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# Estimating the order of vanishing at infinity of Drinfeld quasi-modular forms.

Federico Pellarin

April 5, 2010

*Abstract.* In this paper we study automorphic properties of certain deformations of *Drinfeld quasi-modular forms* (defined in [3]) motivated by the quest of *multiplicity estimates*, important tool in transcendence and algebraic independence, in the realm of analytic functions over algebraically closed, complete fields containing global fields of positive characteristic.

The main consequence of our results on such deformations is a multiplicity estimate for Drinfeld quasi-modular forms. Our result seems inaccessible by dealing directly with iterative higher derivations on Drinfeld quasi-modular forms and requires transcendence constructions in its proof, unlike classical multiplicity estimates in characteristic zero.

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# 1 Introduction, results.

*Foreword about the classical theory.* For  $w, l$  non-negative integers, let  $\widetilde{M}_w^{\leq l}$  be the  $\mathbb{C}$ -vector space generated by the classical *quasi-modular forms* (for  $\mathbf{SL}_2(\mathbb{Z})$ ) which have weight  $w$  and depth  $\leq l$  <sup>(1)</sup>.

In [19], Kaneko and Koike highlight the following hypothesis; for all  $w, l$  such that  $\widetilde{M}_w^{\leq l} \neq (0)$  the image of the function

$$\nu_\infty : \widetilde{M}_w^{\leq l} \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0},$$

which associates to every form its *order of vanishing at infinity* <sup>(2)</sup>, is equal to the interval  $[0, \dots, \dim_{\mathbb{C}} \widetilde{M}_w^{\leq l} - 1]$ .

Writing  $\nu_\infty^{\max}(w, l) = \max\{\nu_\infty(f) : f \in \widetilde{M}_w^{\leq l} \setminus \{0\}\}$ , we obviously get from this hypothesis:

$$\lim_{w \rightarrow \infty} \frac{\dim_{\mathbb{C}} \widetilde{M}_w^{\leq l}}{\nu_\infty^{\max}(w, l)} = 1, \quad l \geq 0 \text{ fixed.} \quad (1)$$

Since it can be easily verified that

$$\lim_{w \rightarrow \infty} \frac{\dim_{\mathbb{C}} \widetilde{M}_w^{\leq l}}{(l+1)(w-l)} = \frac{1}{12},$$

the truth of Kaneko and Koike's hypothesis would imply, for all  $l \geq 0$ :

$$\lim_{w \rightarrow \infty} \frac{\nu_\infty^{\max}(w, l)}{(l+1)(w-l)} = \frac{1}{12}. \quad (2)$$

The conjectural limit (2) yields, for  $w$  big enough depending on  $l$ , a rather sharp upper bound for  $\nu_\infty(f)$ , with  $f \in \widetilde{M}_w^{\leq l} \setminus \{0\}$  confirmed by experimental evidence.

In [4] it was noticed, for  $f \neq 0$  a classical quasi-modular form of weight  $w$  and depth  $\leq l$ , that at least,

$$\nu_\infty(f) \leq \frac{(l+1)(w-l)}{6}. \quad (3)$$

The main theorem of the paper is a slightly weaker analog of inequality (3) in the framework of *Drinfeld quasi-modular forms*; finding a reasonable substitute of Kaneko and Koike's hypothesis in the drinfeldian framework remains an open problem.

*Drinfeldian theory.* Let  $q = p^e$  be a power of a prime number  $p$  with  $e > 0$  an integer, let  $\mathbb{F}_q$  be the finite field with  $q$  elements. Let us write  $A = \mathbb{F}_q[\theta]$  and

<sup>1</sup>A definition can be found in the paper by Kaneko and Zagier [20].

<sup>2</sup>Terminology explained, for example, in [4].

$K = \mathbb{F}_q(\theta)$ , with  $\theta$  an indeterminate over  $\mathbb{F}_q$ , and define an absolute value  $|\cdot|$  on  $K$  by  $|a| = q^{\deg_\theta a}$ ,  $a$  being in  $K$ , so that  $|\theta| = q$ . Let  $K_\infty := \mathbb{F}_q((1/\theta))$  be the completion of  $K$  for this absolute value, let  $K_\infty^{\text{alg}}$  be an algebraic closure of  $K_\infty$ , let  $C$  be the completion of  $K_\infty^{\text{alg}}$  for the unique extension of  $|\cdot|$  to  $K_\infty^{\text{alg}}$ , and let  $K^{\text{alg}}$  be the algebraic closure of  $K$  in  $C$ .

Following Gekeler in [13], we denote by  $\Omega$  the rigid analytic space  $C \setminus K$  and write  $\Gamma$  for  $\mathbf{GL}_2(A)$ , group that acts on  $\Omega$  by homographies. In this setting we have three functions  $E, g, h : \Omega \rightarrow C$ , holomorphic in the sense of [11, Definition 2.2.1], such that, for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$  and  $z \in \Omega$ :

$$\begin{aligned} g(\gamma(z)) &= (cz + d)^{q-1}g(z), \\ h(\gamma(z)) &= (cz + d)^{q+1}\det(\gamma)^{-1}h(z), \\ E(\gamma(z)) &= (cz + d)^2\det(\gamma)^{-1}\left(E(z) - \frac{c}{\tilde{\pi}(cz + d)}\right) \end{aligned} \quad (4)$$

where  $\gamma(z) = (az + b)/(cz + d)$  and

$$\tilde{\pi} := \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} (1 - \theta^{1-q^i})^{-1} \in K_\infty((-\theta)^{\frac{1}{q-1}}) \setminus K_\infty,$$

a choice of a  $(q-1)$ -th root having been made once and for all <sup>(3)</sup>.

The functional equations above tell that  $g, h$  are *Drinfeld modular forms*, of weights  $q-1$ ,  $q+1$  and types  $0, 1$  respectively. More precisely, the function  $g$  is proportional to a variant of an Eisenstein series (constructed by Goss in [15]), while  $h$  is proportional to a variant of a Poincaré series (constructed by Gekeler in [13]).

The function  $E$  is not a Drinfeld modular form. In [13], Gekeler calls it “False Eisenstein series” of weight 2 and type 1; it is often considered as a reasonable substitute of the normalised (complex) Eisenstein series  $E_2$  of weight 2, although it “vanishes at infinity” (see later), because it provides a good way to compute *quasi-periods* of lattices of rank 2 (see [12]).

The  $C$ -algebra  $\widetilde{M} := C[E, g, h]$  has dimension 3. Weights and types of  $E, g, h$  associated to the functional equations (4) determine a graduation of  $\widetilde{M}$  by the group  $\mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z}$ . A *degree* is a couple of integers  $(w, m) \in \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z}$ . By convention, we identify the class  $m$  with its unique representative in the interval  $[0, q-1[$ . A polynomial  $f \in \widetilde{M} \setminus \{0\}$  is a *Drinfeld quasi-modular form* of *weight*  $w$  and *type*  $m$  if it is homogeneous (or isobaric) of degree  $(w, m)$ .

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<sup>3</sup>See [27, Section 2.1], where the notation  $\bar{\pi}$  is adopted; there is an analogy with the number  $2\pi i$ .

The algebra  $\widetilde{M}$  is also filtered by the *depths*. The depth  $l(f)$  of a non-zero polynomial  $f \in \widetilde{M}$  is by definition its degree  $\deg_E f$ . By convention, the zero polynomial of  $\widetilde{M}$  is a quasi-modular form of weight  $w$ , type  $m$  and depth  $l$  for all  $w, m, l$ . In all the following, we denote by  $\widetilde{M}_{w,m}^{\leq l}$  the finite-dimensional  $C$ -vector space of Drinfeld quasi-modular forms of weight  $w$ , type  $m$  and depth  $\leq l$ ; we recall that if  $w \not\equiv 2m \pmod{q-1}$ , then  $\widetilde{M}_{w,m}^{\leq l} = (0)$ . Obviously, if  $f \neq 0$  is quasi-modular, then  $w(f) \geq 2l(f)$ .

Let  $e_{\text{Car}} : C \rightarrow C$  be the *Carlitz exponential function* (see below, (22)) and let us write  $u : \Omega \rightarrow C$  for the “parameter at infinity” of  $\Omega$ , that is, the function  $u(z) = 1/e_{\text{Car}}(\widetilde{\pi}z)$ .

The  $C$ -algebra  $\widetilde{M}$  embeds in  $C[[u]]$  (cf. [13]). If  $f \in \widetilde{M}$ , then, for  $u = u(z)$  with  $|u|$  small enough, we have a converging  $u$ -expansion

$$f(z) = \sum_{i=0}^{\infty} c_i u^i, \quad c_i \in C. \quad (5)$$

Let us write, with  $f$  as in (5),  $\nu_{\infty}(f) := \inf\{i \text{ such that } c_i \neq 0\}$  with the convention  $\inf \emptyset = \infty$ . If  $f \neq 0$ , then  $\nu_{\infty}(f) < \infty$ .

The main result of this paper is the following:

**Theorem** *Let  $w, l, m$  be integers, with  $0 \leq m < q-1$ ,  $l > 0$ , let  $f \in \widetilde{M}_{w,m}^{\leq l}$  be a non-zero Drinfeld quasi-modular form. If*

$$w \geq 4l(2q(q+2)(3+2q)l + 3(q^2+1))^{3/2}, \quad (6)$$

*Then we have the estimate:*

$$\nu_{\infty}(f) \leq 16q^3(3+2q)^2lw. \quad (7)$$

Our Theorem does not overlap with [4, Theorem 1.4] (joint work with Bosser), a rather precise estimate which, however, holds for Drinfeld quasi-modular forms of depth  $\leq q^2$  only.

Our methods imply explicit estimates for the order of vanishing of any non-zero quasi-modular forms, but the unconditional estimates obtained so far are not as precise as (7). With the same hypotheses on  $f$  as the Theorem, but without assuming (6), we will also prove, in the present paper, that:

$$\nu_{\infty}(f) \leq 6q^4(4q+5)^3(q^2+1)l^2w(12 + \log_q w), \quad (8)$$

where  $\log_q$  denotes the logarithm in base  $q$ . Recently, in collaboration with Bosser, we have obtained a better (but still not optimal) unconditional estimate whose proof will appear in another work. In the next subsection we will explain why our Theorem is not “far away from the truth”.

## 1.1 A conjectural upper bound for $\nu_\infty$

Let  $w, m, l$  be such that  $\widetilde{M}_{w,m}^{\leq l} \neq (0)$ . What is the image of the map  $\nu_\infty : \widetilde{M}_{w,m}^{\leq l} \setminus \{0\} \rightarrow \mathbb{Z}$  and how big can  $\nu_\infty(f)$  be, with  $f \in \widetilde{M}_{w,m}^{\leq l}$  non-zero?

If  $\widetilde{M}_{w,m}^{\leq l} \neq (0)$  we write

$$\nu_\infty^{\max}(w, m, l) = \max\{\nu_\infty(f) : f \in \widetilde{M}_{w,m}^{\leq l} \setminus \{0\}\}$$

and we first look at the case  $l = 0$ . In this case, we write  $M_{w,m} = \widetilde{M}_{w,m}^{\leq 0}$ . Let  $M = \bigoplus_{w,m} M_{w,m} = C[g, h]$  be the graded  $C$ -algebra of Drinfeld modular forms, of dimension 2. It is easy to show (the brackets  $[\cdot]$  denote the lower integer part) that if  $\dim_C M_{w,m} \neq 0$ ,

$$\left\lfloor \frac{\nu_\infty^{\max}(w, m, 0)}{q-1} \right\rfloor = \dim_C M_{w,m} - 1, \quad (9)$$

so that, for all  $f \in M_{w,m} \setminus \{0\}$ , the image of  $\nu_\infty$  on the union  $\bigcup_m M_{w,m} \setminus \{0\}$  is equal to the interval  $[0, \dots, \lfloor \frac{w}{q+1} \rfloor]$  and

$$\nu_\infty(f) \leq \frac{w}{q+1}, \quad (10)$$

which also is the best possible bound linear in  $w$  <sup>(4)</sup>.

Different phenomena arise in the vector spaces  $\widetilde{M}_{w,m}^{\leq l}$  for  $l > 0$ . In this case, it is unclear how to extend (9) and compute the image of  $\nu_\infty$ .

*An example and a conjecture.* The dimension of  $\widetilde{M}_{w,m}^{\leq l}$  can be computed inductively with the formulas of [14, p. 33]; one deduces that for  $l, m$  fixed,

$$\lim_{w \rightarrow \infty} \frac{\dim_C \widetilde{M}_{w,m}^{\leq l}}{w-l} = \frac{l+1}{q^2-1}, \quad (11)$$

the limit being taken over the  $w$ 's such that  $w \equiv 2m \pmod{q-1}$ . For example, if  $q = 2$ , one gets the explicit formula  $\dim_C \widetilde{M}_{w,0}^{\leq 1} = \lfloor \frac{2w}{3} \rfloor$  for all  $w$ .

We consider more carefully the case of depth  $\leq 1$ . In [4] (see also Section 6 of the present paper) we have constructed a family of Drinfeld quasi-modular forms  $(x_k)_{k \geq 0}$  with  $x_k \in \widetilde{M}_{q^{k+1},1}^{\leq 1} \setminus M$  *extremal* in the sense that it attains, in the indicated vector space, the biggest possible order of vanishing at infinity; we have also proved that  $\nu_\infty(x_k) = q^k$  for all  $k$ . From this construction, one can in

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<sup>4</sup>Riemann-Roch's Theorem over the rigid analytic curve compactification of  $\Gamma \backslash \Omega$  also implies (10), see [13, (5.14)].

fact furnish a normalised extremal quasi-modular form  $f_{w,m,1}$  in every non-trivial vector space  $\widetilde{M}_{w,m}^{\leq 1}$  (5). If for example  $q = 5, m = 1$ , the sequence  $(f_{4n+2,1,1})_{n \geq 0}$  is:

$$-x_0, \underbrace{-\frac{x_1}{[1]}, -g\frac{x_1}{[1]}, \dots, -g^4\frac{x_1}{[1]}}_{5 \text{ terms}}, \underbrace{-\frac{x_2}{[1][2]}, -g\frac{x_2}{[1][2]}, \dots, -g^{24}\frac{x_2}{[1][2]}}_{25 \text{ terms}}, \dots,$$

where  $[i] := \theta^q - \theta$  (if  $w$  is not of the form  $4n + 2$ , the space  $\widetilde{M}_{w,1}^{\leq 1}$  is trivial). For general  $q$  and  $m$  it can be proved, by using the forms  $x_k$ 's, that the sequences  $(f_{(q-1)n+2m,m,1})_{n \geq 0}$  involve quasi-modular forms which are monomials  $\lambda g^a h^b x_k$  ( $\lambda \in C^\times, a, b, k \geq 0$ ; see Section 6 for details on the normalisations). An accurate study of these forms (that we skip here), implies that for all  $q, m$ :

$$0 < \liminf_{w \rightarrow \infty} \frac{\dim_C \widetilde{M}_{w,m}^{\leq 1}}{\nu_\infty^{\max}(w, m, 1)} < \limsup_{w \rightarrow \infty} \frac{\dim_C \widetilde{M}_{w,m}^{\leq 1}}{\nu_\infty^{\max}(w, m, 1)} < \infty,$$

the limits being taken in sequences with  $w$  such that  $\dim_C \widetilde{M}_{w,m}^{\leq 1} \neq 0$ . This means that there is no close analog of (1) and Kaneko and Koike's hypothesis in the Drinfeldian framework.

The infimum limit precisely occurs in the sequence of spaces  $\widetilde{M}_{q^k+1,1}^{\leq 1}$ , for  $k \geq 0$ . Induction on  $k \geq 0$  starting with the equality  $\dim_C \widetilde{M}_{2,1}^{\leq 1} = 1$  yields the computation of the dimensions: if  $q = 2$  a formula quoted above yields  $\dim_C \widetilde{M}_{2^k+1,1}^{\leq 1} = \lfloor \frac{2(2^k+1)}{3} \rfloor$  and if  $q \neq 2$  we get

$$\dim_C \widetilde{M}_{q^{k+1}+1,1}^{\leq 1} = q \dim_C \widetilde{M}_{q^k+1,1}^{\leq 1} + r_k,$$

where  $r_{2s} = -q + 2$  and  $r_{2s+1} = -2q + 3$ . Hence, for all  $q$ ,

$$\lim_{k \rightarrow \infty} \frac{\dim_C \widetilde{M}_{q^k+1,1}^{\leq 1}}{\nu_\infty^{\max}(q^k + 1, 1, 1)} = \frac{2}{q^2 - 1}.$$

Combining with (11) we find that for all  $w$  big enough with  $\widetilde{M}_{w,m}^{\leq 1} \neq (0)$  and  $f \in \widetilde{M}_{w,m}^{\leq 1} \setminus \{0\}$ ,  $\nu_\infty(f) \leq w - 1$  (that is,  $\leq l(w - l)$  with  $l = 1$ ).

These arguments have been extended with the help of several experiments to some higher depths and seem to justify the following (cf. [4]):

**Conjecture** *Let  $q$  and  $l > 0$  be fixed. For all  $m$ , for all  $w$  big enough such that  $\widetilde{M}_{w,m}^{\leq l} \neq (0)$ , and for all  $f \in \widetilde{M}_{w,m}^{\leq l} \setminus \{0\}$ ,*

$$\nu_\infty(f) \leq l(w - l). \quad (12)$$

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<sup>5</sup>A formal series  $\sum_{i \geq i_0} c_i u^i \in L[[u]]$  ( $L$  being a field) with  $c_{i_0} \neq 0$  is *normalised* if its leading coefficient  $c_{i_0}$  is one.

This upper bound cannot be improved, as the choice of  $f$  varying in the family  $(x_k)$  indicates (for  $k \geq 0$ ), with the  $x_k$ 's the functions introduced in [4]. Other evidences of the truth of this conjecture appear in this paper, notably in relation with *reduced forms* as defined in Section 5.3 (see Remark 27).

## 1.2 Methods of proof

Inequality (3) is a very simple *multiplicity estimate*. Multiplicity estimates are important tools in transcendence and algebraic independence techniques. A much deeper result was obtained by Nesterenko [24, Chapter 10, Theorem 1.3], and was the key tool in his theorem on the algebraic independence of values of normalised Eisenstein series and the function  $e^{2\pi iz}$ . The theory of multiplicity estimates in differential polynomial algebras gave general results when the base field is algebraically closed of zero characteristic; see for example [5, 6, 23].

A sketch of proof of inequality (3) is given in the introduction of [4]. If  $f \in \mathbb{C}[E_2, E_4, E_6]$  is a non-zero (classical) quasi-modular form <sup>(6)</sup>, Ramanujan's differential system implies that  $(d/dz)f$  is again a non-zero quasi-modular form. The bound follows easily remarking that if  $f$  and  $(d/dz)f$  are coprime, then the resultant

$$\mathbf{Res}_{E_2}(f, (d/dz)f) \tag{13}$$

of the polynomials  $f, (d/dz)f \in \mathbb{C}[E_2, E_4, E_6]$  with respect to  $E_2$  is a non-zero modular form whose weight is controlled by elementary considerations and whose order of vanishing at infinity is controlled by the well known suitable analog of (10).

If  $f$  and  $(d/dz)f$  are not coprime, one combines the resultant argument with a variant of the *separation property* of Brownawell and Masser [5, Lemma, p. 212] which holds in this case thanks to existence of the parabolic form  $\Delta = e^{2\pi iz} + \dots$  of weight 12, non-vanishing on the upper-half plane (see Nesterenko's Lemma 5.2 of [24, Chapter 10]).

Reasonable analogues in positive characteristic of differential algebras are the so-called *iterative differential*, or *hyperdifferential* algebras (cf. [22]); very little is known on multiplicity estimates in this framework; something which is really missing in this context is a suitable variant of the separation property.

In a joint paper with Bosser, we proved [3, Theorem 2] that the algebra  $\widetilde{M}$  is hyperdifferential, endowed with a *hyperdifferential structure*  $D$ , that is, a collection of  $C$ -linear operators  $D = (D_n)_{n \geq 0}$  satisfying certain properties among which Leibniz's formula. For this hyperdifferential algebra  $(\widetilde{M}, D)$ , there is not a suitable separation property analog (see [4]).

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<sup>6</sup>The  $E_{2i}$ 's denote normalisations of classical Eisenstein series of weights  $2i$ .



To overcome the above-mentioned difficulties and to prove our Theorem, the main idea of this paper is to work with certain *deformations* of Drinfeld quasi-modular forms, that we call *almost  $A$ -quasi-modular forms* <sup>(7)</sup>, introduced in this paper with an underlying connection to Anderson’s  $t$ -motives, also called  $A$ -motives.

In Section 2 we review and develop tools which have essentially been introduced by Anderson in [1], concerning *rigid analytic trivialisations* of  $A$ -motives associated to rank 2 Drinfeld  $A$ -modules. We use the exposition in [27] as a basis to build the necessary background to proceed further.

While spaces of Drinfeld quasi-modular forms embed in  $C[[u]]$  and are spanned by forms with coefficients in  $\mathbb{F}_q[\theta]$ , the spaces of almost  $A$ -quasi-modular forms we are interested in embed in  $C[[t, u]]$ , with  $t$  a new indeterminate and are spanned by forms with coefficients in  $\mathbb{F}_q[t, \theta]$  <sup>(8)</sup>. When it makes sense, replacing  $t$  by  $\theta$  in an almost  $A$ -quasi-modular form gives a Drinfeld quasi-modular form (care is required to check convergence of our series).

In particular, we will construct a particular almost  $A$ -quasi-modular form  $\mathbf{E} \in \mathbb{F}_q[t, \theta][[u]]$  such that, replacing  $t$  by  $\theta$ , it specialises to the quasi-modular form  $E$ . This and several crucial properties of  $\mathbf{E}$  will be described in Section 3.

The  $\mathbb{F}_q$ -linear *Frobenius* map  $F : C[[t, u]] \rightarrow C[[t, u]]$  (defined by  $F(x) = x^q$  for all  $x \in C[[t, u]]$ ) splits as

$$F = \tau\chi = \chi\tau,$$

where  $\tau$  is *Anderson’s*  $\mathbb{F}_q[[t]]$ -linear map defined by

$$\tau \sum_{m,n \geq 0} c_{m,n} t^m u^n := \sum_{m,n \geq 0} c_{m,n}^q t^m u^{qn}$$

the  $c_{m,n}$ ’s being in  $C$ , and  $\chi$  is *Mahler’s*  $C[[x]]$ -linear map, defined by

$$\chi \sum_{m,n \geq 0} c_{m,n} t^m u^n := \sum_{m,n \geq 0} c_{m,n} t^{qm} u^n,$$

with analogous  $c_{m,n}$ ’s.

The  $C[[t]]$ -algebra of almost  $A$ -quasi-modular forms embeds in  $C[[t, u]]$  but is not stable under the action of  $\tau$  and  $\chi$ . At least, it contains “large” sub-algebras which are stable under the action of  $\tau$ . Thanks to the results of Section 3, we will construct, in Section 4, one of them; a four dimensional sub-algebra  $\mathbb{M}^\dagger$  of almost  $A$ -quasi-modular forms which is at once:

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<sup>7</sup>The reader might find this terminology rather heavy. It has been chosen because in forthcoming works, we will also need to deal with  *$A$ -quasi-modular forms* and  *$A$ -modular forms*.

<sup>8</sup>These forms do not seem to have a counterpart in the classical framework yet.

- Graded by the group  $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$  (a *degree* will be a triple  $(\mu, \nu, m)$ , with  $(\mu, \nu)$  the *weight*,  $m$  the *type*).
- Stable under the action of  $\tau$ , in a way which is compatible with the graduation.
- Endowed with a set of generators contained in  $\mathbb{F}_q[t, \theta][[u]]$ , whose coefficients  $c_n$  of their  $u$ -expansions have the property that the degrees in  $t$  grow “slowly” as  $n$  increases, unlike the degrees in  $\theta$ .

The Theorem will be proved in Section 5 by using the properties above, avoiding resultants such as (13), in two steps. The first step is made by a multiplicity estimate in  $\mathbb{M}^\dagger$  itself (Proposition 24), with the use of resultants like

$$\mathbf{Res}_E(\mathbf{f}, \tau \mathbf{f}), \tag{14}$$

with  $\mathbf{f} \in \mathbb{M}^\dagger$ , which essentially land in  $C[[t]][g, h]$ , after rescaling by a well controlled  $A$ -quasi-modular form. The required variant of Brownawell and Masser’s separation property is easy to obtain, and the use of the grading by  $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$  is essential at this stage.

The second step will use *transcendental techniques* <sup>(9)</sup>. With a variant of Siegel’s lemma we construct a collection of non-trivial auxiliary forms  $\mathbf{f}_{\mu, \nu, m} \in \mathbb{M}^\dagger$  of weight  $(\mu, \nu)$  and type  $m$  with certain technical conditions on  $\mu, \nu \in \mathbb{Z}$  and  $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ . These forms vanish with high order at infinity, but we can bound from above this order by Proposition 24. Let  $f$  be a non-modular Drinfeld quasi-modular form. The proof of our Theorem ends with the study of a second resultant which lands in Drinfeld modular forms:

$$\mathbf{Res}_E(f, \varepsilon(\tau^k \mathbf{f}_{\mu, \nu, m})),$$

making good choice of the parameters  $\mu, \nu, m$ . Here, we also need to choose  $k$  not too big; this choice will be made possible by the crucial property highlighted above, that the  $u$ -expansions of the generators of  $\mathbb{M}^\dagger$  have their coefficients whose degrees in  $t$  grow slowly as the index increases.

The paper ends with Sections 6 and 7, which are independent. In Section 6 we describe a link to extremal quasi-modular forms as defined in [4]. To motivate Section 7, we point out that after having discovered the benefits of the graduation by  $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$  of  $\mathbb{M}^\dagger$ , we have, for a while, searched for similar structures above classical quasi-modular forms. We believe that this theme should be investigated in the sequel. The reader might be interested in the collection of fragments of result we know in this domain, appearing there.

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<sup>9</sup>This explains the presence of the  $\log_q$  factor in (8) and the condition (6).

## 2 Anderson's functions

In this section we recall some tools introduced in [13, Section 2], [1, 2] and described in [27, Section 2 and Section 4.2].

As *A-lattice of rank  $r > 0$*  we mean a free sub- $A$ -module of  $C$  of rank  $r$ , discrete in the sense that, given a compact subset of  $C$ , only finitely many elements of it lie in. Let  $\Lambda \subset C$  be an  $A$ -lattice of rank  $r$  and let us consider, for  $\zeta \in C$ , the *exponential function associated to  $\Lambda$* , defined by the product:

$$e_\Lambda(\zeta) := \zeta \prod_{\omega \in \Lambda \setminus \{0\}} \left(1 - \frac{\zeta}{\omega}\right), \quad (15)$$

which converges for all  $\zeta \in C$ . For  $\lambda \in C^\times$ , the product expansion (15) implies:

$$e_{\lambda\Lambda}(\zeta) = \lambda e_\Lambda(\lambda^{-1}\zeta). \quad (16)$$

There exist elements  $1 = \alpha_0(\Lambda), \alpha_1(\Lambda), \alpha_2(\Lambda), \dots \in C$ , depending on  $\Lambda$  only, such that:

$$e_\Lambda(\zeta) = \sum_{n \geq 0} \alpha_n(\Lambda) \zeta^{q^n}, \quad (17)$$

the series having infinite radius of convergence (cf. [13, 16]).

The construction of the exponential function by (15) is the main tool to prove that the category whose objects are homothety classes of  $A$ -lattices of rank  $r$  and morphisms are inclusions, is dually equivalent to the category whose objects are isomorphism classes of Drinfeld  $A$ -modules of rank  $r$  and morphisms are isogenies (see [13, Section (2.6)] or [27, Section 2]). For  $\Lambda$  as above, there is a Drinfeld  $A$ -module  $\phi_\Lambda$  such that

$$\phi_\Lambda(a)e_\Lambda(\zeta) = e_\Lambda(a\zeta) \quad (18)$$

(for all  $\zeta \in C$  and  $a \in A$ ), which is uniquely determined by its value  $\phi_\Lambda(\theta) \in \mathbf{End}_{\mathbb{F}_q\text{-lin.}}(\mathbb{G}_a(C))$  in  $\theta$ . This value is a polynomial of degree  $r$  in  $\tau$ , which we recall, is the Frobenius endomorphism  $\tau : c \mapsto c^q$ . On the other side, to any Drinfeld  $A$ -module  $\phi$  of rank  $r$ , a lattice  $\Lambda_\phi$  of rank  $r$  can be associated, so that the functors  $\Lambda \mapsto \phi_\Lambda$  and  $\phi \mapsto \Lambda_\phi$  are inverse of each other up to isomorphisms.

Let  $t$  be a new indeterminate. With  $\Lambda$  an  $A$ -lattice of rank  $r > 0$  and  $e_\Lambda$  as in (15), let us consider  $\omega \in \Lambda \setminus \{0\}$  and introduce, following Anderson in [1], the formal series:

$$s_{\Lambda, \omega}(t) := \sum_{i=0}^{\infty} e_\Lambda\left(\frac{\omega}{\theta^{i+1}}\right) t^i.$$

For a positive real number  $r$ , we denote by  $\mathbb{T}_{<r}$  the sub- $C$ -algebra of  $C[[t]]$  whose elements are formal series  $\sum_{i \geq 0} c_i t^i$  that converge for any  $t \in C$  with  $|t| < r$ . We

denote by  $\mathbb{T}_{>0}$  the sub- $C$ -algebra of  $C[[t]]$  whose series converge in some open disk containing 0, and we notice that all the series of  $\mathbb{T}_{<q^q}$  converge at  $t = \theta$ . We also denote by  $\mathbb{T}_\infty$  the sub- $C$ -algebra of series that converge everywhere in  $C$ .

If  $r_1 > r_2 > 0$ , we have

$$\mathbb{T}_{>0} \supset \mathbb{T}_{<r_2} \supset \mathbb{T}_{<r_1} \supset \mathbb{T}_\infty.$$

The *Tate algebra* of formal series of  $C[[t]]$  converging for all  $t$  such that  $|t| \leq 1$  will be also denoted by  $\mathbb{T}$ ; it is contained in  $\mathbb{T}_{<1}$  and contains  $\mathbb{T}_{<1+\epsilon}$  for all  $\epsilon > 0$ .

It is easy to verify that, with  $\Lambda$  and  $\omega \in \Lambda$  as above,  $s_{\Lambda,\omega} \in \mathbb{T}_{<q} \subset \mathbb{T}$ . If  $\Lambda \subset K_\infty^{\text{alg}}$ , it can be proved that  $s_{\Lambda,\omega}(t) \in K_\infty^{\text{alg}}[[t]]$ .

We extend the operator  $\tau$  from  $C$  to  $C[[t]]$  as follows:

$$f = \sum_{n \geq 0} c_n t^n \mapsto \tau f := \sum_{n \geq 0} c_n^q t^n.$$

We will also write  $f^{(k)}$  for  $\tau^k f$ ,  $k \in \mathbb{Z}$  (the operator  $\tau^{-1}$  is well defined). One checks that  $\tau$  sends  $\mathbb{T}_{<r}$  in  $\mathbb{T}_{<r^q}$ . The extension  $\tau$  so constructed defines  $\mathbb{F}_q$ -automorphisms of  $\mathbb{T}_{>0}$ ,  $\mathbb{T}$  and  $\mathbb{T}_\infty$ .

We write  $\overline{A} = \mathbb{F}_q[t]$ ,  $\overline{K} = \mathbb{F}_q(t)$ . If  $a = a(\theta) \in A$  we also write  $\overline{a} = a(t) \in \overline{A}$ . If  $\Lambda$  is an  $A$ -lattice of rank  $r$  and if  $\phi_\Lambda$  is the Drinfeld  $A$ -module of rank  $r$  in (18), then, for all  $a_1, a_2 \in A$  and  $\omega_1, \omega_2 \in \Lambda$ ,

$$\phi_\Lambda(a_1) s_{\Lambda,\omega_1} + \phi_\Lambda(a_2) s_{\Lambda,\omega_2} = s_{\Lambda,a_1\omega_1+a_2\omega_2} = \overline{a}_1 s_{\Lambda,\omega_1} + \overline{a}_2 s_{\Lambda,\omega_2}. \quad (19)$$

These identities, which hold in  $\mathbb{T}$ , are proved in [27, Section 4.2.2].

From (16) it immediately follows that, for  $\lambda \in C^\times$ ,

$$s_{\lambda\Lambda,\lambda\omega}(t) = \lambda s_{\Lambda,\omega}(t). \quad (20)$$

We also have the series expansion (cf. [27, Section 4.2.2])

$$s_{\Lambda,\omega}(t) = \sum_{n=0}^{\infty} \frac{\alpha_n(\Lambda)\omega^{q^n}}{\theta^{q^n} - t}, \quad (21)$$

uniformly convergent in every compact subset of  $C \setminus \{\theta, \theta^q, \dots\}$ , and  $s_{\Lambda,\omega}(t) - \omega/(\theta - t)$  extends to a rigid holomorphic function for  $|t| < q^q$ . We will then often say that  $s_{\Lambda,\omega}$  has a *simple pole of residue*  $-\omega$  in  $t = \theta$ . Notice that other poles occur at  $t = \theta^q, \theta^{q^2}, \dots$ , but we will never need to focus on them in this paper.

*Example: rank one case.* For  $\Lambda = \tilde{\pi}A$  (rank 1), the exponential function (17) is:

$$e_{\text{Car}}(\zeta) = \sum_{n \geq 0} \frac{\zeta^{q^n}}{d_n}, \quad (22)$$

where  $d_0 := 1$  and  $d_i := [i][i-1]^q \cdots [1]^{q^{i-1}}$ , recalling that  $[i] = \theta^{q^i} - \theta$  if  $i > 0$ . The relations (18) become, for all  $a \in A$ ,

$$\phi_{\text{Car}}(a)e_{\text{Car}}(\zeta) = e_{\text{Car}}(a\zeta),$$

where  $\phi_{\text{Car}}$  is Carlitz's module defined by

$$\phi_{\text{Car}}(\theta) = \theta\tau^0 + \tau \in \mathbf{End}_{\mathbb{F}_q\text{-lin.}}(\mathbb{G}_a)$$

(see Section 4 of [13]).

We will write  $s_{\text{Car}} = s_{\tilde{\pi}A, \tilde{\pi}}$ . The function  $s_{\text{Car}}$  has a simple pole in  $\theta$  with residue  $-\tilde{\pi}$ .

By (19) (cf. [27, Section 4.2.5]), the following  $\tau$ -difference equation holds:

$$s_{\text{Car}}^{(1)}(t) = (t - \theta)s_{\text{Car}}. \quad (23)$$

After [11, Theorem 2.2.9],  $\mathbb{T}$  is a principal ideal domain. This property can be used to verify that the subfield of constants  $\mathbb{L}^\tau := \{l \in \mathbb{L}, \tau l = l\}$ , where  $\mathbb{L}$  is the fraction field of  $\mathbb{T}$ , is equal to  $\overline{K} := \mathbb{F}_q(t)$  (see also [25, Lemma 3.3.2]). We deduce, just as in the proof of [25, Lemma 3.3.5], that the  $\tau$ -difference equation  $f^{(1)} = (t - \theta)f$  has, as a complete set of solutions in  $\mathbb{L}$ , the  $\mathbb{F}_q(t)$ -vector space  $\mathbb{F}_q(t)s_{\text{Car}}$ . In fact, for all  $a = a(\theta) \in A$ , we have  $s_{\tilde{\pi}A, a\tilde{\pi}} = \overline{a}s_{\text{Car}}$ .

Comparing with (22) we also point out, for further references in this paper, that (21) becomes in this case:

$$s_{\text{Car}}(t) = \sum_{n=0}^{\infty} \frac{\tilde{\pi}^{q^n}}{d_n(\theta^{q^n} - t)}, \quad |t| < q. \quad (24)$$

## 2.1 Anderson's functions for elliptic Drinfeld modules

We recall and deepen some tools described in [27, Section 4.2.5] (see also [8, 25]). Let  $z$  be in  $\Omega$ , and consider the  $A$ -lattice  $\Lambda = \Lambda_z = A + zA$  of rank 2, with associated exponential function  $e_z = e_\Lambda$ . Let us consider the Drinfeld module  $\phi_z$  defined by

$$\phi_z : \theta \mapsto \phi_z(\theta) = \theta\tau^0 + \tilde{g}(z)\tau^1 + \tilde{\Delta}(z)\tau^2, \quad (25)$$

where  $\tilde{g}(z) = \tilde{\pi}^{q-1}g(z)$ ,  $\tilde{\Delta}(z) = \tilde{\pi}^{q^2-1}\Delta(z)$ , with  $\Delta = -h^{q-1}$ . Then,

$$\phi_z(a)e_z(\zeta) = e_z(a\zeta) \quad (26)$$

for all  $a \in A$  and  $\zeta \in C$  ([13, Section 5], [27, Section 4.2.5], see also [25]).

We can write, for  $\zeta \in C$ ,

$$e_z(\zeta) = \sum_{i=0}^{\infty} \alpha_i(z)\zeta^{q^i}, \quad (27)$$

for functions  $\alpha_i : \Omega \rightarrow C$  with  $\alpha_0 = 1$ . By (26) we deduce, with the initial values  $\alpha_0 = 1, \alpha_{-1} = 0$ , the recursive relations

$$\alpha_i = \frac{1}{[i]}(\tilde{g}\alpha_{i-1}^q + \tilde{\Delta}\alpha_{i-2}^{q^2}), \quad i > 0. \quad (28)$$

This implies that the function  $\alpha_i(z)$  is a modular form of weight  $q^i - 1$  and type 0 for all  $i \geq 0$ . There exist elements  $c_{i,m} \in C$  such that

$$\alpha_i(z) = \sum_{m \geq 0} c_{i,m} u^m, \quad i \geq 0, \quad (29)$$

with convergence for  $z \in \Omega$  such that  $|u|$  is small enough. The following lemma tells that a non-zero disk of convergence can be chosen independently on  $i$ .

**Lemma 1** *We have*

$$c_{i,0} = \frac{1}{d_i} \tilde{\pi}^{q^i - 1}, \quad i \geq 0, \quad (30)$$

and

$$|c_{i,m}| \leq q^{-q^i} B^m, \quad (i, m \geq 0). \quad (31)$$

*Proof.* Let us write  $\tilde{g} = \sum_{i \geq 0} \tilde{\gamma}_i u^i$  and  $\tilde{\Delta} = \sum_{i \geq 0} \tilde{\delta}_i u^i$  with  $\tilde{\gamma}_i, \tilde{\delta}_i \in C$ . The recursive relations (28) imply, for  $i > 1, m \geq 0$  and  $j, k$  non-negative integers:

$$c_{i,m} = \frac{1}{[i]} \left( \sum_{j+qk=m} \tilde{\gamma}_j c_{i-1,k}^q + \sum_{j+q^2k=m} \tilde{\delta}_j c_{i-2,k}^{q^2} \right),$$

from which we deduce at once (30) because  $\tilde{\gamma}_0 = \tilde{\pi}^{q-1}$  and  $\tilde{\delta}_0 = 0$ .

We now need to provide upper bounds for the  $|c_{i,m}|$ 's, with explicit dependence on  $i, m$ .

Looking at [13, Definition (5.7), (iii)], there exists  $B \geq q$  such that, for all  $i \geq 0, \max\{|\tilde{\gamma}_i|, |\tilde{\delta}_i|\} \leq B^i$ . We know that  $\alpha_0 = 1$  and that  $|c_{1,m}| \leq q^{-q} B^m$ . Now,

After induction and the equality  $|[i]| = q^{q^i}$  ( $i > 0$ ), we deduce (31) from these identities.  $\square$

In all the following, we shall write:

$$\mathbf{s}_1(z, t) = s_{\Lambda_z, z}(t), \quad \mathbf{s}_2(z, t) = s_{\Lambda_z, 1}(t).$$

These are functions  $\Omega \times B_q \rightarrow C$ , where, for  $r > 0, B_r$  is the set  $\{t \in C, |t| < r\}$ .

In fact the definition of the functions  $s_{\Lambda, \omega}$  tells that  $\mathbf{s}_1, \mathbf{s}_2 \in \mathbf{Hol}(\Omega)[[t]]$ , where  $\mathbf{Hol}(\Omega)$  denotes the  $C$ -algebra of rigid holomorphic functions  $\Omega \rightarrow C$ . After (27)

and (21) we see that, for any couple  $(z, t) \in \Omega \times B_q$ , the following convergent series expansions hold:

$$\begin{aligned}\mathbf{s}_1(z, t) &= \sum_{i=0}^{\infty} \frac{\alpha_i(z) z^{q^i}}{\theta^{q^i} - t} \\ \mathbf{s}_2(z, t) &= \sum_{i=0}^{\infty} \frac{\alpha_i(z)}{\theta^{q^i} - t}.\end{aligned}$$

Our notations stress the dependence on two variables  $z \in \Omega, t \in B_q$ . For these functions, we will also write, occasionally,  $\mathbf{s}_1(z), \mathbf{s}_2(z)$ , to stress the dependence on  $z \in \Omega$ . We can also fix  $z \in \Omega$  and study the functions  $\mathbf{s}_1(z, \cdot), \mathbf{s}_2(z, \cdot)$ , or look at the functions  $\mathbf{s}_1(\cdot, t), \mathbf{s}_2(\cdot, t) : \Omega \rightarrow \mathbb{T}_{< q}$  with formal series as values. In the next section, we provide the necessary analysis of the functions  $\mathbf{s}_1(z, \cdot), \mathbf{s}_2(z, \cdot)$ . Hence, we fix now  $z \in \Omega$ .

### 2.1.1 The $\mathbf{s}_i$ 's as functions of the variable $t$ , with $z$ fixed.

At  $\theta$ , the functions  $\mathbf{s}_i(z, \cdot)$  have simple poles. Their respective residues are, according to Section 2,  $-z$  for the function  $\mathbf{s}_1(z, \cdot)$  and  $-1$  for  $\mathbf{s}_2(z, \cdot)$ . Moreover, we have  $\mathbf{s}_1^{(1)}(z, \theta) = \eta_1$  and  $\mathbf{s}_2^{(1)}(z, \theta) = \eta_2$ , where  $\eta_1, \eta_2$  are the *quasi-periods* of  $\Lambda_z$  (see [27, Section 4.2.4] and [12, Section 7]).

Let us consider the matrix function:

$$\widehat{\Psi}(z, t) := \begin{pmatrix} \mathbf{s}_1(z, t) & \mathbf{s}_2(z, t) \\ \mathbf{s}_1^{(1)}(z, t) & \mathbf{s}_2^{(1)}(z, t) \end{pmatrix}.$$

By [27, Section 4.2.3] (see in particular equation (15)), we have:

$$\widehat{\Psi}(z, t)^{(1)} = \widetilde{\Theta}(z) \cdot \widehat{\Psi}(z, t), \quad \text{where } \widetilde{\Theta}(z) = \begin{pmatrix} 0 & 1 \\ \frac{t-\theta}{\widetilde{\Delta}(z)} & -\frac{\widetilde{g}(z)}{\widetilde{\Delta}(z)} \end{pmatrix}, \quad (32)$$

yielding the following  $\tau$ -difference linear equation of order 2:

$$\mathbf{s}_2^{(2)} = \frac{t-\theta}{\widetilde{\Delta}} \mathbf{s}_2 - \frac{\widetilde{g}}{\widetilde{\Delta}} \mathbf{s}_2^{(1)}. \quad (33)$$

**Remark 2** By [1], there is a fully faithful contravariant functor from the category of Drinfeld  $A$ -modules over  $K^{\text{alg}}$  to the category of *Anderson's  $A$ -motives* over  $K^{\text{alg}}$ . Part of this association is sketched in [27, Section 4.2.2], where the definition of  $A$ -motive is given and discussed (see also [8]); it is based precisely on Anderson's functions  $s_{\Lambda, \omega}$ . In the language introduced by Anderson,  $\widehat{\Psi}$  is a *rigid analytic trivialisation* of the  $A$ -motive associated to the Drinfeld module  $\phi = \phi_{\Lambda}$ .

We will also use the following fundamental lemma, whose proof depends on Gekeler's paper [12].

**Lemma 3 (“Deformation of Legendre’s identity”)** *We have, for all  $z \in \Omega$  and  $t \in \mathbb{T}_{<q}$ :*

$$\det(\widehat{\Psi}) = \tilde{\pi}^{-1-q} h(z)^{-1} s_{\text{Car}}(t). \quad (34)$$

*Proof.* Let  $f(z, t)$  be the function  $\det(\widehat{\Psi}(z, t)) h(z) \tilde{\pi}^{1+q}$ , for  $z \in \Omega$  and  $t \in B_q$ . We have:

$$f^{(1)}(z, t) = -(t - \theta) \tilde{\Delta}(z)^{-1} \det(\widehat{\Psi}(z, t)) h(z)^q \tilde{\pi}^{q+q^2} = (t - \theta) f(z, t).$$

For fixed  $z \in \Omega$ , we know that  $\mathbf{s}_i^{(k)}(z, \cdot) \in \mathbb{T}_{<q^{q^k}} \subset \mathbb{T}$  for all  $k \geq 0$ . Hence,  $f(z, \cdot) \in \mathbb{T}$  for all  $z \in \Omega$ . By arguments used in the remark on the  $\overline{K}$ -vector space structure of the set of solutions of (23),  $f(z, t)$  is equal to  $\lambda(z, t) s_{\text{Car}}(t)$ , for some  $\lambda(z, t) \in \overline{A}$ ; the matter is now to compute  $\lambda$ , which does not depend on  $z \in \Omega$  as follows easily by fixing  $t = t_0 \in B_q$  transcendental over  $\mathbb{F}_q$  and observing that  $f(z, t_0)$  is holomorphic over  $\Omega$  with values in a discrete set.

Now, for  $z$  fixed as  $t \rightarrow \theta$ ,

$$\lim_{t \rightarrow \theta} \widehat{\Psi}(z, t) = \begin{pmatrix} -\frac{z}{t-\theta} & -\frac{1}{t-\theta} \\ \eta_1 & \eta_2 \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix},$$

$\eta_1, \eta_2$  being the *quasi-periods* (periods of second kind) of the lattice  $A\omega_1 + A\omega_2$  (respectively associated to  $\omega_1$  and  $\omega_2$ ) [12, Section 7, Equations (7.1)], with generators  $\omega_1 = z, \omega_2 = 1$ , where the asterisks denote continuous functions of the variable  $z$ . Hence, we have  $\lim_{t \rightarrow \theta} (t - \theta) \det(\widehat{\Psi}(z, t)) = -z\eta_2 + \eta_1$ . By [12, Theorem 6.2],  $-z\eta_2 + \eta_1 = -\tilde{\pi}^{-q} h(z)^{-1}$ .

At once:

$$\begin{aligned} & -\tilde{\pi}^{-q} h(z)^{-1} = \\ & = \lim_{t \rightarrow \theta} (t - \theta) \det(\widehat{\Psi}(z, t)) \\ & = \lambda(\theta) \tilde{\pi}^{-q-1} h(z)^{-1} \lim_{t \rightarrow \theta} (t - \theta) s_{\text{Car}}(t) \\ & = -\lambda(\theta) \tilde{\pi}^{-q} h(z)^{-1}, \end{aligned}$$

which implies that  $\lambda = \lambda(\theta) = 1$  ( $\theta$  is transcendental over  $\mathbb{F}_q$ ). Our Lemma follows.

□

In the next section, we study the functions  $\mathbf{s}_1, \mathbf{s}_2$  as functions  $\Omega \rightarrow \mathbb{T}_{<q}$ .



### 2.1.2 The $s_i$ 's as functions $\Omega \rightarrow \mathbb{T}_{<q}$ .

We observe, by the definitions of  $s_1, s_2$ , and by the fact, remarked in (28), that  $\alpha_i$  is a modular form of weight  $q^i - 1$  and type 0 for all  $i$ , and by (19), that for all

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma:$$

$$\begin{aligned} s_2(\gamma(z), t) &= \sum_{i=0}^{\infty} (cz + d)^{q^i - 1} \frac{\alpha_i(z)}{\theta^{q^i} - t} \\ &= (cz + d)^{-1} s_{\Lambda_z, cz+d}(z) \\ &= (cz + d)^{-1} (\bar{c}s_1(z, t) + \bar{d}s_2(z, t)). \end{aligned}$$

Similarly,

$$\begin{aligned} s_1(\gamma(z), t) &= \sum_{i=0}^{\infty} (cz + d)^{q^i - 1} \frac{\alpha_i(z)(\gamma(z))^{q^i}}{\theta^{q^i} - t} \\ &= (cz + d)^{-1} s_{\Lambda_z, az+b}(z) \\ &= (cz + d)^{-1} (\bar{a}s_1(z, t) + \bar{b}s_2(z, t)). \end{aligned}$$

Let us write

$$\Sigma(z, t) := \begin{pmatrix} s_1(z, t) \\ s_2(z, t) \end{pmatrix}.$$

We have proved:

**Lemma 4** *For all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ , and for all  $z \in \Omega$ , we have the following identity of series in  $\mathbb{T}_{<q}$ :*

$$\Sigma(\gamma(z), t) = (cz + d)^{-1} \bar{\gamma} \cdot \Sigma(z, t), \quad (35)$$

where  $\bar{\gamma}$  is the matrix  $\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \in \Gamma$ .

### 2.1.3 Behaviour of $s_2$ at the infinity cusp and $u$ -expansion

We use the results of the previous subsections to see how the function  $s_2$  behaves for  $z$  approaching the cusp at infinity of the rigid analytic space  $\Gamma \backslash \Omega$ . Here we will prove two lemmas.

**Lemma 5** *There exists a real number  $r > 0$  such that for all  $(z, t) \in \Omega \times C$  with  $|u| = |u(z)| < r, |t| < r$ , we have:*

$$s_2(z, t) = \tilde{\pi}^{-1} s_{Car}(t) + \sum_{m \geq 1} \kappa_m(t) u^m, \quad (36)$$

where for  $m \geq 1$ ,

$$\begin{aligned}\kappa_m(t) &= \sum_{i \geq 1} \frac{c_{i,m}}{\theta^{q^i} - t} \\ &= \sum_{j \geq 0} t^j \sum_{i \geq 1} c_{i,m} \theta^{-q^i(1+j)} \in \mathbb{T}_{< q^q},\end{aligned}$$

the  $c_{i,m}$ 's being the coefficients in the expansions (29).

*Proof.* For  $z \in \Omega$  such that  $|u| < B^{-1}$  with  $B$  as in (31), and for  $|t| < q$ , (21) yields:

$$\begin{aligned}\mathbf{s}_2(z, t) &= \frac{1}{\theta - t} + \sum_{i \geq 1} \frac{\alpha_i(z)}{\theta^{q^i} - t} \\ &= \frac{1}{\theta - t} + \sum_{i \geq 1} \sum_{m \geq 0} c_{i,m} u^m \frac{1}{\theta^{q^i} - t} \\ &= \frac{1}{\theta - t} + \sum_{i \geq 1} c_{i,0} \frac{1}{\theta^{q^i} - t} + \sum_{m \geq 1} u^m \sum_{i \geq 1} \frac{c_{i,m}}{\theta^{q^i} - t} \\ &= \tilde{\pi}^{-1} \sum_{i \geq 0} \frac{\tilde{\pi}^{q^i}}{d_i} \frac{1}{\theta^{q^i} - t} + \sum_{m \geq 1} \kappa_m(t) u^m \\ &= \tilde{\pi}^{-1} s_{\text{Car}}(t) + \sum_{m \geq 1} \kappa_m(t) u^m,\end{aligned}$$

where, in the second equality we have substituted the  $u$ -expansions of the  $\alpha_i$ 's in our formulas, in the third we have separately considered constant terms, in the fourth equality, we have used (30), in the fifth we have recognised the shape of  $s_{\text{Car}}$  (24), and we have noticed, by using (31), that for all  $t \in C$  such that  $|t| \leq q$ ,  $|\kappa_m(t)| \leq B^m q^{-1}$ .  $\square$

Later, we will need to do some arithmetic with the  $u$ -expansion (36). To this purpose, it is advantageous to set:

$$\mathbf{d}(z, t) := \tilde{\pi} s_{\text{Car}}(t)^{-1} \mathbf{s}_2(z, t),$$

function for which (33) becomes:

$$\mathbf{d} = (t - \theta^q) \Delta \mathbf{d}^{(2)} + g \mathbf{d}^{(1)}. \quad (37)$$

We will need part of the following lemma.

**Lemma 6** *We have*

$$\mathbf{d} = \sum_{i \geq 0} c_i(t) u^{(q-1)i} \in 1 + u^{q-1} \mathbb{F}_q[t, \theta][[u^{q-1}]]. \quad (38)$$

*More precisely,*

$$\mathbf{d} = 1 + (\theta - t)u^{q(q-1)} + (\theta - t)u^{(q^2-q+1)(q-1)} + \dots \in 1 + (t - \theta)u^{q-1} \mathbb{F}_q[t, \theta][[u^{q-1}]],$$

*where the dots  $\dots$  stand for terms of higher order in  $u$ .*

*Let  $i$  be a positive integer, let  $k_i$  be the unique integer such that  $q^{2k_i} \leq i < q^{4k_i}$ . Then,*

$$-\infty \leq \deg_t c_i \leq k_i \leq \log_{q^2} i,$$

*where  $\log_{q^2}$  is the logarithm in base  $q^2$ , with the convention  $\deg_t 0 = -\infty$ .*

*Proof.* For simplicity, we write  $v = u^{q-1}$ . It is clear, looking at Lemma 5, that  $\mathbf{d}$  is a series in  $\mathbb{T}_{<q^q}[[v]]$ . We have the series expansions (cf. [13, Section 10]):

$$\begin{aligned} g &= 1 - [1]v + \dots = \sum_{n=0}^{\infty} \gamma_n v^n \in A[[v]], \\ \Delta &= -v(1 - v^{q-1} + \dots) = \sum_{n=0}^{\infty} \delta_n v^n \in uA[[v]], \end{aligned}$$

We deduce, from (37), that

$$c_m = (t - \theta^q) \sum_{i+q^2j=m} \delta_i c_j^{(2)} + \sum_{i+qj=m} \gamma_i c_j^{(1)}, \quad (39)$$

which yields inductively that  $c_i$  belongs to  $\mathbb{F}_q[t, \theta]$ , because the coefficients of the  $u$ -expansions of  $\Delta$  and  $g$  are  $A$ -integral. The statement on the degrees of the coefficients of the  $c_i$ 's, is also a simple inductive consequence of (39) and the following two facts: that  $\deg_t \delta_i, \deg_t \gamma_i \leq 0$ , and that  $\deg_t c_i^{(k)} = \deg_t c_i$  for all  $i, k$  ( $t$  is  $\tau$ -invariant).

The explicit formula for the coefficients  $c_i$  with  $i \leq q^2 - q + 1$  is an exercise that we leave to the reader, which needs [13, Corollaries (10.3), (10.11)]. The explicit computation can be pushed easily to coefficients of higher order, but we skip it as we will not need these explicit formulas at all in this paper. The fact that the coefficients  $c_i$  belong to the ideal generated by  $t - \theta$  for  $i \geq 1$  follows from the computation of the residues in 2.1.1.  $\square$

### 3 The function $\mathbf{E}$

The function of the title is defined, for  $z \in \Omega$  and  $t \in B_q$ , by:

$$\mathbf{E}(z, t) = -h(z)\mathbf{d}^{(1)}(z, t) = -(t - \theta)^{-1}\tilde{\pi}^q h(z) s_{\text{Car}}^{-1}(t) \mathbf{s}_2^{(1)}(z, t),$$

with  $\mathbf{d}$  the function of Lemma 6. This section is entirely devoted to the description of its main properties. Three Propositions will be proved here.

In Proposition 7 we use the arguments of 2.1.1 to show that, just as  $\mathbf{d}$ ,  $\mathbf{E}$  satisfies a linear  $\tau$ -difference equation of order 2 with coefficients isobaric in  $C[[t]][g, h]$  <sup>(10)</sup>.

In Proposition 8, where we use this time the arguments developed in 2.1.2, we analyse the functional equations relating the values of  $\mathbf{E}$  at  $z$  and  $\gamma(z)$ , where  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ ; they involve the *factors of automorphy*:

$$J_\gamma(z) = cz + d, \quad \mathbf{J}_\gamma(z) = \bar{c} \frac{\mathbf{s}_1(z, t)}{\mathbf{s}_2(z, t)} + \bar{d},$$

with values convergent in  $C[[t]]$ .

Proposition 9 follows from what we did in 2.1.3 and describes the third important feature of the function  $\mathbf{E}$ ; the existence of a  $u$ -expansion in  $\mathbb{F}_q[t, \theta][[u]]$ . For Drinfeld quasi-modular forms, the degree in  $\theta$  of the  $n$ -th coefficient of the  $u$ -expansion grows pretty rapidly with  $n$  in contrast of the classical framework. The function  $\mathbf{E}$  does not make exception to this principle. However, the degree in  $t$  of the  $n$ -th coefficient grows slowly, and this property is used crucially in the proof of the multiplicity estimate. Another important property studied in this section is that  $\mathbf{E}(z, \theta)$  is a well defined function  $\Omega \rightarrow C$  and is equal to Gekeler's function  $E$ .

#### 3.1 linear $\tau$ -difference equations

**Proposition 7** *For all  $z \in \Omega$ , the function  $\mathbf{E}(z, \cdot)$  can be developed as a series of  $\mathbb{T}_{<q^q}$ . Moreover, The following linear  $\tau$ -difference equation holds in  $\mathbb{T}_{<q^q}$ , for all  $z \in \Omega$ :*

$$\mathbf{E}^{(2)} = \frac{1}{t - \theta^{q^2}} (\Delta \mathbf{E} + g^q \mathbf{E}^{(1)}). \quad (40)$$

*Proof.* After having chosen a  $(q-1)$ -th root of  $-\theta$ , let us write, following Anderson, Brownawell, and Papanikolas in [2, Section 3.1.2],

$$\Omega(t) := (-\theta)^{\frac{-q}{q-1}} \prod_{n=1}^{\infty} \left( 1 - \frac{t}{\theta^{q^n}} \right) \in (\mathbb{T}_\infty \cap K_\infty((-\theta)^{\frac{1}{q-1}})[[t]]) \setminus K_\infty(t)^{\text{alg}}.$$

---

<sup>10</sup>This phenomenon holds with more generality and should be compared with a result of Stiller in [28].

It is plain that

$$\Omega^{(-1)}(t) = (t - \theta)\Omega(t).$$

Thanks to the remark on the  $\overline{K}$ -vector space structure of the set of solutions of (23) and after the computation of the constant of proportionality, we get

$$s_{\text{Car}}(t) = \frac{1}{\Omega^{(-1)}(t)}. \quad (41)$$

At once, we obtain that the function  $s_{\text{Car}}$  has no zeros in the domain  $C \setminus \{\theta, \theta^q, \dots\}$  from which it follows that  $((t - \theta)s_{\text{Car}})^{-1} \in \mathbb{T}_{<q^q}$ . Moreover, for all  $z \in \Omega$ , we have  $\mathbf{s}_2 \in \mathbb{T}_{<q}$  so that  $\mathbf{s}_2^{(1)} \in \mathbb{T}_{<q^q}$ . Multiplying the factors that define the function  $\mathbf{E}$ , we then get, for all  $z \in \Omega$ , that  $\mathbf{E}(z, \cdot) \in \mathbb{T}_{<q^q}$ , which gives the first part of the proposition (and in fact, it can be proved that  $\mathbf{d}, \mathbf{E}(z, \cdot) \in \mathbb{T}_\infty$  for all  $z \in \Omega$ , but we skip on this property since it will not be needed in the present paper).

In order to prove the second part of the proposition, we remark, from (37) (or what is the same, (33)), that

$$\mathbf{s}_2^{(3)} = \frac{t - \theta^q}{\widetilde{\Delta}^q} \mathbf{s}_2^{(1)} - \frac{\widetilde{g}^q}{\widetilde{\Delta}^q} \mathbf{s}_2^{(2)}, \quad \text{or equivalently,} \quad \mathbf{d}^{(3)} = \frac{1}{(t - \theta^{q^2})\Delta^q} (\mathbf{d}^{(1)} - g^q \mathbf{d}^{(2)}).$$

By the definition of  $\mathbf{E}$  and the  $\tau$ -difference equation (23) we find the relation:

$$\begin{aligned} \mathbf{E}^{(k)} &= -(t - \theta^{q^k})^{-1} (t - \theta^{q^{k-1}})^{-1} \dots (t - \theta)^{-1} \widetilde{\pi}^{q^{k+1}} h^{q^k} s_{\text{Car}}^{-1} \mathbf{s}_2^{(k+1)}, \\ &= -h^{q^k} \mathbf{d}^{(k+1)} \end{aligned} \quad (42)$$

for  $k \geq 0$ . Substituting the above expression for  $\mathbf{d}^{(3)}$  in it, we get what we expected.  $\square$

### 3.2 Factors of automorphy, modularity

In the next proposition, the function  $\mathbf{E}$  is viewed as a function  $\Omega \rightarrow \mathbb{T}_{<q^q}$  (it can be proved that it defines, in fact, a function  $\Omega \rightarrow \mathbb{T}_\infty$ ). In order to state the proposition, we first need a preliminary discussion.

If  $\omega \notin \theta\Lambda$ ,  $e_\Lambda(\omega/\theta) \neq 0$  and  $s_{\Lambda,\omega}(t) \in \mathbb{T}_{>0}^\times$  (group of units of  $\mathbb{T}_{>0}$ ), so that, for every  $z$  fixed,  $\mathbf{s}_2(z, \cdot)^{-1} \in \mathbb{T}_{>0}^\times$  <sup>(11)</sup>. Hence, we have a well defined map

$$\begin{aligned} \xi &: \Omega \rightarrow \mathbb{T}_{>0}^\times \\ z &\mapsto \begin{pmatrix} \mathbf{s}_1(z,t) \\ \mathbf{s}_2(z,t) \end{pmatrix}, \end{aligned}$$

---

<sup>11</sup>The radius of convergence, in principle depending on  $z$ , seems difficult to compute.

and we can consider the map

$$(\gamma, z) = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \in \Gamma \times \Omega \mapsto \mathbf{J}_\gamma(z) := \bar{c}\boldsymbol{\xi} + \bar{d} \in \mathbb{T}_{>0}.$$

Since  $c, d$  are relatively prime, we have  $cz + d \notin \theta\Lambda_z$  implying that  $\bar{c}\mathbf{s}_1 + \bar{d}\mathbf{s}_2 = s_{\Lambda_z, cz+d} \in \mathbb{T}_{>0}^\times$ . Therefore, for all  $\gamma \in \Gamma$  and  $z \in \Omega$ ,  $\mathbf{J}_\gamma \in \mathbb{T}_{>0}^\times$ .

Moreover, by (35) we have, for all  $\gamma \in \Gamma$  and  $z \in \Omega$ ,

$$\boldsymbol{\xi}(\gamma(z)) = \bar{\gamma}(\boldsymbol{\xi}(z)) \in C((t)), \quad (43)$$

so that, for  $\gamma, \delta \in \Gamma$  and  $z \in \Omega$ ,

$$\mathbf{J}_{\gamma\delta}(z) = \mathbf{J}_\gamma(\delta(z))\mathbf{J}_\delta(z). \quad (44)$$

the map  $\mathbf{J} : \Gamma \times \Omega \rightarrow \mathbb{T}_{>0}^\times$  is our “new” *factor of automorphy*, to be considered together with the more familiar factor of automorphy

$$J_\gamma(z) := cz + d.$$

Let us also write, for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ :

$$\begin{aligned} L_\gamma(z) &= \frac{c}{cz + d}, \\ \mathbf{L}_\gamma(z) &= \frac{\bar{c}}{\bar{c}\mathbf{s}_1 + \bar{d}\mathbf{s}_2}. \end{aligned}$$

We remark that for all  $\gamma \in \Gamma$ ,  $\mathbf{L}_\gamma(z)$  belongs to  $\mathbb{T}_{>0}$  because  $\mathbf{s}_2\mathbf{J}_\gamma(z) \in \mathbb{T}_{>0}^\times$ . Moreover, the functions  $\mathbf{J}_\gamma$  and  $(\theta - t)^{-1}\mathbf{L}_\gamma$  are deformations of  $J_\gamma$  and  $L_\gamma$  respectively, for all  $\gamma \in \Gamma$ . Indeed, we recall that  $(t - \theta)\mathbf{s}_2(z, t) \rightarrow -1$  and  $(t - \theta)\mathbf{s}_1(z, t) \rightarrow -z$  as  $t \rightarrow \theta$ . Hence,  $\lim_{t \rightarrow \theta} \frac{\mathbf{s}_1}{\mathbf{s}_2} = z$ . This implies that

$$\lim_{t \rightarrow \theta} \mathbf{J}_\gamma(z) = J_\gamma(z). \quad (45)$$

In a similar way we see that

$$\lim_{t \rightarrow \theta} (t - \theta)^{-1} \mathbf{L}_\gamma(z) = -L_\gamma(z). \quad (46)$$

We further define the sequence of functions  $(g_k^*)_{k \geq 0}$  by:

$$g_{-1}^* = 0, \quad g_0^* = 1, \quad g_1^* = g, \quad g_k^* = (t - \theta^{q^{k-1}})g_{k-2}^* \Delta^{q^{k-2}} + g_{k-1}^* g^{q^{k-1}}, \quad k \geq 2,$$

so that for all  $k \geq 0$ , we have the identity  $g_k^*(z, \theta) = g_k(z)$ , the function introduced in [13, Equation (6.8)].

We have:

**Proposition 8** For all  $z \in \Omega$ ,  $\gamma \in \Gamma$  and  $k \geq 0$  the following identity of formal series of  $\mathbb{T}_{>0}$  holds:

$$\begin{aligned} \mathbf{E}^{(k)}(\gamma(z), t) &= \det(\gamma)^{-1} J_\gamma(z)^{q^k} \mathbf{J}_\gamma(z) \times \\ &\quad \left( \mathbf{E}^{(k)}(z, t) + \frac{g_k^*(z)}{\tilde{\pi}(t-\theta)(t-\theta^q) \cdots (t-\theta^{q^k})} \mathbf{L}_\gamma(z) \right). \end{aligned} \quad (47)$$

From the deformation of Legendre's identity (34) we deduce that

$$\mathbf{s}_1^{(1)} = \frac{1}{\mathbf{s}_2} (\mathbf{s}_1 \mathbf{s}_2^{(1)} - \tilde{\pi}^{-1-q} h^{-1} s_{\text{Car}}). \quad (48)$$

Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ . Applying  $\tau$  on both left and right hand sides of

$$\mathbf{s}_2(\gamma(z)) = J_\gamma^{-1} \mathbf{J}_\gamma \mathbf{s}_2(z) = J_\gamma^{-1} (\bar{c} \mathbf{s}_1(z) + \bar{d} \mathbf{s}_2(z)), \quad (49)$$

consequence of Lemma 4, we see that

$$\mathbf{s}_2^{(1)}(\gamma(z)) = J_\gamma^{-q} (\bar{c} \mathbf{s}_1^{(1)} + \bar{d} \mathbf{s}_2^{(1)}).$$

We now eliminate  $\mathbf{s}_1^{(1)}$  from this identity and (48), getting identities in  $\mathbb{T}_{>0}$ :

$$\begin{aligned} \bar{c} \mathbf{s}_1^{(1)} + \bar{d} \mathbf{s}_2^{(1)} &= \\ &= \frac{\bar{c}}{\mathbf{s}_2} (\mathbf{s}_1 \mathbf{s}_2^{(1)} - \tilde{\pi}^{-1-q} h(z)^{-1} s_{\text{Car}}(t)) + \bar{d} \mathbf{s}_2^{(1)} \\ &= \mathbf{s}_2^{(1)} \left( \bar{c} \frac{\mathbf{s}_1}{\mathbf{s}_2} + \bar{d} \right) - \tilde{\pi}^{-1-q} h(z)^{-1} s_{\text{Car}}(t) \mathbf{s}_2^{-1} \\ &= \left( \bar{c} \frac{\mathbf{s}_1}{\mathbf{s}_2} + \bar{d} \right) \left( \mathbf{s}_2^{(1)} - \tilde{\pi}^{-1-q} h(z)^{-1} s_{\text{Car}}(t) \frac{\bar{c}}{\bar{c} \mathbf{s}_1 + \bar{d} \mathbf{s}_2} \right), \end{aligned}$$

that is,

$$\mathbf{s}_2^{(1)}(\gamma(z)) = J_\gamma^{-q} \mathbf{J}_\gamma \left( \mathbf{s}_2^{(1)}(z) - \frac{\tilde{\pi}^{-1-q} s_{\text{Car}}(t)}{h(z)} \mathbf{L}_\gamma \right). \quad (50)$$

This functional equation is equivalent to the following functional equation for  $\mathbf{d}^{(1)}$  (in  $\mathbb{T}_{>0}$ ):

$$\mathbf{d}^{(1)}(\gamma(z)) = J_\gamma^{-q} \mathbf{J}_\gamma \left( \mathbf{d}^{(1)}(z) - \frac{1}{\tilde{\pi}(t-\theta)h(z)} \mathbf{L}_\gamma \right). \quad (51)$$

This already implies, by the definition of  $\mathbf{E}$  and the modularity of  $h$ :

$$\mathbf{E}(\gamma(z)) = \det(\gamma)^{-1} J_\gamma \mathbf{J}_\gamma \left( \mathbf{E}(z) + \frac{1}{\tilde{\pi}(t-\theta)} \mathbf{L}_\gamma \right)$$

which is our proposition for  $k = 0$ .

We point out that (35) implies the functional equation, for all  $\gamma \in \Gamma$ :

$$\mathbf{d}(\gamma(z)) = J_\gamma^{-1} \mathbf{J}_\gamma \mathbf{d}(z). \quad (52)$$

The joint application of (52), (49), (51) and (32) and induction on  $k$  imply, for all  $k \geq 0$  and  $\gamma \in \Gamma$ , the functional equation in  $\mathbb{T}_{>0}$ :

$$\mathbf{d}^{(k)}(\gamma(z)) = J_\gamma^{-q^k} \mathbf{J}_\gamma \left( \mathbf{d}^{(k)}(z) - \frac{g_{k-1}^*}{\tilde{\pi} h(z)^{q^{k-1}} (t - \theta)(t - \theta^q) \dots (t - \theta^{q^{k-1}})} \mathbf{L}_\gamma \right), \quad (53)$$

where we have also used the functional equation (23). By (42), we end the proof of the proposition.  $\square$

### 3.3 $u$ -expansions

**Proposition 9** *We have*

$$\mathbf{E}(z, t) = u \sum_{n \geq 0} c_n(t) u^{(q-1)n} \in u \mathbb{F}_q[\theta, t][[u^{q-1}]],$$

where the formal series on the right-hand side converges for all  $t, u$  with  $|t| \leq q$  and  $|u|$  small. The terms of order  $\leq q(q-1)$  of the  $u$ -expansion of  $\mathbf{E}$  are:

$$\mathbf{E} = u(1 + u^{(q-1)^2} - (t - \theta)u^{(q-1)q} + \dots). \quad (54)$$

Moreover, for all  $n > 0$ , we have the following inequality for the degree in  $t$  of  $c_n(t)$ :

$$\deg_t c_n \leq \log_q n,$$

where  $\log_q$  denotes the logarithm in base  $q$  and where we have adopted the convention  $\deg_t 0 = -\infty$ .

*Proof.* This is a simple consequence of Lemma 6 and the definition of  $\mathbf{E}$ .  $\square$

**Remark 10** Let us introduce the function

$$\boldsymbol{\mu} = \tilde{\pi}^{1-q} \mathbf{s}_2^{(1)} / \mathbf{s}_2 \in C[[t, u^{q-1}]].$$

By (32),  $\boldsymbol{\mu}$  satisfies the non-linear  $\tau$ -difference equation:

$$\boldsymbol{\mu}^{(1)} = \frac{(t - \theta)}{\Delta} \boldsymbol{\mu}^{-1} - \frac{g}{\Delta}.$$



Hence,  $\boldsymbol{\mu} = (t - \theta)\Delta^{-1}(\boldsymbol{\mu}^{(1)} + g/\Delta)^{-1}$ . Although not needed in this paper, we point out that this functional equation gives the following continued fraction development, which turns out to be convergent for the  $u$ -adic topology:

$$\boldsymbol{\mu} = \frac{(t - \theta)}{g + \frac{\Delta(t - \theta^q)}{g^q + \frac{\Delta^q(t - \theta^{q^2})}{g^{q^2} + \frac{\Delta^{q^2}(t - \theta^{q^3})}{\dots}}}} \in \mathbb{F}_q[t, \theta][[v]]. \quad (55)$$

This property should be compared with the connection of Atkin's polynomials with certain continued fraction developments in [20, Section 4, 5], or the continued fraction developments described after [18, Theorem 2].

## 4 Bi-weighted automorphic functions

In this section we define almost  $A$ -quasi-modular forms. We will see that they generate a  $\mathbb{T}_{>0}$ -algebra  $\widetilde{\mathcal{M}}$  with natural embedding in  $C[[t, u]]$ . Thanks to the two kinds of factor of automorphy described below,  $\widetilde{\mathcal{M}}$  is also graded by the group  $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$ .

Presently, we do not have full structure information on  $\widetilde{\mathcal{M}}$  but we will show, with the help of the results of Section 3, that  $g, h, \mathbf{E}, \mathbf{F} \in \widetilde{\mathcal{M}}$  with  $\mathbf{F} = \tau\mathbf{E}$ . It will be proved that for this graduation, the degrees (in  $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$ ) of these functions are respectively  $(q-1, 0, 0)$ ,  $(q+1, 1, 0)$ ,  $(1, 1, 1)$  and  $(q, 1, 1)$  and we will show from this that they are algebraically independent over  $C((t))$  (also  $E$  belongs to  $\widetilde{\mathcal{M}}$ , but we will not use it). Since they take values in  $\mathbb{T}_{<q^q}$ , we will study with some detail the four dimensional  $\mathbb{T}_{<q^q}$ -algebra

$$\mathbb{M}^\dagger := \mathbb{T}_{<q^q}[g, h, \mathbf{E}, \mathbf{F}].$$

Proposition 7 implies that  $\tau$  acts on  $\mathbb{M}^\dagger$ : If  $\mathbf{f} \in \mathbb{M}^\dagger$  is homogeneous of degree  $(\mu, \nu, m)$  then  $\tau\mathbf{f}$  is also homogeneous of degree  $(q\mu, \nu, m)$ .

We will see that if  $\mathbf{f} \in \mathbb{M}^\dagger$  is homogeneous of degree  $(\mu, \nu, m)$ , the function

$$\begin{aligned} \Omega &\rightarrow C \\ \varepsilon(\mathbf{f}) : z &\mapsto \mathbf{f}(z)|_{t=\theta} \end{aligned}$$

is a well defined Drinfeld quasi-modular form of weight  $\mu + \nu$ , type  $m$  and depth  $\leq \nu$ . An example is given by Lemma 13:  $\varepsilon(\mathbf{E}) = E$ .

## 4.1 Preliminaries on the functions $J_\gamma$ and $L_\gamma$

Let us consider three matrices in  $\Gamma$ :

$$\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \mathcal{C} = \mathcal{A} \cdot \mathcal{B} = \begin{pmatrix} * & * \\ x & y \end{pmatrix} \in \Gamma. \quad (56)$$

**Lemma 11** *We have the following identities in  $\mathbb{T}_{>0}$ :*

$$\begin{aligned} L_{\mathcal{A}}(\mathcal{B}(z)) &= \det(\mathcal{B})^{-1} J_{\mathcal{B}}(z)^2 (L_{\mathcal{C}}(z) - L_{\mathcal{B}}(z)), \\ \mathbf{L}_{\mathcal{A}}(\mathcal{B}(z)) &= \det(\mathcal{B})^{-1} J_{\mathcal{B}}(z) \mathbf{J}_{\mathcal{B}}(z) (\mathbf{L}_{\mathcal{C}}(z) - \mathbf{L}_{\mathcal{B}}(z)). \end{aligned}$$

*Proof.* We begin by proving the first formula, observing that  $c = \det(\mathcal{B})^{-1}(x\delta - y\gamma)$ :

$$\begin{aligned} J_{\mathcal{B}}(z)^2 (L_{\mathcal{C}}(z) - L_{\mathcal{B}}(z)) &= \\ &= (\gamma z + \delta)^2 \left( \frac{x}{xz + y} - \frac{\gamma}{\gamma z + \delta} \right) \\ &= \det(\mathcal{B}) \frac{c(\delta + \gamma z)}{(c\alpha + d\gamma)z + (c\beta + d\gamma)} \\ &= \det(\mathcal{B}) \frac{c}{\frac{(c\alpha + d\gamma)z + (c\beta + d\gamma)}{\delta + \gamma z}} \\ &= \det(\mathcal{B}) \frac{c}{c \frac{\alpha z + \beta}{\gamma z + \delta} + d} \\ &= \det(\mathcal{B}) L_{\mathcal{A}}(\mathcal{B}(z)). \end{aligned}$$

As for the second formula, we set

$$\tilde{L}_{\mathcal{A}} = \frac{\bar{c}}{\bar{c}\bar{\xi} + \bar{d}}, \quad \bar{\xi} := \frac{s_1}{s_2}.$$

By using (43) and the obvious identity  $\det(\mathcal{B}) = \det(\bar{\mathcal{B}})$ , we compute in a similar way:

$$\begin{aligned} \mathbf{J}_{\mathcal{B}}(z)^2 (\tilde{L}_{\mathcal{C}}(z) - \tilde{L}_{\mathcal{B}}(z)) &= \\ &= (\bar{\gamma}z + \bar{\delta})^2 \left( \frac{\bar{x}}{\bar{x}\bar{\xi} + \bar{y}} - \frac{\bar{\gamma}}{\bar{\gamma}\bar{\xi} + \bar{\delta}} \right) \\ &= \det(\bar{\mathcal{B}}) \frac{\bar{c}}{\bar{c} \frac{\bar{\alpha}\bar{\xi} + \bar{\beta}}{\bar{\gamma}\bar{\xi} + \bar{\delta}} + \bar{d}} \\ &= \det(\mathcal{B}) \tilde{L}_{\mathcal{A}}(\mathcal{B}(z)). \end{aligned}$$

Hence,

$$\tilde{L}_{\mathcal{A}}(\mathcal{B}(z)) = \det(\mathcal{B})^{-1} \mathbf{J}_{\mathcal{B}}(z)^2 (\tilde{L}_{\mathcal{C}}(z) - \tilde{L}_{\mathcal{B}}(z)).$$

But

$$\tilde{L}_{\mathcal{A}}(z) = \mathbf{s}_2(z)\mathbf{L}_{\mathcal{A}}(z),$$

so that

$$\begin{aligned}\tilde{L}_{\mathcal{A}}(\mathcal{B}(z)) &= \mathbf{s}_2(\mathcal{B}(z))\mathbf{L}_{\mathcal{A}}(\mathcal{B}(z)) \\ &= (\bar{\gamma}\mathbf{s}_1(z) + \bar{\delta}\mathbf{s}_2(z))\mathbf{L}_{\mathcal{A}}(\mathcal{B}(z)) \\ &= \mathbf{s}_2(z)J_{\mathcal{B}}(z)^{-1}\mathbf{J}_{\mathcal{B}}(z)\mathbf{L}_{\mathcal{A}}(\mathcal{B}(z)),\end{aligned}$$

where  $\mathbf{s}_1, \mathbf{s}_2$  are considered as functions  $\Omega \rightarrow \mathbb{T}_{>0}$ , from which we deduce the expected identity.  $\square$

## 4.2 Almost $A$ -quasi-modular forms.

We recall that for all  $z \in \Omega$  and  $\gamma \in \Gamma$ , we have  $J_{\gamma}, \mathbf{J}_{\gamma}, L_{\gamma}, \mathbf{L}_{\gamma} \in \mathbb{T}_{>0}$ .

Let  $r$  be a positive real number and  $\mathbf{f} : \Omega \rightarrow \mathbb{T}_{<r}$  a map. We will say that  $\mathbf{f}$  is *regular* if the following properties hold.

1. There exists  $\varepsilon > 0$  such that, for all  $t_0 \in C$ ,  $|t_0| < \varepsilon$ , the map  $z \mapsto \mathbf{f}(z, t_0)$  is holomorphic on  $\Omega$ .
2. For all  $a \in A$ ,  $\mathbf{f}(z + a) = \mathbf{f}(z)$ . Moreover, there exists  $c > 0$  such that for all  $z \in \Omega$  with  $|u(z)| < c$  and  $t$  with  $|t| < c$ , there is a convergent expansion

$$\mathbf{f}(z, t) = \sum_{n,m \geq 0} c_{n,m} t^n u^m,$$

where  $c_{n,m} \in C$ .

**Definition 12 (Almost  $A$ -quasi-modular forms)** Let  $\mathbf{f}$  be a regular function  $\Omega \rightarrow \mathbb{T}_{<r}$ , for  $r$  a positive real number. We say that  $\mathbf{f}$  is an *almost- $A$ -quasi-modular form of weight  $(\mu, \nu)$ , type  $m$  and depth  $\leq l$*  if there exist regular functions  $\mathbf{f}_{i,j} : \Omega \rightarrow \mathbb{T}_{<r}$ ,  $0 \leq i + j \leq l$ , such that for all  $\gamma \in \Gamma$  and  $z \in \Omega$  the following functional equation holds in  $\mathbb{T}_{>0}$ :

$$\mathbf{f}(\gamma(z), t) = \det(\gamma)^{-m} J_{\gamma}^{\mu} \mathbf{J}_{\gamma}^{\nu} \left( \sum_{i+j \leq l} \mathbf{f}_{i,j} L_{\gamma}^i \mathbf{L}_{\gamma}^j \right). \quad (57)$$

The *radius of convergence*  $\rho(\mathbf{f})$  of an almost  $A$ -quasi-modular form  $\mathbf{f} : \Omega \rightarrow \mathbb{T}_{>0}$  is the infimum of the set of the real numbers  $r$  such that the maps  $\mathbf{f}, \mathbf{f}_{i,j}$  appearing in (57) simultaneously are well defined maps  $\Omega \rightarrow \mathbb{T}_{<r}$ .

We will say that  $\mu = \mu(\mathbf{f}), \nu = \nu(\mathbf{f}), m = m(\mathbf{f})$  are respectively the *first weight*, the *second weight* and the *type* of  $\mathbf{f}$ .

### 4.2.1 Some remarks.

It is obvious that in (57),  $\mathbf{f} = \mathbf{f}_{0,0}$  (use  $\gamma =$  identity matrix).

If  $\lambda \in \mathbb{T}_{>0}$ , then the map  $z \mapsto \lambda$  trivially is an almost  $A$ -quasi-modular form of weight  $(0,0)$ , type 0, depth  $\leq 0$ . The radius  $\rho(\lambda)$  is then just the radius of convergence of the series  $\lambda$ .

Examples of almost  $A$ -quasi-modular forms are Drinfeld quasi-modular forms. To any Drinfeld quasi-modular form of weight  $w$ , type  $m$ , depth  $\leq l$  is associated an almost  $A$ -quasi-modular form of weight  $(w, 0)$ , type  $m$ , depth  $\leq l$  whose radius is infinite.

The  $\mathbb{T}_{>0}$ -algebra  $\mathbb{T}_{>0}[g, h]$  is graded by the couples  $(w, m) \in \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z}$  of weights and types, and the isobaric elements are all almost  $A$ -quasi-modular forms with the second weight 0.

The function  $\mathbf{s}_2$  is, by Lemmas 4 and 5, an almost  $A$ -quasi-modular form of weight  $(-1, 1)$ , depth 0, type 0. The radius is  $q$ , by the results of Section 2.1.

If  $\mathbf{f}$  is an almost  $A$ -quasi-modular form of weight  $(\mu, \nu)$ , type  $m$ , depth  $\leq l$  and radius of convergence  $> q$ , then  $\varepsilon(\mathbf{f}) := \mathbf{f}|_{t=\theta}$  is a well defined holomorphic function  $\Omega \rightarrow C$ . It results from (45) and (46) that  $\varepsilon(\mathbf{f})$  is a Drinfeld quasi-modular form of weight  $\mu + \nu$ , type  $m$  and depth  $\leq l$ .

The function  $\mathbf{f} := \mathbf{s}_2$  is not well defined at  $t = \theta$  because its radius of convergence is  $q$ , and we know from (36) that there is divergence at  $\theta$ . However, the function  $\mathbf{f} := (t - \theta)\mathbf{s}_2$ , which is an almost  $A$ -quasi-modular of same weight, type and depth as  $\mathbf{s}_2$ , has convergence radius  $q^q$ . Therefore,  $\varepsilon(\mathbf{f})$  is well defined, and is the constant function  $-1$  by the results of Subsection 2.1.1. From (50) we see that the function  $\mathbf{s}_2^{(1)}$  is not an almost  $A$ -quasi-modular form. The non-zero function  $\varepsilon(\mathbf{s}_2^{(1)})$  is well defined and we have already mentioned the results of Gekeler in [12] that allow to compute  $\varepsilon(\mathbf{s}_2^{(1)})$ .

Let us write  $\phi = \varepsilon(\mathbf{E})$ , which corresponds to a well defined series of  $uC[[u^{q-1}]]$  by (54). We obtain, by using (47), (45) and (46) with  $k = 0$ , that

$$\phi(\gamma(z)) = \det(\gamma)^{-1}(cz + d)^2 \left( \phi(z) - \tilde{\pi}^{-1} \frac{c}{cz + d} \right).$$

This is the collection of functional equations of the Drinfeld quasi-modular form  $E$  (4), whose  $u$ -expansion begins with the term  $u$ . Applying [3, Theorem 1] we obtain:

**Lemma 13** *We have, for all  $z \in \Omega$ :*

$$\varepsilon(\mathbf{E}) = \mathbf{E}(z, \theta) = E(z).$$

It is easy to verify, as a confirmation of this result, that the first coefficients of the  $u$ -expansion of  $\mathbf{E}$  given in (54) agree, substituting  $t$  by  $\theta$ , with the  $u$ -expansion of  $E$  that we know already after [13, Corollary (10.5)]:

$$E = u(1 + v^{(q-1)} + \dots).$$

More generally, Propositions 7, 8 and 9 imply that for all  $k \geq 0$ ,  $\mathbf{E}^{(k)}$  is an almost  $A$ -quasi-modular form of weight  $(q^k, 1)$  type 1 and depth  $\leq 1$  with convergence radius  $\geq q^q$ , so that  $\varepsilon(\mathbf{E}^{(k)})$  is well defined, and is a Drinfeld quasi-modular form of weight  $q^k + 1$ , type 1 and depth  $\leq 1$ . The computation of  $\varepsilon(\mathbf{E}^{(k)})$ , not needed for the proof of our Theorem, is made in Proposition 34.

#### 4.2.2 Grading by the weights, filtering by the depths.

For  $\mu, \nu \in \mathbb{Z}, m \in \mathbb{Z}/(q-1)\mathbb{Z}, l \in \mathbb{Z}_{\geq 0}$ , we denote by  $\widetilde{\mathcal{M}}_{\mu, \nu, m}^{\leq l}$  the  $\mathbb{T}_{>0}$ -module of almost  $A$ -quasi-modular forms of weight  $(\mu, \nu)$ , type  $m$  and depth  $\leq l$ . We have

$$\widetilde{\mathcal{M}}_{\mu, \nu, m}^{\leq l} \widetilde{\mathcal{M}}_{\mu', \nu', m'}^{\leq l'} \subset \widetilde{\mathcal{M}}_{\mu+\mu', \nu+\nu', m+m'}^{\leq l+l'}.$$

We also denote by  $\widetilde{\mathcal{M}}$  the  $\mathbb{T}_{>0}$ -algebra generated by all the almost  $A$ -quasi-modular forms. We prove below that this algebra is graded by the group  $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$ , filtered by the depths (Proposition 18), and contains five algebraically independent functions  $E, g, h, \mathbf{E}, \mathbf{F}$  (Proposition 19).

Let  $\mathcal{K}$  be any field extension of  $\mathbb{F}_q(t, \theta)$ . The key result of this section is the following elementary lemma.

**Lemma 14** *The subset  $\Theta = \{(d, \bar{d}), d \in A\} \subset \mathbb{A}^2(\mathcal{K})$  is Zariski dense.*

*Proof.* Let us assume by contradiction that the lemma is false and let  $\bar{\Theta}$  be the Zariski closure of  $\Theta$ . Then, we can write

$$\bar{\Theta} = \bigcup_{i \in \mathcal{I}} \Theta_i \cup \bigcup_{j \in \mathcal{J}} \widetilde{\Theta}_j,$$

where the  $\Theta_i$ 's are irreducible closed subsets of  $\mathbb{A}^2(\mathcal{K})$  of dimension 1, the  $\widetilde{\Theta}_j$ 's are isolated points of  $\mathbb{A}^2(\mathcal{K})$ , and  $\mathcal{I}, \mathcal{J}$  are finite sets.

From  $\Theta = \Theta + (d, \bar{d})$  for all  $d \in A$  we deduce  $\bar{\Theta} = \bar{\Theta} + (d, \bar{d})$ . The translations of  $\mathbb{A}^2(\mathcal{K})$  by points such as  $(d, \bar{d})$  being bijective, they induce permutations of the sets  $\{\Theta_i\}$  and  $\{\widetilde{\Theta}_j\}$ , from which we easily deduce that  $\mathcal{J} = \emptyset$ . Therefore, the ideal of polynomials  $R \in \mathcal{K}[X, Y]$  such that  $R(\Theta) \subset \{0\}$  is principal, generated by a non-zero polynomial  $P$ .

Now, if  $b \in A$ ,  $m_b(\bar{\Theta}) \subset \bar{\Theta}$ , where  $m_b(x, y) := (bx, \bar{b}y)$ . Hence,  $P(m_b(X, Y)) \in (P)$  and there exists  $\kappa_b \in \mathcal{K}^\times$  such that

$$P(bX, \bar{b}Y) = \kappa_b P(X, Y).$$

Let us write:

$$P(X, Y) = \sum_{\alpha, \beta} c_{\alpha, \beta} X^\alpha Y^\beta,$$

and choose  $b \notin \mathbb{F}_q$ . If  $c_{\alpha, \beta} \neq 0$ , then  $\kappa_b = b^{-\alpha} \bar{b}^{-\beta}$ . If  $P$  is not a monomial, we have, for  $(\alpha, \beta) \neq (\alpha', \beta')$ ,  $c_{\alpha, \beta}, c_{\alpha', \beta'} \neq 0$ , so that  $b^{-\alpha} \bar{b}^{-\beta} = b^{-\alpha'} \bar{b}^{-\beta'}$ , yielding a contradiction, because  $b \notin \mathbb{F}_q$ .

If  $P$  is a monomial, however, it cannot vanish at  $(1, 1) \in \Theta$ ; contradiction.  $\square$

**Lemma 15** *Let us suppose that for elements  $\psi_{\alpha, \beta} \in C((t))$  and for a certain element  $z \in \Omega$  we have an identity:*

$$\sum_{\alpha, \beta} \psi_{\alpha, \beta} \mathbf{J}_\gamma^\alpha \mathbf{J}_\gamma^\beta = 0, \quad (58)$$

in  $C((t))$ , for all  $\gamma = \begin{pmatrix} a & b \\ 1 & d \end{pmatrix} \in \Gamma$  with determinant 1, the sum being finite. Then,  $\psi_{\alpha, \beta} = 0$  for all  $\alpha, \beta$ .

*Proof.* Let us suppose by contradiction the existence of a non-trivial relation (58). We have, with the hypothesis on  $\gamma$ ,  $\mathbf{J}_\gamma = z + d$ ,  $\mathbf{J}_\gamma = \boldsymbol{\xi} + \bar{d} \in C((t))$ , so that the relation of the lemma implies the existence of a relation:

$$\sum_{\alpha, \beta} \ell_{\alpha, \beta} d^\alpha \bar{d}^\beta = 0, \quad d \in A,$$

with  $\ell_{\alpha, \beta} \in \mathcal{K} = C((t))$  not all zero, and all, but finitely many, vanishing. Lemma 14 yields a contradiction.  $\square$

Another useful lemma is the following. The proof is again a simple application of Lemma 14 and will be left to the reader.

**Lemma 16** *If the finite collection of functions  $f_{i,j} : \Omega \rightarrow \mathbb{T}_{>0}$  is such that for all  $z \in \Omega$  and for all  $\gamma \in \Gamma$ ,*

$$\sum_{i,j} f_{i,j}(z) L_\gamma^i \mathbf{L}_\gamma^j = 0,$$

*then the functions  $f_{i,j}$  are all identically zero.*

**Lemma 17** *Let  $\mathbf{f}$  be an almost  $A$ -quasi-modular form of type  $m$  with  $0 \leq m < q - 1$ . Then, with  $v = u^{q-1}$ ,*

$$\mathbf{f}(z) = u^m \sum_{i \geq 0} c_i(t) v^i.$$

*Proof.* It follows the same ideas of the remark on p. 23 of [14]. Let us consider  $\gamma = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma$  with  $\lambda \in \mathbb{F}_q^\times$ . We have  $\gamma(z) = \lambda z$ ,  $\det(\gamma) = \lambda$ ,  $J_\gamma = \mathbf{J}_\gamma = 1$ ,  $L_\gamma = \mathbf{L}_\gamma = 0$ , so that  $\mathbf{f}(\lambda z) = \lambda^{-m} \mathbf{f}(z)$ , for all  $z \in \Omega$ . Now, if  $\mathbf{f} = \sum_i c_i(t) u^i$ , since  $e_{\text{Car}}$  is  $\mathbb{F}_q$ -linear, we get  $u(\lambda z) = \lambda^{-1} u(z)$  and if  $c_i \neq 0$ , then  $i \equiv m \pmod{q-1}$ .  $\square$

**Proposition 18** *The  $\mathbb{T}_{>0}$ -algebra generated by the almost  $A$ -quasi-modular forms is graded by weights and types, hence by the group  $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$ , and filtered by the depths:*

$$\widetilde{\mathcal{M}} = \bigoplus_{(\mu, \nu, m) \in \mathbb{Z}^2 \times \frac{\mathbb{Z}}{(q-1)\mathbb{Z}}} \bigcup_{l=0}^{\infty} \widetilde{\mathcal{M}}_{\mu, \nu, m}^{\leq l}.$$

*Proof.* We begin by proving the property concerning the grading by the group  $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$ . Let us consider distinct triples  $(\mu_i, \nu_i, m_i) \in \mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$ ,  $i = 1, \dots, s$ , non-negative integers  $l_1, \dots, l_s$  and non-zero elements  $\mathbf{f}_i \in \widetilde{\mathcal{M}}_{\mu_i, \nu_i, m_i}^{\leq l_i}$ . Then, we claim that  $\sum_{i=1}^s \mathbf{f}_i \neq 0$ . To see this, we assume by contradiction that for some forms  $\mathbf{f}_i$  as in the proposition, we have the identity in  $\mathbb{T}_{>0}$ :

$$\sum_{i=1}^s \mathbf{f}_i = 0. \quad (59)$$

Recalling Definition 12 (identity (57)), we have, for all  $i = 1, \dots, s$ ,  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ ,  $z \in \Omega$ :

$$\mathbf{f}_i(\gamma(z), t) = \det(\gamma)^{-m_i} J_\gamma^{\mu_i} \mathbf{J}_\gamma^{\nu_i} \sum_{j+k \leq l} \mathbf{f}_{i,j,k}(z, t) L_\gamma^j \mathbf{L}_\gamma^k,$$

for certain functions  $\mathbf{f}_{i,j,k} : \Omega \rightarrow \mathbb{T}_{>0}$ .

Let us suppose first that  $\gamma$  is of the form  $\begin{pmatrix} a & b \\ 1 & d \end{pmatrix}$  with  $ad - b = 1$ . We recall that  $\mathbf{s}_2(z)^{-1} \in \mathbb{T}_{>0}^\times$  for all  $z$ . Therefore, for all  $z \in \Omega$ , (59) becomes the identity of formal series in  $\mathbb{T}_{>0}$ :

$$\sum_{i=1}^s \sum_{j+k \leq l_i} \mathbf{f}_{i,j,k} \mathbf{s}_2^{-k} (z+d)^{\mu_i-j} (\boldsymbol{\xi} + \bar{d})^{\nu_i-k} = 0. \quad (60)$$

By Lemma 15, (60) is equivalent to the relations:

$$\sum_{i,j,k} \phi_{i,j,k} = 0, \quad \text{for all } (\alpha, \beta) \in \mathbb{Z}^2 \quad (61)$$

where  $\phi_{i,j,k} := \mathbf{f}_{i,j,k} \mathbf{s}_2^{-k}$  and the sum runs over the triples  $(i, j, k)$  with  $i \in \{1, \dots, s\}$  and  $j, k$  such that  $\mu_i - j = \alpha$  and  $\nu_i - k = \beta$ , with obvious vanishing conventions on some of the  $\phi_{i,j,k}$ 's.

Let  $\mu$  be the maximum value of the  $\mu_i$ 's, and let us look at the relations (61) for  $\alpha = \mu$ . Since for all  $\mu_i < \mu$  we get  $\alpha = \mu > \mu_i - j$  for all  $j \geq 0$ , for such a choice of  $\alpha$  we get:

$$\sum_{i,k} \phi_{i,0,k} = 0, \quad \text{for all } \beta \in \mathbb{Z}, \quad (62)$$

where the sum is over the couples  $(i, j)$  with  $i$  such that  $\mu_i = \mu$  and  $\nu_i - k = \beta$ . Now, let  $\mathcal{E}$  be the set of indices  $i$  such that  $\mu_i = \mu$  and write  $\nu$  for the maximum of the  $\nu_i$  with  $i \in \mathcal{E}$ . If  $j$  is such that  $\mu_j = \mu$ , and if  $\nu \neq \nu_j$ , then for all  $k \geq 0$ ,  $\nu > \nu_j - k$ , so that for  $\beta = \nu$ , (62) becomes

$$\sum_i \phi_{i,0,0} = 0,$$

where the sum runs this time over the  $i$ 's such that  $(\mu_i, \nu_i) = (\mu, \nu)$ . But  $\phi_{i,0,0} = \mathbf{f}_{i,0,0} = \mathbf{f}_i$  for  $i = 1, \dots, s$ . Since the types of the  $\mathbf{f}_i$ 's with same weights are distinct by hypothesis, Lemma 17 implies that for all  $i$  such that  $(\mu_i, \nu_i) = (\mu, \nu)$ ,  $\mathbf{f}_i = 0$ . This contradicts our initial assumptions and proves our initial claim. Combining with Lemma 16, we end the proof of the proposition.  $\square$

**Proposition 19** *The functions*

$$E, g, h, \mathbf{s}_2, \mathbf{s}_2^{(1)} : \Omega \rightarrow \mathbb{T}_{>0}$$

*are algebraically independent over the fraction field of  $\mathbb{T}_{>0}$ .*

*Proof.* Assume by contradiction that the statement of the proposition is false. Since  $E, g, h, \mathbf{s}_2, \mathbf{s}_2^{(1)} \in \widetilde{\mathcal{M}}$  are almost  $A$ -quasi-modular forms, by Proposition 18, there exist  $(\mu, \nu), m \in \mathbb{Z}$ , and a non-trivial relation (where the sum is finite):

$$\sum_{i,j \geq 0} P_{i,j} E^i \mathbf{s}_2^{(1)j} = 0,$$

with  $P_{i,j} \in \mathbb{T}_{>0}[g, h, \mathbf{s}_2] \cap \widetilde{\mathcal{M}}_{\mu-2i+qj, \nu-j, m-i}^{\leq l}$  (for some  $l \geq 0$ ). By Proposition 18, any vector space of almost  $A$ -quasi-modular forms of given weight and depth is filtered



by the depths. Comparing with the functional equations (50) and [3, Functional equation (11)], and applying Lemma 16, we see that all the forms  $P_{i,j}$  vanish. There are three integers  $\alpha, m, n$  and a non trivial polynomial relation  $P$  among  $g, h, \mathbf{s}_2$ , with coefficients in  $\mathbb{T}_{>0}$ :

$$\sum_{s=0}^n Q_s \mathbf{s}_2^s = 0,$$

where  $Q_s \in \mathbb{T}_{>0}[g, h] \cap \widetilde{\mathcal{M}}_{\alpha+s, 0, m}^{\leq l}$  ( $s = 0, \dots, n$ ), and for some  $s$ ,  $Q_s$  is non-zero. Since  $\nu(Q_s) = 0$  for all  $s$  such that  $Q_s \neq 0$  and  $\nu(\mathbf{s}_2) = 1$ , The polynomial  $P$ , evaluated at the functions  $E, g, h, \mathbf{s}_2, \mathbf{s}_2^{(1)}$  is equal to  $Q \mathbf{s}_2^s$  for  $Q \in \mathbb{T}_{>0}[g, h] \setminus \{0\}$  and  $s \in \mathbb{Z}$ , quantity that cannot vanish because  $g, h$  are algebraically independent over  $\mathbb{T}_{<q^a}$ : contradiction.  $\square$

## 5 Estimating the multiplicity

We prove our Theorem in this section.

### 5.1 Preliminaries

Let us denote by  $\mathbb{M}^\dagger$  the  $\mathbb{T}_{<q^a}$ -algebra  $\mathbb{T}_{<q^a}[g, h, \mathbf{E}, \mathbf{F}]$ , where  $\mathbf{F} := \mathbf{E}^{(1)}$ ; its dimension is 4, according to Proposition 19 and Proposition 8. By Proposition 18, this algebra is graded by the group  $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$ :

$$\mathbb{M}^\dagger = \bigoplus_{(\mu, \nu), m} \mathbb{M}_{\mu, \nu, m}^\dagger,$$

where  $\mathbb{M}_{\mu, \nu, m}^\dagger = \widetilde{\mathcal{M}}_{\mu, \nu, m} \cap \mathbb{M}^\dagger$ .

The operator  $\tau$  acts on  $\mathbb{M}^\dagger$  by Proposition 7. More precisely, we have the homomorphism of  $\mathbb{F}_q[t]$ -modules

$$\tau : \mathbb{M}_{\mu, \nu, m}^\dagger \rightarrow \mathbb{M}_{q\mu, \nu, m}^\dagger.$$

Let us write  $\mathbf{h} = \widetilde{\pi} h s_{\text{Car}}^{-1} \mathbf{s}_2 = \mathbf{h} \mathbf{d}$ .

**Lemma 20** *The formula  $\mathbf{h} = (t - \theta^q) \mathbf{F} - g \mathbf{E}$  holds, so that  $\mathbf{h} \in \mathbb{M}_{q, 1, 1}^\dagger$  and  $\mathbb{M}^\dagger = \mathbb{T}_{<q^a}[g, h, \mathbf{E}, \mathbf{h}]$ .*

*Proof.* From the definition of  $\mathbf{E}$  and (37), we find:

$$\begin{aligned} (t - \theta^q) \mathbf{F} - g \mathbf{E} &= \\ &= -(t - \theta^q) h^q \mathbf{d}^{(2)} + g h \mathbf{d}^{(1)} \\ &= (-h)^q (-h^{q-1})^{-1} (\mathbf{d} - g \mathbf{d}^{(1)}) + g h \mathbf{d}^{(1)} \\ &= \mathbf{h} \mathbf{d} = \mathbf{h}. \end{aligned}$$

This makes it clear that  $\mathbf{h}$  belongs to  $\mathbb{M}_{q,1,1}^\dagger$  and that  $\mathbb{M}^\dagger = \mathbb{T}_{<q^q}[g, h, \mathbf{E}, \mathbf{h}]$ .  $\square$

We denote by  $\varepsilon_{\mu,\nu,m}$  or again  $\varepsilon$  the map which sends an almost  $A$ -quasi-modular form  $\mathbf{f}$  of weight  $(\mu, \nu)$ , type  $m$ , with radius  $> q$  to the Drinfeld quasi-modular form  $\varepsilon(\mathbf{f})$  of weight  $\mu + \nu$ , type  $m$ . This map is clearly a  $C$ -algebra homomorphism.

**Lemma 21** *We have  $\varepsilon(\mathbf{h}) = h$ .*

*Proof.* This follows from the limit  $\lim_{t \rightarrow \theta} s_{\text{Car}}^{-1} \mathbf{s}_2 = \tilde{\pi}^{-1}$  and the definition of  $\mathbf{d}$ .  $\square$

More generally, we have the following result.

**Proposition 22** *For all  $(\mu, \nu), m$ , the map*

$$\varepsilon : \mathbb{M}_{\mu,\nu,m}^\dagger \rightarrow \widetilde{M}_{\mu+\nu,m}^{\leq \nu}$$

*is well defined and the inverse image of 0 is the  $\mathbb{T}_{<q^q}$ -module  $(t - \theta)\mathbb{M}_{\mu,\nu,m}^\dagger$ .*

*Proof.* Let  $\mathbf{f}$  be an element of  $\mathbb{M}_{\mu,\nu,m}^\dagger$ . Then, by Lemma 20,

$$\mathbf{f} = \sum_{i=0}^{\nu} \phi_i \mathbf{h}^{\nu-i} \mathbf{E}^i,$$

where  $\phi_i \in M_{\mu-\nu q+i(q-1), m-\nu} \otimes_C \mathbb{T}_{<q^q}$ . Since  $\lim_{t \rightarrow \theta} s_{\text{Car}}^{-1} \mathbf{s}_2 = \tilde{\pi}^{-1}$ , we have  $\varepsilon(\mathbf{h}) = h$  by Lemma 21. Moreover, by Lemma 13,  $\varepsilon(\mathbf{E}) = E$ , and

$$\varepsilon(\mathbf{f}) = \sum_{i=0}^{\nu} \varepsilon(\phi_i) h^{\nu-i} E^i,$$

so that  $\varepsilon(\mathbf{f}) = 0$  if and only if  $\varepsilon(\phi_i) = 0$  for all  $i$ . But for all  $i$ ,  $\phi_i$  is a polynomial in  $g, h$  with coefficients in  $\mathbb{T}_{<q^q}$ . If  $\varepsilon(\phi_i) = 0$ , then  $\phi_i$  is a linear combination  $\sum_{a,b} c_{a,b} g^a h^b$  with  $c_{a,b} \in \mathbb{T}_{<q^q}$  such that  $c_{a,b}(\theta) = 0$ . Since  $\mathbb{T}_{<q^q} \subset \mathbb{T}$ , it is a principal ideal domain and the last condition is equivalent to  $\phi_i \in (t - \theta)(M \otimes_C \mathbb{T}_{<q^q})$ . Hence,  $\varepsilon(\mathbf{f}) = 0$  if and only if, for all  $i$ ,  $\phi_i \in (t - \theta)(M \otimes_C \mathbb{T}_{<q^q})$ . The proposition follows.  $\square$

## 5.2 Multiplicity estimate in $\mathbb{M}^\dagger$

By Proposition 9,  $\mathbf{E} = u + \dots \in u\mathbb{F}_q[t, \theta][[u^{q-1}]]$ . Hence,

$$\mathbf{E}^{(k)} = u^{q^k} + \dots \in u^{q^k} \mathbb{F}_q[t, \theta][[u^{(q-1)q^k}]], \quad k \geq 0,$$

and there is an embedding  $\mathbb{M}^\dagger \rightarrow \mathbb{T}_{<q^q}[[u]]$ . It will be sometimes useful to fix an embedding of  $\mathbb{T}_{<q^q}$  in  $\mathcal{K}$ , an algebraic closure of  $C((t))$ ; we will then often consider elements of  $\mathbb{M}^\dagger$  as formal series in  $\mathcal{K}[[u]]$  (especially in this subsection). Anderson's operator  $\tau : C((t)) \rightarrow C((t))$  extends in a natural way to an  $\mathbb{F}_q(t)$ -linear operator  $\tau : \mathcal{K} \rightarrow \mathcal{K}$  (we will keep using the notation  $\tau^k f = f^{(k)}$ ). If  $f = \sum_{n \geq n_0} c_n(t)u^n$  is a formal series of  $\mathcal{K}[[u]]$ , then, Anderson's operator further extends as follows:

$$f^{(k)} = \sum_{n \geq n_0} c_n^{(k)}(t)u^{q^k n}, \quad k \in \mathbb{Z}. \quad (63)$$

Let  $f = \sum_{n \geq n_0} c_n(t)u^n$  be in  $\mathcal{K}[[u]]$ , with  $c_{n_0} \neq 0$ . We write  $\nu_\infty(f) := n_0$ . We also set  $\nu_\infty(0) := \infty$ . Obviously,  $\nu_\infty(f^{(k)}) = q^k \nu_\infty(f)$  for all  $k \geq 0$ . We recall that  $\nu_\infty(g) = 0, \nu_\infty(h) = \nu_\infty(\mathbf{E}) = 1$  and  $\nu_\infty(\mathbf{F}) = q$ . Since  $\nu_\infty(\mathbf{s}_2) = 0$ , we also get  $\nu_\infty(\mathbf{h}) = 1$ . In the following, we will write  $\mathbb{M}_{\mu,\nu,m}^\dagger(\mathcal{K}) = \mathbb{M}_{\mu,\nu,m}^\dagger \otimes_{\mathbb{T}_{<q^q}} \mathcal{K}$  and  $M_{w,m}(\mathcal{K}) = M_{w,m} \otimes_C \mathcal{K}$ . It is evident that the  $\mathcal{K}$ -algebra  $\mathbb{M}^\dagger(\mathcal{K}) = \sum_{\mu,\nu,m} \mathbb{M}_{\mu,\nu,m}^\dagger(\mathcal{K})$  is again graded by the group  $\mathbb{Z}^2 \times \mathbb{Z}/(q-1)\mathbb{Z}$ ; similarly for the algebra  $M(\mathcal{K}) = \sum_{w,m} M_{w,m}(\mathcal{K})$ .

We begin with a rather elementary estimate, for  $\mathbf{f} \in \mathbb{M}^\dagger$  of weight  $(\mu, 0)$ .

**Lemma 23** *If  $\mathbf{f} \in \mathbb{M}_{\mu,0,m}^\dagger(\mathcal{K})$  is non-zero, then  $\nu_\infty(\mathbf{f}) \leq \frac{\mu}{q+1}$ .*

*Proof.* A weight inspection shows that  $\mathbb{M}_{\mu,0,m}^\dagger(\mathcal{K}) = \mathcal{K}[g, h]_{\mu,m}$ . We can write  $\mathbf{f} = h^{\nu_\infty(\mathbf{f})} \mathbf{b}$ , with  $\mathbf{b} \in \mathcal{K}[g, h]$  and  $h$  not dividing  $\mathbf{b}$ . Therefore,  $\nu_\infty(\mathbf{f}) \leq \frac{\mu}{q+1}$ .  $\square$

In the next proposition, we study the case of  $\mathbf{f}$  of weight  $(\mu, \nu)$  with  $\nu > 0$ .

**Proposition 24** *Let  $\mathbf{f}$  be a non-zero element of  $\mathbb{M}_{\mu,\nu,m}^\dagger(\mathcal{K})$  with  $\nu \neq 0$ . Then,*

$$\nu_\infty(\mathbf{f}) \leq \mu\nu.$$

It is not difficult to show that the statement of this proposition cannot be improved (this can be checked with the functions  $\mathbf{E}^{(k)}$ ).

Before proving the proposition, we need to state and prove a lemma.

**Lemma 25** *Let  $\mathbf{f} \in \mathbb{M}_{\mu,\nu,m}^\dagger(\mathcal{K})$ ,  $\mathbf{f}' \in \mathbb{M}_{\mu',\nu',m'}^\dagger(\mathcal{K})$ . By Lemma 20,  $\mathbf{f}, \mathbf{f}'$  can be written in a unique way as polynomials in  $\mathcal{K}[g, h, \mathbf{E}, \mathbf{h}]$ . Let  $l, l'$  be the degrees in  $\mathbf{E}$  of  $\mathbf{f}, \mathbf{f}'$  respectively. Then,*

$$\phi := \mathbf{Res}_{\mathbf{E}}(\mathbf{f}, \mathbf{f}') = \mathbf{h}^{\nu l' + \nu' l - ll'} \phi_0,$$

where  $\phi_0 \in M_{w^*, m^*}(\mathcal{K})$ , with

$$w^* = \mu l' + \mu' l - ll' - q(\nu l' + \nu' l - ll'), \quad m^* := ml' + m'l - (\nu l' + \nu' l).$$

*Proof.* With an application of an obvious variant of [26, Lemme 6.1] <sup>(12)</sup> we see that

$$\phi \in \mathbb{M}_{\mu' + \mu'l - l', \nu' + \nu'l - l', m' + m'l - l'}^\dagger(\mathcal{K}).$$

At the same time,  $\phi \in \mathcal{K}[g, h, \mathbf{h}]$ . Since  $\nu(g) = \nu(h) = 0$  and  $\nu(\mathbf{h}) = 1$ , we have  $\phi_0 := \phi / \mathbf{h}^{\nu' + \nu'l - l'} \in M(\mathcal{K})$ . The computation of the weight and type of  $\phi_0$  is obvious, knowing that  $\mu(\mathbf{h}) = q$ .  $\square$

*Proof of Proposition 24.* Let  $\mathbf{f}$  be in  $\mathbb{M}_{\mu, \nu, m}^\dagger(\mathcal{K})$ , with  $\nu > 0$ . Assume first that  $\mathbf{f}$ , as a polynomial in  $g, h, \mathbf{E}, \mathbf{h}$ , is irreducible. If  $\mathbf{f}$  belongs to  $\mathcal{K}[g, h, \mathbf{h}]$  then  $\mathbf{f} = \phi \mathbf{h}^\nu$  with  $\phi \in M_{\mu - q\nu, 0, m - \nu}(\mathcal{K})$  and

$$\begin{aligned} \nu_\infty(\mathbf{f}) &\leq \nu_\infty(\phi) + \nu\nu_\infty(\mathbf{h}) \\ &\leq \frac{\mu - q\nu}{q + 1} + \nu \\ &\leq \frac{\mu + \nu}{q + 1} \\ &\leq \mu\nu. \end{aligned}$$

We now suppose that  $\mathbf{f} \notin \mathcal{K}[g, h, \mathbf{h}]$ ; there are two cases left.

*Case (i).* We suppose that  $\mathbf{f}$  divides  $\mathbf{f}^{(1)} \in \mathbb{M}_{q\mu, \nu, m}^\dagger(\mathcal{K})$  as a polynomial in  $g, h, \mathbf{E}, \mathbf{h}$ . For weight reasons,  $\mathbf{f}^{(1)} = a\mathbf{f}$  with  $a \in M_{\mu(q-1), 0}(\mathcal{K})$  and  $a \neq 0$ . We also have  $\nu_\infty(\mathbf{f}^{(1)}) = q\nu_\infty(\mathbf{f})$  by (63), so that, by Lemma 23,  $(q-1)\nu_\infty(\mathbf{f}) = \nu_\infty(a) \leq (q-1)(q+1)^{-1}\mu$ . Hence, in this case, we get the stronger inequality <sup>(13)</sup>

$$\nu_\infty(\mathbf{f}) \leq \frac{\mu}{q+1}.$$

*Case (ii).* In this case,  $\mathbf{f}$  and  $\mathbf{f}^{(1)}$  are coprime. Since  $\mathbf{f}$  is irreducible,  $\deg_{\mathbf{E}}(\mathbf{f}) = l = \nu > 0$ , so that  $\mathbf{f}, \mathbf{f}^{(1)}$  depend on  $\mathbf{E}$ , and their resultant  $\phi$  with respect to  $\mathbf{E}$  is non-zero. We apply Lemma 25 with  $\mathbf{f}' = \mathbf{f}^{(1)}$ , finding

$$\phi = \mathbf{h}^{\nu^2} \phi_0,$$

with  $\phi_0 \in M_{(q+1)\nu(\mu-\nu), m^*}(\mathcal{K})$ , for a certain  $m^*$  that can be computed with Lemma 25. By Lemma 23 again,  $\nu_\infty(\phi_0) \leq \nu(\mu - \nu)$ . Since  $\nu_\infty(\mathbf{h}) = 1$ ,  $\nu_\infty(\phi) \leq \nu(\mu -$

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<sup>12</sup>The first formula after the statement of the above cited lemma, mistakenly typed, must be replaced with

$$p(R) = p(F) \deg_{X_0}(G) + p(G) \deg_{X_0}(F) - p(X_0) \deg_{X_0}(F) \deg_{X_0}(G).$$

<sup>13</sup>It can be proved that  $\mathbf{f}$  is, in this case, a modular form multiplied by an element of  $\mathcal{K}$ , but we do not need this information here.

$\nu) + \nu^2 = \mu\nu$ . Now, the number  $\nu_\infty(\phi)$  is an upper bound for  $\nu_\infty(\mathbf{f})$  by Bézout identity for the resultant.

We have proved the proposition if  $\mathbf{f} \in \mathbb{M}_{\mu,\nu,m}^\dagger(\mathcal{K})$  is irreducible. If  $\mathbf{f}$  is not irreducible, we can write  $\mathbf{f} = \prod_{i=0}^r \mathbf{f}_i$  with  $\mathbf{f}_0 \in \mathbb{M}_{\mu_0,0,m_0}^\dagger(\mathcal{K})$ ,  $\mathbf{f}_i \in \mathbb{M}_{\mu_i,\nu_i,m_i}^\dagger(\mathcal{K})$  irreducible for all  $i > 0$  with  $\nu_i > 0$ , and  $\sum_i \mu_i = \mu$ ,  $\sum_i \nu_i = \nu$ ,  $\sum_i m_i \equiv m \pmod{q-1}$ . Since  $\nu_\infty(\mathbf{f}) = \sum_i \nu_\infty(\mathbf{f}_i)$ , we get, applying Lemma 23,

$$\nu_\infty(\mathbf{f}) \leq \frac{\mu_0}{q+1} + \sum_{i>0} \mu_i \nu_i \leq \mu\nu.$$

□

### 5.3 Reduced forms

Let  $\mathbf{f}$  be in  $\mathbb{M}^\dagger$ . Since  $\varepsilon(\mathbf{f}) \in \widetilde{M} \subset C[[u]]$ , it is legitimate to compare the quantities  $\nu_\infty(\mathbf{f})$  and  $\nu_\infty(\varepsilon(\mathbf{f}))$ . We have the inequality:

$$\nu_\infty(\mathbf{f}) \leq \nu_\infty(\varepsilon(\mathbf{f})), \quad (64)$$

but the equality is not guaranteed in general, because the leading term of the  $u$ -expansion of  $\mathbf{f}$  can vanish at  $t = \theta$ .

**Definition 26** A function  $\mathbf{f}$  in  $\mathbb{M}^\dagger$  is *reduced* if  $\nu_\infty(\mathbf{f}) = \nu_\infty(\varepsilon(\mathbf{f}))$ , that is, if the leading coefficient of the  $u$ -expansion of  $\mathbf{f}$  does not vanish at  $t = \theta$ .

**Remark 27** If  $f \in \widetilde{M}_{w,m}^{\leq l}$  is a Drinfeld quasi-modular form which is not a modular form, and if there exists  $\mathbf{f} \in \mathbb{M}_{\mu,\nu,m}^\dagger$  reduced with  $f = \varepsilon(\mathbf{f})$  and  $w = \mu + \nu$ ,  $l = \nu$ , then (12) holds applying Proposition 24.

The next lemma provides a tool to construct reduced almost  $A$ -quasi-modular forms, useful in the sequel.

**Lemma 28** Let  $\mathbf{f} \in \mathbb{M}_{\mu,\nu,m}^\dagger$  be such that  $\mathbf{f} = \sum_{n \geq n_0} b_n u^n$ , with  $b_n \in \mathbb{F}_q[t, \theta]$  for all  $n$  and  $b_{n_0} \neq 0$ . Then, for all  $k > \log_q(\deg_t b_{n_0})$ , the function  $\mathbf{f}^{(k)}$  is reduced.

*Proof.* We have  $b_{n_0}^{(k)}(\theta) = b_{n_0}(\theta^{q^{-k}})^{q^k} = 0$  if and only if  $t - \theta^{1/q^k}$  divides the polynomial  $b_{n_0}(t)$  in  $K^{\text{alg.}}[t]$ . This polynomial having coefficients in  $K$ , we have  $b_{n_0}^{(k)}(\theta) = 0$  if and only if the irreducible polynomial  $t^{q^k} - \theta$  divides  $b_{n_0}(t)$ . However, this is impossible if  $k > \log_q(\deg_t b_{n_0})$ . □

## 5.4 Construction of the auxiliary forms.

We recall the  $u$ -expansion of  $\mathbf{E}$  whose existence is proved in Proposition 9:

$$\mathbf{E} = u \sum_{i \geq 0} c_i(t) v^i,$$

where  $c_0 = 1$ ,  $c_i \in \mathbb{F}_q[t, \theta]$  for all  $i > 0$  and  $v = u^{q-1}$ .

**Proposition 29** *The following properties hold.*

- (i) *Let  $\alpha, \beta, \gamma, \delta$  be non-negative integers and let us write  $\mathbf{f} = g^\alpha h^\beta \mathbf{E}^\gamma \mathbf{F}^\delta \in \mathbb{M}_{\mu, \nu, m}^\dagger$ , with  $\mu = \alpha(q-1) + \beta(q+1) + \gamma + q\delta$ ,  $\nu = \gamma + \delta$  and  $\beta + \gamma + \delta \equiv m \pmod{q-1}$ ,  $m \in \{0, \dots, q-2\}$ . Let us write*

$$\mathbf{f} = u^m \sum_{n \geq 0} a_n(t) v^n$$

*with  $a_n \in \mathbb{F}_q[t, \theta]$  (this is possible after Proposition 9 and the integrality of the coefficients of the  $u$ -expansions of  $g, h$ ). Then, for all  $n \geq 0$ ,*

$$\deg_t a_n(t) \leq \nu \log_q \max\{1, n\}.$$

- (ii) *Let  $\lambda$  be a positive real number. Let  $\mathbf{f}_1, \dots, \mathbf{f}_\sigma$  be a basis of monic monomials in  $g, h, \mathbf{E}, \mathbf{F}$  of the  $\mathcal{K}$ -vector space  $\mathbb{M}_{\mu, \nu, m}^\dagger(\mathcal{K})$ . Let  $x_1, \dots, x_\sigma$  be polynomials of  $\mathbb{F}_q[t, \theta]$  with  $\max_{0 \leq i \leq \sigma} \deg_t x_i \leq \lambda$ . Then, writing*

$$\mathbf{f} = \sum_{i=1}^{\sigma} x_i \mathbf{f}_i = u^m \sum_{n \geq 0} b_n(t) v^n$$

*with  $b_n \in \mathbb{F}_q[t, \theta]$  with  $0 \leq m \leq q-2$ , we have, for all  $n \geq 0$ :*

$$\deg_t b_n \leq \lambda + \nu \log_q \max\{1, n\}.$$

*Proof.* Since by definition  $\mathbf{F} = \mathbf{E}^{(1)}$ , we have

$$\mathbf{F} = u^q \sum_{n \geq 0} c_n^{(1)} v^{qn} = u \sum_{r \geq 0} d_r v^r,$$

where  $d_r = 0$  if  $q \nmid r-1$  and  $d_r = c_{(r-1)/q}^{(1)}$  otherwise. Now, the operator  $\tau$  leaves the degree in  $t$  invariant. Therefore, by Proposition 9  $\deg_t d_r \leq \log_q \max\{1, r/q\} \leq \log_q \max\{1, r\}$ .

Let us consider the  $u$ -expansions:

$$\begin{aligned} g &= \sum_{n \geq 0} \gamma_n v^n, & \mathbf{E} &= u \sum_{n \geq 0} c_n v^n, \\ h &= u \sum_{n \geq 0} \rho_n v^n, & \mathbf{F} &= u \sum_{n \geq 0} d_n v^n, \end{aligned}$$

with  $\gamma_n, \rho_n \in A$ ,  $c_n, d_n \in \mathbb{F}_q[t, \theta]$  for all  $n$ , we can write:

$$\mathbf{f} = u^{m'} \sum_{n \geq 0} \kappa_n v^n,$$

where  $m' = \beta + \gamma + \delta$  and for all  $n$ ,  $\kappa_n = \sum \prod_x \gamma_{i_x} \prod_y \rho_{j_y} \prod_s c_{k_s} \prod_z d_{r_z}$ , the sum being over the vectors of  $\mathbb{Z}_{\geq 0}^{\alpha+\beta+\gamma+\delta}$  of the form

$$(i_1, \dots, i_\alpha, j_1, \dots, j_\beta, k_1, \dots, k_\gamma, r_1, \dots, r_\delta)$$

whose sum of entries is  $n$ , and with the four products running over  $x = 0, \dots, \alpha$ ,  $y = 0, \dots, \beta$ ,  $s = 0, \dots, \gamma$  and  $z = 0, \dots, \delta$  respectively. Since the coefficients of the  $u$ -expansions of  $g, h$  do not depend on  $t$  and  $\gamma + \delta = \nu$ , we obtain  $\deg_t \kappa_n \leq \nu \log_q \max\{1, n\}$ .

If  $m' = m + k(q - 1)$  with  $k \geq 0$  integer, and  $0 \leq m < q - 1$ . We can write

$$\mathbf{f} = u^{m'} \sum_{n \geq 0} c'_n v^n = u^m \sum_{n \geq 0} c_n v^n,$$

where  $c_n = c'_{n-k}$ , with the assumption that  $c'_{n-k} = 0$  if the index is negative. The inequalities  $\deg_t c'_n \leq \nu \log_q \max\{1, n\}$  for  $n \geq 0$  imply that  $\deg_t c_n$  is submitted to the same bound, proving the first part of the proposition. The second part is a direct application of the first and ultrametric inequality.  $\square$

#### 5.4.1 Dimensions of spaces

**Lemma 30** *We have, for all  $m$  and  $\mu, \nu \in \mathbb{Z}$  such that  $\mu \geq (q + 1)\nu \geq 0$ ,*

$$\sigma(\mu, \nu) - \nu - 1 \leq \dim_{\mathcal{K}} \mathbb{M}_{\mu, \nu, m}^\dagger(\mathcal{K}) \leq \sigma(\mu, \nu) + \nu + 1,$$

where

$$\sigma(\mu, \nu) = \frac{(\nu + 1) \left( \mu - \frac{\nu(q+1)}{2} \right)}{q^2 - 1}.$$

Therefore, if  $\mu > \frac{\nu(q+1)}{2} + q^2 - 1$ , we have  $\dim_{\mathcal{K}} \mathbb{M}_{\mu, \nu, m}^\dagger(\mathcal{K}) > 0$ .

*Proof.* By [14, p. 33], we know that

$$\delta(k, m) := \dim_C M_{k, m} = \left\lfloor \frac{k}{q^2 - 1} \right\rfloor + \dim_C M_{k^*, m},$$

where  $k^*$  is the remainder of the euclidean division of  $k$  by  $q^2 - 1$ . In the same reference, it is also proved that  $\dim_C M_{k^*,m} = 0$  unless  $k^* \geq m(q+1)$ , case where  $\dim_C M_{k^*,m} = 1$ , so that, in all cases,  $0 \leq \dim_C M_{k^*,m} \leq 1$ .

A basis of  $\mathbb{M}_{\mu,\nu,m}^\dagger(\mathcal{K})$  is given by:

$$(\mathbf{b}_k)_{k=1,\dots,\dim \mathbb{M}_{\mu,\nu,m}^\dagger(\mathcal{K})} = (\phi_{i,s} \mathbf{h}^s \mathbf{E}^{\nu-s})_{s=0,\dots,\nu, i=1,\dots,\sigma(s)}, \quad (65)$$

with, for all  $s$ ,  $(\phi_{i,s})_{i=1,\dots,\sigma(s)}$  a basis of  $M_{\mu-s(q-1)-\nu, m-\nu}$  (hence  $\sigma(s) = \delta(\mu - s(q-1) - \nu, m - \nu)$ ). We have (taking into account the hypothesis on  $\mu$  which implies  $\mu - s(q-1) - \nu > 0$  for all  $0 \leq s \leq \nu$ ):

$$\begin{aligned} \dim \mathbb{M}_{\mu,\nu,m}^\dagger(\mathcal{K}) &= \sum_{s=0}^{\nu} \delta(\mu - \nu - s(q-1), m - \nu) \\ &= \sum_{s=0}^{\nu} \left\lfloor \frac{\mu - s(q-1) - \nu}{q^2 - 1} \right\rfloor + \dim_C M_{(\mu - \nu - s(q-1))^*, m - \nu}. \end{aligned}$$

But

$$\sum_{s=0}^{\nu} \frac{\mu - s(q-1) - \nu}{q^2 - 1} = \sigma(\mu, \nu).$$

Moreover,  $\mu > \frac{\nu(q+1)}{2} + q^2 - 1$  if and only if  $\sigma(\mu, \nu) > \nu + 1$ , from which we deduce the lemma easily.  $\square$

#### 5.4.2 Applying a variant of Siegel's Lemma

We now prove the following:

**Proposition 31** *Let  $\mu, \nu \in \mathbb{Z}_{\geq 0}$  be such that*

$$\mu \geq (q+1)\nu + 2(q^2 - 1) \quad (66)$$

*with  $\nu \geq 1$ , let  $m$  be an integer in  $\{0, \dots, q-2\}$ . There exists an integer  $r > 0$  such that*

$$r \leq 4q\mu\nu \log_q(\mu + \nu + q^2 - 1) + \nu \quad (67)$$

*and, in  $\widetilde{M}_{r,m}^{\leq \nu}$ , a quasi-modular form  $f_{\mu,\nu,m}$  such that*

$$\frac{1}{q(q+1)} \mu \nu^2 \log_q(\mu + \nu + q^2 - 1) \leq \nu_\infty(f_{\mu,\nu,m}) \leq 4q\mu\nu^2 \log_q(\mu + \nu + q^2 - 1). \quad (68)$$

We will need the following variant of Siegel's Lemma whose proof can be found, for example, in [21, Lemma 1] (see also [7]).



**Lemma 32** *Let  $U, V$  be positive integers, with  $U < V$ . Consider a system (69) of  $U$  equations with  $V$  indeterminates:*

$$\sum_{i=1}^V a_{i,j} x_i = 0, \quad (1 \leq j \leq U) \quad (69)$$

where the coefficients  $a_{i,j}$  are elements of  $K[t]$ . Let  $d$  be a non-negative integer such that  $\deg_t a_{i,j} \leq d$  for each  $(i, j)$ . Then, (69) has a non-zero solution  $(x_i)_{1 \leq i \leq V} \in (K[t])^V$  with  $\deg_t x_i \leq Ud/(V - U)$  for each  $i = 1, \dots, V$ .

*Proof of Proposition 31.* We apply Lemma 32 with the parameters  $V = \dim \mathbb{M}_{\mu, \nu, m}^{\dagger}(\mathcal{K})$ ,  $U = \lfloor V/2 \rfloor$ . We know that  $V > 0$  because of (66) and Lemma 30.

If  $\mathbf{f} = \mathbf{b}_i$  as in (65), Writing

$$\mathbf{b}_i = u^m \sum_{j \geq 0} a_{i,j} v^j, \quad a_{i,j} \in A[t] \quad (70)$$

with  $0 \leq m < q - 1$ , Proposition 29 says that for all  $i$  and for all  $j \geq 0$ ,

$$\deg_t a_{i,j} \leq \nu \log_q \max\{1, j\}. \quad (71)$$

Lemma 32 yields polynomials  $x_1, \dots, x_V \in K[t]$ , not all zero, such that if we write

$$\mathbf{f} = \sum_i x_i \mathbf{b}_i = u^m \sum_{n \geq n_0} b_n v^n, \quad 0 \leq m < q - 1 \quad (72)$$

with  $b_n \in K[t]$  for all  $n$  and  $b_{n_0} \neq 0$ , we have the following properties. The first property is the last inequality below:

$$\begin{aligned} m + (q - 1)n_0 = \nu_{\infty}(\mathbf{f}) &\geq m + (q - 1)U \\ &\geq (q - 1)(\sigma(\mu, \nu) - \nu - 1)/2 - 1 \\ &\geq \frac{(\nu + 1)(\mu - \frac{\nu(q+1)}{2} - q^2 + 1)}{2(q + 1)} - 1 \\ &\geq \frac{1}{4(q + 1)}(\nu + 1)\mu - 1, \end{aligned} \quad (73)$$

where we have applied Lemma 30 and (66). The second property is that, in (72),

$$\deg_t b_n \leq 2\nu(\log_q(\mu + \nu + q^2 - 1) + \log_q \max\{1, n\}), \quad n \geq 0, \quad (74)$$

which follows from the following inequalities, with  $d = \nu \log_q \max\{1, U\}$

$$\begin{aligned} \deg_t x_i &\leq Ud/(V - U) \\ &\leq \nu \log_q \max\{1, U\} \\ &\leq \nu \log_q((\sigma(\mu, \nu) + \nu + 1)/2) \\ &\leq \nu(\log_q(\nu + 1) + \log_q(\mu + q^2 - 1) - \log_q(q^2 - 1)) \\ &\leq 2\nu \log_q(\mu + \nu + q^2 - 1), \end{aligned}$$

and Proposition 29.

By Proposition 24, we have  $m + (q - 1)n_0 = \nu_\infty(\mathbf{f}) \leq \mu\nu$  so that  $n_0 \leq \frac{\mu\nu}{q-1}$ , where  $n_0$  is defined in (72). Hence, by (74),

$$\deg_t b_{n_0} \leq 4\nu \log_q(\mu + \nu + q^2 - 1). \quad (75)$$

Lemma 28 implies that for every integer  $k$  such that

$$k \geq \log_q(4\nu) + \log_q \log_q(\mu + \nu + q^2 - 1), \quad (76)$$

the function  $f_k := \varepsilon(\mathbf{f}^{(k)})$  satisfies  $\nu_\infty(f_k) = \nu_\infty(\mathbf{f}^{(k)}) = q^k \nu_\infty(\mathbf{f})$ . Let  $k$  be satisfying (76). We have, by (73), Proposition 24 and (63):

1.  $f_k \in \widetilde{M}_{\mu q^k + \nu, m}^{\leq \nu}$ ,
2.  $\left(\frac{(\nu+1)\mu q^k}{4(q+1)} - 1\right) \leq \nu_\infty(f_k) \leq \mu\nu q^k$ .

Let us define the function

$$\kappa(\mu, \nu) := \lfloor \log_q(4\nu) + \log_q \log_q(\mu + \nu + q^2 - 1) \rfloor + 1$$

and write:  $f_{\mu, \nu, m} := f_{\kappa(\mu, \nu)}$ . This Drinfeld quasi-modular form satisfies the properties announced in the proposition.  $\square$

## 5.5 Proof of the Theorem

Let  $f$  be a Drinfeld quasi-modular form of weight  $w$  and depth  $l$ . We can assume, without loss of generality, that  $f$ , as a polynomial in  $E, g, h$  with coefficients in  $C$ , it is an irreducible polynomial. We can also assume, by (10) and, [4, Theorem 1.4], that  $l > q$ .

Let  $W$  be a real number  $\geq 1$  and let  $\alpha$  be the function of a real variable defined, for  $\mu \geq 0$ , by  $\alpha(\mu) = \mu l \log_q(\mu + Wl + q^2 - 1)$ ; we have  $\alpha(\mu + 1) \leq 2\alpha(\mu)$ . Since (the dash ' is the derivative)  $\alpha'(\mu) \geq l \log_q(Wl + q^2 - 1) > 1$  for all  $l \geq q$  and  $\mu \geq 0$ , for all  $w \geq 0$  integer, there exist  $\mu \in \mathbb{Z}_{\geq 0}$  such that

$$\alpha(\mu) \leq w < \alpha(\mu + 1), \quad (77)$$

and we choose one of them, for example the biggest one. Let us suppose that (6) holds and, at once, set

$$\nu = Wl,$$

with

$$W = q(2 + 4(q + 1)) = 2q(3 + 2q).$$

We define  $\beta(l)$  to be the right hand side of (6), as a function of  $l \geq q$ . Condition (6) implies

$$\mu \geq \frac{\beta(l)}{2l \log_q(\mu + Wl + q^2 - 1)}.$$

Since  $\log_q(x) \leq 2x^{1/2}$  for all  $x \geq 1$  and  $q \geq 2$ , we get

$$(\mu + Wl + q^2 - 1)^{3/2} \geq \frac{\beta(l)}{4l},$$

that is,

$$\mu \geq \left(\frac{\beta(l)}{4l}\right)^{2/3} - Wl - q^2 + 1.$$

But replacing  $\beta(l)$  by its value yields  $\mu \geq (q+1)\nu + 2(q^2 - 1)$ , which is the condition (66) needed to apply Proposition 31.

Let us write  $\mathcal{L} := \log_q(\mu + \nu + q^2 - 1)$  so that  $\alpha(\mu) = \mu\mathcal{L}$ . By Proposition 31, there exists a form  $f_{\mu,\nu,m} \in \widetilde{M}_{r,m}^{\leq \nu}$  such that  $l(f_{\mu,\nu,m}) \leq \nu$  and

$$(q(q+1))^{-1}\mu\nu^2\mathcal{L} \leq \nu_{\infty}(f_{\mu,\nu,m}) \leq 4q\mu\nu^2\mathcal{L} \leq 4(q+1)\mu\nu\mathcal{L} \quad (78)$$

We have two cases.

*Case (i).* If  $f|f_{\mu,\nu,m}$ , then

$$\nu_{\infty}(f) \leq \nu_{\infty}(f_{\mu,\nu,m}) \leq 4q\mu\nu^2\mathcal{L}. \quad (79)$$

*Case (ii).* If  $f \nmid f_{\mu,\nu,m}$ , then  $\rho := \mathbf{Res}_E(f, f_{\mu,\nu,m})$  is a non-zero modular form, whose weight  $w(\rho)$  and type  $m(\rho)$  can be computed with the help of [4, Lemma 2.5] (we do not need an explicit computation of  $m(\rho)$ ):

$$\begin{aligned} w(\rho) &= w\nu + w(f_{\mu,\nu,m})l - 2l\nu \\ &\leq w\nu + 4l(q+1)\mu\nu\mathcal{L} - 2l\nu \\ &\leq \nu(w + 4(q+1)\mu\mathcal{L}) \\ &< \nu(\alpha(\mu+1) + 4(q+1)\mu\mathcal{L}) \\ &< \nu(2\alpha(\mu) + 4(q+1)\mu\mathcal{L}) \\ &< (2 + 4(q+1))\nu\mu\mathcal{L}. \end{aligned} \quad (80)$$

Let us suppose that  $\nu_{\infty}(f) > (q(q+1))^{-1}\mu\nu^2\mathcal{L}$ . Then, by Bézout identity for the resultant and (78),  $\nu_{\infty}(\rho) \geq (q(q+1))^{-1}\mu\nu^2\mathcal{L}$ . At the same time, by (10),  $\nu_{\infty}(\rho) \leq \frac{w(\rho)}{q+1}$ , yielding the inequality  $W < q(2 + 4(q+1))$  which is contradictory with the definition of  $W$ .

Therefore, in case (ii), we have that  $\nu_\infty(f) \leq 4q\mu\nu^2\mathcal{L}$ . Ultimately, we have shown that, in both cases (i), (ii),

$$\begin{aligned}\nu_\infty(f) &\leq 4q\mu\nu^2\mathcal{L} \\ &\leq 4q\mu W^2 l^2 \mathcal{L} \\ &\leq 4qW^2 lw,\end{aligned}$$

which is the estimate (7).  $\square$

We now prove the weaker, but unconditional inequality (8); again, we can suppose, without loss of generality, that  $f$  is an irreducible polynomial in  $E, g, h$ . Let  $f$  be in  $\widetilde{M}_{w,m}^{\leq l}$  be non-zero and irreducible as a polynomial in  $E, g, h$ . Let us set this time

$$\mu = BWw, \quad \nu = Wl,$$

with

$$B = \frac{3}{2}(q^2 + 1), \quad W = q(4(q + 1) + 1).$$

Then, since  $w \geq 2l$ ,

$$\begin{aligned}\mu &\geq 3(q^2 + 1)Wl \\ &\geq (q + 1)Wl + 2(q^2 + 1) \\ &\geq (q + 1)\nu + 2(q^2 + 1),\end{aligned}$$

we can apply Proposition 31 again. As before, there exists a form  $f_{\mu,\nu,m} \in \widetilde{M}_{r,m}^{\leq \nu}$  such that the inequalities (78) hold. Again, we can distinguish two cases, according with  $f$ , if divides or not  $f_{\mu,\nu,m}$ . If  $f$  divides  $f_{\mu,\nu,m}$ , we get (79).

Let us assume that  $f$  and  $f_{\mu,\nu,m}$  are coprime and form their non-vanishing resultant  $\rho := \mathbf{Res}_E(f, f_{\mu,\nu,m})$ , whose weight satisfies (80).

If  $\nu_\infty(f) > (q(q + 1))^{-1}\mu\nu^2\mathcal{L}$ , by Bézout identity for the resultant and (78),  $\nu_\infty(\rho) \geq (q(q + 1))^{-1}\mu\nu^2\mathcal{L}$ . At the same time, by (10), we find the inequality  $W < q(4(q + 1) + 1)$  which is contradictory with the definition of  $W$ .

Hence, in all cases,

$$\begin{aligned}\nu_\infty(f) &\leq 4q\mu\nu^2\mathcal{L} \\ &\leq 4q\mu W^2 l^2 \mathcal{L} \\ &\leq 4qBW^3 l^2 w \log_q(\mu + \nu + q^2 - 1),\end{aligned}$$

which yields the estimate (8).  $\square$

**Remark 33** The dependence on  $l$  in condition (6) can be relaxed, adding conditions on  $q$ . For all  $\epsilon > 0$  there exists a constant  $c > 0$  such that for all  $q > c$ , assuming that  $w \gg_\epsilon l^{2+\epsilon}$ , then, the inequality (7) holds. We do not report the proof of this fact here.

## 6 Link with extremal quasi-modular forms

Here, we would like to describe some links between the present work and the joint work [4]. In [4], we have introduced the sequence of Drinfeld quasi-modular forms  $(x_k)_{k \geq 0}$  with  $x_k \in \widetilde{M}_{q^{k+1},1}^{\leq 1} \setminus M$ , defined by  $x_0 = -E$ ,  $x_1 = -Eg - h$  and by the recursion formula

$$x_k = x_{k-1}g^{q^{k-1}} - [k-1]x_{k-2}\Delta^{q^{k-2}}, \quad k \geq 2,$$

where we recall that  $\Delta = -h^{q-1}$ . In [4, Theorem 1.2], we have showed that for all  $k \geq 0$ ,  $x_k$  is *extremal*, in the sense that  $\nu_\infty(x_k)$  is the biggest possible value for  $\nu_\infty(f)$ , if  $f \in \widetilde{M}_{q^{k+1},1}^{\leq 1} \setminus \{0\}$ . We also computed the order of vanishing:  $\nu_\infty(x_k) = q^k$  for all  $k$ .

**Proposition 34** *For  $k \geq 0$ , we have*

$$\mathbf{E}^{(k)}(z, \theta) = (-1)^{k+1} \frac{x_k(z)}{[1][2] \cdots [k]}, \quad (81)$$

where the empty product is 1 by definition.

*Proof.* By (47) and by the limits (45) and (46) of  $\mathbf{J}_\gamma, \mathbf{L}_\gamma$  that we have computed earlier, for  $k \geq 0$ , the function  $\phi_k(z) := \mathbf{E}^{(k)}(z, \theta)$ , is a well defined Drinfeld quasi-modular form in the space  $\widetilde{M}_{q^{k+1},1}^{\leq 1}$ . By (54),

$$\mathbf{E}^{(k)} = u^{q^k} + \dots$$

Hence,  $\phi_k$  is non-vanishing, and by [4, Theorem 1.2] normalised, extremal, therefore proportional to  $x_k$  for all  $k$ . By [4, Proposition 2.3],

$$x_k = (-1)^{k+1} L_k u^{q^k} + \dots,$$

where  $L_k = [k][k-1] \cdots [1]$  if  $k > 0$  and  $L_0 = 1$ . This proves the proposition.  $\square$

Combining with Proposition 9, we also obtain the following corollary, which gives a proof of the property claimed in [4, Remark 2.4].

**Corollary 35** *Define, for all  $k \geq 0$ ,  $E_k := \phi_k$ . Then,  $E_k$  is normalised in  $A[[u^{q^k}]]$ .*

Another interesting connection with [4] occurs with the sequence  $(\xi_k)_{k \geq 0}$  introduced in [4, Identity (8)]. Let us define:

$$\mathbf{G} := \det \begin{pmatrix} \mathbf{E}^{(1)} & \mathbf{E} \\ \mathbf{E}^q & (\mathbf{E}^{(-1)})^q \end{pmatrix} = \det \begin{pmatrix} \tau^1 \chi^0 & \tau^0 \chi^0 \\ \tau^1 \chi^1 & \tau^0 \chi^1 \end{pmatrix} \mathbf{E},$$

where  $\chi$  is Mahler's map, defined in the introduction. It is straightforward to see that  $\mathbf{G} \in \mathbb{F}_q[t, \theta][[u]]$ . From (54) and a computation we deduce

$$\mathbf{G} = (t - t^q)u^{q^2+1}(1 - v^{q^2-1} - [1]v^{q^2} + \dots).$$

For all  $k \geq 1$ ,  $\mathbf{G}^{(k)} \in \mathbb{F}_q[t, \theta][g, h, \mathbf{E}, \mathbf{E}^{(1)}]$  is reduced and  $\nu_\infty(\mathbf{G}^{(k)}) = (q^2 + 1)q^k$ ; moreover,  $\mathbf{G}^{(k)} \in \mathbb{M}_{(q+1)q^k, q+1, 2}^\dagger$  for all  $k > 0$ . It even seems that the normalisation of this formal series is defined over  $\mathbb{F}_q[t, \theta]$ .

By [4, Theorem 1.3], for all  $k \geq 1$  and  $q \geq 3$ ,  $\varepsilon(\mathbf{G}^{(k)})$  is proportional to  $\xi_k$ . This property seems to hold also for  $k = 0$  by numerical inspection for some small values of  $q$ , but we do not know if  $\mathbf{G}$  itself belongs to  $\mathbb{M}^\dagger$ . It is plausible that it is at least an almost  $A$ -quasi-modular form.

A result of Stiller [28] asserts that, if  $b$  is a non-constant meromorphic modular function for  $\mathbf{SL}_2(\mathbb{Z})$  which generates the field of modular functions and  $f$  is a meromorphic modular form of weight  $k$  for  $\mathbf{SL}_2(\mathbb{Z})$ , then  $f$ , as a function of  $b$ , satisfies a linear differential equation with rational functions of  $b$  as coefficients.

Analogously, It is not difficult to show that every non-zero element

$$\mathbf{f} \in \mathbb{F}_q[t, \theta][g, h, \mathbf{E}, \mathbf{E}^{(1)}]$$

satisfies a non-trivial linear  $\tau$ -difference equation of order  $\leq \nu(\mathbf{f})$ , with coefficients in  $C(t)[g, h]$  (for example, (40) for  $\mathbf{E}$ ). We were unable to explicitly determine such an equation for  $\mathbf{G}$ , leaving open the problem to prove or disprove that  $\mathbf{G}/(t - t^q)$ , normalised, is itself in  $\mathbb{F}_q[t, \theta][[u]]$ .

## 7 Observations in the classical theory

Let  $\mathcal{H}$  be the complex upper-half plane, let  $\Lambda$  be the lattice  $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$  of  $\mathbb{C}$ , with the basis  $\omega_1 = z \in \mathcal{H}, \omega_2 = 1$ . Let us denote by  $\eta_1(z), \eta_2(z)$  the quasi-periods  $\eta(\omega_1)$  and  $\eta(\omega_2)$  respectively, with  $\eta : \Lambda \rightarrow \mathbb{C}$  the quasi-period map associated to the Weierstraß  $\zeta$ -function for  $\Lambda$ . It is well known that  $\eta_2(z) = -\pi^2 E_2(z)/3$ , and that  $\eta_1(z)$  is related to  $\eta_2(z)$  by Legendre's formula

$$z\eta_2(z) - \eta_1(z) = 2\pi i. \tag{82}$$

Let  $\iota : \mathcal{H} \rightarrow \mathbb{P}_1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$  be the map defined by  $\iota(z) = \eta_1(z)/\eta_2(z)$ . From the relation (82) and the basic properties of the quasi-period map, we obtain the

following identities, for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z})$ :

$$\eta_2(\gamma(z)) = (cz + d)^2 \left( \eta_2(z) - 2\pi i \frac{c}{cz + d} \right) \quad (83)$$

$$= (cz + d)(c\iota(z) + d)\eta_2(z) \quad (84)$$

$$= (c\iota(z) + d)^2 \left( \eta_2(z) + 2\pi i \frac{c}{c\iota(z) + d} \right),$$

where the second and the third functional equation hold for  $z$  which is not a zero of  $E_2$ . We have, for all  $\gamma \in \mathbf{SL}_2(\mathbb{Z})$ ,  $\iota(\gamma(z)) = \gamma(\iota(z))$ . Therefore, the map  $\iota : \mathbf{SL}_2(\mathbb{Z}) \times \mathcal{H} \rightarrow \mathbb{P}_1(\mathbb{C})$  satisfies the cocycle condition of a factor of automorphy. Although there is no graduation by  $\mathbb{Z}^2$  here, at least, the map  $\eta_2$  can be elusively considered at once as of “bi-weights  $(2, 0)$ ,  $(1, 1)$  and  $(0, 2)$ ”.

We also notice, from elementary computations, that  $\iota$  sends a certain domain  $\mathcal{D}_\infty \subset \mathcal{H}$  contained in the half-plane  $\Im(z) > 1.91$ , invariant by translations in  $\mathbf{SL}_2(\mathbb{Z})$ , biholomorphically on  $\mathcal{H}$ . The boundary of  $\mathcal{D}_\infty$  is a real curve of  $\mathcal{H}$  which is close, homotopically equivalent, although not equal to a horocycle with center at  $\infty$  (when one draws it, it appears very much as an horizontal line, with small, periodic nutations). The “compactified” boundary  $\partial_\infty$  of  $\mathcal{D}_\infty$  (union of the boundary and the cusp at infinity) contains all the  $z$ 's such that the lattice  $\mathbb{Z} + z\mathbb{Z}$  has its quasi-periods  $\mathbb{R}$ -linearly dependent and in this locus sits exactly one vanishing point for  $\eta_2$ , which is the point at infinity itself. For all  $\gamma \in \mathbf{SL}_2(\mathbb{Z})$ , the set  $\gamma(\partial_\infty) \setminus \gamma(\infty)$  contains exactly one zero of  $\eta_2$  and this shows by the way that every non-empty vertical strip  $a < \Re(z) < b$ ,  $a, b \in \mathbb{R}$ , contains a zero of  $E_2$  extending a result of Heins in [17]. We deduce that  $\iota$  is surjective. Therefore, for  $\gamma$  as above, the map  $(\gamma, z) \mapsto c\iota(z) + d$  is not a factor of automorphy over  $\mathcal{H}$  (a factor of automorphy is everywhere well defined and never vanishes), although it is one on the domain

$$\bigcup_{\gamma \in \mathbf{SL}_2(\mathbb{Z})_\infty \setminus \mathbf{SL}_2(\mathbb{Z})} \gamma(\mathcal{D}_\infty),$$

in which points  $z$  in the boundary such that the quasi-periods of  $\mathbb{Z} + z\mathbb{Z}$  are  $\mathbb{Q}$ -linearly dependent play the role of “cusps” (the union is disjoint, over cosets representatives of the action of the subgroup  $\mathbf{SL}_2(\mathbb{Z})_\infty$  of translations of  $\mathbf{SL}_2(\mathbb{Z})$ )<sup>(14)</sup>.

Now, let us return to our function  $\mathbf{E}$ , which is a deformation of  $E$ . We have

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<sup>14</sup>Zeroes of  $E_2$  have some connection with the vanishing locus of Bergman's kernel and equilibrium points of Green's functions for annuli as pointed out by Falliero and Sebbar in [10].

already noticed the two equalities, for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  this time in  $\Gamma$ :

$$\mathbf{E}(\gamma(z)) = \det(\gamma)^{-1}(cz + d)(\bar{c}\xi + \bar{d}) \left( \mathbf{E}(z) + \frac{1}{\tilde{\pi}(t - \theta)} \mathbf{L}_\gamma \right) \quad (85)$$

$$= \det(\gamma)^{-1}(cz + d)(\bar{c}\xi^{(1)} + \bar{d})\mathbf{E}(z). \quad (86)$$

Replacing  $t$  by  $\theta$  in these identities yields the analog of (83) and (84) in the Drinfeldian framework, which legitimate the following loosely question.

**Question.** *Do there exist a deformation of  $\eta_2$  and a suitable difference operator allowing to furnish a proper analog of identities (85) and (86) in the classical framework?*

An answer could come from  $q$ -difference operators. Linear  $q$ -difference equations seem to be legitimate analogues in zero characteristic of linear  $\tau$ -difference equations. In this direction, Di Vizio has recently studied in [9] the notion of  $G_q$ -function,  $q$ -Gevrey function etc. It is perhaps natural to investigate the analogues (if any) of the functions  $\mathbf{E}$  and allied, in her theory.

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