as well as the constants 0 and 1, and since $\|a^*\| \neq 0$ we may also define $|a| = \frac{\|a\|}{\|a^*\|}$.

Let us check that $(K, 0, 1, -, +, \cdot, |\cdot|)$ is a valued field. For this purpose, we shall use brackets to enclose expressions involving the field operations of K, whereas expressions outside brackets correspond to polynomials over $\mathbf{Z}$. Axiom (Perm) ensures that we need not worry about the order of variables in a polynomial nor about dummy variables, and will be used implicitly throughout.

In order to see that addition is associative, for example, observe that

$$X^*(W - Y - Z - T)^h = Y^* Z^* (W - X - T)^h + W^* T^* (X - Y - Z)^h.$$

Then by (Ult) and the fact that $\|P\| = \|-P\|$, that for all $a, b, c \in K$,

$$\big\|[a + b]^*\big\| \big\|[(a + b) + c] - a - b - c\big\| = 0 \qquad \Longrightarrow \qquad \big\|[(a + b) + c] - a - b - c\big\| = 0.$$

A similar argument yields $\big\|[a+(b+c)]-a-b-c\big\|=0$. It follows from the uniqueness clause of Lemma 1.7 that $[(a+b)+c]=[a+(b+c)]$. Similarly,

$$X^*(W-YZT)^h=Y^*Z^*(W-XT)^h+W^*(XT-YZT)^h.$$

Using also axiom (Prod) we obtain $\big\|[(ab)c]-abc\big\|=0$, and similarly $\big\|[a(bc)]-abc\big\|=0$, concluding that $[(ab)c]=[a(bc)]$.

Proceeding in this manner, we show that $\big\|[P(\bar{a})]-P(\bar{a})\big\|=0$ for every $\bar{a}\in K$, polynomial $P(\bar{X})\in\mathbf{Z}[\bar{X}]$ and ring language term $[P]$ which evaluates to P in rings. In particular $[P(\bar{a})]$ only depends on P and not on the choice of $[P]$, whence it follows that K is a ring. If $a\in K\smallsetminus\{0\}$ then $\|a\|=\|a-0\|>0$, so $b=\frac{1}{a}$ exists. Thus $\big\|[ab]-1\big\|=0=\big\|[1]-1\big\|$, whereby $\big\|[ab]-[1]\big\|=0$ and $[ab]=[1]$, so K is a field.

The identity $\big\|[P(\bar{a})]-P(\bar{a})\big\|=0$ also implies that $\big\|P^*(\bar{a})\big\|\big\|[P(\bar{a})]\big\|=\big\|[P(\bar{a})]^*\big\|\big\|P(\bar{a})\big\|$, or $\big|[P(\bar{a})]\big|=\frac{\|P(\bar{a})\|}{\|P^*(\bar{a})\|}$. By axiom (Prod) it follows that $|[ab]|=\frac{\|a\|\|b\|}{\|a^*\|\|b^*\|}=|a||b|$. Similarly, with axiom (Ult) we have $|[a+b]|=\frac{\|a+b\|}{\|a^*\|\|b^*\|}\leq\frac{\|b^*\|\|a\|+\|a^*\|\|b\|}{\|a^*\|\|b^*\|}=|a|+|b|$. It follows that K is a valued field, and that the interpretation of the symbols $\|P\|$ is as intended, completing the proof. $\blacksquare_{1.8}$

The problem with extending multiplication to the projective line arises with expressions close to $0\cdot\infty$, i.e., when trying to multiply points which are close to 0 with points which are close to ∞. This situation cannot happen when taking powers, and indeed,

Lemma 1.9. *For $n\in\mathbf{Z}$, the operation $x\mapsto x^n$ is uniformly definable in models of MVF. This is under the convention that $0^0=\infty^0=1$, $\infty^n=\infty$ for $n>0$, and $\infty^n=0$, $0^n=\infty$ for $n<0$.*

Proof. Indeed, $\mathbf{a}^n=[a^n:(a^*)^n]$ and $|a^n|\vee|(a^*)^n|=1$. It follows that $d(y,x^n)=\|x^n-y\|$, and similarly $d(y,x^0)=\|y^*\|$, $d(y,x^{-n})=\|1-x^ny\|$. $\blacksquare_{1.9}$

It is natural to ask whether other projective spaces $K\mathbf{P}^n$, for $n>1$, have more (or less) structure than the projective line. In order to give a precise meaning to this question, we should first define the projective spaces as metric structures. It will be most convenient to define the entire family $(K\mathbf{P}^n)_n$ as a single multi-sorted structure $K\mathbf{P}$.

Definition 1.10. The signature $\mathcal{L}_{\mathbf{P}}$ consists of $\aleph_0$ many sorts $\{\mathbf{P}^n\}_{n\in\mathbf{N}}$. They are equipped with the following symbols:

- For each n,m a function symbol $\otimes\colon\mathbf{P}^n\times\mathbf{P}^m\to\mathbf{P}^{n+m+nm}$.
- For each $A\in SL_{n+1}(\mathbf{Z})$ (or in some generating subset), a function symbol $A\colon\mathbf{P}^n\to\mathbf{P}^n$.
- For each n a predicate symbol $\|\cdot\|$ on $\mathbf{P}^n$.

Definition 1.11. Let $(K,|\cdot|)$ be any valued field. We define an $\mathcal{L}_{\mathbf{P}}$-pre-structure $K\mathbf{P}$ as follows:

- The sort $\mathbf{P}^n$ consists of the projective space $K\mathbf{P}^n$, namely the quotient of $K^{n+1}\smallsetminus\{0\}$ by $K^\times$. The equivalence class of $(a_0,\ldots,a_n)$ will be denoted $\mathbf{a}=[a]=[a_i]_i=[a_0,\ldots,a_n]$. We may, and shall, assume that each representative satisfies $\bigvee|a_i|=1$.
- For $n,m\in\mathbf{N}$, we fix some natural isomorphism $K^{n+1}\otimes K^{m+1}\cong K^{(n+1)(m+1)}$, say the one given by $(a\otimes b)_{i+(n+1)j}=a_ib_j$. We then interpret $\otimes$ as the Segre embedding $[a]\otimes[b]=[a\otimes b]=[a_ib_j]_{i\leq n,j\leq m}$.
- For $A\in SL_{n+1}(\mathbf{Z})$, the corresponding function symbol acts on $K\mathbf{P}^n$ naturally via its action on $K^{n+1}\smallsetminus\{0\}$.
- We interpret:

$$\big\|\mathbf{a}\big\|=|a_0|.$$

- The distance on $K\mathbf{P}^n$ is interpreted as:

$$d(\mathbf{a},\mathbf{b})=\bigvee_{i<j<n}|a_ib_j-a_jb_i|.$$

Notice that on $K\mathbf{P}^1$, the interpretation of $\|x\|$ and $d(x,y)$ is consistent with that given in Definition 1.5.

We need check that the distance defined above is indeed an ultra-metric distance function. Clearly it only depends on the equivalence classes $\mathbf{a}$ and $\mathbf{b}$. One checks easily that $d(\mathbf{a},\mathbf{b})=0$ if and only if $\mathbf{a}=\mathbf{b}$.

Symmetry is immediate. We are left with checking the ultra-metric triangle inequality. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in K\mathbf{P}^n$, and fix j_0 such that $|b_{j_0}| = 1$. For all i and k we then have:

$$\begin{aligned}|a_i c_k - a_k c_i| &= |a_i b_{j_0} c_k - a_{j_0} b_i c_k + a_{j_0} b_i c_k - a_{j_0} b_k c_i + a_{j_0} b_k c_i - a_k b_{j_0} c_i| \\ &\leq |c_k||a_i b_{j_0} - a_{j_0} b_i| \vee |a_{j_0}||b_i c_k - b_k c_i| \vee |c_i||a_{j_0} b_k - a_k b_{j_0}| \\ &\leq d(\mathbf{a}, \mathbf{b}) \vee d(\mathbf{b}, \mathbf{c}).\end{aligned}$$

In order to show that $\mathbf{P}^n$ is interpretable in $\mathbf{P}^1$ we shall attempt to repeat the standard trick of covering $\mathbf{P}^n$ with $n+1$ affine charts. The problem is that $\mathbf{A}^n$ is not definable, or even type-definable, in $\mathbf{P}$, so we shall have to make do with $n+1$ copies of $(\mathbf{P}^1)^n$ instead. As above, a point $\mathbf{a} \in \mathbf{P}^1$ is viewed as $[a : a^*]$ where $|a| \vee |a^*| = 1$. It is either equal to $\infty = [1 : 0]$ or else can be identified with $\frac{a}{a^*} \in K$. Agreeing that $|\infty| = \infty$ we have $|\mathbf{a}| \leq 1$ if and only if $|a^*| = 1$. As in Lemma 1.9 we also have $\mathbf{a}^{-1} = [a^* : a]$.

Let $M = \frac{n(n+1)}{2}$. Given a tuple $\bar{\mathbf{a}} = (\mathbf{a}_{ij})_{i<j\leq n} \in (\mathbf{P}^1)^M$ let $\mathbf{a}_{ii} = 1 = [1 : 1]$ and $\mathbf{a}_{ji} = \mathbf{a}_{ij}^{-1}$, and consider the matrix

$$(\mathbf{a}_{ij})_{i,j\leq n} = \begin{pmatrix} 1 & \mathbf{a}_{0,1} & & \cdots & & \mathbf{a}_{0,n} \\ \mathbf{a}_{0,1}^{-1} & 1 & & \cdots & & \mathbf{a}_{1,n} \\ & & 1 & & & \\ \vdots & \vdots & & 1 & & \vdots \\ & & & & 1 & \\ \mathbf{a}_{0,n}^{-1} & \mathbf{a}_{1,n}^{-1} & & \cdots & & 1 \end{pmatrix} \tag{1}$$

Intuitively, we wish to consider such matrices whose rows represent identical points in the standard affine charts for $\mathbf{P}^n$, i.e., such that

$$[1 : \mathbf{a}_{0,1} : \ldots : \mathbf{a}_{0,n}] = [\mathbf{a}_{1,0} : 1 : \mathbf{a}_{1,2} : \ldots : \mathbf{a}_{1,n}] = \ldots = [\mathbf{a}_{n,0} : \ldots : \mathbf{a}_{n,n-1} : 1].$$

These precise identities are meaningless, since some of the $\mathbf{a}_{ij}$ may be ∞, but we may nonetheless express them formally by the system of equations

$$X_{ij} X_{jk} = X_{ik} \qquad (i < j < k \leq n),$$

which are homogenised into

$$X_{ij} X_{jk} X_{ik}^* = X_{ik} X_{ij}^* X_{jk}^* \qquad (i < j < k \leq n).$$

The following asserts that the solutions to these equations form a well-behaved (definable) set, and that this set covers $\mathbf{P}^n$. We recall from [BBHU08] or [Ben10, Fact 1.7] that in continuous logic, a subset $X \subseteq M^n$ is called a *definable set* if it is closed and the distance predicate $d(X, \bar{x})$ is definable. This has several equivalent characterisations, among which the existence of a definable predicate $\varphi(\bar{x})$ such that $d(X, \bar{x}) \leq \varphi(\bar{x})$ and such that the zero set of φ is exactly X. That the latter property implies the former uses quantification, and when dealing with quantifier-free definability the two properties need no longer be equivalent. The latter one is more robust, and in particular can be shown to still hold if we replaced the ambient distance with an equivalent definable one, so it is it we shall use.

Definition 1.12. We shall say that a set X is *quantifier-free definable* if there exists a quantifier-free definable predicate (i.e., a uniform limit of quantifier-free formulae) $\varphi(\bar{x})$ such that, first, X is the zero set of φ, and second, $d(X, \bar{x}) \leq \varphi(\bar{x})$.

Lemma 1.13. *Let $E \subseteq (\mathbf{P}^1)^M$ consist of all tuples satisfying the homogeneous equations above.*

(i) *The set E is quantifier-free definable.*

(ii) *For every tuple $\bar{\mathbf{a}} \in E$ there exists $\ell \leq n$ such that in the ℓth row of the matrix (1) all entries are finite of value ≤ 1.*

(iii) *Let ℓ be as in the previous item, and let $\mathbf{b} \in \mathbf{P}^n$ be the class of the ℓth row, i.e., $\mathbf{b} = [\mathbf{a}_{\ell,0} : \ldots : \mathbf{a}_{\ell,\ell-1} : 1 : \mathbf{a}_{\ell,\ell+1} : \ldots : \mathbf{a}_{\ell,n}]$. Then $\mathbf{b}$ is the unique solution for the following system of homogeneous equations*

$$a_{ij} Y_i = a_{ij}^* Y_j \qquad (i < j).$$

Conversely, every $\mathbf{b} \in \mathbf{P}^n$ arises in this manner (for some $\bar{\mathbf{a}} \in E$).

Proof. We define

$$\varphi(\bar{x}) = \bigvee_{i<j<k} \|x_{ij}x_{jk} - x_{ik}\|.$$

Then E is the zero set of φ, and we claim that $d(\bar{x}, E) \leq \varphi(\bar{x})$, which is enough for the first item. Indeed, assume that $\bar{\mathbf{a}} \notin E$, so $\varphi(\bar{\mathbf{a}}) = r > 0$, and we wish to show that $d(\bar{\mathbf{a}}, E) \leq r$. If $r = 1$ then there is nothing to show. We may therefore assume that $r < 1$. It will be convenient to work with the entire matrix (1) rather than with its upper triangle. Observe that passing to the whole matrix does not change our basic hypothesis, i.e., $\bigvee_{i<j<k\leq n} \|\mathbf{a}_{ij}\mathbf{a}_{jk} - \mathbf{a}_{ik}\| = \bigvee_{i,j,k\leq n} \|\mathbf{a}_{ij}\mathbf{a}_{jk} - \mathbf{a}_{ik}\|$. If we apply a permutation of $n+1$ both to the rows and columns of the matrix, the resulting matrix will still have the same properties (namely $\mathbf{a}_{ij} = \mathbf{a}_{ji}^{-1}$ and $\bigvee_{i,j,k\leq n} \|\mathbf{a}_{ij}\mathbf{a}_{jk} - \mathbf{a}_{ik}\| \leq r$).

We first claim that if $\varphi(\bar{\mathbf{a}}) = r < 1$ then the matrix possesses a row, say the ℓth, such that $|\mathbf{a}_{\ell j}| \leq 1$ for all $j \leq n$. In order to prove the claim it will be enough to show that if the ith row does not have this property, say because $|\mathbf{a}_{ij}| > 1$, then in the jth row there are strictly more entries than in the ith with value ≤ 1. Indeed, assume that $|\mathbf{a}_{ik}| \leq 1$ and let us show that $|\mathbf{a}_{jk}| \leq 1$ as well. By assumption we have

$$|a_{ij}a_{jk}a_{ik}^* - a_{ik}a_{ij}^*a_{jk}^*| = \|\mathbf{a}_{ij}\mathbf{a}_{jk} - \mathbf{a}_{ik}\| \leq r < 1.$$

We also assume that $|a_{ik}^*| = 1$ and $|a_{ij}^*| < 1 = |a_{ij}|$, whereby $|a_{ij}a_{jk}a_{ik}^*| = |a_{jk}|$ and $|a_{ik}a_{ij}^*a_{jk}^*| \leq |a_{ij}^*| < 1$. Since the difference has value < 1 we must have $|a_{jk}| < 1$ as well, so $|a_{jk}^*| = 1$ and $|\mathbf{a}_{jk}| = |a_{jk}| < 1$. In addition we have $|\mathbf{a}_{jj}| = 1 < |\mathbf{a}_{ij}|$, which is one more, so our claim is proved.

We next claim that applying a permutation of rows and columns as described earlier, the entire upper triangle can be assumed to consist of elements of value ≤ 1. Indeed, by the previous claim we may assume that $|\mathbf{a}_{0i}| \leq 1$ for all i and then proceed by induction on n to treat the matrix $(\mathbf{a}_{ij})_{1\leq i,j\leq n}$.

We are now at a situation where $|\mathbf{a}_{ij}| \leq 1$ if $i < j$ (and $|\mathbf{a}_{ij}| \geq 1$ if $i > j$). We observe that if $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{P}^1$ all have values ≤ 1 then the product $\mathbf{ab} = [ab : a^*b^*]$ is well defined and moreover $|ab| \leq 1 = |a^*b^*|$, i.e., the $|ab| \vee |a^*b^*| = 1$. It follows that

$$d(\mathbf{ab}, \mathbf{ac}) = |aba^*c^* - a^*b^*ac| \leq |bc^* - b^*c| = d(\mathbf{b}, \mathbf{c}).$$

Similar observations hold if all values are ≥ 1. We may therefore define

$$\mathbf{c}_{ij} = \prod_{i\leq k<j} \mathbf{a}_{k,k+1}, \qquad \mathbf{c}_{ji} = \mathbf{c}_{ij}^{-1} = \prod_{i\leq k<j} \mathbf{a}_{k+1,k}, \qquad (i \leq j).$$

It is not difficult to check that $\bar{\mathbf{c}} \in E$, and in order to prove the first item all that is left to check is that $d(\bar{\mathbf{c}}, \bar{\mathbf{a}}) \leq r$. Keeping in mind that $d(\mathbf{c}, \mathbf{a}) = d(\mathbf{c}^{-1}, \mathbf{a}^{-1})$, it will be enough to check that $d(\mathbf{a}_{ij}, \mathbf{c}_{ij}) \leq r$ for all $i < j$. We do this by induction on $j - i$. In the base case $j - i = 1$ we have $\mathbf{a}_{ij} = \mathbf{c}_{ij}$. Assume now that $d(\mathbf{a}_{ij}, \mathbf{c}_{ij}) \leq r$. Then

$$\begin{aligned} d(\mathbf{c}_{i,j+1}, \mathbf{a}_{i,j+1}) &\leq d(\mathbf{c}_{i,j+1}, \mathbf{a}_{ij}\mathbf{a}_{j,j+1}) \vee d(\mathbf{a}_{ij}\mathbf{a}_{j,j+1}, \mathbf{a}_{i,j+1}) \\ &= d(\mathbf{c}_{ij}\mathbf{a}_{j,j+1}, \mathbf{a}_{ij}\mathbf{a}_{j,j+1}) \vee \|\mathbf{a}_{ij}\mathbf{a}_{j,j+1} - \mathbf{a}_{i,j+1}\| \\ &\leq d(\mathbf{c}_{ij}, \mathbf{a}_{ij}) \vee r = r. \end{aligned}$$

This concludes the proof of the first item, and we have also proved the second item as a special case of our first claim.

For the third item, the fact that $[\mathbf{a}_{\ell,0} : \ldots : \mathbf{a}_{\ell,\ell-1} : 1 : \mathbf{a}_{\ell,\ell+1} : \ldots : \mathbf{a}_{\ell,n}] \in \mathbf{P}^n$ is a solution is an immediate consequence of the hypothesis that $\bar{\mathbf{a}} \in E$. Conversely, let $\mathbf{b} \in \mathbf{P}^n$ be any solution. Then $b_i = \mathbf{a}_{\ell i}b_\ell$ for all i, and since $\bigvee |b_i| = 1$ we must have $|b_\ell| = 1$. We may therefore assume that $b_\ell = 1$ and we obtain $b_i = \mathbf{a}_{\ell i}$ as desired. Finally, let $\mathbf{b} \in \mathbf{P}^n$, and define $\mathbf{a}_{ij} = [b_j : b_i]$ when at least one of b_i, b_j is non zero and $[1 : 1]$ otherwise. Then $\bar{\mathbf{a}} \in E$ and $\mathbf{b}$ is the associated solution. $\blacksquare_{1.13}$

We recall from [Ben10, Section 1.2] that a map $f\colon X \to Y$ between type-definable subsets of a structure is called *definable* if its graph is type-definable, or equivalently, if composing any definable predicate with f yields a definable predicate (a type-definable set is one which is the intersection of a family of zero sets of formulae, or of definable predicates; as in classical logic, a type-definable set corresponds to a closed set of types, see [Ben10, Section 1.1]). The former characterisation implies that if f is bijective then its inverse is definable as well. In the latter characterisation, it suffices to verify for the distance predicate alone.

Theorem 1.14. *The projective line $K\mathbf{P}^1$ is uniformly quantifier-free biinterpretable with $K\mathbf{P}$, and in fact $K\mathbf{P}^1$ is uniformly definable (rather than merely interpretable) in each of the sorts $K\mathbf{P}^n$ of $K\mathbf{P}$ for $n \geq 1$. More precisely:*

(i) *The $\mathcal{L}_{\mathbf{P}^1}$-structure $K\mathbf{P}^1$ and the sort $\mathbf{P}^1$ of the $\mathcal{L}_{\mathbf{P}}$-structure $K\mathbf{P}$ are quantifier-free definable in one another, meaning that a predicate $\varphi\colon (K\mathbf{P}^1)^m \to [0,1]$ is quantifier-free definable in $K\mathbf{P}^1$ if and only if it is quantifier-free definable in $K\mathbf{P}$.*

(ii) *For every $n \geq 1$ there exist a quantifier-free definable subset $D_n \subseteq \mathbf{P}^n$ and a definable bijection $\theta_n\colon D_n \to \mathbf{P}^1$ such that for every quantifier-free definable predicate $\varphi\colon (\mathbf{P}^1)^m \to [0,1]$, the predicate $\varphi \circ (\theta_n)\colon (D_n)^m \to [0,1]$ is quantifier-free definable as well.*

(iii) *For every n there exist a quantifier-free definable subset $E_n \subseteq (\mathbf{P}^1)^{M(n)}$ and a definable surjection $\rho_n\colon E_n \to \mathbf{P}^n$ such that for every quantifier-free definable predicate $\varphi\colon \mathbf{P}^{n_0} \times \cdots \times \mathbf{P}^{n_{m-1}} \to [0,1]$, the predicate $\varphi \circ (\rho_{n_0}, \ldots, \rho_{n_{m-1}})\colon E_{n_0} \times \cdots \times E_{n_{m-1}} \to [0,1]$ is quantifier-free definable as well.*

(iv) *The predicates defining D_n and E_n, as well as the translation schemes from quantifier-free predicates in one sort or structure to another are uniform, i.e., do not depend on K.*

Proof. The first item is easy, keeping in mind that it is enough to show that every atomic formula in one structure is quantifier-free definable in the other.

For the second item, we let $D_n = \{[a_0 : a_1 : 0 : \ldots : 0] \colon [a_0 : a_1] \in \mathbf{P}^1\}$. It is not difficult to check that $d(\mathbf{b}, D_n) = \bigvee_{2 \leq i \leq n} |b_i|$ which is definable by a quantifier-free formula. The map $\theta_n\colon [a_0 : a_1 : 0 : \ldots : 0] \mapsto [a_0 : a_1]$ is definable since its graph is given by

$$\theta_n(x) = y \quad \Longleftrightarrow \quad \|x_0 y_1 - x_1 y_0\| = 0.$$

We leave it to the reader to check that the pull-back of every atomic formula in $\mathbf{P}^1$ is quantifier-free definable in $\mathbf{P}^n$.

For the third item most of the work has already been done in Lemma 1.13. We take $M(n) = \frac{n(n+1)}{2}$ and define E_n as in the Lemma. Then we have already seen that E_n is quantifier-free definable and constructed the surjection $\rho_n\colon E_n \to \mathbf{P}^n$. Again we leave it to the reader to check that the pull-back of an atomic formula from $\prod \mathbf{P}^{n_i}$ to $\prod E_{n_i}$ is quantifier-free definable.

Everything we did (or left to the reader) is independent of the field K, whence follows the uniformity. $\blacksquare_{1.14}$

It follows that the class of structures $K\mathbf{P}$ is elementary as well. Moreover, if we prove that some theory extending MVF eliminates quantifiers (as we shall, in Theorem 2.4 below) it will follow that the corresponding $\mathcal{L}_{\mathbf{P}}$-theory eliminates quantifiers as well.

2. The theory of algebraically closed metric valued fields

Definition 2.1. We define $ACMVF$, the theory of algebraically closed metric valued fields, to consist of MVF along with the following additional axioms

$$\exists y \, \|y\| = \tfrac{1}{2}$$
$$\exists y \, \|P(\bar{x}, y)\| = 0 \qquad\qquad (\deg_Y(P) \geq 1)$$

As usual, the existential quantifier should be understood in the approximate sense. In the case of the first axiom, it may indeed happen that in a model of $ACMVF$ the value $\frac{1}{2}$ never occurs. For the second axiom, the approximate witnesses must accumulate near at least one of finitely many roots, so a root must exist in the (complete) model.

Lemma 2.2. *The models of $ACMVF$ are precisely the projective lines over complete, algebraically closed, non trivially valued fields.*

Proof. One direction is clear. For the other, given an algebraically closed field equipped with a non trivial valuation, the set of values must be dense in $\mathbf{R}$ and in particular contain $\frac{1}{2}$ in its closure. $\blacksquare_{2.2}$

Fact 2.3. *Let $K \subseteq L$ be an extension of valued fields, where K is complete, and let $a \in L$ be algebraic over K of degree n and with irreducible polynomial $P(X) \in K[X]$. Then $|a|^n = |P(0)|$.*

Theorem 2.4. *The theory $ACMVF$ eliminates quantifiers. It is therefore the model completion of MVF.*

Proof. Let both $K\mathbf{P}^1, F\mathbf{P}^1 \vDash ACMVF$ be somewhat saturated, and let $\theta\colon A \to B$ be a valuation-preserving isomorphism of relatively small sub-fields $A \subseteq K$ and $B \subseteq F$. First of all we may assume that A and B are complete. Second, any extension of the isomorphism to an algebraic isomorphism of their algebraic closure will preserve the valuation, so we may further assume that A and B are algebraically closed (of course, the algebraic closure need not be complete, so we would have to pass to the completion again).

Let now $c \in K$ be transcendental over A. The quantifier-free type of c over A is determined by the mapping assigning to each $P(X) \in A[X]$ the value $|P(c)|$. Since A is algebraically closed, it suffices to know this for linear polynomials, i.e., to know $|c - a|$ for all $a \in A$.

For our purposes it will be enough to show that for every finite tuple $a_0, \ldots, a_{n-1} \in A^n$ and every $\varepsilon > 0$ there exists $d \in F$ such that $\big||c - a_i| - |d - \theta a_i|\big| < \varepsilon$ for $i < n$. Let $r = \min_{i<n} |c - a_i|$. Possibly decreasing ε and re-arranging the tuple $\bar{a}$, we may assume that there is k such that $|c - a_i| = r$ if $i < k$ and $|c - a_i| > r + \varepsilon$ if $k \leq i < n$. It will therefore be enough to find $d \in F$ such that $\big|r - |d - \theta a_i|\big| < \varepsilon$ for $i < k$ (since then $|d - \theta a_i| = |a_0 - a_i| = |c - a_i|$ follows for $k \leq i < n$). We consider two cases:

Case I: If $|c| > r$, we choose $d_0 \in F$ such that $r < |d_0| < \min(r + \varepsilon, |c|)$ (such d_0 exists since the set of values is dense in $\mathbf{R}$), and let $d = d_0 + \theta a_0$. Then $|d - \theta a_i| = |d_0|$ for all $i < k$.

Case II: If $|c| \leq r$, then $|a_i| \leq r$ for all $i < k$. Since B is algebraically closed, so is its residue field. In particular, the residue field is infinite, so we may choose $b_{\leq k} \in B$ such that $|b_i| = 1$ for all $i \leq k$ and $|b_i - b_j| = 1$ for all $i < j \leq k$. We may also choose $e \in F$ such that $r - \varepsilon < |e| < r$. We claim that there is $j \leq k$ such that for all $i < k$: $|b_j e - \theta a_i| \geq |e|$. Indeed, otherwise, by the pigeonhole principle we can find $i < j \leq k$ such that $|b_i e - b_j e| < |e|$, whereby $|b_i - b_j| < 1$, contrary to our assumption. Let d be this $b_j e$. Since $|a_i| \leq r$ and $|d| < r$, we must have $|e| \leq |d - \theta a_i| \leq r$ for all $i < k$.

This concludes the proof that K and F correspond by an infinite back and forth. It follows that $ACMVF$ eliminates quantifiers. It is also clearly a companion of MVF and therefore it is its model completion. $\blacksquare_{2.4}$

Remark 2.5. Let $MVF_{\mathbf{Z}}$ denote the theory MVF along with axioms saying that the set of non zero values is contained in some fixed infinite discrete group, say $e^{\mathbf{Z}}$. This can be expressed by the axiom $\|x^*\| \in e^{-\mathbf{N}} \cup \{0\}$. In models of this theory both the valuation ring and its complement are type-definable, so they are in fact definable. The maximal ideal is definable as well, so we may refer to the residue field directly as an imaginary sort. Similarly, for every n, the set of field elements of value e^{-n} is definable.

Let $ACMVF_{\mathbf{Z}}$ consist in addition of axioms saying that the value e^{-1} is attained, that every element of value e^{kn} has an nth root and that every irreducible monic polynomial over the valuation ring with free term 1 has a root. Then $ACMVF_{\mathbf{Z}}$ eliminates quantifiers, and it is the model completion of $MVF_{\mathbf{Z}}$. The argument is similar to that given for Theorem 2.4.

Corollary 2.6. *The following is an exhaustive list of the completions of $ACMVF$:*

(i) *Characteristic $(0,0)$: $|p| = 1$ for all prime p.*
(ii) *Characteristic $(0,p)$: $|p| = \alpha$ for some prime p and $0 < \alpha < 1$.*
(iii) *Characteristic (p,p): $p = 0$ for some prime p.*

Proof. It is known (e.g., from [Art67]) that every model of $ACMVF$ falls into one of these categories and that none of them is empty. Since each of the listed theories determines $|n|$ for each $n \in \mathbf{Z}$, by quantifier elimination they are complete. $\blacksquare_{2.6}$

The space of completions consists therefore of a family of segments $[0,1]$, one for each prime p, with all the 1 points identified (the $(0,0)$ case). This is essentially the zero dimensional Berkovich space over $\mathbf{Z}$, just without the segment corresponding to Archimedean valuations, which we chose to exclude. Similarly,

Corollary 2.7. *Let K be a model of model of $ACMVF$, let $A \subseteq K$, and let K_0 be the complete sub-field generated by A. Then the space of 1-types over A in the sort $\mathbf{P}^n$ is precisely the n-dimensional projective analytic Berkovich space over K_0.*

Let us give a slightly different characterisation of types (or more precisely, of 1-types) which will be useful for counting them.

Definition 2.8. Let K be a valued field and let $\mathcal{C}$ and $\mathcal{C}'$ be two chains of closed balls in $K\mathbf{P}^1$. Say that $\mathcal{C}$ and $\mathcal{C}'$ are *mutually co-final* if each ball in one chain contains some ball belonging to the other. This is an

equivalence relation, and by a *sphere* over $K\mathbf{P}^1$ we mean an equivalence class of such a chain. The set of all spheres will be denoted $\mathrm{Sph}(K\mathbf{P}^1)$.

Let $S, S' \in \mathrm{Sph}(K\mathbf{P}^1)$ be spheres, say represented by $\mathcal{C}$ and $\mathcal{C}'$. We define the radius of S as $\mathrm{rad}(S) = \inf_{B\in\mathcal{C}} \mathrm{rad}(B)$. We define the Hausdorff distance between S and S' as the limit of Hausdorff distances between balls in $\mathcal{C}$ and $\mathcal{C}'$:

$$d_H(S, S') = \lim_{\substack{B\in\mathcal{C},\mathrm{rad}(B)\to\mathrm{rad}(S)\\ B'\in\mathcal{C}',\mathrm{rad}(B')\to\mathrm{rad}(S')}} d_H(B, B').$$

It is not difficult to see that for closed balls B and B',

- $d_H(B, B') = 0$ if and only if $B = B'$,
- if $B \supsetneq B'$ then $d_H(B, B') = \mathrm{rad}(B)$, and
- if $B \cap B' = \varnothing$ then $d_H(B, B') = d(B, B')$.

Notice that every sphere admits a countable representative. The field K is complete if and only if every sphere of radius zero contains a point. If every sphere contains a point then K is called *spherically complete*.

Theorem 2.9. *Let $K\mathbf{P} \vDash ACMVF$. Then:*

(i) *Let $S \in \mathrm{Sph}(K\mathbf{P}^1)$ be a sphere, say the class of $\mathcal{C} = \{\overline{B}(a_n, r_n)\}_{n\in\mathbf{N}}$, and let $r = \inf r_n$ denote its radius. Then the set of conditions*

$$\{\|x - a_n\| \le r_n\}_{n\in\mathbf{N}} \cup \{\|x - a\| \ge r\}_{a\in K\mathbf{P}^1} \tag{2}$$

axiomatises a complete type $p_S(x) \in \mathrm{S}_1(K)$ which depends only on S.

(ii) *The mapping $S \mapsto p_S$ is an isometric bijection $\big(\mathrm{Sph}(K\mathbf{P}^1), d_H\big) \cong \big(\mathrm{S}_1(K), d\big)$, where the distance between two types is the minimal distance between realisations.*

Proof. Let us first show that (2) is consistent for every S. Possibly passing to a sub-sequence, and possibly applying the isometry $a \mapsto a^{-1}$ to $\mathbf{P}^1$, we may assume that $|a_n| \le 1$ for all n. Let $L = K(\alpha)$ where α is transcendental over K. Then we may extend the valuation to L so that for every polynomial $P(X) = \sum_{k\le m} b_k X^k \in K[X]$ we have $|P(\alpha)| = \bigvee_k \mathrm{rad}(S)^k |b_k|$. In particular, $|\alpha| = \mathrm{rad}(S) \le 1$. Further extending to a model of $ACMVF$ we may assume that $L\mathbf{P}^1 \succeq K\mathbf{P}^1$. Let $c_n = a_n + \alpha$. For $a \in K\mathbf{P}^1$ we have $\|c_n - a\| = 1 \ge \mathrm{rad}(S)$ if $|a| > 1$ and $\|c_n - a\| = |c_n - a| = |a_n - a| \vee \mathrm{rad}(S)$ otherwise. For $m < n$ we also have $\|c_n - a_m\| = |\alpha + (a_n - a_m)\| \le r_m$. Thus (2) is finitely consistent and therefore consistent.

By quantifier elimination and the fact that K is algebraically closed, the type of an element α over $K\mathbf{P}^1$ is determined by $|\alpha - a|$ as a varies over K, or equivalently, by $\|\alpha - a\|$ as a varies over $K\mathbf{P}^1$. Let S be the sphere consisting of all balls $\overline{B}\big(a, d(a, \alpha)\big)$, $a \in K\mathbf{P}^1$. Then S only depends on $\mathrm{tp}(\alpha/K)$, and conversely, $p_S = \mathrm{tp}(\alpha/K)$. This yields the bijection $\mathrm{Sph}(K\mathbf{P}^1) \to \mathrm{S}_1(K)$.

It is left to show that this bijection is isometric. So let S and S' be two distinct spheres and let α and β realise p_S and $p_{S'}$, respectively. Assume first that $B \cap B' \ne \varnothing$ for all $B \in S$ and $B' \in S'$. Then $\mathrm{rad}(S) \ne \mathrm{rad}(S')$ (since else the spheres coincide), say $\mathrm{rad}(S) > \mathrm{rad}(S)$. Then $d_H(S, S') = \mathrm{rad}(S) = d(\alpha, \beta)$. On the other hand, if there are $B \in S$ and $B' \in S'$ which are disjoint then $d_H(S, S') = d(B, B') = d(\alpha, \beta)$ again. $\blacksquare_{2.9}$

Corollary 2.10. *The theory $ACMVF$ is strictly stable (i.e., stable non super-stable).*

Proof. Let K be a model. Since every sphere has a countable representative, a quick calculation yields that there are at most $|K|^{\aleph_0}$ spheres, and therefore types, over K. Thus the theory is stable.

On the other hand, for every $0 < r < r' < 1$, every ball of radius r' contains $|K|$ many distinct balls of radius r. Thus a refinement of our earlier calculation yields that there exist precisely $|K|^{\aleph_0}$ distinct spheres of radius r. The distance between any two such spheres is at least r, so the theory is not super-stable. $\blacksquare_{2.10}$

Remark 2.11. Here we assume the reader has some familiarity with the notion of perturbations of metric structures and its uses, as introduced in [Ben08b], or, in a somewhat simpler fashion, in [Ben08c]. Extensions of perturbations to types over parameters, and λ-stability up to perturbation, are also discussed in [Ben08c]. For example, it is shown in [BB09] that the theory of atomless probability algebras with a generic automorphism, even though it is strictly stable, is $\aleph_0$-stable up to arbitrarily small perturbations of the automorphism.

Omitting many details, let us consider a theory T and a set of parameters $A \subseteq \mathcal{M} \vDash T$. We define $\mathcal{L}(A)$ to consist of the base language $\mathcal{L}$ together with, for each $a \in A$, a unary predicate $P_a(x)$ for the distance $d(a,x)$. Thus "a model of T containing A" is essentially the same as a model of $T(A) = \mathrm{Th}_{\mathcal{L}(A)}(\mathcal{M})$, and types over A are just types of $T(A)$ over $\varnothing$. Roughly speaking, a perturbation of a model of $T(A)$ consists of modifying the interpretation of the symbols of $\mathcal{L}$ (usually with some small uniform bound on the extent of the modification, prescribed by a *perturbation system*), in such a manner that the end result is again a model of $T(A)$, and that the predicates P_a, representing the parameters, remain unchanged.

In $ACVMF$, when $A = K$ is a model, a 1-type $\mathrm{tp}(b/K)$ is entirely determined by the map $a \mapsto P_a(b)$, so a perturbation cannot change 1-types over K at all (even if it does change, to some small extent, the distance and/or algebraic structure of an extension of K containing the realisation).

It follows that even up to perturbation, in the sense of the articles cited above, $ACMVF$ is strictly stable, i.e., λ-stable up to perturbation only when $\lambda = \lambda^{\aleph_0}$.

The same argument does not work for $ACMVF_{\mathbf{Z}}$, since there a strictly decreasing sequence of radii must necessarily go to zero, and it follows that the theory is $\aleph_0$-stable. This is hardly surprising, since equal characteristic models of $ACMVF_{\mathbf{Z}}$ are just something of the form $K = k((X))$. They are therefore interpretable in the valuation ring $k[[X]]$ which is in turn interpretable (as a metric structure) in k, a plain strongly minimal algebraically closed field.

It is an easy fact that if the union of two disjoint type-definable sets is definable then each of the two sets is definable as well. The following is a useful extension of this fact.

Lemma 2.12. *Let X and Y be two type-definable sets such that both $X \cup Y$ and $X \cap Y$ are definable. Then X and Y are definable as well.*

Proof. It will be enough to show that X is definable, and for this, it will be enough to show that for every $\varepsilon > 0$, the ε-neighbourhood $B(X,\varepsilon)$ contains a logical neighbourhood of X.

Since Y is type-definable and $X \cap Y$ definable, the properties $d(x,Y) \leq \delta$ and $d(x, X \cap Y) \geq \varepsilon$ are type-definable. By compactness there exists $\delta > 0$ such that $\big(x \in X$ **and** $d(x, X \cap Y) \geq \varepsilon$ **and** $d(x,Y) \leq \delta\big)$ is contradictory. We may further assume that $\delta \leq \varepsilon$. We claim that the desired neighbourhood of X is the given by the property

$$\Big(d(x,Y) > \delta \textbf{ and } d(x, X \cup Y) < \delta\Big) \textbf{ or } d(x, X \cap Y) < \varepsilon.$$

Indeed, this is an open property, and it holds for every $x \in X$ by choice of δ. Assume this property holds for x. If $d(x, X \cap Y) < \varepsilon$ then $d(x,X) < \varepsilon$ as well. Otherwise, $d(x, X \cup Y) < \delta$ and $d(x,Y) > \delta$ imply that $d(x,X) < \delta \leq \varepsilon$, and the proof is complete. $\blacksquare_{2.12}$

The following generalises the fact that a definable image of a definable set is definable.

Lemma 2.13. *Let X be a definable set, Y and $Z \subseteq X$ type-definable sets, and let $f\colon X \smallsetminus Z \to Y$ be a bijection. Assume furthermore that f is definable, in the sense that there exists a type-definable set $R \subseteq X \times Y$ such that $R \cap \big((X \smallsetminus Z) \times Y\big)$ is the graph of f. Then Y is definable as well.*

Proof. Since Y is type-definable, the property $d(y,Y) \leq r$ is type definable. It will therefore be enough to show that $d(y,Y) \geq r$ is a type-definable property for all r. Let $\pi(x)$ be the partial type defining Z, and let $\varphi \in \pi$. For each $x \in X$, either $f(x)$ is well defined or $\varphi(x) = 0$, so either way $d(y, f(x)) \wedge \varphi(x)$ is well defined, and we claim that it is a definable predicate. Indeed, $d(y,f(x)) \wedge \varphi(x) \geq s$ if and only if there exists w such that $R(x,w)$ and $d(y,w) \wedge \varphi(x) \geq s$, and similarly for $\leq s$. Since X is definable, we obtain a definable predicate

$$\psi_\varphi(y) = \inf_{x \in X} \big[r \dot{-} d(y, f(x))\big] \wedge \varphi(x).$$

We conclude by observing that $d(y,Y) \geq r$ is defined by the partial type $\{\psi_\varphi\}_{\varphi \in \pi}$. $\blacksquare_{2.13}$

Recall:

Fact 2.14 (Noether's Normalisation Lemma)**.** *Let A be an integral domain, finitely generated over a field k. Then there exist algebraically independent elements $x_0, \ldots, x_{d-1} \in A$ such that A is integral over $k[x_0, \ldots, x_{d-1}]$.*

Moreover, if k is infinite and $A = k[y_0, \ldots, y_{n-1}]$ then each x_i can be taken to be a k-linear combination of the y_j.

Let V be a projective variety of dimension d defined over an infinite field k. Let $y = [y_0 : \ldots : y_n]$ be a generic point of V. Let $x_0, \ldots, x_d$ be a transcendence basis for $k[\bar{y}]$ consisting of k-linear combinations of $\bar{y}$, as per Noether's Normalisation Lemma. Then $[x_0 : \ldots : x_d : y_0 : \ldots : y_n]$ is the generic point of a projective variety isomorphic to V.

Proposition 2.15. *Let $K\mathbf{P}^1 \vDash ACMVF$. Then every Zariski closed set $V \subseteq K\mathbf{P}^n$ is definable.*

Proof. Since a finite union of definable sets is definable, we may assume that V is a variety, say of dimension d. Clearly every algebraic morphism is definable, and recall that the image of a definable set by a definable mapping is definable as well. It follows that we may replace V with any isomorphic projective variety. Therefore, using Noether's Normalisation Lemma we may assume that the homogeneous prime ideal defining V is $I(V) \subseteq K[X_0, \ldots, X_d, Y_0, \ldots, Y_{n-1}]$, where $K[\bar{X}] \cap I(V) = 0$ and for each $j < n$ there exists a homogeneous polynomial $f_j \in I(V) \cap K[\bar{X}, Y_j]$ which is monic in Y_j. Possibly replacing V with an isomorphic variety we may further assume that all the coefficients in each f_j have value ≤ 1. Thus we may express $|f_j(x_0, \ldots, x_d, x_{d+j+1})|$ as an atomic formula $\|f_j(x)\|$ in the free variable $x = [x_0 : \ldots : x_n] \in \mathbf{P}^n$ and with parameters in K. We may further assume that all the f_j have common degree m.

As a first approximation, let $J = \langle f_j \rangle_{j<n} \subseteq I(V)$ be the generated homogeneous ideal, and let us show that $V(J)$ is definable. Clearly $V(J)$ is the zero set of the formula $\bigvee_{j<n} \|f_j(x)\|$, and it will be enough to show that $d(x, V(J)) \leq \bigvee_{j<n} \|f_j(x)\|^{\frac{1}{m}}$. So let us fix $x \in \mathbf{P}^n$. For $j < n$, let

$$g_j(Y_j) = f_j(x_0, \ldots, x_d, Y_j) = \prod_{k<m} (Y_j - \gamma_j^k) \in K[Y_j]$$

We may assume that for each $j < n$, the root $\gamma_j^0 = \gamma_j$ is closest to x_{d+j+1} among all the roots of g_j. Let

$$y = [x_0 : \ldots : x_d : \gamma_0 : \ldots : \gamma_{n-1}] = \left[\tfrac{x_0}{s} : \ldots : \tfrac{x_d}{s} : \tfrac{\gamma_0}{s} : \ldots : \tfrac{\gamma_{n-1}}{s}\right] \in V(J),$$

where s is chosen of maximal value among $x_0, \ldots, x_d, \gamma_0, \ldots, \gamma_{n-1}$. A quick calculation yields, for $i \leq d$ and $j < n$,

$$|x_i y_{d+j+1} - x_{d+j+1} y_i| = |\tfrac{x_i}{s}||\gamma_j - x_{d+j+1}| \leq |g_j(x_{d+j+1})|^{\frac{1}{m}} = \|f_j(x)\|^{\frac{1}{m}},$$

and for $i, j < n$,

$$\begin{aligned} |x_{d+i+1} y_{d+j+1} - x_{d+j+1} y_{d+i+1}| &= |\tfrac{1}{s}||\gamma_j x_{d+i+1} - \gamma_i x_{d+j+1}| \\ &\leq |\tfrac{\gamma_j}{s}||x_{d+i+1} - \gamma_i| \vee |\tfrac{\gamma_i}{s}||\gamma_j - x_{d+i+1}| \\ &\leq (\|f_i(x)\| \vee \|f_j(x)\|)^{\frac{1}{m}}. \end{aligned}$$

Thus $d(x, V(J)) \leq d(x, y) \leq \bigvee_{j<n} \|f_j(x)\|^{\frac{1}{m}}$, as desired.

By construction, $V(J)$ is of dimension $\leq d$, and can be decomposed as $V(J) = V \cup W$ where $W \subseteq \mathbf{P}^n$ is a Zariski closed as well and $\dim(V \cap W) < d$. By induction on the dimension we may assume already known that $V \cap W$ is definable. We may now apply Lemma 2.12 and conclude that V is definable. $\blacksquare_{2.15}$

Corollary 2.16. *Every complete variety is interpretable in $ACMVF$.*

Proof. By Chow's Lemma, if W is a complete variety then it is the image of a projective variety V by a morphism. In other words, it is a definable quotient of a definable set, and therefore interpretable. $\blacksquare_{2.16}$

In particular, this means that a complete variety W is endowed with the quotient structure it inherits from the definable set V. This does not depend on the choice of V.

Question 2.17. Characterise all definable sets over K. Notice that since every compact set is definable, there are definable sets which are not projective varieties, e.g., any set of the form $\{a_n\}_n \cup \{0\}$ where $|a_n| \to 0$. More generally, every metrisable totally disconnected compact space can be embedded in $K\mathbf{P}^1$, and a characterisation of definable sets will have to allow for them.

Let $\{V_\alpha\}_{\alpha \in A}$ be a family of projective varieties, and assume that for every $\varepsilon > 0$ there is a finite $A_0 \subseteq A$ such that $\bigcup_{\alpha \in A} V_\alpha$ is contained in the ε-neighbourhood of $\bigcup_{\alpha \in A_0} V_\alpha$. Then $X = \overline{\bigcup_{\alpha \in A} V_\alpha}$ is a definable set. Every Zariski closed set and every compact set are of this form. Are there any other definable sets?

Question 2.18. Let A be any semi-normed ring. Let $\mathcal{L}_{\mathbf{P}}(A)$ consist of a constant symbol in the sort $\mathbf{P}^1$ for each member of A, and let $ACMVF(A)$ be the $\mathcal{L}_{\mathbf{P}}(A)$-theory consisting of $ACMVF$ along with axioms saying that $1 = 1_A$, $a+b = (a+_A b)$, $a \cdot b = (a \cdot_A b)$ and $|a| \leq |a|_A$ (i.e., $\|a\| \leq |a|_A$ if $|a|_A < 1$ and $\|a^*\| \geq |a|_A^{-1}$ otherwise).

Assuming that $I \subseteq A[X_0, \ldots, X_n]$ is a homogeneous ideal, is $V(I)$ uniformly definable in $ACMVF(A)$?

3. Real closed and ordered metric valued fields

We shall now seek to understand the metric valued analogue of the theory of real closed fields. First of all, we observe that the class of metric valued fields which are, as pure fields, formally real, is *not* elementary. Indeed, such fields can be constructed with $1 + a^2$ of arbitrarily small (non zero) valuation, and in an ultraproduct we would obtain $1 + a^2 = 0$. Thus $|1 + x^2|$ must be bounded away from zero, which, in a real closed field (and more generally, in a field where a sum of squares is a square), implies $|1 + x^2| \geq 1$.

Definition 3.1. We say that a valued field $(K, |\cdot|)$ is a *formally real valued field*, or that that $|\cdot|$ is a *formally real valuation* on K, if its residue field is formally real. If in addition K is real closed (as a pure field) then we say that it is a *real closed valued field*.

We recall that a field ordering (possibly partial) is one in which sums and products of positive elements, as well as all squares, are positive. A *valued field ordering* is one in which, in addition, the valuation ring is convex.

Lemma 3.2. *Let $(K, |\cdot|)$ be a valued field. Then the following are equivalent.*

(i) *The valued field $(K, |\cdot|)$ is formally real (as a valued field).*
(ii) *For all $x_0, \ldots, x_{n-1} \in K$:*
$$\left|\sum x_i^2\right| = \bigvee |x_i|^2.$$
(iii) *For all $x_0, \ldots, x_{n-1} \in K$:*
$$\left|1 + \sum x_i^2\right| \geq 1.$$

Similarly, a field K equipped with a valuation $|\cdot|$ and an ordering $\leq$ is an ordered valued field if and only if for every $x, y \geq 0$: $|x + y| = |x| \vee |y|$.

Proof. Easy. $\blacksquare_{3.2}$

A formally real valued field is formally real as a plain field, and conversely, a field K is formally real if and only if the trivial valuation on K is formally real.

Lemma 3.3. *Let $(K, |\cdot|)$ be a complete valued field. Then the following are equivalent.*

(i) *The valued field $(K, |\cdot|)$ is real closed (as a valued field).*
(ii) *The valued field $(K, |\cdot|)$ is formally real (as a valued field) and maximal as such among its algebraic valued field extensions.*

Proof. One direction is immediate. For the other, we already know that $(K, |\cdot|)$ is a formally real valued field, and it is left to show that it is real closed as a pure field. Indeed, let K_1/K be any proper algebraic field extension, which we may assume to be finite. We may then equip K_1 with an extension of the valuation (which is moreover unique since K is complete). Let k_1/k denote the corresponding residue field extension. Then $(K_1, |\cdot|)$ is not formally real, whereby k_1 is not formally real. On the other hand, k_1/k is an algebraic extension, so k_1 is algebraically closed. Since $(K_1, |\cdot|)$ is complete, as a finite extension of a complete valued field, by Hensel's Lemma we have $i \in K_1$, and in particular K_1 is not formally real. This completes the proof. $\blacksquare_{3.3}$

Lemma 3.4. (i) *A real closed valued field admits a unique ordering (as a valued field), namely its unique ordering as a pure real closed field: $x \geq 0$ if and only if x is a square.*
(ii) *Every formally real valued field embeds in a real closed valued field.*
(iii) *A valued field $(K, |\cdot|)$ is formally real if and only if it admits an ordering (as a valued field).*

Proof. For the first item, all we need to check is that valuation ring is convex in the unique field ordering, which is more or less immediate from the definition. The second item follows from Lemma 3.3. For the third and last item, one direction follows from the previous item, the other directly from the definitions. $\blacksquare_{3.4}$

In order to express in $\mathcal{L}_{\mathbf{P}^1}$ that the valuation is formally real one needs to take into account the homogenisation, yielding

$$\left\|\sum x_i^2\right\| = \bigvee \|x_i\|^2 \prod_{j\neq i} \|x_j^*\|^2. \tag{FR}$$

Working in the projective space $\mathbf{P}^n$ one can express this slightly more elegantly as

$$\left\|\sum x_i^2\right\| = 1, \tag{FR'}$$

where the sum is now over the homogeneous coordinates of a single point x.

Definition 3.5. We define $FRMVF$, the theory of formally real metric valued fields, to consist of MVF along with the axiom (FR). We define $RCMVF$, the theory of real closed metric valued fields, to consist, in addition, of the axioms

$$\begin{aligned}
&\exists y\, \|y\| = \tfrac{1}{2},\\
&\exists y\, \|x^2 - y^4\|,\\
&\exists y\, \left\|y^{2n+1} + \sum_{i\leq 2n} x_i y^i\right\|.
\end{aligned}$$

As in the discussion following the definition of $ACMVF$, the existential quantifiers are approximate, but in the case of the second and third axiom they imply exact existence.

Proposition 3.6. *Models of $FRMVF$ ($RCMVF$) are the projective lines over complete formally real (real closed and non trivial) valued fields.*

Ordered metric valued fields will be considered in an expanded language $\mathcal{L}_{o\mathbf{P}^1} \supseteq \mathcal{L}_{\mathbf{P}^1}$ which we now define. First, we wish to introduce a predicate $\langle x\rangle$, equal to zero if and only if x is positive or zero. Since ∞ is neither strictly positive not strictly negative, and may be arbitrarily close both to positive and to negative field elements, we require $\langle\infty\rangle = 0$. One natural definition (which later turns out to be correct) is $\langle x\rangle = \|x\| \wedge \|x^*\|$ for negative x, so in particular we have a natural identity $\langle x\rangle = \langle x^{-1}\rangle$. Since our language contains no function symbols, it will be convenient to go further and add, for each polynomial $P \in \mathbf{Z}[\bar{X}]$, a predicate

$$\langle P(\bar{x})\rangle = \begin{cases} 0 & P(\bar{x}) \geq 0,\\ \|P(\bar{x})\| \wedge \|P^*(\bar{x})\| & \text{otherwise.}\end{cases}$$

In particular, if any x_i is equal to ∞ and $\deg_{X_i} P > 0$ then $\langle P(\bar{x})\rangle = 0$ by the "otherwise" clause. Using the assumption that K is an ordered valued field one verifies that all the new predicates are 1-Lipschitz. In what follows, it will be convenient to keep in mind that $\|P\| \wedge \|P^*\| = \big(|P| \wedge 1\big)\|P^*\|$.

Definition 3.7. We define $OMVF$, the theory of ordered metric valued fields, to consist of MVF along with

$$\langle P\rangle \wedge \langle -P\rangle = 0 \tag{Tot}$$

$$\langle P\rangle \vee \langle -P\rangle = \|P\| \wedge \|P^*\| \tag{AS}$$

$$\langle P+Q\rangle\|P^*Q^*\| \leq \langle Q\rangle\|P^*(P+Q)^*\| \vee \langle P\rangle\|Q^*(P+Q)^*\| \tag{CA}$$

$$\langle -PQ\rangle \geq \langle P\rangle\langle Q\rangle \tag{CM}$$

We leave it to the reader to check that if K is an ordered valued field then the associated $\mathcal{L}_{o\mathbf{P}^1}$-structure is a model of $OMVF$, and conversely, that every model of $OMVF$ arises uniquely in this fashion.

For any field K, let $\mathrm{Sq}^K = \{x^2\}_{x\in K\mathbf{P}^1} \subseteq K\mathbf{P}^1$ (where $\infty^2 = \infty$). For $P(\bar{X}) \in \mathbf{Z}[\bar{X}]$ we consider the following definable predicate

$$\langle P(\bar{x})\rangle^{\mathrm{Sq}} = \inf_y \|P(\bar{x}) - y^2\|.$$

Lemma 3.8. *For every model $K\mathbf{P}^1 \vDash MVF$ we have*

$$\langle P(\bar{x})\rangle^{\mathrm{Sq}} = \begin{cases} 0 & P(\bar{x}) \in \mathrm{Sq}, \\ \|P(\bar{x})\| \wedge \|P^*(\bar{x})\| & \textit{otherwise.} \end{cases}$$

In particular, if $x_i = \infty$ and $\deg_{X_i} P > 0$ then $\langle P(\bar{x})\rangle^{\mathrm{Sq}} = 0$.

Proof. Clearly, if $\bar{x} \in K$ and $P(\bar{x}) \in \mathrm{Sq}$ then $\langle P(\bar{x})\rangle^{\mathrm{Sq}} = 0$. Also, we observe that $\|P(\bar{x}) - 0^2\| = \|P(\bar{x})\|$ and $\|P(\bar{x}) - \infty^2\| = \|P^*(\bar{x})\|$. Thus $\langle P(\bar{x})\rangle^{\mathrm{Sq}} \leq \|P(\bar{x})\| \wedge \|P^*(\bar{x})\|$, and in particular $\langle P(\bar{x})\rangle^{\mathrm{Sq}} = 0$ if $x_i = \infty$ and $\deg_{X_i} P > 0$. It is left to consider the case where $\bar{x} \in K$ and $P(\bar{x}) \notin \mathrm{Sq}$. Indeed, assume that $\langle P(\bar{x})\rangle^{\mathrm{Sq}} < \|P(\bar{x})\| \wedge \|P^*(\bar{x})\|$. Then there is $z \in K^*$ such that $\|P(\bar{x}) - z^2\| < \|P(\bar{x})\| \wedge \|P^*(\bar{x})\|$, or equivalently $|P(\bar{x}) - z^2|\|z^*\|^2 < |P(\bar{x})| \wedge 1$. If $|z| \leq 1$ then $|P(\bar{x}) - z^2| < |P(\bar{x})|$, whereby $|P(\bar{x})| = |z^2|$; and if $|z| > 1$ then $|P(\bar{x}) - z^2| < \|z^*\|^{-2} = |z^2|$, and again $|P(\bar{x})| = |z^2|$. Either way we get $|\frac{P(\bar{x})}{z^2} - 1^2| < 1 = |\frac{P(\bar{x})}{z^2}|$, and by Hensel's Lemma, $P(\bar{x}) \in \mathrm{Sq}$, contrary to our assumption. $\blacksquare_{3.8}$

Lemma 3.9. *In any metric valued field the set* Sq *is closed and $d(x, \mathrm{Sq}) = \langle x\rangle^{\mathrm{Sq}}$. In particular,* Sq *is uniformly definable across all complete valued fields.*

Proof. It is easy to see that $\|x - z^2\| = d(x, z^2)$ (compare with Lemma 1.9), whereby $d(x, \mathrm{Sq}) = \langle x\rangle^{\mathrm{Sq}}$. By Lemma 3.8, if $x \notin \mathrm{Sq}$ then $d(x, \mathrm{Sq}) = \|x\| \wedge \|x^*\| > 0$, so Sq is closed. $\blacksquare_{3.9}$

Proposition 3.10. *Let $K \vDash RCMVF$. Then K admits a unique expansion to a model of $OMVF$, given by $\langle P\rangle = \langle P\rangle^{\mathrm{Sq}}$.*

Theorem 3.11. *The $\mathcal{L}_{o\mathbf{P}^1}$-theory $ORCMVF = RCMVF \cup OMVF$ is complete and admits quantifier elimination. The theory $RCMVF$ is model complete.*

Proof. Completeness and model completeness follow quite easily from quantifier elimination, so we only prove the latter. For this, we shall prove that sufficiently saturated models admit an infinite back-and-forth. Using the uniqueness of the real closure of an ordered field, and proceeding as in the proof of Theorem 2.4, we reduce to the case where $K\mathbf{P}^1$ and $F\mathbf{P}^1$ are two sufficiently saturated models, $A \subseteq K$ and $B \subseteq F$ are relatively algebraically closed complete sub-fields, and $\theta\colon A \to B$ is an isomorphism. In particular, A and B are real closed valued fields.

Now let $c \in K \smallsetminus A$. Its quantifier-free type is determined by the value and sign of $P(c)$ as $P(X)$ varies over $A[X]$. Since A is real closed, every polynomial decomposes as a product of linear factors $X - a$ and irreducible quadratic factors $(X - a)^2 + b$, $b > 0$ (and $a, b \in A$). In the second case we have $(c - a)^2 + b > 0$ and $|(c - a)^2 + b| = |c - a|^2 \vee b$. Thus, the quantifier-free type of c is determined by the value and sign of $c - a$ as a varies over A. In order to find $d \in F$ with the corresponding quantifier-free type over B, it is enough to show that for every $\varepsilon > 0$ and every finite family $a_0, \ldots, a_{n-1} \in A$ there is $d \in F$ such that $d \leq \theta a_i \iff c \leq a_i$ and $\big||d - \theta a_i| - |c - a_i|\big| < \varepsilon$. We may assume that $a_i < a_{i+1}$ for $i < n - 1$.

If $c > A$ then the valuation on A is necessary trivial. In this case we may take $d \in F$ to be any positive element with the same value as c (or at least close enough). The case $c < A$ is treated similarly. Otherwise, there is i for which $a_i < c_i < a_{i+1}$. Translating by a_i and dividing by a_{i+1} we may assume that $a_i = 0$ and $a_{n+1} = 1$. It will then be enough to find $0 < d < 1$ such that $\big||d| - |c|\big|, \big||1 - d| - |1 - c|\big| < \varepsilon$, and the rest will follow. Possibly replacing c with $1 - c$, we may further assume that $|c| \leq |1 - c| = 1$. If $|c| < 1$, just take for d any positive element whose value is close enough to $|c|$, and if $|c| = 1$ choose d so that $|d|$ is close enough to $1 - \varepsilon/2$. This completes the proof. $\blacksquare_{3.11}$

Theorem 3.12. *The theory $RCMVF$ is dependent.*

Proof. It is enough to show that every formula $\varphi(x, \bar{y})$, where x is a single variable, is dependent (this is shown in [Ben09] along the lines of the proof for classical logic in [Poi85]; a simplified argument appears in Adler [Adl], and it translates quite effortlessly to continuous logic). It is therefore enough to show that

if $(\bar{b}_n)_n$ is an indiscernible sequence then $\big(\varphi(a,\bar{b}_n)\big)_n$ converges for every a. By quantifier elimination, we may assume that φ is an atomic $\mathcal{L}_{o\mathbf{P}^1}$-formula, namely of the form $\|P(x,\bar{y})\|$ or $\langle P(x,\bar{y})\rangle$. Since the type $p=\mathrm{tp}(\bar{b}_n)$ is constant, and since every field element which is algebraic over $\bar{b}_n$ is definable over $\bar{b}_n$ (because of the linear ordering), we may express $\|P(x,\bar{b}_n)\|$ and $\langle P(x,\bar{b}_n)\rangle$ as continuous combinations of things of the form $|x-f(\bar{b}_n)|$ and $\langle x-f(\bar{b}_n)\rangle$, where f stands for a partial $\varnothing$-definable function whose domain contains p (as in the proof of the previous theorem). For each such function, the sequence $\big(f(\bar{b}_n)\big)_n$ is indiscernible as well, so in particular monotone, and it follows that $|a-f(\bar{b}_n)|$ and $\langle a-f(\bar{b}_n)\rangle$ converge. This completes the proof. $\blacksquare_{3.12}$

Alternatively, we may define $\mathcal{L}_{o\mathbf{P}}$ to consist of $\mathcal{L}_{\mathbf{P}}$ augmented with one predicate symbol $\langle\cdot\rangle$ for each sort $\mathbf{P}^n$, $n\geq 1$, interpreted in an ordered valued field by

$$\langle[a_0:\ldots:a_n]\rangle=\begin{cases}0 & a_0a_1\geq 0,\\ |a_0|\wedge|a_1| & \text{otherwise.}\end{cases}$$

We observe that this does not depend on the choice of representatives (as long as $\bigvee|a_i|=1$, as usual) and this is compatible with the interpretation of $\langle x\rangle$ on $\mathbf{P}^1$ we introduced earlier. One can extend Theorem 1.14, showing that for an ordered valued field K, the $\mathcal{L}_{o\mathbf{P}^1}$-pre-structure $K\mathbf{P}^1$ and the $\mathcal{L}_{o\mathbf{P}}$-pre-structure $K\mathbf{P}$ are quantifier-free biinterpretable, and this uniformly in K.

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